STELLAR MODELS WITH ISOTHERMAL CORES AND POINT-SOURCE ENVELOPES

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ABSTRACT

Stellar models, consisting of isothermal cores and point-source envelopes, are considered, in which the ratio of the mean molecular weight in the core to the mean molecular weight in the envelope varies from 1 to 2. It is found that the upper and lower limit of the mass and radius of the core is a decreasing function of μ_c/μ_e. The evolution of main-sequence stars consequent to the burning of hydrogen in the central regions is also examined.

1. *Introduction.*—The construction of stellar models formed by an isothermal core surrounded by a point-source envelope was discussed by L. R. Henrich and S. Chandrasekhar[1] and by M. Schönberg and S. Chandrasekhar[2] for two values of the ratio μ_c/μ_e, namely, $\mu_c/\mu_e = 1$ and $\mu_c/\mu_e = 2$. It is now proposed to determine the effect of intermediate values on the fraction of the mass and radius inclosed in the convective core, the total radius and luminosity of the configuration, and the ratio of the central to the mean density.

2. *The equations of the models.*—The equation of state of the isothermal core with negligible radiation pressure is

$$P = K_2 \rho \,, \tag{1}$$

where

$$K_2 = \frac{k}{\mu H} T \,. \tag{2}$$

With the foregoing equation of state, the equation of equilibrium,

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho \,, \tag{3}$$

can be reduced to the form

$$\frac{1}{\eta^2}\frac{d}{d\eta}\left(\eta^2\frac{d\phi}{d\eta}\right) = e^{-\phi} \,, \tag{4}$$

where

$$r = \beta\eta \,; \qquad \rho = \lambda_2 e^{-\phi} \,; \qquad \beta = \left[\frac{K_2}{4\pi G\lambda_2}\right]^{1/2} \,, \tag{5}$$

and λ_2 is an arbitrary constant.

The mass of the material inclosed up to a radius η is given by

$$M(\eta) = 4\pi\beta^3\lambda_2\eta^2\frac{d\phi}{d\eta} \,. \tag{6}$$

The equations of equilibrium for the point-source envelope are used in the standard form given in a recent paper.[3]

[1] *Ap. J.*, **94**, 525, 1941.

[2] *Ap. J.*, **96**, 161, 1942.

[3] Harrison, *Ap. J.*, **103**, 196, 1946.

3. *The equations of fit.*—Let the interface occur at $\xi = \xi_i$ and $\eta = \eta_i$. Our equations of fit then are

$$-\tfrac{2}{5}M \frac{\xi^2}{\sigma} \frac{d(\sigma\theta)}{d\xi} = 4\pi\beta^3\lambda_2\eta^2 \frac{d\phi}{d\eta}, \tag{7}$$

$$R\xi = \beta\eta\ , \tag{8}$$

$$\rho_0\sigma = \lambda_2 e^{-\phi}x\ , \tag{9}$$

and

$$\frac{k}{\mu H}\rho_0 T_0 \sigma\theta = \frac{k}{\mu H} T\lambda_2 e^{-\phi}, \tag{10}$$

where, in equation (9)

$$x = \frac{\rho_{\text{env}}}{\rho_{\text{core}}} \simeq \frac{\mu_{\text{env}}}{\mu_{\text{core}}}. \tag{11}$$

This system of equations can be reduced to one involving only two homology invariant combinations of σ, θ, and their derivatives σ', θ', and ξ. If we raise equation (8) to the third power, multiply by equation (9), and divide by equation (7), we are left with

$$-\frac{5}{2}\frac{\sigma^2\xi}{(\sigma\theta)'} = \frac{\eta e^{-\phi}}{\phi'}x\ , \tag{12}$$

which we re-write in the form

$$-\frac{5}{2}\frac{\sigma^2\xi}{(\sigma\theta)'} = u(\xi) = U(\eta) = \frac{\eta e^{-\phi}}{\phi'}x\ . \tag{13}$$

