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ON THE DYNAMICAL STABILITY OF STARS

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ABSTRACT

It is well known that if the ratio of specific heats, Γ, in a star is a constant and smaller than $\frac{4}{3}$, the star is dynamically unstable. In this paper the case in which Γ is variable in the star is discussed in some detail, especially for the homogeneous and standard models. It is shown that if Γ becomes smaller than $\frac{4}{3}$ in a part of the star, that part has to be very extensive to render the star unstable. For example, if $\Gamma = \frac{5}{3}$ in the central part of a star and is as small as 1 in the outer part, this has to extend to a depth at which the temperature is of the order of half the central temperature. Peculiar forms of $\Gamma(r)$ could increase the instability considerably, but it is doubtful whether any of these could be of physical importance.

1. *General considerations.*—It has been known for a long time[1] that a star with a constant ratio of specific heats, Γ, smaller than $\frac{4}{3}$ is dynamically unstable. If Γ is variable, this should apply to a certain mean value of Γ.[2] This becomes clearer if we consider a formula such as the one given by the author in a recent paper.[3] For a small radial adiabatic deformation, such that

$$\frac{\delta r}{r_0} = \xi\,(r_0)\, e^{i\sigma t},$$

σ^2 is given by

$$\sigma^2 = -\frac{\int_0^R (3\Gamma - 4)\,\xi\, d\Omega_0 + 3\int_0^R \frac{P_0}{\rho_0}\,\xi\, r_0\,\frac{d\Gamma}{d r_0}\, dm}{\int_0^R \xi\, d I_0}, \tag{1}$$

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1 Ritter, *Anwendungen des mechanisches Wärmetheorie auf kosmologische Probleme*, 1879. The result derived there concerns a homogeneous star. For a more general result see, e.g., S. Rosseland, *Pub. Oslo U. Obs.*, No. 1, p. 20; or S. Chandrasekhar, *An Introduction to the Study of Stellar Structure*, 1939, p. 52, and references given there.

2 Cf. L. Biermann and T. G. Cowling, *Zs. f. Ap.*, **19**, 1, 1939 (first part of the paper, where a formula of type [1], except for the term in $d\Gamma/dr_0$, is used).

3 P. Ledoux, *Ap. J.*, **102**, 56, 1945.

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where the suffix zero refers to equilibrium values. Furthermore,

$$\Omega_0 = -\int_0^R \frac{Gm(r)\,dm}{r_0} = -3\int_0^R P_0 dV_0 \quad \text{and} \quad I_0 = \int_0^R r_0^2 dm$$

represent, respectively, the gravitational potential energy and the moment of inertia with respect to the origin, and ξ is the solution of the differential equation

$$\frac{d^2\xi}{dr_0^2} + \left[\frac{4-\mu}{r_0} + \frac{1}{\Gamma}\frac{d\Gamma}{dr_0}\right]\frac{d\xi}{dr_0} + \left[\frac{\sigma^2\rho_0}{\Gamma P_0} - \frac{\mu}{r_0^2}\left(3-\frac{4}{\Gamma}\right) + \frac{3}{\Gamma r_0}\frac{d\Gamma}{dr_0}\right]\xi = 0\,, \quad (2)$$

satisfying the boundary conditions

$$\xi r_0 = 0 \quad \text{at} \quad r_0 = 0 \quad (3)$$

and

$$\delta P = -\Gamma P_0\left[3\xi + r_0\frac{d\xi}{dr_0}\right] = 0 \quad \text{at} \quad r_0 = R\,. \quad (4)$$

In equation (2), $\mu = Gm(r)\rho_0/r_0P_0$, and Γ is the general adiabatic exponent defined by $dQ = dU - Pd\rho/\rho^2 = 0$, where the variation of the internal energy dU is expressed in terms of P and ρ. In a star, Γ is, in general, a function of the ratio of the pressure of radiation to the total pressure $(1-\beta)$, of the degree of ionization, and of the number of degrees of freedom of the particles.[4]

Equation (2) reduces to Eddington's equation for small radial pulsations if Γ is constant. Formula (1) can also be written as

$$\sigma^2 = \frac{9\int_0^R \xi\Gamma P_0 dV_0 + 4\int_0^R \xi\, d\Omega_0 + 3\int_0^R \Gamma P_0 r_0 \frac{d\xi}{dr_0}\, dV_0}{\int_0^R \xi\, dI_0}\,, \quad (5)$$

if we integrate the last term of the numerator by parts.

If σ^2, as given by equation (1) or equation (5), is positive, a small radial oscillation will result, which will be damped out or not according to the condition of vibrational stability.[5] But if σ^2 is negative, the star will be violently unstable, as the small disturbance considered will grow exponentially with the time.

Cases have been considered in which, because of a very small abundance of hydrogen and helium, Γ becomes smaller than $\frac{4}{3}$ in an appreciable part of the star.[6] The stability of these models has been investigated[2] by a general method, using the minimal property of the total available energy of the star in case of stability. However, the rigorous application of such a condition is rather difficult, and it seems worth while to re-examine the problem from the point of view of the sign of σ^2.

If Γ is a constant and if ξ is the amplitude of the fundamental mode, then it readily follows from equation (1) that the star is unstable for this mode if Γ is smaller than $\frac{4}{3}$.

The case of the harmonics, however, is not so simple. If Γ_c^n denotes the critical value of Γ for the harmonic of order n, then, on the basis of the general theory of equations of type (2), we can expect that Γ_c^n will be smaller, the higher the order n of the harmonic considered. We shall see that this is true in the examples which have so far been discussed in some detail. Thus, in general, we may expect that if Γ is a constant for the whole

[4] R. H. Fowler and E. A. Guggenheim, *M.N.*, **85**, 961, 1925.

