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THE DYNAMICS OF THE INTERSTELLAR MEDIUM

III. GALACTIC DISTRIBUTION

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ABSTRACT

Measurements of the rotational velocity in NGC 3115 show that the random stellar velocities in this elliptical galaxy are of the order of 200 km/sec; other elliptical systems with masses of 10^{10} M⊙ will also contain stars with the same velocity dispersion. The random velocities of interstellar atoms and dust particles in equilibrium, however, are less than 20 km/sec. Analysis of electron captures and free-free transitions, taking into account the variation of the *g*-factors with energy, shows that atoms of appreciable abundance will, in fact, reach such a low-velocity equilibrium within 10^7–10^8 years; collisions between dust particles will similarly reduce dust velocities within 10^5–10^6 years. Unless interstellar particles are subject to unknown forces, the velocities of these particles in the more massive elliptical and spherical galaxies must be much less than those of the stars.

It follows that any interstellar matter in such stellar systems must be highly concentrated to the center or to the equatorial plane. For a rigorously spherical system the total mass of such matter cannot exceed 4×10^{-4} times the mass of all the stars. In an elliptical system the mass is unrestricted, but if the mass of interstellar matter exceeds that of the stars, then half of such matter must lie within 3 parsecs from the equatorial plane. In any case the light observed from spherical and elliptical systems of large mass must be direct starlight, not diffuse or scattered light.

The equatorial layer of interstellar matter in an elliptical galaxy may in theory contain as much or more matter than the stars in the system. The determination by Oort of the stellar density distribution in the elliptical nebula NGC 3115, and in particular the high-luminosity gradient which he found close to the equatorial plane, suggest that a large fraction of the mass of this system may be in the form of dark matter in the equatorial plane. Matter in so dense a layer would presumably be quite different in its physical state from interstellar matter in our own Galaxy and would perhaps condense into faint stars of low random velocity. Further observations are necessary to decide whether a rotating massive disk of dark matter may play an important part in the structure of some elliptical systems.

The presence of interstellar atoms and dust particles in spiral galaxies is known from direct observation. The question naturally arises as to whether interstellar particles may be present also in elliptical and globular galaxies. The diffuse, nonstellar appearance of most elliptical systems has frequently led to the suggestion that most of the light from such galaxies might consist of diffuse radiation, scattered by small particles.

Direct observations on this point are difficult and by themselves not very conclusive. Fortunately, the equilibrium of interstellar particles in a galactic system is susceptible to rather simple theoretical analysis. Given the forces that act on the particles, it should be possible to demonstrate which equilibrium configurations are possible. It has been shown in the first two papers of this series[1] that the interstellar medium behaves as a

[1] L. Spitzer, Jr., *Ap. J.*, **93**, 369, 1941; **94**, 232, 1941.

perfect gas and that the direct effects of radiation pressure may be neglected for most purposes. The problem is therefore reduced to the equilibrium of a gas in a gravitational field.

The most important quantity in such a problem is the ratio of the random stellar velocities to the velocities of the interstellar particles. Stellar velocities in elliptical galaxies are unfortunately rather uncertain. For one system, NGC 3115, direct measurements of the rotational velocity by Humason[2] are available; the observed values range from 80 km/sec near the center to 450 km/sec farther out. It is clear that since this nebula has a considerable extension perpendicular to its galactic plane, the random stellar velocities must be an appreciable fraction of the rotational velocities and are probably in the neighborhood of 200 km/sec.

For other systems the only relevant data are the random velocities of nebulae in clusters, which indicate a mass of roughly 10^{10} $M_\odot$ or more for an elliptical system.[3] Since the radius[4] containing half the mass of an elliptical system is at most 1000 parsecs, the virial theorem gives a mean square stellar velocity of 200 km/sec. Nebular masses are so uncertain, however, that this value for the random stellar velocities cannot be assumed for all elliptical systems.

Nevertheless, it is of considerable interest to investigate the effects associated with such high galactic masses. The following analysis will therefore be developed for galaxies in which the random stellar velocities are assumed to be of the order of 200 km/sec. The results of the analysis may be applied with some certainty to NGC 3115. It is highly probable that they are relevant to many other elliptical and globular galaxies as well. Whether or not most of the elliptical and globular galaxies have such high masses and such high random stellar velocities cannot be decided until more observational data are available.

In contrast to the assumed velocities of 200 km/sec for stars, the equilibrium random velocities of interstellar particles are quite low, some 20 km/sec for protons and electrons at 10,000°, less for other atoms, and negligibly small for dust particles. Since equilibrium will be reached in much less than 10^9 years, we may infer that the root mean square velocity of interstellar particles does not exceed 20 km/sec. It follows that the root mean square stellar velocities in the elliptical galaxies considered here are some ten or more times as great as the velocities of interstellar particles.

A necessary consequence of this large difference in velocities is that any interstellar matter in such a system must be almost entirely concentrated toward the equatorial plane or, in the case of a spherical system, toward the center. A factor of ten in velocities corresponds to a factor of one hundred in kinetic energy per unit mass. Hence a star in the equatorial plane of an elliptical galaxy will have at least one hundred times as much energy per unit mass as any atoms or dust particles in the vicinity and will clearly be able to rise much farther from the equatorial plane before the gravitational attraction of the galaxy pulls it back.

The analysis proceeds in three separate parts in the following sections. First, the rate of dissipation of energy for atoms or dust particles at high velocities must be determined in order to exclude the possibility that the interstellar medium could have remained at a temperature of millions of degrees during 10^9 years. Second, the equilibrium of a gas within a spherical gravitational field is examined, and a maximum total mass is found for the interstellar matter in a spherical galaxy. In the case of a typical globular cluster the situation is quite different, since the mean square stellar velocities are comparable with the atomic ones. It is doubtful, however, whether some of the smaller clusters have a sufficient gravitational force to hold atoms at all.

[2] *Report of the Director, Mt. W. Obs., 1936–37*, p. 31.

[3] S. Smith, *Ap. J.*, **83**, 23, 1936; F. Zwicky, *Ap. J.*, **86**, 217, 1937.

[4] E. Hubble, *The Realm of the Nebulae*, p. 178, Yale University Press, 1936.

