LARGE-SCALE MOTION IN STARS

GUNNAR RANDERS

ABSTRACT

The most general type of stationary internal motion in a star of rotational symmetry has been studied from a dynamical point of view, and some general theorems have been obtained. The period of a stationary meridional circulation is related to the effective viscosity in the star. Restrictions on the possible distributions of angular velocity follow from the boundary conditions imposed by the kinematics, as well as by the dynamics, of the system.

I. INTRODUCTION

In stable, nonrotating stars we have no reason to expect large-scale currents in the interior. The only kind of motion we may expect is a disorderly mixing of material between adjacent layers in a convective zone or a slow diffusion through the star of certain elements, due to the transmutations which proceed in the central regions.

In a rotating star conditions are different. There will be a difference in temperature between the poles and the equator and a corresponding difference in density and pressure. The rotation also introduces new dynamical effects, stabilizing or destabilizing, according to conditions. Especially interesting from the dynamical point of view is the case of *differential* rotation (i.e., angular velocity varying through the star) in a star of viscous material.

The possibility of internal currents existing in rotating stars has often been emphasized. Because of the remoteness from direct observation, however, no observational information has been obtained, and, because of the mathematical difficulties involved in attacking the problem hydrodynamically, only qualitative suggestions can be obtained from the theory. The two main starting-points for previous discussions of internal currents in stars have been (1) von Zeipel's theorem and (2) the differential rotation observed in the sun.

The theorem of von Zeipel[1] gives a condition to be fulfilled by the energy production in a star in radiative equilibrium, rotating as a rigid body. Since the condition most probably cannot be fulfilled, it was suggested by E. A. Milne that the rotation would be differential. A. S. Eddington[2] discussed this possibility and came to the conclusion that a violation of von Zeipel's theorem would result in a state of differential rotation combined with meridional currents. In this state the viscous forces arising from the differential rotation would be balanced by the transport of momentum by the meridional currents, so that the state of motion is a stationary one. Similar conclusions from von Zeipel's theorem were also given by H. Vogt.[3] B. Gerasimovič[4] suggested that the stars might assume pure axial rotation with a distribution of angular velocity of the form

$$W = c_1 + \frac{c_2}{R^2} ,$$

where R is the distance from the axis. Such a distribution would give no resultant viscous force in the medium. It is, however, obvious that such a distribution would have nonvanishing viscous stresses at the surface, so that forces would have to be applied at the surface to keep the motion going.

[1] *Seeliger Festschrift*, p. 144, 1924.

[2] *Observatory*, **48**, 73, 1925.

[3] *A.N.*, **223**, 229, 1925.

[4] *Observatory*, **48**, 148, 1925.

An important question to be answered is whether the viscous forces are strong enough to affect appreciably the internal motion of stars. It has been generally agreed that the viscous forces due to large-scale laminar motion in a star would be so insignificant that the present differential rotation of the sun might be a remnant from a primordial state of motion.[5] Jeans[6] found the radiative viscosity to be more important than the material and suggested that the motion in the outer parts of a star may be entirely governed by the "braking effect" of the outflowing radiation.

Rosseland[7] pointed out that there are both observational and theoretical indications that the motion in the sun is turbulent. The eddy motion may, according to experiments, easily increase the effective viscosity of a large-scale motion by a factor of one million. Hence he concluded that meridional currents are necessary for maintaining the differential rotation in the sun.

Rosseland's argument was questioned by L. Biermann,[8] according to whom the thermal stability of the sun, if in radiative equilibrium, would be sufficient to overcome the tendency to turbulence. The obviously turbulent structure observed on the solar surface can be explained as being due to the narrow convection zone which according to Unsöld[9] extends through a few hundred kilometers below the photosphere.

