THE RELATION OF COHESION TO ROCHE'S LIMIT

Harold Jeffreys

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Summary

Roche's proof of the existence of a critical distance for a satellite, within which it would be broken up by tidal action, assumed the satellite fluid. It is shown that a solid satellite, or a small solid body making a casual near approach, would not be broken up even close to the surface of a planet unless its diameter was less than a critical value, which is unexpectedly large, of the order of 200 km. It appears in particular that small bodies of the sizes of the majority of the asteroids could not have been formed by close approach to Jupiter, and that fragments of ice consistent with the maximum possible thickness of Saturn's rings could not have been formed by disruption of a solid mass by tidal action.

It was proved by Roche that a small liquid satellite moving in a circular orbit about a primary, the periods of rotation and revolution being equal, would take an approximately ellipsoidal form provided that it was not too close to the primary. If, however, the mean distance was less than about twice the radius of the primary (depending somewhat on the ratio of the densities) there would be no possible permanent form and the satellite would be broken up. Many extensions of Roche's argument have been given, notably by Darwin and Jeans; the theory has been given for gaseous satellites and the restriction that the satellite is to be small has been removed. On the whole the modifications give surprisingly little change in Roche's main result. But the hypotheses all suppose the satellite to be fluid, and the corresponding problem for a solid satellite seems to have escaped explicit solution. The solution can actually be derived easily from that of the straining of an elastic sphere.

If an incompressible spherical body with density ρ, radius a, surface gravity g, and rigidity μ is deformed by a gravitational potential $k_n K_n$, where K_n is a solid harmonic, the stress components are given by

$$p_{ik} = -p_n \delta_{ik} K_n + \mu \left\{ 2(A+Br^2)\frac{\partial^2 K_n}{\partial x_i \partial x_k} + (2B+C)\left(x_k \frac{\partial K_n}{\partial x_i} + x_i \frac{\partial K_n}{\partial x_k} \right) + 2C\delta_{ik} K_n \right\}, \tag{1}$$

where

$$\mathrm{B} = -\frac{n+3}{2n}C\ ; \quad A = \frac{n+2}{2(n-1)}Ca^2\ ; \quad \epsilon_n = \frac{2n+1}{2(n-1)}Ca^n, \tag{2}$$

$$C\left\{ \frac{2n^2+4n+3}{n}\mu + g\rho a \right\} = \rho k_n, \tag{3}$$

$$p_n = \mu\{(4n+2)B + 2C\} + \rho\left(k_n + \frac{3g\epsilon_n}{(2n+1)a^{n-1}} \right). \tag{4}$$

The radial displacement at the surface is $\epsilon_n K_n / a^{n-1}$, but is not required in the present problem. Gravity between parts of the small body is of minor importance in comparison with rigidity; even for a body as large as the Earth the terms in the coefficient of C are nearly as 6 to 1 for $n=2$. We shall therefore neglect terms in g.

Compressibility is also of small importance, affecting the results by only a few per cent.

Now take the body to be at distance c from the centre of a mass M and to rotate with an angular velocity ω in a plane through the radius vector. Then the deformation is derived from a gravitational potential

$$U=\frac{fM}{2c^3}(2x^2-y^2-z^2)+\tfrac{1}{6}\omega^2(x^2+y^2-2z^2). \tag{5}$$

We consider two cases. The body may be a wandering asteroid, making an accidental close approach to Jupiter or Mars. In that case it may be supposed that ω is far too small to produce disruption by itself; in some cases stress-differences due to rotation will have already been removed by plastic adjustment. In these cases we can take ω zero.

The body may however be a satellite that is being made to approach its primary by tidal friction or a resisting medium. If tidal friction in the satellite itself is strong or the satellite is triaxial like the Moon, the relation

$$\omega^2=fM/c^2 \tag{6}$$

will be maintained throughout the motion. Thus we should take

$$U=\frac{fM}{2c^3}(2x^2-y^2-z^2)\;; \qquad U=\frac{fM}{6c^3}(7x^2-2y^2-5z^2), \tag{7}$$

according as we are considering a small body making a single approach to a larger one or a satellite gradually approaching its primary.