[5] Cf., e.g., T. G. Cowling, *M.N.*, **94**, 768, 1934; **96**, 42, 1935; **98**, 528, 1938; and P. Ledoux, *Ap. J.*, **94**, 537, 1941.

[6] L. Biermann, *Zs. f. Ap.*, **18**, table on 356, 1939; also n. 2.

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star, it is through the fundamental mode that the stability of the star will first be endangered when we go along a series of models with decreasing Γ.

Considering, next, the general case of variable Γ, we shall first re-write equation (2) in the form

$$\frac{d}{d r_0}\left(p \frac{d\xi}{d r_0}\right) - q\xi + \sigma^2 r_0^4 \rho_0 \xi = 0 , \tag{2'}$$

where

$$p = \Gamma P_0 r_0^4 \quad \text{and} \quad q = \left[(3\Gamma - 4)\, Gm\,(r)\; r_0 \rho_0 - 3\, r_0^3 P_0 \frac{d\Gamma}{d r_0}\right] .$$

Now let Γ_m be the minimum value of Γ along r. Denoting by a suffix m the values corresponding to Γ_m, we have $p_m \leqslant p$ and, if $d\Gamma/dr_0$ is everywhere negative or zero, $q_m \leqslant q$. Hence $\sigma_n^2 \geqslant (\sigma_n^2)_m$.

If $d\Gamma/dr_0$ is positive in some regions of the star, we cannot say anything rigorous about the relative values of q and q_m. But, in general, the second term in q is small compared to the first one, and we can expect that in most cases the preceding inequalities will hold. Thus if Γ_m is greater than Γ_c^n, the star will be stable toward the harmonics of order n or greater, and the fundamental mode remains the most dangerous one for the stability of the star. Since $\Gamma_c^0 = \frac{4}{3}$, the star will be stable if Γ is greater than $\frac{4}{3}$ all over the star.

However, there might be exceptions; for, writing equation (1) in the form

$$\sigma^2 \int_0^R \xi\, d I_0 = -\int_0^R 4\pi \xi\, r_0^3 \frac{d}{d r_0}[\,(3\Gamma - 4) P_0]\, d r_0 , \tag{1'}$$

let us consider again the fundamental mode. The pressure P is essentially positive and decreases from the center to the surface, where it can be taken as being zero. If $(3\Gamma - 4)$ is everywhere positive ($\Gamma > \frac{4}{3}$), the star will certainly be stable if $(3\Gamma - 4)P_0$ decreases everywhere with increasing r.

But if $(3\Gamma - 4)P_0$, for instance, remains constant, then $\sigma^2 = 0$, and the star is on the verge of instability, although Γ is greater than $\frac{4}{3}$ everywhere. Of course, this particular case implies an infinite Γ at the surface, and physically it has no meaning.

In fact, if $(3\Gamma - 4)$ is everywhere positive, we can admit for physical reasons only the range of Γ, $0 < 3\Gamma - 4 \leqslant 1$; and $(3\Gamma - 4)P_0$ will be an oscillating function in r, each oscillation corresponding to the ionization of a new shell of electrons of some fairly abundant element. If $(3\Gamma - 4)P_0$ starts at the center with its maximum value (this is likely as long as the central temperature remains high), these oscillations will be superposed on a generally decreasing curve. And if there should be n regions of increasing $(3\Gamma - 4)P_0$, there will be $(n + 1)$ regions of decreasing $(3\Gamma - 4)P_0$—and one of them just near the surface. Starting from the surface, we can associate them by pairs (decreasing, increasing), and the decreasing one will have a greater weight than the increasing one, as the corresponding r and ξ will be greater.

Therefore, on this account, the right-hand member of equation (1′) will tend to be positive; but, furthermore, we are still left with a region of decreasing $(3\Gamma - 4)P_0$ at the center. Thus it is very unlikely that this case could lead to instability.

If $(3\Gamma - 4)P_0$ starts by increasing at the center, then we shall have as many regions in which $(3\Gamma - 4)P_0$ increases as regions where it decreases. The same reasoning would apply, but in this case the argument loses some of its force, as we have no extra region of decreasing $(3\Gamma - 4)P_0$ near the center. Thus cases in which an important ionization takes place just at the center might require a more careful analysis.

However, for the time being, we shall assume that the fundamental mode is the most dangerous for the stability of the star and that it is stable if Γ is everywhere greater than $\frac{4}{3}$.

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2. *Approximate formula for σ^2.*—Now we shall consider the case in which Γ becomes smaller than $\frac{4}{3}$ in some part of the star. For the fundamental mode of pulsation one usually gets quite a good approximation[3] for σ^2 by assuming ξ to be a constant in formula (5). In that case, equation (5) becomes

$$\sigma^2 = -\frac{(3\overline{\Gamma} - 4)\,\Omega_0}{I_0}, \tag{6}$$

where

$$\overline{\Gamma} = \frac{\int_0^R \Gamma P_0 d V_0}{\int_0^R P_0 d V_0}. \tag{7}$$

Thus in a first approximation the sign of σ^2 and consequently the stability or instability of the star will be determined by the value of Γ, averaged with respect to $\int_0^{r_0} P_0 dV_0$ over the whole star. As the pressure P decreases rapidly with increasing r, the external layers, where Γ is most likely to become smaller than $\frac{4}{3}$, will have to be very extensive to influence $\overline{\Gamma}$ appreciably.