The stabilizing effect of the thermal stratification is wholly dependent upon the deviation of the actual temperature gradient from the adiabatic one. The heat transport effected by the turbulence itself will tend to decrease this deviation. The assumption of a temperature gradient corresponding to purely radiative energy transport, therefore, already affects the possibility of turbulence existing in this region. Biermann's result, depending upon his assumption about the temperature gradient, although plausible, is therefore not to be considered as absolutely conclusive. In any case, the sunspots seem to indicate instability of much deeper origin than that of the narrow Unsöld layer. Also, it should be remembered that the sun is a very slowly rotating star. Most early-type stars (O, B, A) of the main sequence have much higher rotational velocities, in some cases obviously approaching the limit of stability. In such stars the conditions for turbulence are much more favorable.

Whether we have to deal with laminar or with turbulent viscosity, it is of interest to study in detail the properties of such steady meridional motion in rotating stars. The theoretical treatment, however, very soon encounters the practically insurmountable difficulties of solving the general hydrodynamical equations for a viscous gas. The best one can do at present is to gain as much general information as possible from the form of the equations governing the motion. This is the purpose of the present work, which may be regarded as an extension of earlier work by the author.[10]

We are going to consider the most general state of steady motion possible in a rotationally symmetrical star of viscous material. Such motion may always be considered as consisting of a differential rotation about the axis of symmetry combined with stationary currents circulating in the meridional plane.

II. ROTATIONALLY SYMMETRICAL MOTION IN A VISCOUS FLUID

We shall use the following notations:

ρ = density
p = total pressure
Φ = gravitational potential

[5] T. Wilsing, *A.N.*, **127**, 233, 1891; E. Wilczynski, "Hydronamische Untersuchungen" (diss. Friedrich Wilhelm University [Berlin: Mayer & Müller, 1897]).

[6] *M.N.*, **86**, 328, 444, 1926.

[7] *M.N.*, **89**, 49, 1928.

[8] *Zs. f. Ap.*, **5**, 117, 1932.

[9] *Zs. f. Ap.*, **1**, 138, 1930.

[10] *Astroph. Norvegica*, **3**, 97, 1939.

V = the velocity vector, with rectangular components V_x, V_y, V_z
μ = the coefficient of viscosity, considered as constant
∇^2 = the Laplacian operator

The equation of motion in a viscous fluid has the form

$$\rho \frac{dV}{dt} = -\rho\nabla\Phi - \nabla p + \mu\{\tfrac{1}{3}\nabla \operatorname{div} V + \nabla^2 V\} . \tag{1}$$

The symbol d/dt is the individual derivative $(\partial/\partial t + V\nabla)$. We introduce cylindrical co-ordinates R, φ, z, defined by

$$x = R\cos\varphi , \qquad y = R\sin\varphi , \qquad z = z . \tag{2}$$

The z-axis is the axis of symmetry. Differentiating the second of equations (2) with respect to time, we obtain the component V_y of the velocity expressed by the cylindrical components V_R and V_φ,

$$V_y = V_R \sin\varphi + V_\varphi \cos\varphi . \tag{3}$$

For the y-component of ∇, we have

$$\frac{\partial}{\partial y} = \sin\varphi \frac{\partial}{\partial R} + \cos\varphi \frac{\partial}{\partial\varphi} . \tag{4}$$

Because of the rotational symmetry we can put $\partial/\partial\varphi = 0$. Hence

$$\frac{\partial}{\partial y} = \sin\varphi \frac{\partial}{\partial R} . \tag{5}$$

By substituting equations (3) and (5) in the y-component of equation (1) and collecting terms in $\sin\varphi$ and $\cos\varphi$, we get

$$\left.\begin{aligned} &\rho\left\{\frac{\partial V_R}{\partial t} + V\nabla V_R - \frac{V_\varphi^2}{R}\right\}\sin\varphi + \rho\left\{\frac{\partial V_\varphi}{\partial t} + V\nabla V_\varphi + \frac{V_R V_\varphi}{R}\right\}\cos\varphi \\ &= \left\{-\rho\frac{\partial\Phi}{\partial R} - \frac{\partial p}{\partial R} + \mu\left(\frac{1}{3}\frac{\partial}{\partial R}\operatorname{div} V + \nabla^2 V_R - \frac{V_R}{R^2}\right)\right\}\sin\varphi + \mu\left\{\nabla^2 V_\varphi - \frac{V_\varphi}{R^2}\right\}\cos\varphi . \end{aligned}\right\} \tag{6}$$