When we divide equation (7) by equation (10), multiply by equation (9), and divide by equation (8), we have

$$-\frac{2}{5}\frac{(\sigma\theta)'}{\sigma\theta}\xi = \tfrac{2}{5}\eta\phi' x\ , \tag{14}$$

which we re-write in the form

$$-\frac{2}{5}\frac{(\sigma\theta)'}{\sigma\theta}\xi = v(\xi) = V(\eta) = \tfrac{2}{5}\eta\phi' x\ . \tag{15}$$

4. *The construction of the models.*—Our fitting conditions are those which are usually used, namely, that the mass, radius, and pressure are continuous at the interface; and we have also made allowance for the difference in the mean molecular weights of the core and envelope by making the density discontinuous at the interface (cf. eq. [11]). The physical characteristics of the models are determined by the method used in the previous investigation with the same two sets of integrations of the point-source envelope.[4]

5. *The physical characteristics of the models.*—From equation (15) we readily obtain the formula for the radius R,

$$R = \tau\frac{\mu_c H}{k}\frac{GM}{T_c}, \tag{16}$$

where

$$\tau = \frac{1}{v_i}\frac{2}{5}\frac{\psi_i}{\xi_i}\frac{\mu_e}{\mu_c}. \tag{17}$$

The variation of R will thus be governed only by the factor $\tau\mu_c$ if T_c is assumed to remain constant. This factor is tabulated in Table 1.

[4] Harrison, *op. cit.*, p. 198.

TABLE 1

PHYSICAL CHARACTERISTICS OF THE COMPOSITE MODEL FOR VARIOUS VALUES OF x

$Q\times10^3$	ξ_i	ψ_i	$\tau\mu_c$	$\frac{L_0}{(\tau\mu_c)^{\frac{1}{2}}}\times10^{-25}$	$\rho_c/\bar{\rho}\times10^{-2}$
			(a) $x=1$		
1.410....	0.1591	0.1777	0.7743	0.8863	0.805
1.547....	.1612	.1937	0.7802	0.9681	0.977
1.776....	.1660	.2315	0.7857	1.108	1.14
1.947....	.1674	.2491	0.7936	1.209	1.31
2.451....	.1673	.2857	0.8181	1.498	1.84
2.750....	.1656	.3001	0.8381	1.662	5.35
3.086....	.1622	.3144	0.8626	1.837	6.47
3.885....	.1588	.3426	0.9124	2.249	7.81
6.157....	.1445	.3673	1.025	3.362	8.79
8.698....	.1224	.3682	1.199	4.391	27.6
8.698....	.0867	.2716*	1.496	3.931	552
6.157....	.0919	.2341*	1.355	2.924	190×10
6.157....	0.1074	0.2720*	1.239	3.058	287×10^3
			(b) $x=1/1.2$		
1.547....	0.1214	0.1111	0.8920	0.9053	0.991
1.776....	.1250	.1390	0.9078	1.031	1.31
1.947....	.1264	.1540	0.9193	1.122	1.51
2.451....	.1268	.1851	0.9589	1.384	2.15
2.750....	.1260	.2000	0.9829	1.535	2.60
3.086....	.1237	.2146	1.016	1.693	3.30
3.885....	.1193	.2380	1.079	2.069	5.07
6.157....	.1061	.2685	1.243	3.053	13.2
8.698....	0.0860	0.2702	1.500	3.926	53.4
			(c) $x=1/1.4$		
1.776....	0.1032	0.1023	0.984	0.990	1.66
1.947....	.1040	.1139	1.002	1.075	1.93
2.451....	.1036	.1394	1.056	1.319	2.86
2.750....	.1023	.1515	1.089	1.457	3.56
3.086....	.0999	.1614	1.130	1.606	4.47
3.885....	.0948	.1832	1.216	1.948	7.82
3.885....	.0596	.1251*	1.497	1.756	180×10
3.885....	0.0703	0.1401*	1.401	1.815	106×10^3
			(d) $x=1/1.6$		
1.776....	0.0896	0.0838	1.036	0.964	2.11
1.947....	.0897	.0932	1.060	1.045	2.53
2.451....	.0883	.1149	1.127	1.277	4.00
2.750....	.0867	.1251	1.169	1.406	5.20
3.086....	.0835	.1350	1.225	1.542	7.25
3.885....	.0769	.1503	1.341	1.855	14.5
3.885....	.0520	.1160*	1.585	1.706	520
3.086....	.0536	.0972*	1.463	1.410	166×10
3.086....	0.0624	0.1064*	1.378	1.454	958×10^2