3. *Applications of formula (6).*—As examples we shall consider the cases of the homogeneous and standard models, and for the sake of simplicity we shall further divide the star into two parts: a central region, where $\Gamma_i = \frac{5}{3}$, and an outer part, where $\Gamma_e = 1$, an extreme case very favorable to instability.

If r_c is the radius of the sphere separating the two regions, we know that, for $r_c = R$, σ^2 is positive and the star is stable; for $r_c = 0$, σ^2 is negative and the star is unstable. As σ^2 is an eigen-value of equation (2), which is of the Sturm-Liouville type, it will vary continuously with the coefficients of equation (2), and when r_c decreases from R to zero, σ^2 will decrease continuously, passing through the value zero and becoming negative. Thus σ^2 will have one zero in the interval $0 < r < R$, which will separate the stable from the unstable configurations.

In a first approximation we can determine the critical value r_c corresponding to $\sigma^2 = 0$ from a formula of the type (6), which in this case can be written as

$$I_0 \sigma^2 = 0 = 9\Gamma_i \int_0^{r_c} P_0 d V_0 + 9\Gamma_e \int_{r_c}^{R} P_0 d V_0 + 4\Omega_0 . \tag{8}$$

In the case of the homogeneous model,

$$\Omega_0 = -\tfrac{16}{15}\,\pi^2 G \rho^2 R^5 \qquad \text{and} \qquad P = \frac{2\pi G}{3}\,\rho^2 (R^2 - r^2)\,; \tag{9}$$

and our condition becomes, after introducing the numerical values of Γ_i and Γ_e,

$$\frac{2}{5}\left(\frac{r_c}{R}\right)^5 - \frac{2}{3}\left(\frac{r_c}{R}\right)^3 + \tfrac{2}{15} = 0 .$$

This equation has one root in the interval $0 \leqslant r_c/R \leqslant 1$. We find

$$\frac{r_c}{R} = 0.64 ,$$

or, in terms of mass or temperature,

$$\frac{m_c}{M} \simeq 0.26 \qquad \text{and} \qquad \frac{T_c}{T_0} \simeq 0.6$$

if T_0 is the central temperature (small masses).

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For the standard model $\Omega_0 = -3GM^2/2R$, and in Emden's variables the condition is

$$(3\Gamma_e - 4) + \frac{3}{2}\frac{6.8969}{(2.018)^2}(\Gamma_i - \Gamma_e)\int_0^{z_c} u^4 z^2 dz = 0 . \tag{10}$$

We find that equation (10) leads to the following values:

$$z_c \simeq 1.6 \quad \text{or} \quad \frac{r_c}{R} \simeq 0.23 , \quad \frac{m_c}{M} \simeq 0.35 , \quad \frac{T_c}{T_0} \simeq 0.7 .$$

Thus in both cases we see that, even with a Γ as small as 1 in the outer part, this must be very extensive before the stability of the star is endangered; in fact, the outer part contains appreciably more than half the mass.

However, our approximation ξ = constant, on which equation (8) is based, is rather crude. If we go back to equation (5), we see that if ξ increases with r, the mean value of Γ should, in fact, be smaller than the one defined by equation (7), and this would increase the instability of the star. But, on the other hand, the third term of the numerator would be positive and contribute to the stability.

To gain a definite idea as to the magnitude of these effects a rigorous method of treating the problem will be developed in the following section and applied to the two cases considered here.

4. *Rigorous treatment in the case of a discontinuity of* Γ.—We shall consider again a star composed of two parts, separated by a surface of discontinuity of Γ (sphere r_c), and we shall distinguish by the suffixes i and e the values relating to the internal and the external parts, respectively.

We have to solve equation (2); but in this case, apart from the boundary conditions (3) and (4), we shall have two more conditions at the surface of discontinuity of Γ: a kinematical condition which reduces to

$$A\,(\xi_i)_{r_c} = B\,(\xi_e)_{r_c} \tag{11}$$

and a dynamical condition,

$$(\delta P_i)_{r_c} = (\delta P_e)_{r_c} ,$$

or, explicitly,

$$A\Gamma_i\left(3\,\xi_i + r\,\frac{d\xi_i}{dr}\right)_{r_c} = B\Gamma_e\left(3\,\xi_e + r\,\frac{d\xi_e}{dr}\right)_{r_c} , \tag{12}$$

where ξ_i and ξ_e already satisfy equations (3) and (4), respectively.

The condition that the homogeneous system (11) and (12) admits solutions other than the trivial ones $A = B = 0$ is that its determinant vanish. We must have

$$\left|\begin{matrix} \xi_i & \xi_e \\ \Gamma_i\left(3\,\xi_i + r\,\dfrac{d\xi_i}{dr}\right) & \Gamma_e\left(3\,\xi_e + r\,\dfrac{d\xi_e}{dr}\right) \end{matrix}\right|_{r=r_c} = 0 . \tag{13}$$

As ξ_i and ξ_e are functions of σ^2, equation (13) provides us with an equation to determine its value. However, in general, equation (2) does not admit of analytical solutions which are explicit in σ^2.

The direct procedure would then be for a given value of r_c to choose a value of σ^2, compute by numerical integration the corresponding values of ξ_i and ξ_e, and determine whether they satisfy condition (13). Repeat this until a correct value of σ^2 has been found. If it is positive, move r_c toward the center and start all over again, and so on

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until a critical value of r_c, r_c', is found such that, for $r_c > r_c'$, $\sigma^2 > 0$, and, for $r_c < r_c'$, $\sigma^2 < 0$. This would be a very arduous task.

