

ON THE CHEMICAL COMPOSITION OF THE SUN FROM ITS INTERNAL CONSTITUTION

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ABSTRACT

The variation in the guillotine factor having been taken into consideration, the chemical composition of the sun has been determined on the assumption that it consists of hydrogen, helium, and some combination of oxygen and Russell mixture. The abundances of hydrogen and helium as functions of the "oxygen" abundance have been determined.

I. THE GUILLOTINE FACTORS

1. *Introduction.*—By using both the mass-luminosity relation and the energy-generation equation, M. Schwarzschild¹ was able to determine the hydrogen and the helium content of the sun on the assumption that the rest of the composition consists of Russell mixture. Making the additional assumption that the chemical composition of the sun is uniform, Schwarzschild found a hydrogen-metal ratio which appears incompatible with the analysis of stellar atmospheres.

By the introduction of a third parameter this discrepancy between theory and observations might be partially removed. The additional parameter which may be used for this purpose is the abundance of elements of the oxygen group (*C, N, O, Fe, Ne*). The importance of such elements arises from the fact that at a temperature of 20×10^6 K these atoms are beginning to be stripped to their bare nuclei, and their contribution to the opacity will decrease very rapidly for increasing temperatures. Also, the analysis of stellar atmospheres would appear to indicate that the abundances of these elements are underestimated in the Russell mixture.

In order to carry out an investigation of such stellar models, it was necessary, first of all, to compute opacity tables for various combinations, by weight, of Russell mixture and oxygen.

2. *The method.*—The calculations of opacity were carried out according to the theory and notation given by B. Strömgren.² We then have

$$\chi_n^{(z)} = \frac{2\pi^2 e^4 m_e Z^2}{n^2 h^2}, \quad (1)$$

$$D_n = D_{i.c.} \frac{2 \chi_n}{n kT} \frac{e^{\chi_n/kT}}{1 + \exp\left(\frac{\chi_n}{kT} - \frac{\psi}{kT}\right)}, \quad (2)$$

$$D(x) = D(x_i, x_{i+1}), \quad (3)$$

$$\tau(x_i, x_{i+1}) = \frac{\phi(x_i) - \phi(x_{i+1})}{D(x_i, x_{i+1})}, \quad (4)$$

and

$$\tau = \sum \tau(x_i, x_{i+1}). \quad (5)$$

The results of these calculations are given in Table 1 for various combinations of oxygen and Russell mixture. Some of these tables are incomplete, since only the values which were actually needed were computed.

¹ *Ap. J.*, **104**, 203, 1946.

² *Zs. f. Ap.*, **4**, 118, 1932.

In order to determine the guillotine factors for any combination of oxygen and Russell mixture, it was necessary to compute tables for oxygen and Russell mixture separately (Tables 2 and 3); then, by a suitable combination of these two tables, the guillotine factor for any combination could be computed. As an example, we have reproduced here sample tables for a combination of 60 per cent oxygen and 40 per cent Russell mixture for a temperature of 20×10^6 K. Table 4 was calculated by this method. It was found that an accuracy in the third decimal place in the D_n 's could be reached; this is sufficient for our purposes.

It is evident from Table 5 that the absorption edges for this particular combination of oxygen and Russell mixture are given by multiplying all the D_n 's for Russell mixture by 0.5 and adding 0.5 times the D_n 's for pure oxygen.

II. THE STELLAR MODELS

3. *Introduction.*—We can derive some knowledge regarding the relative abundances of H , He , O , and Russell mixture if we assume that, according to the conditions found in the sun's atmosphere, the Russell mixture is replaced by some combination of oxygen and Russell mixture. The assumptions made for these models are as follows:

1. The composition (and consequently the molecular weight, μ) is constant throughout the star.
2. The radiation pressure is negligible.
3. The star consists of a convective core and an envelope in radiative equilibrium.
4. The entire energy generation from nuclear sources occurs within the convective core.

Since all the models are computed in exactly the same manner, we shall give the complete theory for only one model, namely, that for which the heavy elements are represented by a combination of 60 per cent oxygen and 40 per cent Russell mixture.

4. *Equations of the model for the outside.*—We have the usual envelope equations

$$\frac{dP}{dr} = -\frac{GM(r)}{r^2} \rho, \quad (6)$$

$$\frac{dM(r)}{dr} = 4\pi\rho r^2, \quad (7)$$

$$\frac{d}{dr} \left(\frac{1}{3} aT^4 \right) = -\frac{L}{4\pi cr^2} \frac{\kappa_0 \rho^2 T^{-3.5}}{\tau}, \quad (8)$$

where the various symbols have their usual meaning and τ is the guillotine factor, which varies throughout the star. In order to be able to proceed with the problem, we must determine τ as a function of the density and temperature; but, since we have no knowledge of this distribution until the model has been computed, we must make a guess. In so far as there is not a great deal of difference in the guillotine factors as we pass along the sequence of tables that have been computed, it was decided to use the density-temperature distribution as it had been computed for another model. For the 60 per cent oxygen and 40 per cent Russell mixture model we used the temperature-density distribution as computed by M. Schwarzschild.³ This method proved successful. We then have ρ , T throughout the star, and also the hydrogen content X , so that we can compute $\log_e G(T)/N_e$. Interpolation in the tables then yields τ .

A graphical method was used to determine the variation in τ . The best two lines (see Fig. 1) were drawn through a plot of $\log \tau$ against $\log \rho$, thus giving two forms for the guillotine factor, one for the outside and one for the inside. For the model under consideration we have, outside,

$$\tau = 1.7266\rho^{-0.0126}, \quad \log \rho \leq +0.0261, \quad (9)$$

inside,

$$\tau = 1.6757\rho^{+0.4849}, \quad \log \rho \geq +0.0261. \quad (10)$$

³ *Op. cit.*, p. 205.