None of the quantities inside the braces depends upon φ. Consequently, the terms in $\sin\varphi$ and $\cos\varphi$ must vanish separately, and we get the two equations

$$\rho\left\{\frac{\partial V_R}{\partial t} + V\nabla V_R - \frac{V_\varphi^2}{R}\right\} = -\rho\frac{\partial\Phi}{\partial R} - \frac{\partial p}{\partial R} + \frac{\mu}{3}\frac{\partial}{\partial R}\operatorname{div} V + \mu\left(\nabla^2 V_R - \frac{V_R}{R^2}\right) \tag{7}$$

and

$$\rho\left\{\frac{\partial V_\varphi}{\partial t} + V\nabla V_\varphi + \frac{V_R V_\varphi}{R}\right\} = \mu\left(\nabla^2 V_\varphi - \frac{V_\varphi}{R^2}\right) , \tag{8}$$

which are the R- and φ-components of the equation of motion (1). The z-component is the same as before:

$$\rho\left\{\frac{\partial V_z}{\partial t}+V\nabla V_z\right\}=-\rho\frac{\partial\Phi}{\partial z}-\frac{\partial p}{\partial z}+\frac{\mu}{3}\frac{\partial}{\partial z}\operatorname{div}V+\mu\nabla^2V_z\,. \tag{9}$$

In the case of stationary motion we can put $\partial/\partial t=0$ and have, finally,

$$\rho\left\{V\nabla V_R-\frac{V_\varphi^2}{R}\right\}=-\rho\frac{\partial\Phi}{\partial R}-\frac{\partial p}{\partial R}+\frac{\mu}{3}\frac{\partial}{\partial R}\operatorname{div}V+\mu\left(\nabla^2V_R-\frac{V_R}{R^2}\right), \tag{10}$$

$$\rho\left\{V\nabla V_\varphi+\frac{V_RV_\varphi}{R}\right\}=\mu\left(\nabla^2V_\varphi-\frac{V_\varphi}{R^2}\right), \tag{11}$$

$$\rho V\nabla V_z=-\rho\frac{\partial\Phi}{\partial z}-\frac{\partial p}{\partial z}+\frac{\mu}{3}\frac{\partial}{\partial z}\operatorname{div}V+\mu\nabla^2V_z\,. \tag{12}$$

III. THE φ-EQUATION

To solve the system of equations of motion (10), (11), and (12) we would have to make use of the Poisson equation for the gravitational potential, as well as of the energy equation and the equation of continuity, since Φ, p, and ρ occur in addition to the velocity components. This is in practice a nearly impossible task. We shall, therefore, first consider the φ-component of the equation of motion (11), which does not contain Φ and p, and see what information we can obtain about the motion from this equation, in connection with the equation of continuity, which also involves ρ and V only.

Instead of the linear velocity V_φ in the φ-direction we shall introduce the angular velocity W. We have

$$V_\varphi=WR\,. \tag{13}$$

The Laplacian operator has the form

$$\nabla^2=\frac{\partial^2}{\partial R^2}+\frac{1}{R}\frac{\partial}{\partial R}+\frac{\partial^2}{\partial z^2}\,. \tag{14}$$

By means of equations (13) and (14) we obtain from equation (11)

$$\rho\{RV\nabla W+2V_RW\}=\mu\left(R\frac{\partial^2W}{\partial R^2}+3\frac{\partial W}{\partial R}+R\frac{\partial^2W}{\partial z^2}\right). \tag{15}$$