In the third and last section the equilibrium of dust and atoms in systems possessing angular momentum is investigated. Here again the relative concentration of the interstellar medium may be evaluated, and as in a spherical system it may be shown that the amount of interstellar matter throughout most of a massive elliptical galaxy must be quite small. The effect which a thin, dense equatorial layer of dark matter can produce on the distribution of stars is also examined. A comparison of the predicted effects with observational data in NGC 3115 suggests that this galaxy may, in fact, possess a large amount of dark matter concentrated to its equatorial plane.

1. DISSIPATION OF ENERGY

In the general case of a gas at high temperature the dissipation of energy will proceed in many ways. To obtain an upper limit for the time required to dissipate large energies, we shall here consider the two types of interstellar media which are the most likely to remain at an initially high temperature—a medium consisting wholly of hydrogen atoms and one composed exclusively of dust particles. In each case the rate of loss of energy will be computed; possible mechanisms for an offsetting increase in energy will be considered later. Since dust and atoms are considered wholly separately, the following results are independent of the equipartition of kinetic energy between dust and atoms which appears when both types of particles are present and which was discussed quantitatively in paper I. It is clear that if both dust and atoms are present together the rate of energy dissipation will be increased from the values found below. This strengthens the conclusion that interstellar particles cannot long remain at high velocities.

A medium composed wholly of hydrogen atoms will be considered first. If the average kinetic energy per atom exceeds the ionization energy E_0, the gas will be largely ionized, since the cross-section for ionization by electronic or atomic impact is much greater than that for electron capture. The resultant assembly of protons and free electrons will come to equipartition of kinetic energy fairly rapidly. It was shown in paper I that for protons and electrons with an assumed density of 1 atom per cm^3 and a kinetic temperature of 10,000°, deviations from equipartition of energy fell to $1/e$ their initial value in 0.48 years. Since this time varies as the cube of the velocity, it is evident that even for velocities of 600 km/sec, corresponding to a temperature of 14,000,000°, the time of equipartition is only 2.7×10^4 years. We may therefore use with confidence the appropriate Maxwellian velocity distributions for both protons and electrons.

A gas of protons and electrons may lose energy by radiation in two ways—electron captures and free-free transitions. Probabilities for these processes are known from wave-mechanical theory. It is frequently assumed in astrophysical investigations that the Gaunt "g-factor" is equal to unity. This is a valid approximation for low energies but may be badly off when the energies are very high. Accurate values of g have been exhaustively determined by Menzel and Pekeris.[5] The equations given by Bethe,[6] based on the analyses of Sommerfeld, Stobbe, and others, are less complete but are in a more convenient form for the present purpose. The rate of loss of energy will be computed from these equations for two processes—recapture of electrons directly in the ground state and radiation of energy in free-free transitions. If other processes are important, they will, of course, increase the rate of energy dissipation.

Let E and v be the energy and velocity, respectively, of an electron, and let E_0 be the ionization energy of hydrogen. The energy radiated in a single recombination into the ground state will be a function of the initial relative velocity V; if V were the same for all encounters, the number of such recombinations per second would be $\pi\sigma_c^2 n_p n_e V$, where $\pi\sigma_c^2$, a function of V, is the cross-section for the capture process, and n_p and n_e are the numbers of protons and electrons per cm^3, respectively. Since the protons have a root

[5] *M.N.*, **96**, 77, 1935.

[6] *Handb. d. Phys.*, **24**, Part I, 488, 1933.

mean square velocity only 1/43 as great as the electrons, we may without serious error replace the relative velocity V by the electron velocity v; with this approximation the energy radiated in a single encounter becomes $E+E_0$. To find the rate of change of $\bar{E}$, the mean energy of protons and electrons, the energy loss per unit time must be averaged over all values of v and divided by $n_p + n_e$, or $2n_e$. This yields the equation

$$\frac{d\bar{E}}{dt} = -\frac{3}{2}\left(\frac{3}{\pi}\right)^{1/2}\frac{n_p}{m^{1/2}\bar{E}^{3/2}}\int_0^\infty E\,(E+E_0)\,e^{-3E/2\bar{E}}\pi\sigma_c^2 dE\,, \tag{1}$$

where m is the mass of the electron.

The capture cross-section $\pi\sigma_c^2$ for the lowest quantum state may be written as

$$\pi\sigma_c^2 = A\frac{E_0^{5/2}}{(E+E_0)\,E^{3/2}}\vartheta_1\left(\frac{E_0}{E}\right), \tag{2}$$

where

$$A = \frac{2^6 e^2 h}{3\,m^2 c^3} = 1.44\times 10^{-21}\text{cm}^2\,. \tag{3}$$

If we let k^2 equal E_0/E, then $\vartheta_1(E_0/E)$ in equation (2) may be written in the form

$$\vartheta_1\,(k^2) = \frac{2\pi k}{1+k^2}\,\frac{\exp\,(-4k\cot^{-1}k)}{1-\exp\,(-2\pi k)}\,. \tag{4}$$

When k is very large or very small we have the expansions

$$\vartheta_1\,(k^2) = \frac{2\pi e^{-4}}{k}\left(1+\frac{1}{3k^2}\,,\,\ldots\right), \tag{5}$$

$$\vartheta_1\,(k^2) = 1-\pi k\,\ldots\,. \tag{6}$$

Since the Gaunt g-factor is given by

$$g = 4\,(3)^{1/2}\left(\frac{E_0}{E}\right)^{1/2}\vartheta_1\left(\frac{E_0}{E}\right), \tag{7}$$

it is evident that when E_0/E is large, g is equal to its familiar value, $8\pi 3^{\frac{1}{2}}e^{-4}$, or 0.797. When E_0/E is small and ϑ_1 is unity, the value of g is very much less than unity. Values of $\vartheta_1(E_0/E)$ are given in Table 1.