TABLE 1-Continued
 95% Oxygen 5% Russell Mixture 90% Oxygen 10% Russell Mixture 80% Oxygen 20% Russell Mixture

T _x 10 ⁶	95% Oxygen 5% Russell Mixture					90% Oxygen 10% Russell Mixture					80% Oxygen 20% Russell Mixture								
	2	3	4	5	6	∞	2	3	4	5	6	∞	2	3	4	5	6	∞	
0.525				0.595	0.565	0.548				0.521	0.494	0.474				0.425	0.391	0.364	0.276
0.714			0.442	0.441	0.417	0.384	0.658	0.501	0.413	0.332	0.281	0.217	0.349	0.606	0.423	0.274	0.255	0.235	0.279
0.933	0.687	0.550	0.452	0.371	0.326	0.260	0.814	0.663	0.501	0.380	0.292	0.176	1.356	0.732	0.546	0.405	0.308	0.210	0.294
1.0	0.862	0.737	0.641	0.522	0.483	0.483	0.950	0.812	0.663	0.543	0.456	0.352	1.363	0.850	0.678	0.549	0.467	0.389	0.364
1.25	1.011	0.999	0.914	0.752	0.683	0.683	0.980	0.844	0.740	0.671	0.636	0.636	1.374	0.877	0.708	0.583	0.504	0.430	0.360
1.67	1.043	0.934	0.851	0.790	0.757	0.757	1.058	0.928	0.827	0.759	0.694	0.694	1.402	0.945	0.790	0.671	0.591	0.528	0.425
2.0	1.128	1.024	0.939	0.874	0.799	0.799	1.138	1.009	0.904	0.812	0.764	0.764	1.412	1.024	0.872	0.756	0.680	0.610	0.326
2.5	1.216	1.115	1.019	0.944	0.860	0.860	1.192	1.066	0.947	0.869	0.797	0.797	1.423	1.075	0.924	0.804	0.727	0.659	0.286
3.0	1.279	1.169	1.067	0.984	0.894	0.894	1.239	1.098	0.977	0.895	0.822	0.822	1.436	1.120	0.963	0.838	0.758	0.690	0.217
4.0	1.364	1.259	1.124	1.041	0.964	0.964	1.265	1.123	0.992	0.911	0.843	0.843	1.449	1.142	0.981	0.855	0.777	0.713	0.193
5.0	1.393	1.261	1.146	1.068	1.000	1.000	1.287	1.134	1.009	0.932	0.869	0.869	1.464	1.155	0.997	0.872	0.797	0.739	0.161
6.0	1.404	1.267	1.153	1.078	1.014	1.014	1.464	1.294	1.137	1.014	0.939	0.878	1.349	1.166	0.999	0.876	0.803	0.748	0.193
6.25	1.413	1.274	1.159	1.086	1.025	1.025	1.475	1.301	1.142	1.019	0.945	0.888	1.356	1.170	1.003	0.880	0.810	0.757	0.204
7.0	1.451	1.285	1.169	1.098	1.041	1.041	1.491	1.312	1.150	1.028	0.958	0.905	1.374	1.176	1.007	0.888	0.822	0.774	0.270
8.0	1.445	1.293	1.173	1.102	1.047	1.047	1.506	1.320	1.154	1.032	0.964	0.915	1.383	1.179	1.009	0.890	0.832	0.790	0.292
9.0	1.460	1.301	1.178	1.105	1.050	1.050	1.521	1.330	1.158	1.035	0.968	0.918	1.391	1.182	1.007	0.891	0.832	0.787	0.304
10.0	1.470	1.308	1.181	1.108	1.052	1.052	1.531	1.337	1.162	1.037	0.970	0.921	1.399	1.185	1.008	0.892	0.832	0.790	0.314
11.0	1.475	1.311	1.183	1.108	1.052	1.052	1.536	1.340	1.163	1.038	0.970	0.921	1.402	1.187	1.008	0.891	0.831	0.790	0.326
12.0	1.490	1.322	1.191	1.116	1.058	1.058	1.550	1.350	1.170	1.043	0.974	0.925	1.412	1.192	1.010	0.892	0.832	0.791	0.332
13.0	1.507	1.338	1.206	1.130	1.073	1.073	1.567	1.365	1.181	1.053	0.984	0.934	1.423	1.202	1.014	0.894	0.835	0.792	0.346
14.0	1.524	1.355	1.224	1.150	1.095	1.095	1.581	1.378	1.194	1.065	0.997	0.951	1.436	1.209	1.021	0.901	0.841	0.800	0.352
15.0	1.543	1.376	1.250	1.180	1.127	1.127	1.597	1.395	1.211	1.084	1.017	0.970	1.449	1.221	1.033	0.912	0.853	0.812	0.360
16.0	1.562	1.401	1.280	1.214	1.166	1.166	1.613	1.413	1.232	1.109	1.045	1.000	1.464	1.236	1.048	0.929	0.870	0.830	0.364
17.0	1.583	1.428	1.315	1.254	1.211	1.211	1.630	1.433	1.257	1.139	1.079	1.037	1.479	1.252	1.080	0.951	0.895	0.857	0.370
18.0	1.604	1.456	1.352	1.297	1.259	1.259	1.646	1.453	1.284	1.173	1.117	1.078	1.495	1.271	1.090	0.978	0.924	0.888	0.376
19.0	1.625	1.486	1.390	1.342	1.308	1.308	1.663	1.475	1.313	1.175	1.158	1.124	1.511	1.291	1.116	1.009	0.958	0.925	0.380
20.0	1.646	1.516	1.429	1.386	1.357	1.357	1.679	1.497	1.344	1.248	1.201	1.170	1.528	1.313	1.144	1.044	0.996	0.966	0.386
21.0	1.668	1.575	1.505	1.471	1.449	1.449	1.711	1.543	1.407	1.526	1.288	1.253	1.562	1.359	1.206	1.118	1.078	1.052	0.393
22.0	1.728	1.632	1.576	1.549	1.532	1.532	1.742	1.588	1.470	1.403	1.372	1.352	1.596	1.407	1.270	1.195	1.162	1.140	0.400

TABLE 1-Continued
 70% Oxygen 30% Russell Mixture 60% Oxygen 40% Russell Mixture 50% Oxygen 50% Russell Mixture