However, as we are mainly interested in the critical value of r_c, we can obtain it much more simply by putting $\sigma^2 = 0$ in equation (2) and treating r_c as a parameter, which can then be determined by equation (13). In this way only one numerical integration will be required.

a) The homogeneous model.—Writing $x = r/R$ and using relation (9), we can write equation (2) as[7]

$$\frac{d^2\xi}{dx^2}+\frac{4-6x^2}{x(1-x^2)}\frac{d\xi}{dx}+J\xi=0\,, \tag{14}$$

where

$$J=\frac{3\sigma^2}{2\pi G\rho\Gamma}-2\alpha \qquad \text{and} \qquad \alpha=\left(3-\frac{4}{\Gamma}\right).$$

If we suppose that $\sigma^2 = 0$, then

$$J=-2\alpha\,. \tag{15}$$

Equation (14) has two regular singularities, one at $r = 0$ $(x = 0)$ and one at $r = R$ $(x = 1)$. The roots of the indicial equation at $x = 0$ are 0 and -3. The general solution would then have the form

$$\xi_i = A\lambda_i(x) + A'x^{-3}[\phi_i+\lambda_i(x)\,lg\,x]\,, \tag{16}$$

where $\lambda_i(x)$ is holomorphic and ϕ_i regular in the vicinity of $x = 0$. As the solution (16) must satisfy the boundary condition (3), we must take $A' = 0$, and we are left with

$$\xi_i = A\lambda_i(x)\,.$$

T. E. Sterne, in the paper we have already referred to, has shown that $\lambda_i(x)$ is of the form

$$\lambda_i(x)=\sum_{k=0}^{\infty}a_{2k}x^{2k}\,, \tag{17}$$

where the coefficients are determined in accordance with the relations

$$a_0=1 \qquad \text{and} \qquad a_{2k+2}=a_{2k}\frac{2k(2k+5)-J}{(2k+2)(2k+5)}\,. \tag{18}$$

If Γ is a constant over the whole star, then this solution should also satisfy condition (4), which is possible only if J has one of the values

$$J_k=2k(2k+5)\,; \tag{19}$$

for the series (17) will then terminate with the term $a_{2k}x^{2k}$ and will represent the eigenfunction of order k; the corresponding eigen-value of σ^2 can be deduced from equation (19) and the definition of J. For any other value of J than those given by equation (19) the series does not terminate. However, it is converging for $0 \leqslant x < 1$ and diverging only at $x = 1$.

Another point brought out clearly by Sterne's analysis is that, while the fundamental mode becomes unstable for $\Gamma < \frac{4}{3}$, the higher harmonics continue to be stable. Indeed, as we may directly verify from equation (19), the first harmonic becomes unstable only if $\Gamma < \frac{2}{5}$ and the second one if $\Gamma < \frac{4}{21}$.

[7] T. E. Sterne, *M.N.*, **97**, 582, 1937.

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At $r = R$, the indicial equation has the double root 0. We can again, from the boundary condition (4), show that the solution of equation (14) will be of the form $\xi_e = B\lambda_e(x)$; and, expanding $\lambda_e(x)$ as a series in powers of $(1 - x)$, we obtain from equation (14)

$$\lambda_e (x) = \sum_{k=0}^{\infty} b_k (1 - x)^k , \tag{20}$$

with the following recurrence formula for the coefficients:

$$k^2 b_k = \tfrac{1}{2} [(k - 1)(k + 4) - J] (b_{k-1} - b_{k-2}) + (k + 1) [(k - 1) b_{k-1} + b_{k-2}] . \tag{21}$$

If we take

$$J = (k - 1)(k + 4) \tag{22}$$

and if we admit that this value of J also satisfies the relation

$$(k - 1) b_{k-1} + b_{k-2} = 0 , \tag{23}$$

so that $b_k = 0$, it is easy to prove that b_{k+1} and all the succeeding coefficients also vanish and the series will terminate with the term $b_{k-1}(1 - x)^{k-1}$.

However, as we can readily verify with the first few terms, the relation (23) will be satisfied only if k is odd. The series will accordingly terminate with an even power of $(1 - x)$. If Γ is a constant for the whole star, our solution (20) must also satisfy equation (3), and this is possible only if the series terminates, that is, for the values of J given by equation (22) for k odd. One verifies that these J's are identical with those given by equation (19). Thus, starting from the surface, we recover the same eigen-values as those given by Sterne. This is, of course, what we should expect. For any value of J different from the ones given by equation (22), the series (20) converges for $0 < x \leqslant 1$; but it diverges at $x = 0$.

In our case the values of J are fixed by equation (15). With the values $\Gamma_i = \frac{5}{3}$, $\Gamma_e = 1$, which we have already adopted, $J_i = -1.2$ and $J_e = +2$, and the solutions ξ_i and ξ_e will not terminate. However, as our condition (13) has to be applied at a point $0 < x < 1$, we need not be concerned with the possible divergence of the series we have referred to above.

Computing the numerical values of the coefficients a_{2k} and b_k for this particular case and introducing the corresponding solutions (17) and (20) into equation (13), we obtain the following values of x_c or r_c/R:

$$(x_c)_2 = 0.575 , \quad (x_c)_4 = 0.717 , \quad (x_c)_6 = 0.725 , \quad (x_c)_8 = 0.727 ,$$

which correspond, respectively, to approximations limited to terms of degree 2, 4, 6, and 8. These values converge so very rapidly that we may adopt the last value as precise enough for our purpose.