TABLE 1

CORRECTION FACTOR ϑ_1

E/E_0	ϑ_1	E/E_0	ϑ_1	E/E_0	ϑ_1
0.01	0.012	1.0	0.14	100	0.74
.02	.016	2.0	.20	200	.81
.04	.023	4.0	.29	400	.86
.06	.029	6.0	.34	600	.88
.08	.034	8.0	.39	800	.89
.10	.038	10.0	.42	1,000	.90
.20	.054	20.0	.52	4,000	.95
.40	.080	40.0	.63	10,000	.97
.60	.102	60.0	.68	40,000	.98
0.80	0.120	80.0	0.72	100,000	0.99

Equations (2) and (4) may be substituted into equation (1) to give an integral formula for $d\bar{E}/dt$. Since this integral can apparently not be evaluated in a simple analytical form, we may approximate by taking $\vartheta_1(E_0/E)$ outside the integral sign with its value for some appropriate value of E. This obviously gives asymptotically correct results as $\bar{E}$ approaches infinity and $\vartheta_1(k^2)$ becomes equal to unity. If this procedure is also to give correct results when $E_0/\bar{E}$ is large, and $\vartheta_2(k^2)$ is approximately given by equation (5), then it is readily shown that the appropriate value of E is $2\bar{E}/3\pi$. This value is relatively small, since, when $E_0/\bar{E}$ is not large, most of the energy losses come from the recapture of low-velocity electrons. With this approximation, then, $d\bar{E}/dt$ becomes

$$\frac{d\bar{E}}{dt} = -\frac{3\,A\,n_p E_0^{5/2}}{2^{1/2} m^{1/2} \bar{E}} \vartheta_1 \left(\frac{3\pi E_0}{2\bar{E}}\right). \tag{8}$$

This method of procedure, although correct when $\bar{E}$ is very large or very small, is not very accurate when $E_0/\bar{E}$ is near unity; the resultant error in equation (8) should be less than some 20 per cent at the most, however, which is as accurate as we shall need here. Since E_0 varies as the square of the ionic charge, it is evident that a few highly ionized elements will very greatly increase the rate of radiation.

To visualize conveniently the order of magnitude of the effects involved, one may define a time of dissipation T_d by the relationship

$$T_d = \frac{\bar{E}}{\left|\frac{d\bar{E}}{dt}\right|}. \tag{9}$$

The quantity T_d so defined is the length of time required for the complete dissipation of energy by radiation if the rate of dissipation remained constant as $\bar{E}$ decreased. Since actually $|d\bar{E}/dt|$ increases as $\bar{E}$ decreases, the actual time required for the cooling of a gas at high temperature is somewhat less than T_d. If we insert numerical values into equation (8) and substitute into equation (9), we have for the time of dissipation arising from recombinations in the ground state

$$T_d\,(\text{rec}) = \frac{6.6\times 10^4}{n_p \vartheta_1\,(4.7E_0/\bar{E})}\left(\frac{\bar{E}}{E_0}\right)^2 \text{ years}. \tag{10}$$

Values of T_d(rec) for n_p equal to unity and for various root mean square proton velocities are given in Table 2.

TABLE 2

VALUES OF THE TIME OF DISSIPATION

v Km/Sec	100	200	400	600	1000
$\bar{E}/E_0$	3.82	15.3	61	137	382
T_d(rec)	8.0×10^6	6.0×10^7	5.3×10^8	2.6×10^9	1.4×10^{10}
T_d(*f-f*)	5.3×10^6	1.1×10^7	2.1×10^7	3.2×10^7	5.3×10^7
T_d(dust)	1.1×10^6	5.5×10^5	2.8×10^5	1.8×10^5	1.1×10^5

NOTE

Values of T_d give the time in years required to dissipate the entire kinetic energy of the assembly if the rate of dissipation remained constant and if no other process were acting.

The free-free transitions may be similarly treated, except that here we must integrate over all possible transitions as well as over all velocities. The exact equations become very complicated, and it will suffice to take the asymptotic form for small $E_0/\bar{E}$, which becomes

$$\frac{d\bar{E}}{dt}=\frac{3}{8\pi}\left(\frac{3}{\pi}\right)^{1/2}\frac{A\,n_p E_0}{m^{1/2}\bar{E}^{3/2}}\int_0^{\infty}E\,e^{-3E/2\bar{E}}\vartheta_2\left(\frac{E_0}{E}\right)dE\,, \tag{11}$$

where the various symbols have the same meanings as before. If we again let k^2 equal E_0/E, we have

$$\vartheta_2\left(k^2\right)=2\pi k\int_1^{\infty}\frac{1}{1-\exp\left(-2\pi u k\right)}\log\frac{u+1}{u-1}\frac{du}{u^2}\,. \tag{12}$$

These results may be derived either from the formula given by Bethe, which is valid only for large electron energies, or from the asymptotic result given by Menzel and Pekeris,[7]

$$g=\frac{2\cdot 3^{1/2}l}{1-\exp\left(-2\pi l\right)}\log\frac{l+k}{l-k}\,, \tag{13}$$

where k^2 equals E_0/E as usual and l equals E_0/E'; E and E' are the initial and final energies of the electron, respectively. The quantity u in equation (12) is the ratio of l to k.

When E is large, $2\pi(E_0/E)^{\frac{1}{2}}$ is small, and for most of the relevant range of integration in equation (12) the denominator may be replaced by $2\pi uk$. The resultant integral may then be evaluated exactly, and we find that ϑ_2 is unity. Equation (11) can then be integrated, and we have

$$\frac{d\bar{E}}{dt}=-\frac{A\,n_p E_0\bar{E}^{1/2}}{2\pi\left(3\pi\right)^{1/2}m^{1/2}}\,. \tag{14}$$

This is equivalent to a formula given by Menzel,[8] provided that $\bar{g}$ is set equal to $2\cdot 3^{\frac{1}{2}}/\pi$, or 1.10. Thus we see that the average value of g is very nearly unity, despite the fact that g varies from a small quantity to infinity over the range of integration. When $\bar{E}$ is small, g may be set equal to unity, as may be seen from the series expansions given by Menzel and Pekeris.[5] We may therefore assume that equation (14) is valid for intermediate velocities as well as for high velocities.

If we define T_d as in equation (9), the time of dissipation arising from free-free transitions becomes

$$T_d\left(f\text{-}f\right)=2.7\times 10^6\frac{1}{n_p}\left(\frac{\bar{E}}{E_0}\right)^{1/2}\text{years}\,. \tag{15}$$

Values of $T_d(f\text{-}f)$ are shown in Table 2 for various values of the root mean square proton velocity; the number of protons per cm^3 is again set equal to unity. It is evident from this table that, for most velocities, recaptures in the ground state are of negligible importance in the dissipation of energy. This is presumably true for recaptures in other lower states as well. The condition for continuity at the series limit indicates that recaptures in the higher quantum states will behave in the same way as the free-free transitions. Such recaptures will, of course, decrease the time of dissipation even further from the values given in Table 2.

Lastly, the dissipation of energy by direct encounters between dust particles must be computed. Let us consider an encounter between two particles of equal mass, m_d, such that the center of gravity of the two particles is at rest and their original relative velocity is V_1. Let the perpendicular distance between the original paths be equal to p_1. Then the total initial energy and angular momentum of the two particles is $\frac{1}{4}m_d V_1^2$ and $\frac{1}{2}p_1 m_d V_1$.