Tx10 ⁻⁶	2	3	4	5	6	∞	2	3	4	5	6	∞	2	3	4	5	6	∞		
0.625					0.272	0.209						0.272	0.212						0.275	0.216
0.714					0.266	0.194						0.262	0.193						0.263	0.198
0.833					0.272	0.192						0.262	0.190						0.258	0.200
1.0				0.412	0.322	0.236						0.297	0.220						0.282	0.210
1.25				0.448	0.306	0.237						0.274	0.204						0.252	0.188
1.67				0.403	0.353	0.318						0.288	0.267						0.270	0.230
2.0				0.364	0.325	0.293						0.320	0.243						0.244	0.205
2.5			0.369	0.282	0.242						0.250	0.164							0.190	0.138
3.0			0.338	0.235	0.177						0.316	0.100							0.175	0.131
4.0			0.374	0.248	0.143		0.543				0.320	0.149							0.190	0.131
5.0			0.474	0.325			0.632				0.423	0.093							0.175	0.131
6.0			0.780	0.456			0.728				0.531	0.093							0.175	0.131
6.25							0.751				0.558	0.236							0.175	0.131
7.0							0.814				0.631	0.270							0.175	0.131
8.0		0.091	0.972				0.882				0.710	0.450							0.175	0.131
9.0		1.153	0.944				0.927				0.759	0.502							0.175	0.131
10.0		1.198					0.969				0.814	0.556							0.175	0.131
11.0		1.233					0.988				0.870	0.594							0.175	0.131
12.0		1.254					1.003				0.927	0.577							0.175	0.131
12.5		1.273					1.006				0.928	0.584							0.175	0.131
13.0							1.009				0.931	0.592							0.175	0.131
14.0							1.012				0.933	0.605							0.175	0.131
15.0							1.011				0.933	0.614							0.175	0.131
16.0							1.011				0.933	0.618							0.175	0.131
16.7							1.012				0.930	0.621							0.175	0.131
17.0							1.012				0.930	0.621							0.175	0.131
18.0							1.013				0.929	0.621							0.175	0.131
19.0							1.022				0.931	0.622							0.175	0.131
20.0							1.023				0.935	0.627							0.175	0.131
21.0							1.032				0.943	0.636							0.175	0.131
22.0							1.043				0.954	0.650							0.175	0.131
23.0							1.057				0.981	0.650							0.175	0.131
24.0							1.073				0.988	0.669							0.175	0.131
25.0							1.092				0.911	0.695							0.175	0.131
26.0							1.112				0.937	0.725							0.175	0.131
28.0							1.157				0.995	0.839							0.175	0.131
30.0							1.205				1.058	0.945							0.175	0.131

TABLE 1-Continued

Tx10 ⁻⁶	25% Oxygen 75% Russell Mixture						100% Russell Mixture					
	2	3	4	5	6	∞	2	3	4	5	6	∞
0.625					0.293	0.236					0.320	0.263
0.714					0.274	0.216					0.296	0.239
0.833					0.264	0.204					0.279	0.226
1.0					0.270	0.208					0.273	0.217
1.25					0.213	0.165					0.246	0.162
1.67					0.214	0.173					0.188	0.144
2.0				0.223	0.189	0.143				0.218	0.162	0.109
2.5				0.194	0.140					0.184	0.123	0.074
3.0			0.284	0.166	0.104					0.160	0.097	0.000
4.0			0.282	0.146	0.072		0.476	0.269	0.138	0.069	0.013	
5.0		0.542	0.326	0.180			0.508	0.293	0.154	0.115	0.014	
6.0		0.613	0.404	0.289			0.564	0.316	0.219	0.142	0.080	
6.25		0.631	0.426				0.580	0.374	0.239	0.163	0.102	
7.0	0.939	0.684	0.490				0.628	0.431	0.302	0.229	0.170	
8.0	0.989	0.745	0.562				0.685	0.498	0.373	0.302	0.245	
9.0	1.026	0.784					0.722	0.542	0.419	0.349	0.295	
10.0	1.054	0.825					0.761	0.578	0.452	0.381	0.328	
11.0	1.069	0.841					0.777	0.593	0.467	0.399	0.348	
12.0	1.083	0.855					1.023	0.790	0.605	0.481	0.415	0.368
12.5	1.086	0.857					1.026	0.791	0.606	0.483	0.419	0.373
13.0	1.088						1.027	0.792	0.608	0.486	0.424	0.379
14.0							1.029	0.793	0.609	0.491	0.431	0.390
15.0							1.029	0.790	0.606	0.492	0.435	0.396
16.0							1.030	0.788	0.604	0.492	0.438	0.400
16.7							1.030	0.786	0.603	0.492	0.438	0.402
17.0							1.031	0.786	0.602	0.491	0.437	0.401
18.0							1.032	0.785	0.599	0.490	0.437	0.402
19.0							1.038	0.794	0.600	0.491	0.440	0.404
20.0							1.042	0.789	0.601	0.492	0.441	0.406
21.0							1.050	0.794	0.606	0.497	0.446	0.412
22.0							1.060	0.803	0.615	0.506	0.455	0.422
23.0							1.071	0.814	0.638	0.520	0.470	0.438
24.0							1.084	0.828	0.642	0.537	0.489	0.458
25.0							1.098	0.844	0.662	0.560	0.513	0.483
26.0							1.114	0.862	0.685	0.586	0.542	0.513
28.0							1.147	0.905	0.738	0.648	0.608	0.583
30.0							1.182	0.952	0.798	0.718	0.684	0.662

TABLE 2
100% Oxygen

x \ ψ	D _n						φ(x)	Δ
	2kT	3kT	4kT	5kT	6kT	∞ kT		
0.0315	0.0142	0.0154	0.0159	0.0161	0.0162	0.0162
0.0553	0.0341	0.0374	0.0383	0.0387	0.0389	0.0390
0.1253	0.1219	0.1331	0.1377	0.1395	0.1402	0.1406
0.4998	1.3473	1.5228	1.5997	1.6298	1.6411	1.6477	196.5
∞							196.5	
D(x)	2.5185	2.7087	2.7916	2.8241	2.8365	2.8435		
τ	78.05	72.56	70.39	69.58	69.28	69.10		

TABLE 3
Russell Mixture

x	D _n						D(x)			
	2kT	3kT	4kT	5kT	6kT	∞kT	ψ(x)	Δ	ψ=2kT	τ(x _i , x _i)
0.0315	0.0047	0.0051	0.0053	0.0054	0.0054	0.0054				
0.0553	0.0114	0.0123	0.0127	0.0129	0.0130	0.0130				
0.0623	0.0072	0.0079	0.0082	0.0082	0.0083	0.0082				
0.0938	0.0037	0.0041	0.0042	0.0042	0.0043	0.0042				
0.1113	0.0180	0.0196	0.0203	0.0206	0.0206	0.0207				
0.1253	0.0410	0.0448	0.0464	0.0470	0.0472	0.0473				
0.1666	0.0094	0.0103	0.0107	0.0108	0.0109	0.0109				
0.2001	0.0087	0.0096	0.0100	0.0101	0.0102	0.0101				
0.2499	0.0683	0.0754	0.0784	0.0795	0.0800	0.0802				
0.3122	0.0450	0.0499	0.0520	0.0528	0.0532	0.0533				
0.3332	0.0218	0.0242	0.0252	0.0256	0.0258	0.0259				
0.3753	0.0380	0.0424	0.0443	0.0450	0.0453	0.0455	0.0002	0.0001	1.277
0.4998	0.4490	0.5076	0.5329	0.5433	0.5469	0.5492	0.0003	0.0012	1.726	0.001
0.5558	0.1306	0.1486	0.1565	0.1596	0.1608	0.1614	0.0015	0.0037	1.857	0.002
0.7498	0.1029	0.1198	0.1274	0.1305	0.1317	0.1324	0.0052	0.0055	1.960	0.003
1.000	0.9935	1.197	1.295	1.335	1.350	1.359	0.0107	0.0394	2.953	0.013
1.250	0.7408	0.9292	1.025	1.066	1.082	1.091	0.0501	0.0394	3.694	0.011
1.500	0.6971	0.9155	1.034	1.087	1.108	1.120	0.0895	2.958	4.391	0.674
3.000	2.703	5.026	7.348	8.857	9.541	10.05	3.048	24.82	7.094	3.499
5.000	17.60	44.22	99.80	185.5	271.2	371.0	27.87	168.6	24.69	6.829
∞							196.5			
τ										11.032