Comparing this value

$$\frac{r_c}{R} = 0.727 \quad \text{or} \quad \frac{m_c}{M} = 0.384 \quad \text{or} \quad \frac{T_c}{T_0} = 0.47$$

with those given by our first approximation, we see that the instability is somewhat greater than is disclosed by equation (6). This corresponds to the fact, illustrated in Figure 1, that ξ increases slightly from the center to the surface, $d\xi/dr$ experiencing a discontinuity when the compressibility changes.

b) The standard model.—The case of the standard model can be treated in the same way, except that we cannot obtain series converging all over the interior of the star. But, before discussing this problem, we shall first determine whether in this case also the gen-

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eral proposition on the stability of the star toward its different modes is satisfied. The exact verification would require numerical integrations. However, we can avail ourselves of the method developed in a previous paper.[8] It was shown there that successive approximations for σ^2 can be obtained by solving determinants of order 2, 4, 6, etc. Because of the form given there to those determinants it is evident that in a given approximation (determinant of rank $2j$) the condition for one or more of the roots σ^2 to be zero is that the minor of rank j in the lower left corner be zero. This provides an equation in $\alpha = (3 - 4/\Gamma)$, which enables us to find successive approximations for the critical Γ_c^n.

In this way we obtain for the first harmonic the critical value $\Gamma_c^1 = 0.891$ in a first approximation and $\Gamma_c^1 = 0.922$ in a second approximation. For the second harmonic, the first approximation gives $\Gamma_c^2 = 0.537$. Of course, this does not provide upper limits for the critical Γ_c; but, as the method converges fairly rapidly as far as the values of σ^2 are

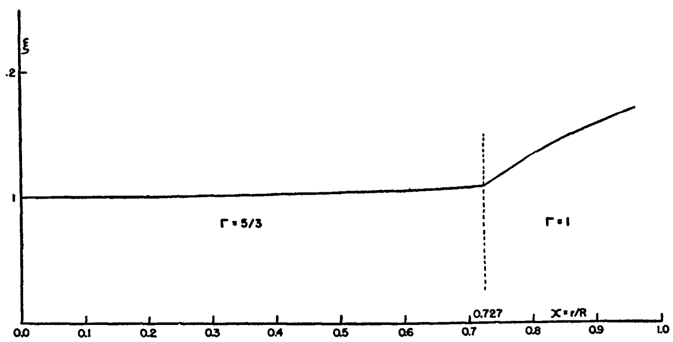

FIG. 1.—Variation of the amplitude with the radius in the homogeneous model for a discontinuity of Γ as indicated.

concerned, we can assume that the same is true of the Γ_c and that their order at least is correct: $\frac{4}{3} > \Gamma_c^1 > \Gamma_c^2$. In the case of Γ_c^1 it is even safe to assume that its réal value will be smaller than 1.

Returning to equation (2), we can re-write it in terms of Emden's variables for the case $\sigma^2 = 0$ in the form

$$\frac{d^2\xi}{dz^2} + \frac{d\xi}{dz}\left[\frac{4}{z} + \frac{4}{u}\frac{du}{dz}\right] + \xi\left[\frac{4\alpha}{z}\frac{1}{u}\frac{du}{dz}\right] = 0\,,$$

where α has the value $\alpha_i = 0.6$ for $z < z_c$ and $\alpha_e = -1$ for $z > z_c$. With the help of well-known series for u, we can obtain series for ξ_i and ξ_e in the neighborhood of $r = 0$ and $r = R$, satisfying conditions (3) and (4), respectively. With these we can start our numerical integrations, one at the center and one at the surface.

The necessary calculations are rather light, as it turns out that the solution which we have to extend furthest is ξ_e. This is the simplest, too, as ξ_e and $d\xi_e/dz$ vary very slowly in a large part of the star. The simplest way to solve equation (13) in this case seems to be to compute its value at different points and then interpolate between its positive and negative values. This shows that the critical value of z_c is of the order of 2.45.

Figure 2 represents the corresponding solution ξ, which exhibits the same peculiarities as the ones discussed in the case of the homogeneous model. However, here ξ in-

[8] P. Ledoux and C. L. Pekeris, *Ap. J.*, **94**, 124, 1941.

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creases more rapidly, as should have been expected from the greater central condensation of the standard model. And, owing to this, the correct values,

$$\frac{r_c}{R} \simeq 0.36 , \qquad \frac{m_c}{M} \simeq 0.66 , \qquad \text{and} \qquad \frac{T_c}{T_0} \simeq 0.47 ,$$

deviate more from the first approximation than in the case of the homogeneous model.

Of course, a varying Γ may mean also a varying mean molecular weight, $\bar{\mu}$, which would result in a slightly different distribution of density. Also the small value of Γ adopted in the external layers would lead to convection, which would again affect the distribution of mass. However, the results just obtained for two models with widely different central condensations show that small deviations, such as those referred to above,

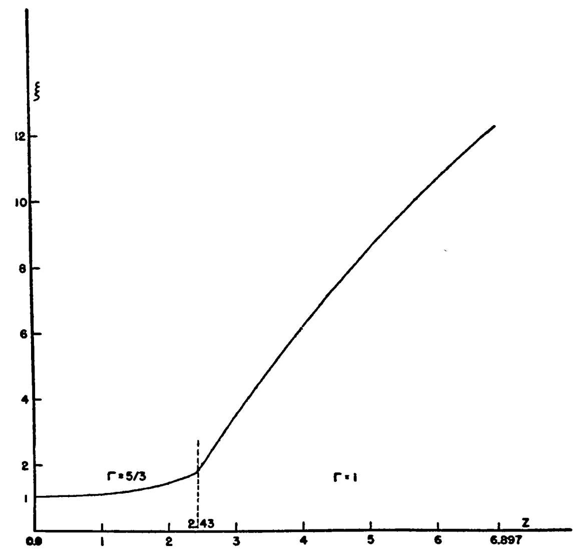

FIG. 2.—Variation of the amplitude with the radius in the standard model for a discontinuity of Γ as indicated.

would affect very little the critical values, especially the critical temperature T_c. Thus we can conclude that, even if Γ_e has its smallest possible value $\Gamma_e = 1$ in the external layers, these layers have to extend to a depth at which the temperature is approximately half the central temperature before the star becomes unstable.