[7] Ref. 5, eqs. (1.47) and (1.49).

[8] *Ap. J.*, **85**, 330, 1937, eq. (30); a factor K/h has been omitted from this equation.

If V_2 is the relative velocity of separation and p_2 the value of p after collision, the energy loss per particle becomes

$$\Delta E = \frac{1}{8} m_d \left(V_1^2 - V_2^2 \right) \tag{16}$$

while the conservation of angular momentum yields

$$p_1 V_1 = p_2 V_2 . \tag{17}$$

The elimination of V_2 from equations (16) and (17) gives

$$\Delta E = \frac{1}{8} m_d V_1^2 \left(1 - \frac{p_1^2}{p_2^2} \right) . \tag{18}$$

It is obvious that the total loss of energy must be the same if the center of gravity is not at rest, and we may therefore use equation (18) for the energy loss in every collision.

The maximum loss of energy occurs when p_2 is a maximum. Since p_2 can scarcely exceed 2σ, the geometrical diameter of each particle, we may replace p_2 by 2σ in equation (18) and multiply the resultant expression by ζ, the ratio of the actual energy loss to the maximum loss possible. To find the rate of decrease of the mean kinetic energy $\bar{E}$, we must multiply ΔE by $2\pi p_1 dp_1 V_1 n_d$, where n_d is the number of dust particles per cm^3, and then integrate over p_1 from zero to 2σ. This yields

$$\frac{d\bar{E}}{dt} = -\frac{\pi}{4} \bar{\zeta} \sigma^2 m_d V_1^3 n_d , \tag{19}$$

where a mean value of ζ has been taken.

Finally, we must determine the mean value of the factor V_1^3 in equation (19). If the distribution of dust-particle velocities is assumed to have spherical symmetry, then

$$\overline{V_1^2} = 2 \overline{v_d^2} , \tag{20}$$

where $\overline{v_d^2}$ is the mean square velocity of the individual particles. If the distribution of dust velocities were Maxwellian, we should have

$$\overline{V_1^3} = \frac{2^{7/2}}{3^{3/2} \pi^{1/2}} \left(\overline{V^2} \right)^{3/2} . \tag{21}$$

Although there is apparently no reason why the velocities of very rapidly moving dust particles should obey the Maxwellian distribution, equation (21) should provide an adequate approximation to the true state of affairs, particularly since the numerical factor in equation (21) should not differ much from unity in any case. If we let m_d equal $4\pi\sigma^3 d/3$, where d is the density of matter within the particle, and let n_d equal ρ_d/m_d, where ρ_d is the density of dust in grams/cm^3, we may combine equations (19), (20), and (21) to find

$$\frac{1}{\bar{E}} \frac{d\bar{E}}{dt} = -\frac{4}{(3\pi)^{1/2}} \frac{\rho_d \bar{\zeta}}{\sigma d} \left(\overline{v_d^2} \right)^{1/2} \tag{22}$$

where now $\bar{\zeta}$ denotes an average over V_1 as well as over p_1. With the same definition of T_d as before, we find that if σ equals 10^{-5}cm, d equals 7.8, and ρ_d is 1.7×10^{-24}, equation (22) gives

$$T_d \text{ (dust)} = \frac{1.1 \times 10^{12}}{\bar{\zeta} \left(\overline{v_d^2} \right)^{1/2}} \text{ years} . \tag{23}$$

The quantity $\bar{\zeta}$ is not likely to be very small when $\overline{v_d^2}$ is large. For the rapid collisions we are concerned with here, the energy of impact is very much greater than the binding energy of the particles, which is normally the source of the elastic forces. Hence such rapid collisions will probably be completely inelastic, and two such particles will in all likelihood vaporize when their velocity of impact is several hundred kilometers per second. To find a lower limit on T_d, however, $\bar{\zeta}$ has been set equal to 0.1 in the computation of the values in Table 2.

It is evident from Table 2 that interstellar matter of appreciable density will have lost its high energy in 10^7 years or less. There is no evidence for any mechanism which can avert this loss of kinetic energy. Ionization by radiation provides at most an energy comparable to E_0 for each such recombination; ionization by electron impact is likely to be more frequent, and this involves no net change in $\bar{E}$. Interaction with stars will be small, particularly at such high velocities. For a particle to be deflected 90° by a star of solar mass, moving with a relative velocity of 200 km/sec, the undisturbed path of the particle must come within 4 solar radii of the sun's surface. Even though this distance may be considerably increased by radiation pressure from the brighter stars, such close encounters are rare and may be neglected.

The role of cosmic rays is somewhat more obscure, since little is known about the numbers of low-energy particles, which would be of importance in this connection. Particles of the energy and abundance observed terrestrially may be shown to be unimportant. The number of particles entering the earth's atmosphere per cm^2 per sec is[9] 0.09. Since the cross-section of a light nucleus for cosmic-ray impact is roughly $10^{-25}cm^2$, the mean life between encounters is 10^{18} years. Since the mean energy acquired per encounter is at most 10^{10} e.v., this corresponds to a gain of 1 e.v. per particle in 10^8 years, which is insufficient to offset a loss of several hundred e.v. per particle in 10^7 years.

If the mass of interstellar matter in a galaxy is comparable with that of the stars, it is difficult to find any mechanism which will maintain the particles of the medium in a state of high kinetic energy. An energy loss of 400 e.v. in 10^7 years, corresponding to an initial proton velocity of roughly 300 km/sec, is a loss of 1 erg per gram per second; and to maintain this in a medium of appreciable mass requires an energy source comparable to that of all the stars. There is no evidence whatever for such an additional source of energy. We may conclude that, on the basis of present theory, the mean square velocity of interstellar particles cannot possibly be as high as 100 km/sec if the density of such matter is appreciable; in all probability interstellar matter is in kinetic equilibrium with a velocity temperature between 5,000° and 20,000°, and with velocities of not more than 20 km/sec.

This analysis refers, of course, to purely random velocities. These results, therefore, do not exclude the possibility that the interstellar medium could be supported by large-scale currents with velocities much greater than the microscopic random velocities of the individual particles. If such currents are assumed to be irregular, as in the case of turbulent stellar atmospheres, the energy of macroscopic motion would soon be dissipated in all probability. When two such currents, with velocities far exceeding the velocity of pressure waves in the medium, encountered each other, the directed macroscopic velocities would tend to be converted into random microscopic motions. The only current system which could apparently be stable in 10^9 years or more would be a simple rotation of the medium about the center of the system—a case which is, of course, relevant in an elliptical galaxy. With this exception one may assume that large-scale currents play no direct part in the equilibrium of the interstellar medium.