*This intersection corresponds to one for which ψ_i has already reached its maximum and thus is not a physically reliable solution.

TABLE 1—*Continued*

$Q\times10^3$	ξ_i	ψ_i	$\tau\mu_c$	$\frac{L_0}{(\tau\mu_c)^{\frac{1}{2}}}\times10^{-25}$	$\rho_c/\bar{\rho}\times10^{-2}$
			(e) $x=1/1.8$		
1.947....	0.0795	0.0811	1.106	1.024	3.36
2.451....	.0770	.0998	1.189	1.243	5.82
2.750....	.0738	.1072	1.250	1.360	8.16
3.086....	.0705	.1164	1.313	1.489	12.8
3.086....	.0484	.0924*	1.516	1.386	451
2.750....	.0485	.0813*	1.458	1.260	111×10
2.451....	.0509	.0747*	1.378	1.155	247×10
2.451....	0.0569	0.0793*	1.324	1.178	258×10²
			(f) $x=1/2$		
1.776....	0.0720	0.0654	1.115	0.930	3.57
1.947....	.0714	.0730	1.149	1.004	4.54
2.451....	.0677	.0893	1.247	1.213	9.01
2.750....	.0638	.0955	1.319	1.324	14.3
3.086....	.0569	.1006	1.429	1.427	36.3
3.086....	.0501	.0940*	1.498	1.394	105
2.750....	.0454	.0790*	1.493	1.244	360
2.451....	0.0454	0.0712*	1.442	1.128	698

For a homologous family of stellar configurations based on Kramers' law of opacity, the luminosity is given by a formula of the form

$$L=\frac{L_0}{Q\mu^{7.5}}\frac{M^{5.5}}{R^{0.5}}\mu_e^{7.5}\times1.809\times10^{-28}, \tag{18}$$

where Q is a parameter defined as in a previous paper[5] and L_0 can be determined from the particular value of Q which satisfies the equations of fit. Combining equations (17) and (18), we have

$$L=\frac{L_0}{(\tau\mu_c)^{0.5}}\left(\frac{k}{GH}\right)^{0.5}\frac{1}{Q\mu^{7.5}}T_c^{0.5}M^5\mu_e^{7.5}\times1.809\times10^{-28}. \tag{19}$$

It is clear from equation (19) that, if the central temperature is assumed constant, the variation in the luminosity will be governed by the factor $L_0(\tau\mu_c)^{-0.5}$; this factor is tabulated in Table 1.

The ratio of the mean to the central density is also readily found from equations (5) and (6),

$$\frac{\rho_c}{\bar{\rho}}=\frac{\psi_i\eta_i}{3\,\xi_i^3\phi_i'}. \tag{20}$$

This quantity is also tabulated in Table 1.

6. *Conclusions.*—The curves in Figure 1 illustrate the variation of the radius and luminosity of a star for different values of μ_c/μ_e and constant central temperature for the composite model and the generalized Cowling model.[6] The upper and lower figures on each curve are ψ_i and ξ_i, respectively. It is clear from an inspection of the curves that

[5] *Ibid.*, p. 196.

[6] Harrison, *Ap. J.*, 96, 343, 1944.

the upper and lower limit of the mass and radius of the core is a decreasing function of μ_c/μ_e; the maximum value is reached for $\mu_c/\mu_e = 1$ and the minimum value for $\mu_c/\mu_e = 2$.