Multiplying by R this can again be written

$$\rho V\nabla(WR^2)=\mu\operatorname{div}(R^2\nabla W)\,. \tag{16}$$

The equation of continuity has the form

$$\operatorname{div}\rho V=0 \tag{17}$$

for stationary motion. Equation (16) can therefore be expressed in the form

$$\operatorname{div} \{R^2(\rho V W - \mu \nabla W)\} = 0 . \tag{18}$$

For simplicity we shall denote the momentum vector $\rho \boldsymbol{V}$ by $\boldsymbol{P}$:

$$\boldsymbol{P} = \rho \boldsymbol{V} . \tag{19}$$

By equation (17) the divergence of $\boldsymbol{P}$ vanishes. $\boldsymbol{P}$ can therefore be written as the curl of a vector $\boldsymbol{A}$:

$$\boldsymbol{P} = \operatorname{curl} \boldsymbol{A} . \tag{20}$$

On the other hand, we can also express the whole bracket in equation (18) as the curl of a vector $\boldsymbol{B}$:

$$WR^2 \operatorname{curl} \boldsymbol{A} - R^2 \mu \nabla W = \operatorname{curl} \boldsymbol{B} . \tag{21}$$

The z- and R-components of equation (21) are

$$WR \frac{\partial R A_\varphi}{\partial R} - \mu R^2 \frac{\partial W}{\partial z} = \frac{1}{R} \frac{\partial R B_\varphi}{\partial R} \tag{22}$$

and

$$-WR \frac{\partial R A_\varphi}{\partial z} - \mu R^2 \frac{\partial W}{\partial R} = -\frac{1}{R} \frac{\partial R B_\varphi}{\partial z} . \tag{23}$$

We shall denote RA_φ by A, and RB_φ by B. Multiplying equation (22) by RdR, equation (23) by Rdz, and subtracting, we obtain

$$WR^2 dA - \mu R^3 \left[\frac{\partial W}{\partial z} dR - \frac{\partial W}{\partial R} dz \right] = dB . \tag{24}$$

Because of the rotational symmetry, it is sufficient to consider the conditions in a fixed meridional plane. In this plane the expression in the bracket represents the vector product of $\nabla W = (\partial W/\partial R, \partial W/\partial z)$ and $d\boldsymbol{r} = (dR, dz)$. We therefore have

$$WR^2 dA - dB = \mu R^3 [\nabla W \times d\boldsymbol{r}] \tag{25}$$

or

$$WR^2 dA - dB = \mu R^3 \frac{\partial W}{\partial n} ds , \tag{26}$$

where $\partial W/\partial n$ is the gradient along the left-hand normal to $d\boldsymbol{r}$ and ds is the length of the displacement $d\boldsymbol{r}$. Equation (26) is equivalent to the original φ-equation (11).

IV. BOUNDARY CONDITIONS

Consider the component $\boldsymbol{p}$, in the meridional plane, of the momentum vector $\boldsymbol{P}$. A line in the meridional plane having at each point the direction of the local $\boldsymbol{p}$ describes a "meridional" streamline of the motion. In other words, if ds is a displacement along a meridional streamline, ds and p are parallel, or

$$[p \times ds] = 0 \quad \text{along a streamline} . \tag{27}$$

We therefore have

$$\begin{aligned} p_z dR - p_R dz &= (\text{curl } A)_z dR - (\text{curl } A)_R dz \\ &= \frac{1}{R}\frac{\partial A}{\partial R} dR + \frac{1}{R}\frac{\partial A}{\partial z} dz = 0 , \end{aligned}$$

or simply

$$dA = 0 \text{ along a streamline} . \tag{28}$$

Since we are considering stationary symmetrical currents in a star, it is obvious that the meridional streamlines must be closed. Actually, the outer boundary of the star must itself be a streamline, and so must the axis of rotation, since no flow can cross it without disturbing the symmetry. The closed curve in the meridional plane, composed by the outer boundary and the axis, is therefore the boundary streamline of the motion, and we can without loss of generality put