In both cases it is found that the greatest stress-difference is $|p_{11}-p_{33}|$ evaluated at the origin, that is, at the centre of the small body, and the respective results are

$$\frac{24}{19}\frac{fM}{c^3}\rho a^2\;; \qquad \frac{32}{19}\frac{fM}{c^3}\rho a^2. \tag{8}$$

The different hypotheses about the rotation therefore do not affect the order of magnitude of the result.

As a specimen we consider a small satellite with the density of the Moon moving near the Earth's surface. The critical stress-differences for the rocks are about 10^9 dynes/cm.^{2}. If ρ_0 is the density of the Earth we have

$$M/c^3=\tfrac{4}{3}\pi\rho_0, \tag{9}$$

and the second of (8) gives for rupture the condition

$$\tfrac{4}{3}\pi\;.\;\frac{32}{19}\;.\;6{\cdot}66\times10^{-8}\rho_0\rho a^2>10^9, \tag{10}$$

whence, with $\rho_0=5{\cdot}5$, $\rho=3$ g./cm.^{3},

$$a\geqslant 1{\cdot}1\times10^7\text{ cm.}=110\text{ km.}$$

In other words, a satellite, and *a fortiori* an asteroid, if of rocky constitution, could graze the Earth's surface without rupture if its diameter were less than 220 km. This value is unexpectedly large, since it is well known that a column of rock a few kilometres in height would be crushed under its own weight. The difference arises from the fact that in the argument leading to equation (8) the centre of the body is free and the resultant attraction of the Earth is used up in maintaining the orbital motion. For the column of rock the resultant attraction has to be balanced by internal stresses over the base.

For the approach of a rocky body to Jupiter the result must be multiplied by about 2, since ρ_0 is then about $1{\cdot}3$ g./cm.^{3}. Thus if all the asteroids were once one body, this would have been large enough to be broken by a very near approach to Jupiter; but no number of such approaches could have produced some thousands of bodies with diameters of a few kilometres.

Another interesting case is the approach of a sphere of ice to Saturn. I am informed by Mr B. B. Roberts, of the Scott Polar Research Institute, that ice cliffs usually break off at a height of about 100 feet. This would indicate a strength of the order of 3×10^6 dynes/cm.^{2}; the strength at the temperature prevailing near Saturn may be greater. Allowing for the changes in (10) we find that the critical radius for an ice sphere, if it is to be broken up by tidal action near Saturn's surface, is of the order of 3×10^6 cm. At the mean distance of the rings, of course, this would have to be increased by a factor of 3 or 4. Hence if an ice satellite ever revolved about Saturn at the mean distance of the rings, and was broken up by tidal action, its diameter was over 200 km.; and the fragments would cease to be broken up further when their diameters had been brought down to this value. Since the ring appears equally bright all round and its maximum possible thickness is given as 10 km. we can conclude that it was not formed by disruption of a solid satellite.

After the above was written, Dr M. Perutz informed me that laboratory tests on ice give an average tensile strength of 11·8 kg./cm.^{2} = $1{\cdot}16 \times 10^7$ dynes/cm.^{2}, at -5 to -10 deg. C., but remarks that individual pieces may fail at half to twice this value. In any case the estimates just given are not too low.

This investigation is supplementary to the usual theory of Roche's limit, not a substitute for it. With sufficiently large bodies it would be possible for the present criterion to be satisfied for distances outside Roche's limit. Subsequent developments would then depend on whether the material was of a type to undergo elastic failure by fracture or flow. In the former case the satellite would presumably be broken into two immediately (that is, within the time needed for a fracture wave to traverse the diameter, probably not more than a few hours, and possibly minutes). In the latter case it would adjust itself by flow towards a state of hydrostatic pressure, and ultimately the resulting deformation would make Roche's criterion applicable again; the body would be permanently distorted but not broken up.

If the present criterion for elastic failure is not satisfied there will be no disruption even if the distance is within Roche's limit. If the distance is within Roche's limit and the criterion is satisfied, the behaviour will differ according as failure is by immediate fracture or flow. For fracture, since a non-zero cohesive force is suddenly removed, the parts will separate with a non-zero difference of velocity towards the primary, and unless the rupture occurs at an apse the parts will have different energies per unit mass, and will therefore proceed in orbits with different mean distances. For plastic flow, the body will be distorted as a quasi-fluid, but will not reach a steady state since this state would itself contradict Roche's theorem. Several complications, such as the ratio of the masses of the parts and the rate of plastic adjustment, would affect the details and it does not appear possible to say without further information whether the outcome would be two independent satellites, two revolving about each other, or the equilateral triangle configuration.