2. SPHERICAL SYSTEMS

The equilibrium of interstellar matter in a spherical galaxy is simplified when the mean square velocity of the interstellar particles is much less than that of the stars. As

[9] I. S. Bowen, R. A. Millikan, and H. V. Neher, *Phys. Rev.*, **53**, 217, 1938.

we shall see below, the total mass of the interstellar medium in such a case must be much less than that of the stars and will therefore not affect appreciably the structure of the stellar system. The problem therefore reduces to the equilibrium of a gas in a potential field which arises partly from a known distribution of stars, partly from the gas itself.

The specific analysis is complicated by two factors. In the first place, the structure of the spherical galaxies is not well known. Second, the mass of the medium complicates the equations. General limits will therefore be discussed for the most part. Where effects produced by the precise structure of a galaxy are of interest, the Emden polytrope for $n = 5$ will be used. This is known to give a good approximation for globular clusters, and even for galaxies it should give some indication of the general effects to be expected when the mass of the system is highly concentrated toward the center.

First, let us derive the form of the virial theorem which will be useful. The equation of motion in the x-direction for a particle of mass m_p may be written

$$m_p \frac{d^2 x}{d t^2} = F_x \, . \tag{24}$$

A subscript p will be used throughout to denote interstellar particles, either dust or atoms. If we multiply equation (24) by x, we find

$$\frac{1}{2} m_p \frac{d^2 x^2}{d t^2} - m_p \left(\frac{d x}{d t} \right)^2 = x F_x \, . \tag{25}$$

If it is assumed that the system comprises a large number of identical particles in a steady state, we may average over all the particles, in which case the second derivative in equation (25) vanishes. The contribution of the atomic interaction forces to xF_x may be neglected, on the average, compared to the external and gravitational forces. If we assume also that both the distribution of velocities and the potential field possess spherical symmetry, then equation (25) becomes

$$\overline{v_p^2} = - \overline{r F_r} / m_p \, , \tag{26}$$

where F_r is the radial force, and $\overline{v_p^2}$ is the mean square velocity of the particles in question.

For a gas of small mass in a cluster of uniform stellar density ρ_{s0} the radial force is given by

$$\frac{F_r}{m_p} = - \frac{4 \pi G \rho_{s0}}{3} r \, . \tag{27}$$

The assumption that ρ_{s0} is constant is legitimate when the medium is concentrated toward the core of the system. From equation (26) we have in this case

$$\overline{v_p^2} = \frac{4 \pi G \rho_{s0}}{3} \overline{r_p^2} \, , \tag{28}$$

where a subscript p is used to indicate that r^2 as well as v^2 is averaged over all the particles (atoms or dust) in the medium.

For the stars the force per unit mass is less than that given by equation (27), since the density decreases outward. From equation (28) and the corresponding inequality for $\overline{v_s^2}$, we have, therefore, the inequality

$$\frac{\overline{r_p^2}}{\overline{r_s^2}} < \frac{\overline{v_p^2}}{\overline{v_s^2}} \, . \tag{29}$$

When we come to extend this result to a specific stellar structure, such as the Emden polytrope $n = 5$, we find that $\overline{r_s^2}$ is infinite. The analysis is therefore best given in terms of $r_{s\frac{1}{2}}$ and $r_{p\frac{1}{2}}$, the radii containing half the stellar mass and half the mass of the medium, respectively. The analysis involves a determination of the mean potential energy of the polytrope.[10] The final result is

$$\frac{r_{p1/2}}{r_{s1/2}} = 0.414 \left(\frac{\overline{v_p^2}}{\overline{v_s^2}} \right)^{1/2} . \tag{30}$$

We may infer that the ratio of the size of the interstellar system to that of the stellar one is in general somewhat less than the ratio of the root mean square velocities.

The neglect of the total mass M_p of the interstellar particles is valid, provided that the density of the medium is everywhere less than that of the stars. When this is not the case, when ρ_p is greater than ρ_{s0}, it is evident that the gravitational attraction of the medium on itself will be greater than that of the stars on the medium. We may replace equation (28) by the more general formula

$$\overline{v_p^2} = \frac{4\pi G \rho_{s0}}{3} \overline{r_p^2} + \beta \frac{G M_p}{(\overline{r_p^2})^{1/2}} , \tag{31}$$

where β is a constant of unit order of magnitude. The second term in equation (31) is the one which usually appears in applications of the virial theorem and is equal but opposite in sign to the potential energy per unit mass of a sphere of gas in its own gravitational field.

It is evident from equation (31) that if M_p is fixed there is a value of $\overline{r_p^2}$ such that $\overline{v_p^2}$ is a minimum. Similarly, if $\overline{v_p^2}$ is fixed, there is a value of $\overline{r_p^2}$ at which M_p is a maximum. This maximum value of M_p is given by

$$\{M_p(\max)\}^2 = \frac{1}{9\pi\beta^2 \rho_{s0}} \left(\frac{\overline{v_p^2}}{G} \right)^3 , \tag{32}$$

which may be written as

$$\left\{ \frac{M_p(\max)}{M_s} \right\}^2 = \frac{4}{27\beta^2} \left(\frac{\overline{v_p^2}\, r_{s1/2}}{G M_s} \right) \frac{3 M_s}{4\pi r_{s1/2}^3 \rho_{s0}} . \tag{33}$$

It is clear that $3M_s/4\pi r_{\frac{1}{2}}^3$ is twice the mean stellar density interior to $r_{s\frac{1}{2}}$; this will be less, as a rule, than the central density ρ_{s0}. Since $GM_s/r_{s\frac{1}{2}}$ is roughly equal to $\overline{v_s^2}$ and β is approximately unity, we have

$$\frac{M_p(\max)}{M_s} \leqslant 0.38 \left(\frac{\overline{v_p^2}}{\overline{v_s^2}} \right)^{3/2} . \tag{34}$$

If we turn again to the polytrope $n = 5$, we find that in this special case $3M_s/4\pi r_{s\frac{1}{2}}^3$ is equal to 0.447 ρ_{s0}, while $GM_s/r_{s\frac{1}{2}}$ is equal to 2.60 $\overline{v_s^2}$. These two factors reduce the constant in equation (34) from 0.38 to 0.060. This indicates that equation (34) gives a generous upper limit on M_p. If, in accordance with the results of section 1, we assume that the ratio of root mean square velocities for stars and particles is roughly equal to ten, it

[10] It is sometimes stated that the potential energy of the polytrope $n=5$ is infinite, since a factor $n-5$ appears in the usual formula for the total potential energy. But the radius R of the system is also infinite in this case; the energy is, in fact, finite, and may be determined by a direct integration.

follows from equation (34) that the maximum possible mass of interstellar matter in a spherical galaxy is at most 4×10^{-4} of the stellar mass and that any such matter must be highly concentrated toward the center of the galaxy.