TABLE 3-Continued

x	D(x)									
	ψ=3kT	τ _i	ψ=4kT	τ _i	ψ=5kT	τ _i	ψ=6kT	τ _i	ψ=∞kT	τ _i
0.3753	1.306	1.318	1.322	1.324	1.325
0.4998	1.813	0.001	1.851	0.001	1.865	0.001	1.871	0.001	1.874	0.001
0.5558	1.962	0.002	2.007	0.002	2.025	0.002	2.032	0.002	2.035	0.002
0.7498	2.082	0.003	2.134	0.002	2.156	0.002	2.164	0.002	2.168	0.002
1.000	3.279	0.012	3.430	0.011	3.490	0.011	3.514	0.011	3.527	0.011
1.250	4.208	0.009	4.454	0.009	4.556	0.009	4.596	0.008	4.618	0.008
1.500	5.123	0.577	5.488	0.539	5.644	0.524	5.704	0.518	5.738	0.516
3.000	10.15	2.445	12.84	1.933	14.50	1.712	15.24	1.629	15.79	1.572
5.000	54.37	3.101	112.6	1.497	200.0	0.843	286.4	0.589	386.8	0.436
τ		6.150		3.994		3.104		2.760		2.548

TABLE 4
60% Oxygen 40% Russell Mixture

x	D(x)											
	ψ=2kT	τ _i	ψ=3kT	τ _i	ψ=4kT	τ _i	ψ=5kT	τ _i	ψ=6kT	τ _i	ψ=∞kT	τ _i
0.4998	2.122	2.260	2.322	2.344	2.353	2.359
0.5558	2.197	0.002	2.335	0.002	2.400	0.002	2.424	0.002	2.434	0.002	2.440	0.002
0.7498	2.239	0.002	2.395	0.002	2.463	0.002	2.490	0.002	2.500	0.002	2.506	0.002
1.000	2.736	0.014	2.994	0.013	3.110	0.013	3.158	0.012	3.175	0.012	3.185	0.012
1.250	3.106	0.013	3.458	0.011	3.623	0.011	3.690	0.011	3.716	0.011	3.731	0.010
1.500	3.454	0.856	3.916	0.755	4.140	0.714	4.234	0.699	4.270	0.693	4.291	0.689
3.000	4.806	5.164	6.429	3.861	7.814	3.176	8.662	2.865	9.040	2.746	9.316	2.664
5.000	13.60	12.40	28.54	5.907	57.71	2.922	101.4	1.663	144.5	1.166	194.8	0.866
τ		18.448		10.551		6.840		5.254		4.632		4.245

The formulae for the other models are given in Table 6.

The envelope equations can now be reduced to a nondimensional form by the transformations

$$r = R\xi, \quad \rho = \rho_0\sigma, \quad T = T_0\theta, \quad M(r) = M\psi, \quad (11)$$

where

$$\rho_0 = \frac{M}{4\pi R^3}, \quad T_0 = \frac{\mu H}{k} \frac{GM}{r}. \quad (12)$$

Then we have

$$\frac{d(\sigma\theta)}{d\xi} = -\frac{\psi\sigma}{\xi}; \quad \frac{d\psi}{d\xi} = \sigma\xi^2, \quad (13)$$

$$\frac{d\theta}{d\xi} = -\frac{Q\sigma^{2.0126}}{\theta^{6.5}\xi^2}, \quad (14)$$

where

$$Q = \frac{3\kappa_0 L}{16\pi a c} \frac{\rho_0^{2.0126}}{1.7266 T_0^{7.5} R}. \quad (15)$$

TABLE 5
60 PER CENT OXYGEN AND 40 PER CENT
RUSSELL MIXTURE

Element	Relative Weight	Relative Contribution to Opacity $\propto \frac{Z^2}{A} \times$ Relative Weight
O.....	24/40	0.5000
O.....	8/40	.1667
Na, Mg.....	4/40	.1250
Si.....	1/40	.0416
K, Ca.....	1/40	.0417
Fe.....	2/40	0.1250

5. *Construction of the model.*—Near $\xi = 1$ equations (13) and (14) can be approximately integrated; for, in the immediate neighborhood of the boundary of the star, $M(r)$ remains constant to a high degree of accuracy. Accordingly, letting ψ have its boundary value, which we shall denote by ψ_1 , equations (13) can be re-written in the form

$$\frac{d(\sigma\theta)}{d(\theta^4)} = \frac{\psi_1\theta^{3.5}}{4Q\sigma^{1.0126}}; \quad \frac{d(\theta^4)}{d\xi} = -\frac{4Q\sigma^{2.0126}}{\xi^2\theta^{3.5}}. \quad (16)$$

Now let

$$\sigma = \frac{\psi_1^{60/4 \cdot 0252}}{4^{1/2 \cdot 0126} Q^{32/4 \cdot 0252}} s; \quad \theta = \frac{\psi_1^4}{Q^2} t, \quad (17)$$