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THE EFFECTS OF COLLISIONS ON SATURN'S RINGS

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Summary

If Saturn's rings were several particles thick the damping effect of collisions on the inclinations and eccentricities would reduce the rings in less than a year to a state where the particles were piled one on another. Dissipation by friction and impact would not cease at this stage. Its later effects would extend the ring inwards and outwards in its own plane, until it was nowhere more than one particle thick and the spacing was just enough for collisions to be avoided. The time needed to attain this state is estimated to be of the order of 10^6 years for particles of diameter 1 cm. and less for larger ones. Some comments are made on Maxwell's and Goldsbrough's criteria for stability.

The outstanding observational facts about Saturn's rings are as follows. (1) Spectroscopic evidence shows that the velocity at any point is nearly that of a particle in a circular orbit at the same distance. (2) The reflecting power is high, that of Ring B exceeding that of the planet and that of Ring A approaching it. (3) Nevertheless the rings are not quite opaque, stars having occasionally been seen through them. (4) The reflecting power is appreciably lower when Saturn is at quadrature than at opposition, even when allowance is made for the difference of distance. (5) The ring is extremely thin.

Maxwell showed that a set of satellites moving in one circle about the planet would be stable, and that all the other suggested types of constitution that he considered would be unstable. Since his essay his results have usually been quoted as the chief evidence for the meteoric constitution, but they are not quite decisive, because in considering a liquid or gaseous ring he supposed it to be in uniform rotation like a rigid body. Previous work of Laplace and Kowalewsky had shown that a ring in such a state of motion would have a thickness comparable with its width, and in view of our present knowledge of the thickness of the ring this hypothesis scarcely merits further consideration. But a fluid ring could be arbitrarily thin if we abandon the hypothesis of rigid-body rotation, which in itself would suggest a very high viscosity.* The stability of a fluid ring with variable rotation has not, I think, been discussed. The best way of presenting the case for a meteoric constitution at present, I think, is to rely on the observational data (1), (4) and (5) above. (1) and (5) limit us to the meteoric theory or to a gas or liquid with the velocities mainly controlled by gravity and not viscosity. A gas can be excluded, since the distribution of density normal to the plane would satisfy the usual laws for a gas. But in the small normal field that must exist in a mass 13 km. thick at most the density could not build up sufficiently to give great scattering of light. A liquid, again, would have a smooth surface and give regular reflection. Images of the ball of Saturn and of stars would be formed in the ring, and could not have escaped observation. Accordingly, quite apart from the mechanical arguments, which are incomplete, I think that optical considerations alone are

* For an analogous problem *cf.* Jeffreys, *The Earth*, 1929, pp. 49–52, Cambridge.

enough to show that the meteoric theory is the only tenable one. Further, the meteoric theory explains (4), as has been shown by H. Seeliger.*

Nevertheless the meteoric theory needs considerable mechanical investigation before it can be regarded as complete. Maxwell's essay was confined to a set of particles in a single circle, and recent investigations of perturbations have considered only disturbances from such a circle. Attractions between particles at different distances are ignored. But in any case, since the masses of the satellites are small, and that of the ring also small, all mutual gravitation is of the second order of small quantities. But in a system of freely moving solid bodies we may expect collisions to be frequent and to give discontinuous changes of momentum of the first order of small quantities. A preliminary treatment of these has been given †, but further investigation is needed.