The existence of a maximum mass for the medium may be explained simply. Let us consider an assemblage of interstellar particles whose mean square velocity is fixed and whose distribution in the cluster satisfies the equations of equilibrium; let us examine the sequence of configurations which is obtained as the total mass M_p is gradually increased. When M_p is zero the value of $\overline{r_p^2}$ will satisfy equation (28). As the mass of the medium is gradually increased, the self-attraction of the medium begins to become important. At first this may be offset by a contraction in the medium, reducing the attraction of the stars on the interstellar matter. Eventually, however, a point is reached at which a further contraction increases the self-attraction exactly as much as it decreases the attraction of the stars. At this point no increase in M_p is possible without also increasing $\overline{v_p^2}$—this is the point of maximum mass for a fixed mean square velocity. When M_p has a value less than this critical one, equation (31) will have two relevant roots for $\overline{r_p^2}$, one corresponding to a diffuse configuration in which the gravitational attraction of the stars is predominant, the other, a dense system subject primarily to its own self-attraction.

The relevance of this maximum mass is restricted by the fact that the analysis holds only for systems possessing no angular momentum whatever. Most spherical galaxies probably have some angular momentum, and in such a case interstellar matter could contract to the equatorial plane, forming a flattened, rapidly rotating disk of dark matter of the type discussed in the next section. The total mass of such matter could not be very great, however, since the gravitational attraction of a very massive layer of dark matter would impart some ellipticity to the observable stellar distribution. In any case we may conclude that interstellar dust and atoms play a wholly subordinate role in the constitution of spherical stellar systems of large mass and large random stellar velocity.

In systems of small mass, such as globular clusters, the random stellar velocities are of the same order as the velocities of the particles. In such systems the total mass of dust particles present could in theory be very large, provided that not more than a small number of atoms was present. An appreciable number of atoms would slow down the dust particles as is shown in paper I and lead eventually to a collapse of the medium, unless the mass of atoms present exceeded sufficiently the mass of dust to support the dust particles.

For a primarily gaseous medium, on the other hand, a new restriction appears. If the mean square atomic velocities were slightly greater than those of the stars and M_s were assumed to be much greater than M_p, the atoms would apparently leave the system entirely. According to equation (26), the average value of rF_r must vary directly with the mean square velocity. While it is easy to find a distribution of matter for which this average will be very much decreased, given the gravitational field of a particular stellar system, there is a limit to the amount this quantity can be increased. Since most of the stars are near the region in which rF_r is a maximum, the average value of this quantity for any distribution of matter will not much exceed its average value for the stars. For the polytrope $n=5$, for instance, the maximum value of rF_r is only 1.31 times its average value for the entire polytrope. It follows that interstellar particles with a mean square velocity some 20 or 30 per cent greater than the velocity of the stars and a total mass considerably less than that of the stars cannot possibly be in equilibrium within a cluster.

A disequilibrium of this sort would not lead to an immediate escape of the medium, provided that the mean square particle velocities were less than the mean square velocity of escape. In fact, the medium would expand adiabatically to reach a new equilibrium with lower mean square velocities of the particles. But in the case of atoms the kinetic temperature is presumably determined essentially by the color temperature of the ionizing radiation and other factors; if these various factors tended always to give the atoms a mean square velocity appreciably greater than that of the stars and if the gravitational attraction of the medium upon itself were unimportant, such an atomic medium would gradually expand and leave the cluster. If the mass of the interstellar atoms exceeded that of the stars, this difficulty would not arise. In such a case the medium would hold itself together under its own attraction, and the observed stellar system would be imbedded in a globular atomic cluster of much greater mass.

3. ELLIPTICAL SYSTEMS

The analysis of the preceding section must be modified for systems which possess angular momentum and axial symmetry. In such a case the interstellar medium will be confined to the equatorial plane, forming a highly flattened disk imbedded in the more extended elliptical aggregation of stars. If the initial distribution of the medium is assumed to be the same as that of the stars, then, as the random velocities of the interstellar particles decreased, there would be some contraction toward the galactic center as well as toward the equatorial plane. The constancy of angular momentum during such a contraction would lead to an increase of the centrifugal force, which would remain proportional to the inverse cube of the distance from the axis of rotation. A new equilibrium would therefore be reached in which the increased centrifugal force just balanced the attraction toward the galactic center; the medium would, of course, be revolving about the galactic center more rapidly at any point than the stars in the immediate neighborhood.

The relative contraction necessary to attain such an equilibrium may be computed on various assumptions; the ratio of the equilibrium and initial distances from the axis of rotation for a given element of matter is roughly equal to the ellipticity of the nebula or of the fourth root of the ellipticity, depending on whether the potential is that of a point mass or of a homogeneous spheroid. This subject is intimately connected with the uncertain problem of the origin of the galaxies. We shall therefore not consider the problem of the density distribution of interstellar matter with increasing distance along the equatorial plane but rather the distribution in directions perpendicular to this plane. Such an analysis rests on firm ground, since it depends only on the conditions of equilibrium and not on any assumed initial properties.

In the following analysis, distance from the equatorial plane will be denoted by z; the distance of a point from the axis of rotation will be denoted by R. As before, subscripts p and s will be used to distinguish quantities pertaining to the interstellar particles—dust or atoms—from those pertaining to the stars. Regions near the center of elliptical systems will not be discussed here, since these will be closely analogous to spherical galaxies; we shall consider, rather, the structure of the galaxy when R is relatively large.