5. *Extension of these results to cases in which* $\Gamma_e \neq 1$.—Of course, Γ_e in practice will not reach such a small value as $\Gamma_e = 1$; and Figure 3 (full curve) shows how the critical ratio, r_c/R, as given by our approximate formula (10), varies with Γ_e for the standard model.

We know the correct critical values for $\Gamma_e = 1$, and another integration in the case of $\Gamma_e = 1.2$ gives

$$\frac{r_c}{R} \simeq 0.22 , \qquad \frac{m_c}{M} = 0.33 , \qquad \frac{T_c}{T_0} = 0.71 .$$

On the other hand, as Γ_e tends toward $\frac{4}{3}$, r_c/R tends toward zero, and ξ tends to become a constant, so that the correct critical values will approach more and more the approximate ones and their variation with Γ_e cannot differ very much from the representation given by the dashed curve. For $\Gamma_e = 1.3$, the external layers have to reach a depth at

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which the temperature is of the order of 88 per cent of the central temperature ($r_c/R = 0.13$; $m_c/M = 0.09$) before the star becomes unstable.

6. *Extension to cases in which* $\Gamma_i \neq \frac{5}{3}$.—On the other hand, until now we have always supposed that $\Gamma_i = \frac{5}{3}$, which corresponds to a negligible radiation pressure, $[(M/M_{\odot})\bar{\mu}^2 \leqslant 1]$. If the mass increases, Γ_i will decrease and tend toward $\frac{4}{3}$ for very large masses.

With $\Gamma_e = 1$, our approximate equation (10) gives the results summarized in Table 1.

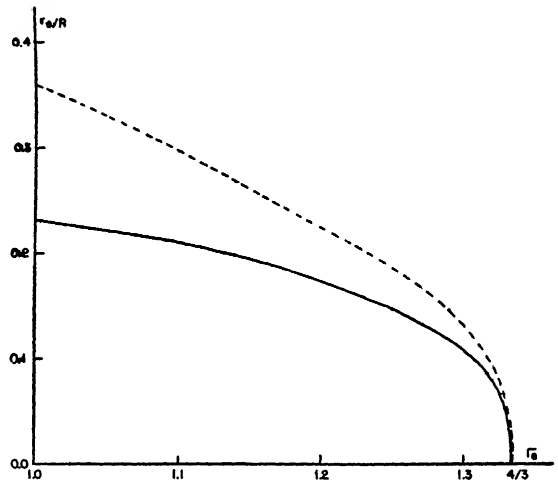

FIG. 3.—Variation of the critical ratio r_c/R with Γ_e in the case of the standard model and $\Gamma_i = 5/3$

TABLE 1

$(M/M_{\odot})\bar{\mu}^2$	Γ_i	r_c/R	m_c/M	T_c/T_0	$(M/M_{\odot})\bar{\mu}^2$	Γ_i	r_c/R	m_c/M	T_c/T_0
1............	5/3	0.23	0.35	0.7	220.6........	1.379	0.41	0.78	0.38
9.14.........	1.538	.27	.46	.62	402.0........	1.366	.43	.82	.36
18.63.........	1.481	.29	.52	.58	1705.9........	1.350	0.49	0.88	0.30
47.69.........	1.429	0.33	0.60	0.51					

These results are also plotted in Figure 4 (full curve). Of course, the real critical values will be different; but we know the correct value for $\Gamma_i = \frac{5}{3}$, and another integration for $\Gamma_i = 1.363$ gives

$$\frac{r_c}{R} \simeq 0.49\,, \qquad \frac{T_c}{T_0} \simeq 0.29\,, \qquad \text{and} \qquad \frac{m_c}{M} \simeq 0.89\,.$$

Again the correct and approximate curves should approach one another as Γ_i tends toward $\frac{4}{3}$, so that we can again tentatively draw a dashed curve, as in Figure 4, to represent the real variations of r_c/R.

Thus, even for large masses, the external layers, where Γ_e is as small as 1, must still be very extensive. For instance, if $(M/M_{\odot})\bar{\mu}^2 = 400$, we find from the dashed curve in Figure 4 that Γ_e must be equal to 1 as far as

$$r_c \simeq 0.48R\,, \qquad \text{where} \qquad T_c \simeq 0.31T_0$$

before the star reaches the limit of stability.

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7. *Small* Γ*'s in the vicinity of the center.*—In contrast to our discussion in the previous sections, we can contemplate the opposite situation and suppose that $\Gamma_i < \frac{4}{3}$ and $\Gamma_e > \frac{4}{3}$. If we take $\Gamma_i = 1$ and $\Gamma_e = \frac{5}{3}$, the two terms in equation (10) simply change sign, and thus in a first approximation we still get the critical values

$$r_c/R \simeq 0.23 \;, \qquad m_c/M \simeq 0.35 \;, \qquad \text{and} \qquad T_c/T_0 \simeq 0.7 \;.$$

As the integrand of the second term in equation (10) has a well-marked maximum around $z = 1.4$, the Γ's corresponding to that region will have the greatest weight in the evaluation of $\bar{\Gamma}$, and therefore it is in that region that a small Γ could have the greater effect.