The essential point is that the high opacity of the rings shows that on an average most rays of light striking the ring at angles up to 27° meet a particle on their way. But every particle of the ring must cross the mean plane of the ring twice in each revolution. Consequently we may expect it to undergo at least two collisions on the way. The point is that the opacity and the frequency of collisions depend on the same function of the number and size of the particles, namely the total surface of the particles per unit area in the plane of the ring; and if the departures of the particles from steady motion in circles can be treated as random the opacity shows that collisions will have a dominating effect. But collisions between solids are essentially non-conservative; at each collision the relative velocity is reduced by imperfection of restitution and by friction, usually by a fraction in the neighbourhood of $\frac{1}{2}$. Hence relative velocities between neighbouring particles will be rapidly annulled, with a time of relaxation not more than the orbital period, about a day. If for instance the particles were originally in an anchor ring, an average normal to the plane of revolution intersecting several particles, the velocities normal to this plane would be practically annihilated in a year. Further, the radial velocities corresponding to the orbital eccentricities would be removed at the same rate. But there is little to alter the mean motions. Hence the state reached in a year would be a peculiar one. The ring would be thin, but at any distance several particles would be piled one on another, in permanent contact. Such a state could last with little change for a long time, because the particles would acquire such rotations that there would be little difference of velocity at the points of contact. Nevertheless dissipation would not be altogether abolished. Detailed treatment is beyond the present resources of statistical mechanics, for even in the absence of dissipation the problem would be that of a fluid with the molecular spacing comparable with the molecular dimensions. But motion in and out, or up and down, would persist, though its actual amount would be only that needed to maintain rolling. Interchange between motions in different directions remains possible, since the lines joining centres of bodies in contact might be in any direction in relation to the mean plane of revolution. We have in fact a case of kinetic theory where there is no independent agitation; such agitation as there is would be parasitic on the general motion.

The outstanding cause of further dissipation would be the differences in orbital period. For simplicity take the particles to be spheres of radius a, and

* *Abh. Bayer. Akad.*, **18**, 1–72, 1893. For full discussion *cf.* E. Schönberg, *Handb. Astrophys.*, **2**, part I, 130–170, 1929.

† Jeffreys, *M.N.*, **77**, 89–92, 1916.

consider two sets moving in circles of radii r, $r+b$, where $b<2a$. In free motion the ratio of the periods is $1-\frac{3}{2}b/r$. Hence if the spacing in longitude is also b, a particle on the inner circle will encounter on an average $\frac{3}{2}$ particles in a revolution and share momentum with them. If v is the orbital velocity at distance r, m the mass of a particle, the outward transfer of angular momentum at a collision is of order $-mbr\,dv/dr$, the numerical factor being less than 1 but probably more than 0·1. Hence a particle transfers outwards, per unit time, an angular momentum of order

$$-\frac{3mbv}{4\pi}\frac{dv}{dr}.$$

As the particle occupies an area $2ab$, this is equivalent to a tangential stress

$$-\lambda\frac{mv}{ar}\frac{dv}{dr},$$

where λ will probably be between 0·01 and 0·1. Using the relation

$$v^2=\mu/r$$

we have for the tangential stress

$$-\lambda m\mu/2ar^3,$$

and for the total rate of outward transfer of angular momentum in a belt where the normal section is c,

$$\pi\lambda\mu cm/ar^2.$$

The total angular momentum in a ring, if a, b, c are constant, is

$$\tfrac{1}{3}\mu^{\frac{1}{2}}\frac{cm}{ab}[r^{\frac{3}{2}}].$$

Comparing these results, we see that the time that would be needed to transfer the whole of the angular momentum from the inner to the outer half of the mass would be of order

$$\frac{2}{3\pi\lambda}\frac{r^{\frac{1}{2}}[r^{\frac{3}{2}}]}{nb},$$

where n is the mean motion of a particle at the mean distance of the rings. If we take them as having their present extent, and $b=1$ cm., this quantity is of the order of 10^6 years. In the early state that we are considering, when the values of r would differ by much less, the time would be shorter. It would also be shortened by taking b larger, as we should have to do if we are to maintain the hypothesis that the particles were produced by tidal disruption of a solid body. The conclusion to draw is therefore that though the evolution in this state would be slow compared with the damping out of inclinations and eccentricities, it would still be rapid on a cosmogonical scale.

The nature of the changes is clear; on account of the steady outward transfer of angular momentum the outer parts of the ring would be driven outwards and the inner parts inwards, so that the ring would become thinner. The process would stop when the relation $b<2a$ is no longer satisfied. That is, we start from the state of close packing in three dimensions, and arrive at one where the particles are just widely enough spaced to avoid collisions altogether. They follow each other around in circles, the circles being spaced at intervals slightly greater than the diameters of the particles; and the ring is nowhere more than one particle thick.

The result is, I think, consistent with Seeliger's conclusion that a distance between particles decidedly more than their diameters is needed to explain the reduction of albedo near quadrature. It cannot be many times the diameters, since the high albedo even when the rings are open to their fullest extent shows that a straight line from the Earth to the ring must usually intersect at least one particle. If we were in a position to see the ring normally a much smaller fraction of the area would appear occupied than from our actual viewpoint.