The analysis depends on whether F_z, the gravitational force in the z-direction, arises primarily from the central mass of the system or from the neighboring matter in the equatorial plane. If the former is the case, then both particles and stars will move in the same external potential field, which for small values of z/R varies as z^2. It is clear that in such a case.

$$\frac{\overline{z_p^2}}{\overline{z_s^2}} = \frac{\overline{v_p^2}}{\overline{v_s^2}} . \tag{35}$$

The total mass of the interstellar medium per unit area of the equatorial plane, as well as the total stellar mass per unit area, is quite unrestricted in such a case, provided only that the total density is always less than the central mass of the galaxy, M_c, divided by

$4\pi R^3$; this is the usual condition that the attraction of the central mass be greater than the attraction of the local matter.

When the total density of stars and medium is greater than $M_c/4\pi R^3$, however, we may neglect the rest of the galaxy and as a first approximation consider only the one-dimensional problem, in which the gravitational potential Φ is determined by the equation

$$\frac{d^2\Phi}{dz^2} = 4\pi G(\rho_s + \rho_p), \tag{36}$$

while the condition of equilibrium gives

$$\frac{d}{dz}\left(\frac{1}{3}\rho_s \overline{v_s^2}\right) = -\frac{d\Phi}{dz}\rho_s, \tag{37}$$

together with an equation for the interstellar particles, identical with equation (37) except that a subscript p replaces the subscript s. One may assume that $\overline{v_s^2}$ and $\overline{v_p^2}$ are not functions of z. This neglects primarily the dispersion in stellar masses, since the chief effect of such a dispersion is to increase $\overline{v_s^2}$ with z.

The solution of these equations depends on ρ_{p0} and ρ_{s0}, the densities of particles and stars when $z=0$. Let us assume first that ρ_{p0} is much less than ρ_{s0}. Then we may neglect ρ_p in equation (36); if we divide equation (37) by ρ_s, then differentiate with respect to z, and use equation (36) to eliminate $d^2\Phi/dz^2$, we have

$$\frac{d}{dz}\left(\frac{1}{\rho_s}\frac{d\rho_s}{dz}\right) = -\frac{12\pi G}{\overline{v_s^2}}\rho_s. \tag{38}$$

If we make the substitutions

$$\rho_s = \rho_{s0}\Lambda(\xi), \tag{39}$$

$$z = \xi\left(\frac{\overline{v_s^2}}{12\pi G\rho_{s0}}\right)^{1/2} \tag{40}$$

the equation assumes a dimensionless form. With the boundary conditions that $\Lambda(0)=1$ and that $\Lambda'(0)=0$, the solution of equation (38) becomes

$$\Lambda(\xi) = \text{sech}^2(2^{-1/2}\xi). \tag{41}$$

The mass $\mathcal{M}_s(\xi)$ per unit area between $-\xi$ and $+\xi$ is

$$\mathcal{M}_s(\xi) = \left(\frac{2\overline{v_s^2}\rho_{s0}}{3\pi G}\right)^{1/2} \tanh(2^{-1/2}\xi). \tag{42}$$

From equations (40) and (42) it follows that $z_{s\frac{1}{2}}$, the value of z such that half the stellar mass per unit area lies between $z_{s\frac{1}{2}}$ and $-z_{s\frac{1}{2}}$, is given by

$$z_{s1/2} = 0.274\frac{\mathcal{M}_s}{\rho_{s0}} \tag{43}$$

where $\mathcal{M}_s$ equals $\mathcal{M}_s(\infty)$ and is simply the total stellar mass above and below a unit area of the equatorial plane. Since $\tanh(\infty)=1$, equation (42) gives $\mathcal{M}_s$ in terms of $\overline{v_s^2}$ and ρ_{s0}

For the interstellar particles, on the other hand, we may neglect the change in ρ_s and set $d\Phi/dz$ equal to $4\pi G\rho_{s0}z$. The equation of hydrostatic equilibrium for the particles then yields

$$\log\frac{\rho_p}{\rho_{p0}} = -\frac{6\pi G\rho_{s0}}{\overline{v_p^2}}z^2 . \tag{44}$$

The mass per unit area leads to the usual error function; if we determine the value of $z_{p\frac{1}{2}}$ by integrating equation (44) and if we take $z_{s\frac{1}{2}}$ from equation (43), substituting from equation (42) for M_s, we have

$$\frac{z_{p\frac{1}{2}}}{z_{s\frac{1}{2}}} = 0.87\left(\frac{\overline{v_p^2}}{\overline{v_s^2}}\right)^{1/2} . \tag{45}$$

Here, as in equation (35) and in the globular galaxies, the ratio of the dimensions of the two systems is roughly equal to the ratio of the root mean square velocities.

The chief difference between the elliptical and the globular systems is the absence of any maximum mass for the interstellar medium. No matter how great the mass of interstellar matter, it is always possible to make $z_{p\frac{1}{2}}$ so small that the mean square velocity is arbitrarily small. If ρ_{p0} should exceed ρ_{s0}, however, equation (44) is no longer applicable. To find the distribution of the medium in this case, a new solution of the equations of equilibrium must be found. If ρ_{p0} is considerably greater than ρ_{s0}, then ρ_p will exceed ρ_s in most of the region in which the density of interstellar matter is appreciable. One may in such a case assume that the distribution of the medium is independent of the presence of stars and set ρ_s equal to zero in equation (36) for the gravitational potential Φ. The resultant equations for ρ_p are identical with equations (38)–(43)—derived for ρ_s on the assumption that ρ_p could be neglected—provided that a subscript p is substituted for the subscript s throughout.

When the equilibrium of the stars is considered in this case, however, the neglect of ρ_s in the determination of Φ will not yield an adequate approximation, if M_s exceeds M_p. In the general case a new solution of equation (38) must be found for ρ_s such that $\Lambda'(0)$ equals a finite value, determined by the gravitational attraction of the thin layer of interstellar dust and atoms. In the limiting case in which M_p is greater than M_s, we may, in fact, neglect ρ_s in the determination of Φ. For all values of z this gives $d\Phi/dz = 2\pi G M_p$. In such a case we have for the stars the usual isothermal distribution in a constant-force field, which gives

$$\log\frac{\rho_s}{\rho_{s0}} = -\frac{6\pi G M_p}{\overline{v_s^2}}z . \tag{46}$$

The value of $z_{s\frac{1}{2}}$ in this case may be found by a simple integration of equation (46).