$$A = 0 \text{ along surface and axis} . \tag{29}$$

Besides the kinematic boundary condition (eq. [29]) we have also conditions to be satisfied by the viscous stresses at the surface. The star is kept together by its own gravitation and has no rigid boundaries inclosing the material. Consequently, the viscous stress across the outer surface must vanish. Let Π_{ik} denote the stress tensor $(i, k = z, R, \varphi)$, and n_j a unit vector normal to the surface. We then have the conditions

$$\sum \Pi_{ik} n_k = 0 \quad \text{at the surface} . \tag{30}$$

Consider the condition (30) with $i = \varphi$,

$$\sum_k \Pi_{\varphi k} n_k = 0 . \tag{31}$$

The normal n_k has only two nonvanishing components, n_z and n_R (it lies in a meridional plane). We further have[11]

$$\Pi_{\varphi z} = \mu R \frac{\partial W}{\partial z} \tag{32}$$

and

$$\Pi_{\varphi R} = \mu R \frac{\partial W}{\partial R} . \tag{33}$$

[11] Milne-Thomson, *Theoretical Hydrodynamics*, p. 520. London: Macmillan & Co., Ltd., 1938.

The condition (31), therefore, says that the vector ∇W is perpendicular to the surface normal or, what is the same, that ∇W is parallel to the outer surface. If $d\mathbf{r}$ is a displacement along the boundary streamline, we have, consequently,

$$[\nabla W \times d\mathbf{r}] = 0 . \tag{34}$$

It then follows immediately from equation (25) that dB is zero along the surface. It is seen by equation (25) that dB vanishes also on the axis ($R = 0$). We therefore have the second boundary condition

$$B = 0 \quad \text{along surface and axis} . \tag{35}$$

If we consider stars which are symmetrical on both sides of the equatorial plane—as we shall always do—then the intersection of the equatorial plane with the meridional plane is a streamline, so that $A = 0$ along this line. From reasons of symmetry $\partial W/\partial n$ vanishes along the equatorial line, hence, according to equation (26), we have $dB = 0$, and consequently $B = 0$ along the equatorial line. Now we need consider only one quadrant of the meridional section of the star, and along the boundary of this quadrant we have $A = B = 0$.

V. CONCLUSIONS FROM THE φ-EQUATION

We shall denote the quantity WR^2, which represents the "circulation" around the axis, by Ω,

$$\Omega \equiv WR^2 . \tag{36}$$

By equation (26) we then have

$$\Omega dA - dB = \mu R^3 \frac{\partial W}{\partial n} ds , \tag{37}$$

with the boundary conditions

$$A = B = 0 \quad \text{along the boundary of a meridional quadrant} .$$

We shall first prove a very general theorem, which is valid for any stationary, rotationally symmetrical motion of a viscous fluid, independent of boundary conditions. Consider the distribution of Ω over a meridional section. We can then prove theorem (a): *The axial circulation Ω cannot have a maximum (or minimum) at any point outside the axis or the surface.* To prove this we shall introduce Ω instead of W in equation (37). We have

$$\left.\begin{aligned} R^3 \frac{\partial W}{\partial n} ds &= R \frac{\partial \Omega}{\partial n} ds - RW \frac{\partial (R^2)}{\partial n} ds \\ &= R \frac{\partial \Omega}{\partial n} ds - 2\Omega \frac{\partial R}{\partial n} ds . \end{aligned}\right\} \tag{38}$$

Remembering that n is along the left-hand normal to ds, we have

$$\frac{\partial R}{\partial n} = -\frac{dz}{ds} . \tag{39}$$

By equations (38) and (39),

$$R^3 \frac{\partial W}{\partial n} ds = R \frac{\partial \Omega}{\partial n} ds + 2\Omega dz . \tag{40}$$