Some suggestions arise from these results that may be relevant to the question of the sizes of the particles and to the stability. H. Struve * has inferred from the failure to detect secular perturbations due to the rings that the total mass is not more than 1/27000 of that of Saturn, say 2×10^{25} g. Supposing the whole area of the rings covered to a thickness $2a$ by matter of density 1, this gives $a < 1 \cdot 8 \times 10^4$ cm. This is less than we have inferred from the hypothesis of tidal disruption of a solid, and of course also less than the maximum thickness inferred from observation, which is of the order of 10 km.; and we seem to be driven to the additional hypothesis that the bodies produced in this way were broken up further by collisions in the early stage or to think of some different explanation altogether, such as that the rings were formed by direct condensation from the gaseous state as in the formation of snowflakes. Allowance for the fact that the whole surface need not be covered would not bridge the gap. The collisions in the later stage would be quite gentle, since the relative velocity of neighbouring particles would be of the order of their diameter in a day—much less than the velocity of fall of snowflakes.

According to Maxwell's theory, a set of particles moving in a circle would be stable only if $mp^3 < 2 \cdot 3M$, where p is the number of particles and M is the mass of Saturn, $5 \cdot 6 \times 10^{29}$ g. With spacing equal to the diameters we have

$$\frac{mp^3}{M} = \left(2\pi \, . \frac{1 \cdot 4 \times 10^{10}}{2a}\right)^3 \frac{4}{3}\pi a^3 / 5 \cdot 6 \times 10^{29}$$

which is very roughly 6. The result is independent of a; but if we allow for the spacing being greater than $2a$ Maxwell's criterion will be satisfied by the particles in each ring separately. It would not, however, have been satisfied in the earlier stages. But the stability in any case needs re-examination, on account of the effects of the attractions between particles in different circles.

If there was a division in the ring while the particles were still in contact, the tendency of the transfer of angular momentum would always be to fill it up. Consequently it is likely that the divisions in the ring have been formed since collisions became rare.

Goldsbrough has recently published † a detailed discussion of the perturbations of a circle of satellites by an independent satellite, and concludes that for certain ranges of distance the perturbations would produce instability; he finds that these ranges, for perturbations by Mimas, show interesting correspondences with the boundaries of the rings, Cassini's division, and Encke's division. The theory is very intricate and it seems hypercritical to suggest that it is not intricate enough, since it neglects the attractions between particles at different distances and the reaction on Mimas, whose mass is at any rate much less than the maximum possible mass of the ring. Goldsbrough works in terms of a parameter νT_s, which also

* H. Struve, *Publ. Obs. cent. Nicolas*, **11**, 228, 1898.
† *Phil. Trans.* A, **239**, 183–216, 1946.

appears in Maxwell's stability condition. Cassini's division, in particular, corresponds to the inequalities

$$2 \leqslant \Omega \leqslant \frac{2}{1-\frac{7}{4}\nu T_s}$$

and ω/ω', the ratio of the mean motions of the particles and Mimas, is given by

$$\frac{\omega}{\omega'} = \frac{\Omega}{\Omega - 1}.$$

From the width of the division he derives a value of νT_s, which is consistent with Maxwell's criterion. His argument is then that a circle of particles not satisfying this condition and moving in the division would be perturbed until some of its members collided with particles in the main rings; the circle would then be broken up. But it seems to me that this only shows that for each departure of Ω from 2 there is a critical value of νT_s, below which the circle would be safe. The division would not be sharp; the brightness would fall off continuously towards $\Omega = 2$, but would vanish only at this value.

On Goldsbrough's theory the inner edge of Ring B and the outer edge of Ring A correspond to instabilities at $\Omega = \frac{3}{2}$ and $\Omega = 3$. It is natural to take the former as an indication that particles near the inner edge of Ring B would overshoot the danger zone and join the crape ring; but if so there will be a systematic loss of angular momentum which, as far as I can see, would be compensated by a gain by Mimas, which would thus be driven further off. But if we accept this explanation for the crape ring we should expect another crape ring outside Ring A. This is not definitely confirmed, though some observers have suggested it.