and

$$K = \frac{\psi_1^3}{Q^2}. \quad (18)$$

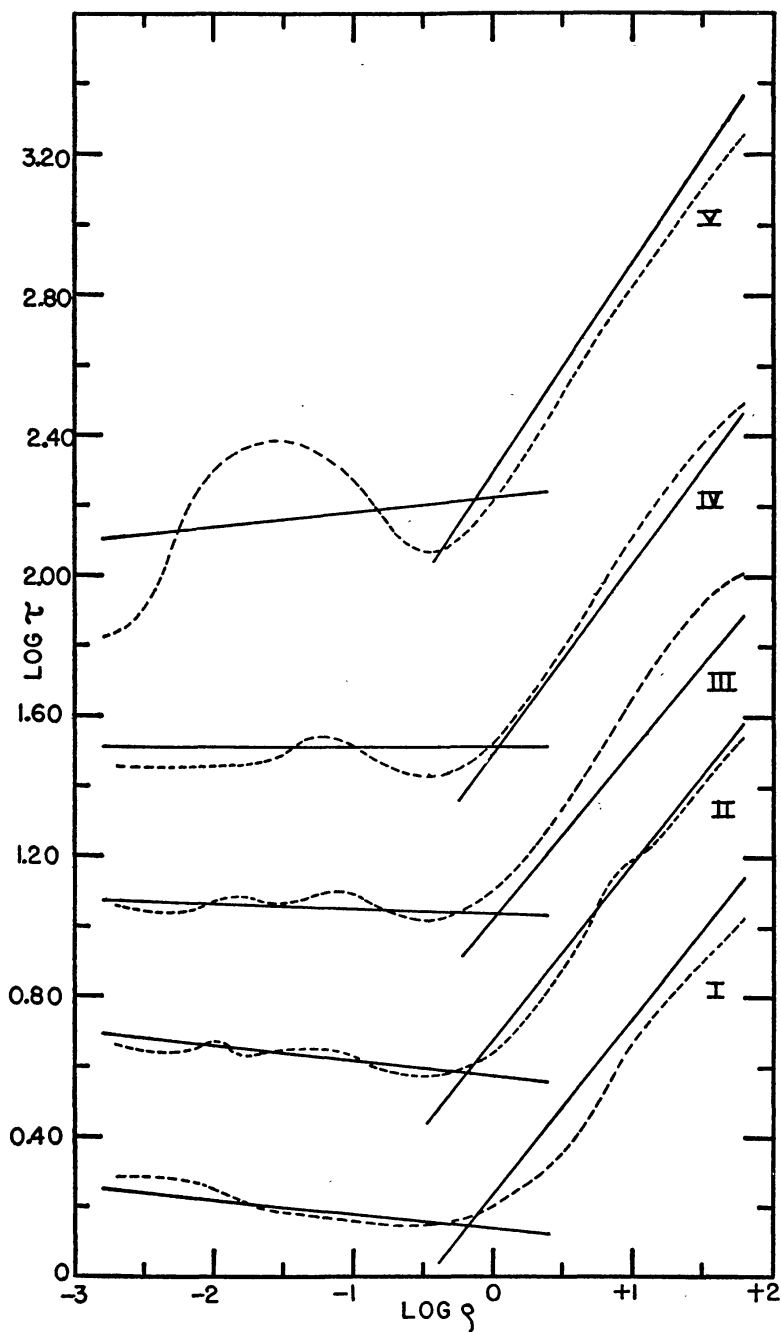


FIG. 1.—The curves illustrate the variation in the guillotine factor as a function of the density. The full lines are the assumed distribution, and the dotted curves are computed from the integrations. The curves are *I*, 100 per cent Russell mixture; *II*, 50 per cent oxygen and 50 per cent Russell mixture; *III*, 60 per cent oxygen and 40 per cent Russell mixture; *IV*, 70 per cent oxygen and 30 per cent Russell mixture; *V*, 100 per cent oxygen. The ordinates are the values for *I*; for each of the other curves subtract the following amounts, *II*, 0.40; *III*, 0.80; *IV*, 1.20; *V*, 1.60.

Introducing these substitutions into equations (16) and making the further substitutions

$$t = \frac{u^2}{K^2}; \quad s^{1.0063} = \frac{zu^6}{K^7}, \quad (19)$$

we have

$$0.124803uz \frac{dz}{du} = 1.004718u - z^2 \quad (20)$$

and

$$\frac{du}{d\xi} = -\frac{K}{8} \frac{z^2}{\xi^2 u^2}. \quad (21)$$

The solutions of equations (20) and (21), satisfying the boundary condition that z and u tend to zero simultaneously, are

$$z^2 = 0.945704u; \quad \frac{1}{\xi} = 1 + \frac{4.229653u^2}{K}. \quad (22)$$

TABLE 6

Model	Outside	Inside
100% Russell mixture.....	1.3753 $\rho^{-0.0404}$	1.7274 $\rho^{+0.5045}$
50% oxygen 50% Russell mixture.....	1.4962 $\rho^{-0.0404}$	1.9063 $\rho^{+0.5045}$
70% oxygen 30% Russell mixture.....	2.0854 $\rho^{+0.0007}$	1.9588 $\rho^{+0.5444}$
100% oxygen.....	4.2658 $\rho^{+0.0348}$	5.1286 $\rho^{+0.5580}$

The corresponding solutions for σ and Θ are

$$\sigma = \frac{0.488436u^{6.459307}Q^{5.962436}}{\psi_1^{5.962436}}, \quad (23)$$

$$\Theta = \frac{u^2 Q^2}{\psi_1^2}.$$

The solutions thus started will have to be continued by numerical methods. For this purpose it is convenient to introduce the variable

$$a = \frac{1}{\xi} \quad (24)$$

and to re-write equations (13) and (14) in the forms

$$\frac{d(\sigma\Theta)}{da} = \psi\sigma, \quad \frac{d\psi}{da} = -\frac{\sigma}{a^4}, \quad (25)$$

and

$$\frac{d\Theta}{da} = \frac{Q\sigma^{2.0126}}{\Theta^{6.5}}. \quad (26)$$

6. *Equations of the model for the inside.*—When the point is reached at which $\log \rho = 0.0261$, it is necessary to use a new set of equations and to make a fit at the interface. We have, as before, equations (6), (7), and (8), except that τ now has the form (10). By using the transformations

$$\rho = \rho_0 a^{-3} f = \rho_0 \sigma; \quad r = R a l = R \xi, \quad (27)$$

$$T = T_0 a^{-1} t = T_0 \Theta; \quad M(r) = 4\pi r^3 \rho_0 \psi = M\psi,$$

equations (6), (7), and (8) become

$$\frac{d(ft)}{dl} = -\frac{\psi f}{l^2}, \quad \frac{d\psi}{dl} = l^2 f, \quad (28)$$

$$\frac{dt}{dl} = -\frac{Q_0 f^{1.5151}}{l^{6.5} l^2}, \quad (29)$$

where

$$Q_0 = \frac{3 \kappa_0 L}{16 \pi a c} \frac{\rho_0^{1.5151} \alpha^{1.9547}}{1.6757 T_0^{7.5} R}. \quad (30)$$

7. *Construction of the model from the interface.*—It was found that it was possible to use an analytical development to continue the integration across the interface, provided that very small intervals for the radius factor are used.