If we take $\Gamma = 1$ in an interval between z_1 and z_2, having $z = 1.4$ as its mid-point and $\Gamma = \frac{5}{3}$ everywhere else, we find that the star is on the verge of instability in a first approximation, when $z_1 \simeq 0.9$ and $z_2 \simeq 1.9$. Thus for instability the region of $\Gamma = 1$ must extend from $r/R \simeq 0.13$ to $r/R \simeq 0.28$; accordingly, it occupies a spherical shell of thickness $R/7$ and has a mass of about $0.4\ M$; the temperatures at its boundaries are, respectively, 88 and 61 per cent of the central temperature.

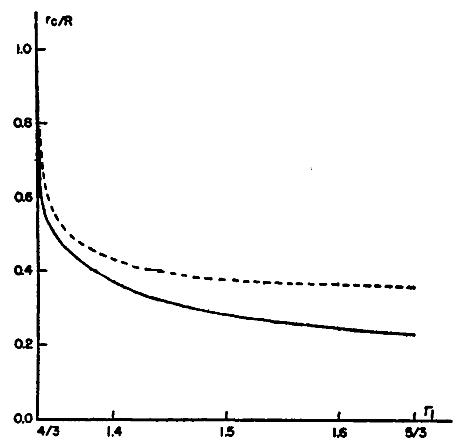

FIG. 4.—Variation of the critical ratio r_c/R with Γ_i in the case of the standard model and $\Gamma_e = 1$

8. *Numerical applications.*—No detailed application has been carried out, but often the results given in the preceding sections enable one to decide whether a star is stable or not. In general, it appears that the conclusions of L. Biermann and T. G. Cowling[2] will be confirmed.

As an example we shall treat the case of the sun. R. H. Fowler and E. A. Guggenheim[4] have shown that, quite generally in the interior of a star, only two consecutive states of ionization of a given element need be considered at the same time. For a given temperature T and electron concentration N_e, there will be approximately as many atoms in one state as in the other if the difference of ionization energy between them is of the order of ψ, as defined by

$$e^{\psi/kT} = \frac{2\,(2\pi m_e kT)^{3/2}}{h^3 N_e}\,. \tag{24}$$

Under these circumstances we can also assume, following Biermann, that all the states with ionization potentials $\chi \leqslant \psi - kT$ will be ionized and those with ionization potential $\chi \leqslant \psi + kT$ will be un-ionized.

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For the sun the standard model gives a fairly good representation, and, for an abundance of hydrogen by weight X of the order of $\frac{1}{3}$, it leads to a central temperature of 2×10^7 degrees and a central density of the order of 75 gm/cm^3. Under these conditions the ionization will already be well advanced, and in a first approximation one can write

$$N_e = \frac{\rho}{m^H}\left(\frac{1+X}{2}\right). \tag{25}$$

With this value of N_e, equation (24) gives for ψ a value of the order of 4400 ev. at the center of the sun. Looking at a table of ionization potentials, one realizes immediately that no important ionization will set in here. Thus, near the center of the sun, the matter can be considered as behaving like a monatomic gas, and, as the radiation pressure can be neglected, we can take $\Gamma = \frac{5}{3}$.

At the point where the temperature drops to half, its central value ψ is still of the order of 3300 ev., and we can again take $\Gamma = \frac{5}{3}$. But we should require Γ to be equal to 1 here and in the rest of the star to bring it to the verge of instability, and this is obviously impossible.

Even for a vanishing abundance of hydrogen, if we take the following mixture to represent the relative abundance by number of atoms of the other elements: $O:Mg:Fe = 8:3:1$, the same considerations as before show that the sun would still be stable.

These results can easily be extended to the other stars of the main sequence. However, for stars of very large masses the pressure of radiation becomes important; and, as that case was not considered by Biermann and Cowling, we shall treat an example. Let us take an extreme case—the Trumpler stars—and let us consider especially NGC 6871.5, for which $M \simeq 400\, M_\odot$ and $R \simeq 16.6\, R_\odot$. If we suppose that it is built on the standard model and composed of pure hydrogen, the central temperature, T_0, is of the order of 10^8 degrees and the central density of the order of 6.6 gm/cm^3. The ratio β of the gas pressure to the total pressure is 0.375. Under these conditions, since hydrogen is very easily ionized, we can assume that, in an extensive region starting from the center, the adiabatic exponent for the gas is $\frac{5}{3}$ and the combined exponent Γ for the matter and radiation will be 1.4. From Figure 4 we see, then, that for instability Γ should be equal to 1 in a region extending from the surface to $r \simeq 0.43R$, where $T \simeq \frac{1}{3}T_0$. But at this temperature the hydrogen is still completely ionized, and the region where Γ is equal to 1.4 will extend much farther toward the surface. Thus the star is stable.

If the star does not contain any hydrogen, the central conditions remain more or less the same, as the increase of $\bar{\mu}$ is practically compensated by the decrease of β, which becomes $\beta = 0.1$. At the center the gas can again be considered as monatomic, and $\Gamma = 1.35$. With this value of Γ in the central part we see from Figure 4 that, for instability, Γ should be equal to 1 from the surface to a point $r \simeq 0.54R$, where $T \simeq (\frac{1}{4})T_0$. But, for that temperature and the corresponding density, ψ is still of the order of 20,000 ev., and Γ will retain its value 1.35 much further, and the star is still stable.

However, it is known from other evidence[9] that the standard model gives a very poor representation of the internal structure of the Trumpler stars and a homogeneous model probably would be better. The problem is then a little complicated by the fact that, for such a model, β varies with depth; but for pure hydrogen one can again determine that the star is stable.