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ON THE THEORY OF GLOBULAR STAR CLUSTERS

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Summary

The theorem is proved that a spherical system of stars contracting under the influence of its own gravitational field passes at one and only one moment, before reaching the state of greatest condensation, through a quasi-stationary state in which its structure corresponds to a solution of the equations of hydrostatic equilibrium.

The theorem is then applied to observational data discussed in a preceding paper.

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1. *Introduction.*—In a preceding paper * I derived from the latest densitometric observations, which had revealed wide envelopes of faint stars surrounding the clusters of bright stars, that the structure of the globular star clusters corresponds very nearly to that of an isothermal core, containing the most massive stars, surrounded by an "adiabatic" atmosphere of light stars. Obviously a cluster cannot remain in this state for an unlimited time, because in the end isothermy should prevail and the cluster be scattered over an infinite space. The present structure therefore represents only an intermediate phase. The following paper adds considerable support to the theory of the preceding paper by showing that every cluster necessarily passes, in the course of its evolution, through such an intermediate phase. It will be proved that a spherical cluster of stars, contracting under its own gravitational field from a very diluted initial state, must pass, *before* reaching the state of greatest condensation, through a distinguished state, called here *quasi-stationary*, not because in this state the contraction comes to a standstill but because in this phase the internal structure of the cluster remains practically unchanged for a long time and resembles during this phase the structure to be expected, if the hydrostatic equations of equilibrium were directly applicable.

The following theory applies therefore only to the contracting phase in the evolution of a cluster which probably is only a relatively short phase of its total lifetime, though not necessarily short in comparison with the present age of the universe. It is an essential feature of the theory that purely *dynamical* considerations are used to explain the present structure of the globular star clusters, despite its close resemblance to the structure of a gas-sphere. This is essential, because it has been generally agreed † that the principles of the kinetic theory of gases are not applicable to globular star clusters. Owing to this, their actually observed structure remained hitherto unexplainable.

The following dynamical considerations show that the contracting phase of a spherical cluster of stars leads to a state in which its structure corresponds to a solution of the differential equations of hydrostatic equilibrium, without defining the actual structure more closely. The observations indicate that the solution attained is that of an isothermal core surrounded by an adiabatic atmosphere.

* *M.N.*, **105**, 237, 1945.
† ten Bruggencate, *Sternhaufen*, p. 97, Berlin, 1927.

After having reached this quasi-stationary state the cluster continues to contract, with only slight changes of its internal structure, until, owing to the increasing density of the core, encounters of stars become more and more frequent, and statistical considerations are needed to explain the further development which ends in isothermal scattering.

It appears to me of particular significance that a purely dynamical theory can be given which links organically the initial dynamical phase of contraction to the final statistical phase. The quasi-stationary state, considered here, is, so to speak, the branch-point connecting the two phases.

2. The following considerations start from a schematic model of a globular star cluster, supposed to be a finite spherical agglomeration of stars. In the differential equation from which the Virial Theorem is derived,

$$\tfrac{1}{4}d^2I/dt^2 = T + \tfrac{1}{2}V, \tag{1}$$

the cluster is characterized at every moment, t, by the value I of its moment of inertia:

$$I = \int_0^R \frac{dM(r)}{dr} r^2 dr,$$

where $dM(r) = 4\pi\rho(r)r^2dv$, $\rho(r)$ the density at the distance r from the centre, R the finite surface radius. T is the kinetic energy, V the potential energy of the cluster for a Newtonian gravitational field. We make the assumption that the cluster has been slowly contracting under its own gravitational field, all the time remaining spherical. Stars which have been lost by "evaporation" during this contraction shall be disregarded; thus the total mass of the cluster M remains constant. The cluster is said to expand, if $dI/dt > 0$ and to contract when $dI/dt < 0$; expansion being accompanied by a decrease, contraction by an increase in $|V|$.

Equation (1) is the "equation of motion" of the dynamical system, i.e. the globular cluster; it has the energy integral

$$T + V = E, \tag{2}$$

in which the energy constant E is supposed to be negative. With the help of (2) the kinetic energy T may be eliminated from (1), yielding the equation

$$d^2I/dt^2 = -2V + 4E. \tag{3}$$

Since V is an unknown function of r, not many results have hitherto been drawn from equation (3), apart from the Virial Theorem. It will be shown, however, that by giving to (3) a new formulation and by performing one integration, important conclusions can be drawn which apply to the evolution of globular star clusters.