Equations (28) can be re-written in the form

$$\frac{d(ft)}{d(t^4)} = \frac{\psi t^{3.5}}{4Q f^{0.5151}}, \quad (31)$$

and

$$\frac{d(t^4)}{dl} = -\frac{4Q_0 f^{1.5151}}{l^{3.5} l^2}. \quad (32)$$

Now let

$$f = \frac{\psi^{60/3.0302}}{4^{1/1.5151} Q_0^{32/3.0302}} x, \quad t = \frac{\psi^4}{Q_0^2} y, \quad (33)$$

and

$$M = \frac{\psi^3}{Q_0^2}. \quad (34)$$

Introducing these substitutions into equations (31) and (32) and making the further substitutions

$$y = \frac{g^2}{M^2}; \quad x^{0.757550} = \frac{X g^6}{M^7}, \quad (35)$$

we have

$$0.133065 g X \frac{dX}{dg} = 0.806430 g - X^2. \quad (36)$$

and

$$\frac{dg}{dl} = -\frac{M X^2}{8 g^2 l^2}. \quad (37)$$

Hence the solutions of equations (36) and (37) are

$$X^2 = 0.756124 a; \quad \frac{1}{l} = 1 + \frac{0.529011 g^2}{M}. \quad (38)$$

The corresponding solutions for f and t are

$$f = \frac{0.333043 g^{8.580291} Q^{7.920269}}{\psi^{7.920269}}, \quad (39)$$

$$t = \frac{g^2 Q^2}{\psi^2}.$$

It is convenient to introduce the variable

$$b = \frac{1}{l} \quad (40)$$

and to re-write equations (28) and (29) in logarithmic form.

8. *Fitting at the interface.*—Now the conditions at the interface are that the values of the temperature, pressure, and mass should be identical. The constants α and Q_0 can be determined from equations (27) and (39). The integrations can then be continued inward by means of equations (28) and (29). It is thus seen that for any specified Q the equations can be integrated, starting from the boundary and going inward. However, as we go along such a solution, a point is generally reached at which the radiative gradient breaks down. This point, ξ_i , where the convective gradient sets in, is defined by the place where the effective polytropic index,

$$n = \frac{\psi t^{6.5}}{Q_0 f^{1.5151}} - 1, \quad (41)$$

becomes $\frac{3}{2}$. The solution interior to ξ_i must be described by a Lane-Emden function $\theta_{3/2}$. For an initially arbitrary choice Q , the solution obtained by the integration cannot be fitted to $\theta_{3/2}$. The condition that the fit be made determines Q and hence Q_0 . Now the conditions at the interface are that the values of the temperature, pressure, and mass of the core should be identical at the interface, along with the condition that the effective polytropic index n attain the value 1.5 at the interface.

TABLE 7

Model	log Q	log Q_0	u_e	v_e	ξ_c	ψ_c	θ	σ_i
100% Russell mixture	-4.9968	-4.7081	2.6748	0.3762	0.1011	0.0970	0.8305	262.5
50% oxygen	-4.9968	-4.7081	2.6748	.3762	.1011	.0970	.8305	262.5
60% oxygen	-5.1310	-4.7017	2.6680	.3844	.0996	.0943	.8272	251.9
70% oxygen	-4.9888	-4.4075	2.6812	.3685	.0908	.0893	.8367	317.5
100% oxygen	-4.7526	-4.3496	2.6468	0.4099	0.0846	0.0984	0.8439	418.0

Introducing the homology invariant quantities u and v , defined as in Chandrasekhar's *Stellar Structure* (pp. 146 ff.), we have for the conditions at the interface

$$u_i = \frac{4\pi p_e(r_i) r_i^3}{M_e(r_i)} = u_e = \left(\frac{v}{\psi} \frac{1}{b^3} \right)_{r=r_i} \quad (42)$$

and

$$v_i = \frac{2GM_e(r_i)}{5r_i T_e(r_i)} \frac{\mu H}{k} = v_e = \left(\frac{2\psi b}{5t} \right)_{r=r_i}, \quad (43)$$

where e refers to the envelope and i to the core.

All calculations were carried out logarithmically to six decimal places. When the point was reached at which the radiative gradient broke down, u and v were calculated. For the regions of the star inside $r = r_i$ the *British Association Mathematical Tables* (Vol. II) were used.

9. *Results.*—From the integrations we find the values given in Table 7. We must have

$$\sigma_i = \sigma_c \theta_i^{3/2}. \quad (44)$$

From equation (44) we obtain

$$\sigma_c = 334.9. \quad (45)$$

Also, since

$$\bar{\rho} = \frac{M}{\frac{4}{3}\pi R^3} = 3\rho_0 \quad (46)$$

and

$$\rho_c = \rho_0 \sigma_c, \quad (47)$$

we find

$$\rho_c = 111.6\bar{\rho}. \quad (48)$$

Moreover,

$$\sigma_i \Theta_i = K \theta_c^{5/2} \theta_i^{5/2} \quad (49)$$

or

$$K = 0.0251 \quad (50)$$

and

$$\sigma_c \Theta = K \sigma_c^{5/3}, \quad (51)$$

or

$$\Theta_c = 1.210. \quad (52)$$

Finally, we have

$$\rho_c = 111.6 \bar{\rho}, \quad (53)$$

$$T_c = T_0 \Theta_c = 27.98 \times 10^6 \frac{\mu M}{R}, \quad (54)$$

$$\rho_c = 157.5 \frac{M}{R^3}, \quad (55)$$

where M and R are expressed in solar units.

10. *Mass-luminosity equation.*—If the eigenvalue Q found by the integration is introduced into equation (15), we obtain the mass-luminosity relation. The expression for the mean molecular weight μ is

$$\frac{1}{\mu} = 2X + \frac{3}{4}Y + \frac{1}{2}(1 - X - Y), \quad (56)$$

where X is the hydrogen content and Y the helium content. We also have

$$\kappa_0 = 3.90 \times 10^{25} (1 + X)(1 - X - Y). \quad (57)$$

Expressing L , M , and R in solar units and introducing the above expressions for μ and κ_0 , we have for the mass-luminosity equation

$$[2X + \frac{3}{4}Y + \frac{1}{2}(1 - X - Y)]^{7.5} (1 + X)(1 - X - Y) = 1.76 \frac{M^{5.4874}}{LR^{0.4622}}. \quad (58)$$

11. *Energy-output equation.*—The second equation necessary for the determination of X and Y is derived by integrating the energy production over the entire core. The rate of energy generation by the carbon cycle is given by

$$\epsilon = \frac{Eh}{m_{14}t} (1 - X - Y) \frac{X}{0.35} \frac{\rho}{80} \left(\frac{T}{2 \times 10^7} \right)^{17} = \epsilon_0 \rho T^{17}, \quad (59)$$

where

$E = 4 \times 10^{-5}$ ergs = total energy released per cycle ;

$m_{14} = 2 \times 10^{-23}$ gm = mass of one N^{14} nucleus ;

$t = 4 \times 10^6$ years = time for a complete reaction if $\rho = 80$, $T = 2 \times 10^7$ K, and $X = 0.35$;

$h = 0.02$ = carbon plus nitrogen content of the Russell mixture .