Hence we can write equation (37) as follows:

$$\Omega d(A - 2\mu z) - dB = \mu R \frac{\partial \Omega}{\partial n} ds . \tag{41}$$

Now, suppose Ω had a maximum at a certain point in the z, R plane. The level lines of the function Ω will then be closed curves encircling the maximum. Integrating equation (41) along one of these closed level lines, we get

$$\Omega \oint d(A - 2\mu z) - \oint dB = \mu \oint R \frac{\partial \Omega}{\partial n} ds , \tag{42}$$

since Ω is constant along the line. The left-hand side vanishes because the integrands are complete differentials. The integrand on the right-hand side is, however, always positive when we integrate in the counterclockwise direction, because Ω is increasing toward the maximum on the left-hand side as we pass along the level line. The right-hand side, therefore, cannot vanish, and the assumption of a maximum leads to a contradiction. The same argument applies to a minimum.

The theorem (a) is obviously related to the stability criteria for rotating fluids studied by Lord Rayleigh and by G. I. Taylor.[12] It should be noticed that the theorem is valid for any rotationally symmetrical motion of a viscous fluid, without restrictions on pressure, density, or temperature.

Passing now to the case of a star, i.e., a fluid without rigid boundaries, we shall prove the additional theorem (b): *The axial circulation in a viscous star cannot have a maximum (or minimum) along any line other than the boundary of a quadrant (in a meridional section).* Suppose Ω had a maximum value along a certain level line. Since there is no point-maximum in the region, the level lines are not closed and must cross the boundary of the quadrant twice, when entering and when leaving. We now integrate equation (41) along the level line in question, crossing the quadrant from boundary to boundary,

$$\Omega \int dA - 2\mu\Omega \int dz - \int dB = \mu \int R \frac{\partial \Omega}{\partial n} ds . \tag{43}$$

Since A and B always vanish at the boundary, the first and third integrals vanish. Because the line along which we integrate is a line of maximum value of Ω, the normal gradient $\partial\Omega/\partial n$ is zero. Consequently, the integral on the right-hand side vanishes also. The value of the second integral on the right is

$$-2\mu\Omega(z_2 - z_1) , \tag{44}$$

where z_1 and z_2 are ordinates of the points of entering and leaving the quadrant of the Ω-line. However, z_1 can never be equal to z_2. This is because no Ω-line cuts the axis (it is an Ω-line itself), while no two points on the outer surface have the same z. Further, an Ω-line cannot both enter and leave through the equatorial plane since then, because of the equatorial symmetry, the Ω-line will be closed and Ω will have a maximum. Conse-

[12] *Phil. Trans. London*, **223**, 289, 1922.

quently, the assumption of a level line of maximum Ω leads to a contradiction. The possibility of a line of minimum Ω is obviously disproved by the same argument. It should be noticed that both proofs fail at the axis where R and Ω vanish.

From theorems (*a*) and (*b*) we can now conclude that *in a viscous star the circulation* Ω *must increase uniformly from the axis* (*where it is zero*) *outward.*

We have excluded the possibility that the equatorial radius is itself a line of constant Ω, because it would imply that the whole equatorial plane was nonrotating (since $\Omega = 0$ at center). Because of the equatorial symmetry we can further conclude that all Ω-lines rise perpendicularly from the equatorial radius, the Ω-value increasing as we pass outward along the radius (Fig. 1).

We shall now consider the case when the meridional motion consists of a single large-scale current in each hemisphere. The meridional streamlines then form a set of closed