3. The following special assumptions have to be made:

(*a*) During the contraction the value of the potential energy

$$V = -\int_0^R \kappa r^{-1} M(r) dM$$

shall be a *monotonic decreasing* function of the time t; for $t = 0$, i.e. at the initial stage of the evolution, I is supposed to have been very large and T small;

(*b*) The density

$$\rho = \frac{dM(r)}{4\pi r^2 dr}$$

shall be a regular function of r only; likewise the mean square velocity $\overline{c^2}$ of the stars' velocities with regard to their common centre of gravity. Then we may define also the quantity $p(r) = \frac{1}{3}\rho \, . \, \overline{c^2}$ as a regular function of r only; p measures in hydrostatics the pressure, but need be treated here as no more than a convenient parameter. No assumption is made concerning a special velocity distribution.

(*c*) There is no reason to doubt that the two preceding assumptions correspond to the physical conditions obtaining in a globular cluster. The special, but none the less extremely general, assumption that the expression

$$f(r) = -\frac{1}{\rho}\frac{dp}{dr} - \frac{\kappa M(r)}{r^2}$$

does not change its sign within the range of all values of $0 \leqslant r \leqslant R$ is also introduced for mathematical convenience. This assumption represents perhaps only a first approximation to physical conditions but merely implies that during the contraction the cluster is at every moment contracting at every point of its whole volume.

4. Equation (1) can be brought into a new form by introducing first into the expression of the kinetic energy

$$T = \int_0^R \tfrac{1}{2}\rho\overline{c^2}\, dv$$

the expressions

$$p = \tfrac{1}{3}\rho\, c^2 \quad \text{and} \quad dv = 4\pi r^2\, dr.$$

This gives

$$T = 6\pi \int_0^R pr^2\, dr,$$

which integrated by parts yields

$$T = 6\pi[\tfrac{1}{3}pr^3]_0^R - 2\pi \int_0^R r^3\, dp.$$

The first term vanishes, because $r = 0$ at the lower limit and $p = 0$ at the upper limit $r = R$; hence,

$$T = -2\pi \int_0^R r^3\, dp.$$

The potential energy, on the other hand,

$$V = -\int_0^R \kappa r^{-1} M(r) dM$$

can be written

$$V = -4\pi \int_0^R \kappa \frac{M(r)}{r^2} \rho r^3\, dr.$$

Introducing these new expressions for T and V into equation (1) produces

$$\frac{1}{4}\frac{d^2I}{dt^2} = -2\pi \int_0^R \left[\frac{1}{\rho}\frac{dp}{dr} + \kappa\frac{M(r)}{r^2}\right] r^3\rho\, dr. \qquad (1.1)$$

5. In the preceding paper we put the bracket expression on the right-hand side of (1.1) equal to zero, thus obtaining an equation mathematically identical with the equation of hydrostatic equilibrium. By borrowing Emden's mathematical solution of this equation, we thus obtained a structure for the globular clusters

consistent with observations. We now show by using the equation of motion of the cluster in the forms (1.1) and (3) that during its evolution each cluster *must* pass through a state in which d^2I/dt^2 vanishes. In such a case the bracket-expression in (1.1) vanishes and the cluster's structure satisfies the equation of hydrostatic equilibrium

$$\frac{1}{\rho}\frac{dp}{dr} = -\frac{\kappa M(r)}{r^2}.$$

More precisely, it will be proved that, preceding the moment of greatest condensation, there is one and only one instant $t=t_1$ at which $d^2I/dt^2=0$. In order to prove this theorem we start from equation (3):

$$d^2I/dt^2 = -2V+4E.$$

The first term on the right-hand side is positive; the second term is a negative constant.

After performing one integration we obtain

$$\frac{dI}{dt} = -2\int_0^t V(\tau)d\tau + 4Et + c,$$

where c is a new constant of integration.

We shall first consider the case $c=0$; the cluster is then supposed to have been contracting from the beginning solely under its own gravitational field. For sufficiently small values of t, dI/dt and d^2I/dt^2 will both be negative and the cluster will contract.