To compare the results for interstellar matter and for stars in the case when M_p exceeds M_s, we take the value of $z_{p\frac{1}{2}}$ found when equations (42) and (43) are applied to dust and atoms rather than to stars and divide this by the value of $z_{s\frac{1}{2}}$ found from equation (46) above. The resultant ratio is

$$\frac{z_{p\frac{1}{2}}}{z_{s\frac{1}{2}}} = 1.57\frac{\overline{v_p^2}}{\overline{v_s^2}} . \tag{47}$$

We see that when the mass of interstellar matter exceeds that of the stars, the relative concentration varies as the energy rather than as the velocity. It is obvious that for intermediate cases, in which ρ_{p0} exceeds ρ_{s0} but M_p is less than M_s, the ratio of $z_{p\frac{1}{2}}$ to $z_{s\frac{1}{2}}$ will follow a relationship intermediate between equations (45) and (47).

Numerical values for $z_{p\frac{1}{2}}$ may be computed from these results. If the ratio of root mean square velocities is set equal to ten as before and if $z_{s\frac{1}{2}}$ equals 200 parsecs, the value of $z_{p\frac{1}{2}}$ equals 18 parsecs when ρ_{p0} is less than ρ_{s0}, and 3.1 parsecs when the total mass of interstellar matter is greater than that of the stars. If the mass of interstellar matter is an appreciable fraction of the mass of the system, $z_{p\frac{1}{2}}$ will clearly not exceed a few parsecs.

The density of the medium in this latter case would be at least 10^{-20} gm/cm^3, and possibly very much greater. With so great a density it is doubtful whether the electron temperature could remain as high as 10,000°, and it is likely that the root mean square velocity would fall, increasing the density even further. Such a medium would probably be unstable and would perhaps condense into stars, meteorites, or large dark bodies. But these processes of condensation would not increase $\overline{v_p^2}$, and the thin massive layer of matter, surrounded by a vastly more extended stellar envelope, would still remain. The probable state of matter in the equatorial plane will not be discussed in detail here.

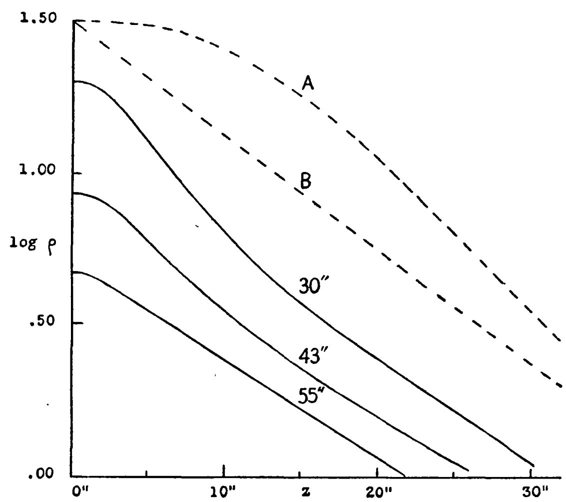

FIG. 1.—Stellar density distribution in NGC 3115. The logarithm of the observed luminosity density, ρ, is plotted against z, the perpendicular distance from the equatorial plane in seconds of arc. The two dashed curves are the theoretical results if ρ is a function of z only, for the two cases: A. A uniform isothermal gas under its own gravitational attraction. B. The same gas under the attraction of a massive equatorial layer of dark matter.

It should be pointed out that such a dark layer is equivalent in its effect to a distribution of very faint dwarf stars with small peculiar motions. In fact, the hypothetical equatorial layer of interstellar matter discussed here might quite possibly condense into stars of absolute magnitude +15 or fainter. The random velocities of stars formed in this way would obviously be small. From an observational standpoint it would be very difficult to distinguish a layer of such stars from a layer containing bodies roughly the size of meteorites; the phrase "dark matter" will be used to denote all such possibilities.

If a massive equatorial layer of this type should exist within an elliptical galaxy, it would lead to directly observable results. The mass deduced from the rotational velocities of the stars would be greater than the mass derived from the luminosity of the system. More important, the distribution of stars on each side of the equatorial plane would follow equation (46) rather than the quite different density function in equation (41). The former distribution is striking in its finite derivative at the equatorial plane. Unfortunately, this effect would be observable only if the plane of the galaxy were parallel to the line of sight from the earth.

These characteristics are remarkably close to the description of NGC 3115, analyzed in detail by Oort.[11] The luminosity distribution in this elliptical system of class E7 is completely in disaccord with the mass distribution as deduced from the observed constant angular velocity.[2] Furthermore, the logarithmic gradients of luminosity density in the z-direction have a nearly constant slope very near to the equatorial plane. The values of log ρ as a function of z for various values of R are shown in Figure 1. These were taken from Figure 7 in Oort's paper; R and z are expressed in seconds of arc. The curved dashed line A shows the distribution to be expected from equation (41), when the stars themselves are primarily responsible for the gravitational attraction. The straight dashed line B represents the distribution given by equation (46), when the equatorial layer of dark matter is responsible for most of the gravitational force on the stars.

It will be noted that the observed curves in Figure 1 correspond to case B more closely than to case A. The observed logarithmic gradients depart from linearity for very small values of z, but a slight inclination of the equatorial plane to the line of sight would account for this effect. The curvature evident for larger values of z could be explained partly by a dispersion in stellar velocities and partly by the deviation of the potential field from the one-dimensional case discussed here. It seems difficult to account for the observational values of log ρ when z is moderately small, if only luminous stars are assumed to be present. The evidence for an equatorial layer of dark matter in NGC 3115 is suggestive but not conclusive. Further examination of elliptical galaxies, and particularly of the density gradients near the equatorial plane, would be necessary to decide the issue.

The central conclusion of the present paper, however, does not concern the possible existence of such a dark layer. The chief result is that, in NGC 3115 and probably in most of the elliptical and globular galaxies as well, the amount of interstellar matter throughout most of the volume of these systems—in the regions surrounding most of the stars—must be negligible. Unless there are unsuspected forces acting on interstellar particles, the light which comes from elliptical and globular systems of large mass must be almost entirely direct starlight, not diffuse or scattered light.

This view is supported by the failure to find space reddening or interstellar absorption lines in these objects. The lack of interstellar emission lines in elliptical objects.[12] is also consistent with this view, although the absence of early-type stars provides an alternative explanation for these observations. The chief support for these conclusions, however, is the essentially theoretical argument presented here. Better determinations of nebular masses would decide to what extent these results may be applied to all elliptical and globular galaxies.

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[11] *Ap. J.*, **91**, 273, 1940.

[12] N. U. Mayall, *Pub. A.S.P.*, **51**, 282, 1939.