By using Figure 1 we can trace the evolution of main-sequence stars. During the relatively early stages of the evolution of a star, as a result of hydrogen combustion in the center, there will be either the formation of an isothermal core or a shrinkage of the convective core. In the latter case the evolution would take place along the curve for the generalized Cowling model (also shown in Fig. 1) in the direction of decreas-

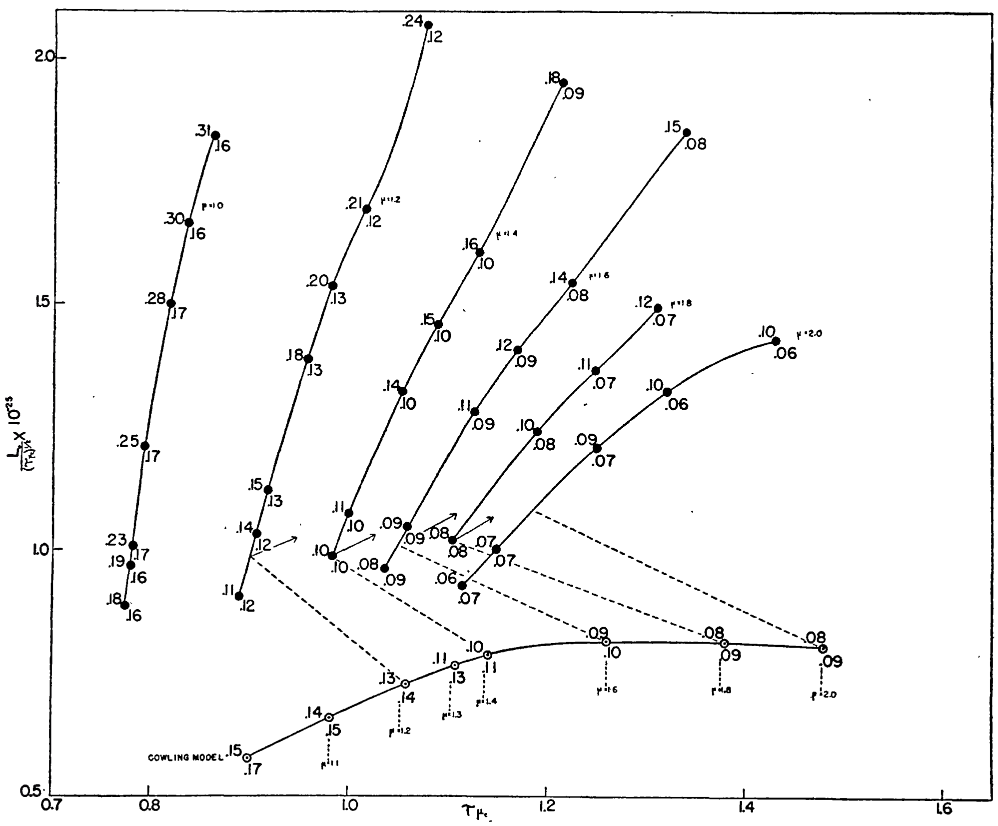

Fig. 1.—The curves illustrate the variation in the luminosity for constant central temperature as a function of the radius for stellar models with isothermal cores and point-source envelopes and the generalized Cowling model. The dotted lines and arrows trace the evolution of main-sequence stars.

ing ψ_i and increasing μ_c/μ_e. The growth of an isothermal core at the center would slow down, and finally stop, convection when the fraction of the stellar mass inside it exceeded a value depending on μ_c/μ_e as shown at the lower end of each curve (●). The transition from one model to another would take place somewhat along the dotted lines as shown in the figure; this transition can take place only at points on the isothermal curves where the mass of the core is equal to the mass of the core of the generalized Cowling model. In the course of the evolution the radius slowly contracts, thus giving rise to an increase in the luminosity which is necessary for the transition. The evolution then continues along a sequence formed by isothermal cores surrounded by point-source envelopes evolving across these curves in the direction of increasing μ_i.

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