But since $-V$ is a monotonic increasing function of the time t, the mean value

$$-\overline{V} = -\frac{1}{t}\int_0^t V d\tau$$

is also a monotonic increasing function of the time and thus there must exist a value $t=t_0$ for which

$$-2\int_0^{t_0} V d\tau = -2\overline{V}_{t_0} t_0 = -4Et_0$$

and for which therefore

$$dI/dt = 0.$$

Beyond this moment t_0 we do not investigate the contraction of the cluster.

Again, in accordance with the "mean value theorem for integrals" there must exist a value t_1, $0<t_1<t_0$ for which

$$-V(t_1)t_0 = -\int_0^{t_0} V\,d\tau,$$

so that

$$\frac{dI}{dt} = -2V(t_1)t_0 + 4Et_0 = 0$$

or

$$t_0[-2V(t_1)+4E]=0.$$

Consequently for $t=t_1$,

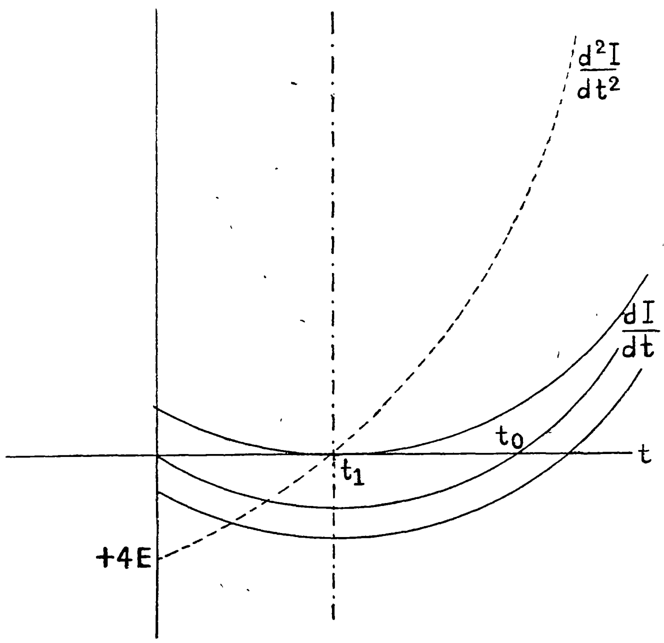
$$d^2I/dt^2 = -2V(t_1)+4E=0,$$

and therefore also in accordance with (1.1)

$$\frac{1}{\rho}\frac{dp}{dr} + \kappa\frac{M(r)}{r^2} = 0.$$

At this epoch, preceding the moment of greatest condensation, the cluster passes through a quasi-stationary state.

6. The general case $c \neq 0$ is best treated in a geometrical way. In the diagram the abscissae measure the time t; the dotted curve illustrates in a schematized way the values of d^2I/dt^2 as a function of the time; the other curves give the values of dI/dt, each curve corresponding to a different value for the constant of integration c. Since d^2I/dt^2 is a monotonic increasing function and starts for $t=0$ from the negative value $4E$, the curve for d^2I/dt^2 must pass through the t-axis. This occurs, as has been proved in the preceding paragraph, for $t=t_1$. The dI/dt curve, for $c=0$, has on the other hand for $t=t_1$ a horizontal tangent and passes the t-axis at a point $t_0 > t_1$. Similarly all dI/dt curves have, when $c \neq 0$ at $t=t_1$, a horizontal tangent; dI/dt reaches here its lowest value and from there on the value rises again.



For negative values of c (only such values are compatible with contraction in the initial phase) the curves pass in one and only one point $t=t_0>t_1$ through the t-axis. When $c>0$, but sufficiently small, the curves pass twice through the t-axis, once for $t<t_1$ and once for $t>t_1$. This is the case for increasing values of c until for $c=c_0$, where

$$c_0 = \int_0^{t_1} [2V(\tau) - 4E]\, d\tau,$$

the t-axis itself becomes at $t=t_1$ a tangent to the dI/dt curve. We consider here only the cases $c \neq 0$; in all such cases there is one and only one instant, preceding the epoch of greatest condensation, at which the cluster passes through a quasi-stationary state, i. e. a state in which its density distribution corresponds to a state of hydrostatic equilibrium.

7. Although the theorem just proved refers to a simplified schematic model of a globular star cluster, it is general enough to be applied to the evolution of actual globular clusters.

The observations indicate that the globular clusters are in or near a quasi-stationary state consisting of an isothermal core surrounded by an adiabatic atmosphere. We may therefore conclude that they are approaching or have