The integration over the core gives

$$\frac{L}{M} = \frac{\epsilon_0 \rho T_c^{17}}{52}. \quad (60)$$

The energy-output equation can then be written in the form

$$\frac{[2X + \frac{3}{4}Y + \frac{1}{2}(1 - X - Y)]^{17}}{(1 - X - Y)X} = \frac{5419M^{19}}{LR^{20}} \quad (61)$$

III. DISCUSSION OF THE MODELS

12. *Summary of the results.*—The main features of the models are summarized in Tables 8 and 9.

TABLE 8

MODEL	ρ_c	$T_c \times 10^{-6}$	$\xi_{\text{interface}}$	$\psi_{\text{interface}}$	μ	WEIGHT			
						H	He	O	Metals
100% Russell mixture...	163.1	19.83	0.5024	0.9652	0.715	0.54	0.36	0.050	0.050
50% oxygen.....	163.1	19.74	.5024	.9652	.712	.54	.35	.082	.028
60% oxygen.....	157.5	20.12	.4710	.9478	.719	.53	.37	.080	.020
70% oxygen.....	196.2	19.85	.4534	.9484	.687	.59	.30	.094	0.016
100% oxygen.....	253.6	18.92	0.4666	0.9672	0.594	0.77	0.12	0.110

TABLE 9

MODEL	NUMBERS		
	H/He	H/O	H/Metals
100% Russell mixture.....	6.0	173	373
50% oxygen.....	6.17	108	677
60% oxygen.....	5.73	106	914
70% oxygen.....	7.87	105	1234
100% oxygen.....	25.7	112

In so far as there is an uncertainty in the half-life of N^{14} used in the energy-generation equation, we have used two values, namely, 4×10^6 and 1×10^6 years. This results in another set of abundances, which are shown in Table 10.

TABLE 10

MODEL	μ	WEIGHT				NUMBERS		
		H	He	O	Metals	H/He	H/O	H/Metals
100% Russell mixture...	0.675	0.598	0.338	0.032	0.032	7.08	299	645
50% oxygen.....	.671	.605	.328	.050	.017	7.38	194	1228
60% oxygen.....	.675	.597	.343	.048	.012	6.96	199	1716
70% oxygen.....	.645	.653	.279	.058	0.010	9.36	180	2253
100% oxygen.....	0.559	0.845	0.085	0.070	39.8	193

The temperature-density distribution throughout each model is shown in Table 11.

It is readily seen from Figure 1 that the variation in the guillotine factor as computed from the temperature-density distribution determined from the integrations agrees very well with the formulae for τ .

TABLE 11

ξ	Temperature degrees K										Density grams/cc#				
	100%	70%	60%	50%	0%	100%	70%	60%	50%		100%	70%	60%	50%	
0.9949	16.19	18.98	19.96	19.97	20.00	12.09	22.47	30.12	33.71	(+3)	(-9)	(-9)	(-9)	(-9)	
.9082	32.19	37.74	39.68	39.72	39.77	23.96	37.44	47.08	46.20	(+4)	(-5)	(-5)	(-5)	(-5)	
.8900	39.38	46.15	48.54	48.58	48.65	46.44	46.44	50.20	87.74	(+4)	(-5)	(-5)	(-5)	(-5)	
.8634	50.42	59.10	62.16	62.22	62.30	10.53	16.11	20.05	19.29	(+4)	(-4)	(-4)	(-4)	(-4)	
.7962	81.58	95.63	10.05	11.06	11.07	51.68	76.95	94.51	89.04	(+5)	(-4)	(-4)	(-4)	(-4)	
.7802	89.76	10.52	11.06	11.07	11.08	70.92	10.50	12.88	12.08	(+5)	(-3)	(-3)	(-3)	(-3)	
.7416	11.09	13.01	13.68	13.89	13.71	14.27	20.93	25.60	23.80	(+5)	(-3)	(-3)	(-3)	(-3)	
.6702	15.67	18.36	19.32	19.32	19.34	44.62	64.18	77.99	71.14	(+5)	(-3)	(-3)	(-3)	(-3)	
.6436	17.63	20.66	21.73	21.73	21.76	65.92	94.06	11.38	10.35	(+5)	(-2)	(-2)	(-2)	(-2)	
.6056	20.73	24.28	25.50	25.55	25.58	11.23	15.87	19.07	17.31	(+5)	(-2)	(-2)	(-2)	(-2)	
.5916	22.89	26.81	28.16	28.21	28.25	15.58	21.87	26.22	23.69	(+5)	(-2)	(-2)	(-2)	(-2)	
.5585	25.14	29.44	30.92	30.97	31.01	21.22	29.58	35.38	31.84	(+5)	(-2)	(-2)	(-2)	(-2)	
.5363	27.47	32.16	33.78	33.93	33.88	28.42	39.38	46.96	42.12	(+5)	(-2)	(-2)	(-2)	(-2)	
.5024	31.42	36.69	38.66	38.66	38.72	44.25	60.24	72.04	64.14	(+5)	(-2)	(-2)	(-2)	(-2)	
.4710	35.56	40.39	43.63	66.47	82.09	10.62	(+5)	(-2)	(-2)	(-2)	(-2)	
.4666	36.18	40.94	70.34	85.72	(+5)	(-2)	(-2)	(-2)	(-2)	
.4534	44.53	11.22	(+5)	(-1)	(-1)	(-1)	(-1)	
.4480	38.22	45.22	46.73	45.71	45.77	89.97	12.00	16.05	13.17	(+5)	(-2)	(-2)	(-2)	(-2)	
.4302	40.43	47.56	49.46	48.40	48.47	11.57	15.11	18.07	16.82	(+5)	(-1)	(-1)	(-1)	(-1)	
.4131	42.73	50.29	52.25	51.19	51.26	14.79	19.14	22.75	21.34	(+5)	(-1)	(-1)	(-1)	(-1)	
.3815	47.48	55.72	58.05	56.92	57.00	23.58	29.91	35.20	33.51	(+5)	(-1)	(-1)	(-1)	(-1)	
.3590	51.32	60.06	62.69	61.55	61.63	33.14	41.30	48.24	46.08	(+5)	(-1)	(-1)	(-1)	(-1)	
.3310	56.70	66.14	69.15	67.94	68.03	51.16	62.53	71.70	69.22	(+5)	(-1)	(-1)	(-1)	(-1)	
.3053	62.40	72.56	75.86	74.59	74.70	77.11	91.73	10.35	10.10	(+5)	(-1)	(-1)	(-1)	(-1)	
.2815	68.36	79.21	84.06	81.54	81.66	11.35	13.16	14.54	14.33	(+5)	(-1)	(-1)	(-1)	(-1)	
.2596	74.54	86.08	89.87	88.61	88.74	16.25	18.38	19.84	19.74	(+5)	(-1)	(-1)	(-1)	(-1)	
.2394	80.96	93.11	97.05	95.76	95.90	22.74	24.99	26.31	26.45	(+5)	(-1)	(-1)	(-1)	(-1)	
.2208	87.84	10.02	10.28	10.30	10.31	30.90	33.06	33.92	34.44	(+5)	(-1)	(-1)	(-1)	(-1)	
.2036	94.23	10.74	11.14	11.02	11.03	41.17	42.54	42.51	43.33	(+5)	(-1)	(-1)	(-1)	(-1)	
.1877	10.10	11.45	11.96	12.43	12.44	53.21	53.26	52.94	53.56	(+5)	(-1)	(-1)	(-1)	(-1)	
.1731	10.77	12.15	12.55	12.43	12.44	72.81	64.61	62.36	68.28	(+5)	(-1)	(-1)	(-1)	(-1)	
.1472	12.09	13.51	13.87	13.76	13.78	98.33	89.70	81.24	85.02	(+5)	(-1)	(-1)	(-1)	(-1)	
.1357	12.73	14.16	14.97	14.38	14.40	11.49	10.19	90.66	94.86	(+5)	(+1)	(+1)	(+1)	(+1)	
.1252	13.34	14.78	15.09	14.98	15.00	13.13	11.36	99.15	10.41	(+6)	(+1)	(+1)	(+1)	(+1)	
.1154	13.94	15.37	15.66	15.55	15.57	14.72	12.33	10.69	11.22	(+6)	(+1)	(+1)	(+1)	(+1)	
.1064	14.51	15.94	16.21	16.10	16.12	16.21	13.39	11.19	11.95	(+6)	(+1)	(+1)	(+1)	(+1)	
.1011	14.96	16.29	16.54	16.40	16.45	17.09	13.90	11.75	12.35	(+6)	(+1)	(+1)	(+1)	(+1)	
.0996	14.96	16.39	16.64	16.49	16.54	17.34	14.05	11.86	12.45	(+6)	(+1)	(+1)	(+1)	(+1)	
.0982	15.06	16.48	16.73	16.58	16.63	17.57	14.18	11.95	12.55	(+6)	(+1)	(+1)	(+1)	(+1)	
.0908	15.57	16.61	17.17	16.99	17.05	18.71	14.94	12.43	13.03	(+6)	(+1)	(+1)	(+1)	(+1)	
.0846	15.99	16.95	17.52	17.33	17.40	19.66	15.48	12.82	13.42	(+6)	(+1)	(+1)	(+1)	(+1)	
.0800	15.99	17.22	17.78	17.58	17.65	19.69	15.87	13.11	13.72	(+6)	(+1)	(+1)	(+1)	(+1)	
.0600	16.69	18.32	18.77	18.48	18.55	21.80	17.40	14.21	14.79	(+6)	(+1)	(+1)	(+1)	(+1)	
.0400	18.07	19.14	19.50	19.17	19.25	23.68	18.58	15.04	15.62	(+6)	(+1)	(+1)	(+1)	(+1)	
.0200	18.69	19.66	19.95	19.60	19.68	24.91	19.35	15.57	16.13	(+6)	(+1)	(+1)	(+1)	(+1)	
0.	18.92	19.95	20.12	19.74	19.83	25.36	19.62	15.75	16.31	(+6)	(+1)	(+1)	(+1)	(+1)	