For vanishing hydrogen content, the central temperature is of the order of 3.2×10^7 degrees, $\beta \simeq 0.06$, and $\psi \simeq 28{,}000$ ev., so that we can still take for Γ the value corresponding to a mixture of monatomic gas and radiation. We find $\Gamma \simeq 1.343$. At the point where $T = \frac{1}{2}T_0$ and $r \simeq 0.96R$, ψ is still of the order of 12,500 ev., and for the mixture considered above we can still neglect ionization and $\Gamma = 1.4$. Computing Γ at

[9] Chandrasekhar, *op. cit.*, p. 313.

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different points inside that region, we find that it remains very close to its central value, as far as $r \simeq 0.8R$, so that the average taken with respect to $\int_0^{r_0} PdV$ cannot be very different from it, say $\Gamma = 1.35$. Using equation (6), we find that Γ must be smaller than 1 in the external region to reach instability. However, we know that the instability is somewhat greater than revealed by equation (6); and Γ, although greater than 1, will be fairly small in that external region, so that probably we could not have much greater masses built on such a model except if they contain an appreciable amount of hydrogen.

In that respect it seems that the condition of vibrational stability is more restrictive. It has been shown,[10] for example, that a star built on the standard model, for which Kramer's law of opacity and Bethe's law of generation of energy are valid, would become unstable when the quantity $(M/M_\odot)\bar{\mu}^2$ is somewhat greater than 100. Since in the case of NGC 6871.5, $M/M_\odot$ is of the order of 400, we should have to take an extremely large abundance of hydrogen to avoid instability.

A change of model in the sense of a greater homogeneïty would not help, since in that case the amplitude ξ would increase less rapidly from the center to the surface and the stabilizing terms which arise near the surface would have less weight in the integral expressing the condition of vibrational stability.

For stars of very small masses falling in the region of low hydrogen content in the Hertzsprung-Russell diagram, it does not seem either that any instability of the kind considered here will appear. For instance, in the case considered by Biermann[11] in connection with his theory of the nova phenomena, $M \simeq 0.5M_\odot$, $R \simeq \frac{1}{5}R_\odot$, the radiative equilibrium might become unstable from the surface to a point where the temperature is of the order of $10^{6.5}$ degrees to 10^7 degrees. But this is of the order of only one-tenth of the central temperature, which in this case is $T_0 \simeq 10^8$ degrees. From this point to the point where $T = \frac{1}{2}T_0$, Γ is certainly greater than 1. Thus, although the radiative equilibrium can be unstable fairly deep, the star remains dynamically stable, at least for a radial perturbation.

Of course, as Biermann and Cowling[2] have shown, it is for large radius that dynamical instability appears most easily; and past a certain value of the radius (for a given mass) one can determine the minimum abundance of hydrogen necessary to keep the configuration stable. The method developed here could be used to obtain more precise values of the critical radius or of the minimum amount of hydrogen, but it would require detailed computation.

9. *Special cases.*—This discussion has left out some special cases, but it is doubtful whether they have any physical interest. For instance, we limited ourselves to real values of σ^2. But in the case of the Roche's model, T. E. Sterne[7] has shown that all modes are unstable for $\Gamma < \frac{4}{3}$, all the corresponding σ^2 then having an imaginary part.

As the coefficients of equation (2) are essentially real, σ^2 can have only an imaginary part if the same is true of ξ. But we can study ξ in the vicinity of the singularities of equation (2) and see whether, for models having a physical meaning, the appearance of imaginary values is possible. For such models ρ, P, and Γ remain finite and different from zero for $0 \leqslant r < R$. We can also suppose that $d\Gamma/dr$ remains finite in the same interval.

Then equation (2) can have singularities only at $r = 0$ and $r = R$. At $r = 0$, $\mu = Gm(r)\rho/Pr$ tends toward zero as r^2 and $(1/\Gamma)(d\Gamma/dr)$ remains finite. The coefficient of $d\xi/dr$ will therefore tend toward infinity as $4/r$. In the same way the coefficient of ξ will tend toward infinity as $1/r$. Thus $r = 0$ is a regular singularity, and the corresponding indicial equation, $\theta(\theta - 1) + 4\theta = 0$, has real roots only, and the solution will be essentially real near $r = 0$. At $r = R$, the singularities of equation (2) will depend on

[10] P. Ledoux, *Ap. J.*, **94**, 537, 1941.

[11] *Op. cit.*, p. 344.

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the behavior of the ratio ρ/P and the quantity $d\Gamma/dr$. As long as we use equation (2), it seems natural in this connection to take the same point of view as the one adopted in the theories of the internal structure of stars, namely, that P, ρ, and T tend toward zero at $r = R$ and that Kramer's law of opacity is valid.

We may then verify[12] that ρ/P will tend toward infinity as $8/(1 - r/R)$ and that $d\Gamma/dr$, which ultimately varies as $d\beta/dr$, remains finite. Thus $r = R$ is also a regular singularity, and the indicial equation, $\theta(\theta - 1) + 8\theta = 0$, can have only real roots. Thus ξ will be real everywhere, and so will σ^2.

Finally, if a star approaches dynamical instability (σ becomes small), there will be a point at which the usual method of perturbation used to obtain the condition of vibrational stability will cease to be applicable, since the perturbation σ' of σ (due to the non-adiabatic processes) will no longer be small compared to σ and the terms in σ'^2 will not be negligible compared to $\sigma\sigma'$ or σ^2.

It would probably be interesting to study the interaction of these two types of instability and try to obtain a more general criterion of stability.

[12] S. Chandrasekhar, *M.N.*, **96**, 647, 1936; also J. Tuominen, *Ann. Acad. Sci. Fenn.*, Ser. A, Vol. **48**, No. 16, 1938.

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