*The density distribution in the model for 0% Oxygen is the same as in the model for 50% Oxygen.

With regard to Figure 2, the undetermined portions of each curve were established by determining the hydrogen and helium content for other combinations of oxygen and Russell mixture. It was found that one of the integrations for another mixture could be used for this purpose if the constants in the formulae for τ were suitably changed. It should be noted, however, that this method cannot be applied if it is necessary to change the exponent of ρ or to move the lines in a horizontal direction.

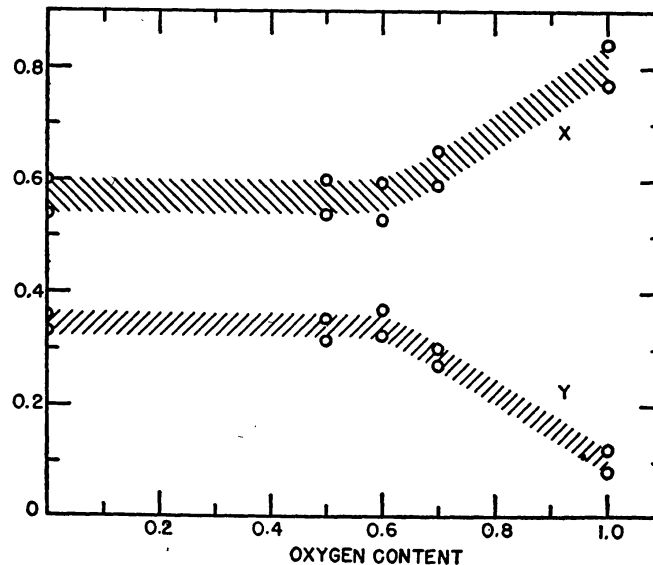


FIG. 2.—The curves illustrate the variation in the hydrogen and the helium content for various combinations of oxygen and Russell mixture as shown in Tables 8 and 10. The ordinates represent the percentage of hydrogen or helium, and the abscissae represent the percentage of oxygen in the mixture of oxygen and Russell mixture.

TABLE 12

Element	Weight (Per Cent)	Numbers
Hydrogen.....	70	1600
Helium.....	28	160
Oxygen group.....	1.4	2
Metals.....	0.4	0.315

13. *Conclusions.*—Table 12 was compiled by Dr. J. L. Greenstein from spectroscopic results. Comparing these values with the values determined from the integrations, we find a fairly good agreement for the H/He and $H/metals$ ratios for a mixture of about 70 per cent oxygen and 30 per cent Russell mixture; however, our oxygen abundance is perhaps a little too high. This high value of the oxygen abundance follows even if we assume that the metals are composed of 100 per cent Russell mixture. An important conclusion we may draw now is that there is no divergence in the absolute abundance of hydrogen and the important H/He ratio as determined from stellar atmospheres and the theory of the internal constitution.

It is again a pleasure to record my indebtedness to Dr. S. Chandrasekhar for his helpful advice.