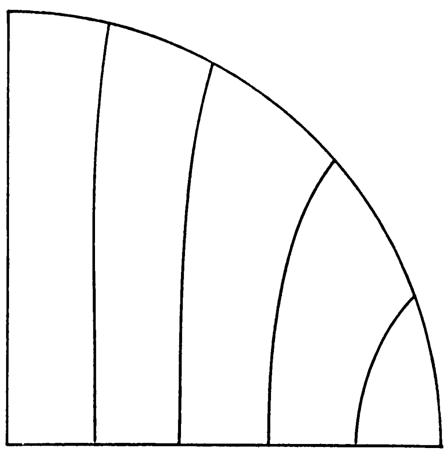

FIG. 1

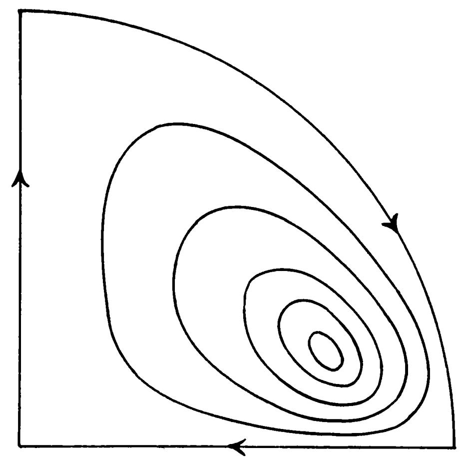

FIG. 2

curves filling the whole region of a quadrant. The direction of the current along the boundary of a quadrant will be[13, 14] from the center to the pole, from the pole along the surface to the equator, and inward along the equatorial radius (Fig. 2). Integrating counterclockwise around a streamline we then must have

$$\oint p dr < 0 , \tag{45}$$

or, introducing A,

$$\oint \frac{1}{R}\left(\frac{\partial A}{\partial R}\, dz - \frac{\partial A}{\partial z}\, dR\right) = -\oint \frac{1}{R}\,[\nabla A \times dr] = -\oint \frac{\partial A}{\partial n}\, ds < 0 , \tag{46}$$

where $\partial A/\partial n$, as usual, denotes the gradient along the left-hand normal to ds. By equation (46) it follows that

$$\frac{\partial A}{\partial n} > 0 , \tag{47}$$

which means that A increases from the boundary inward. Since A is zero at the boundary it therefore follows that A is positive in the upper right quadrant.

[13] A. S. Eddington, *M.N.*, **90**, 54, 1929.

[14] S. Rosseland, *Astrophysik auf atomtheoretisher Grundlage*, p. 91. Berlin: J. Springer, 1931.

We shall now prove theorem (*c*): *If the angular velocity changes uniformly as we proceed through the star in directions parallel to the axis or to the equatorial plane, then the angular velocity increases inward in both cases.*

To prove theorem (*c*) we integrate equation (37) upward along a line $R=\text{const}$ from the equatorial plane to the boundary. We get

$$\int \Omega dA = -\mu \int R^3 \frac{\partial W}{\partial R} dz \qquad (48)$$

or, by partial integration and remembering that A vanishes at both limits,

$$-\int A \frac{\partial \Omega}{\partial z} dz = -\mu \int R^3 \frac{\partial W}{\partial R} dz\,. \qquad (49)$$

Since A is positive, it follows from equation (49) that

$$\frac{\partial W}{\partial z} \quad \text{and} \quad \frac{\partial W}{\partial R} \quad \text{have similar signs}\,. \qquad (50)$$

Now we integrate equation (37) outward along $z=\text{const}$ from axis to boundary, and,

$$\int \Omega dA = \mu \int R^3 \frac{\partial W}{\partial z} dR\,, \qquad (51)$$

or, as before,

$$-\int A \frac{\partial \Omega}{\partial R} dR = \mu \int R^3 \frac{\partial W}{\partial z} dR\,, \qquad (52)$$

which may again be written

$$\int \left(AR^2 \frac{\partial W}{\partial R} + \mu R^3 \frac{\partial W}{\partial z} \right) dR = -\int AW \cdot 2RdR\,. \qquad (53)$$

The integrand on the right-hand side is positive. Consequently, the left-hand side is negative, and from the condition (50) it follows that

$$\frac{\partial W}{\partial z} \quad \text{and} \quad \frac{\partial W}{\partial R} \quad \text{are both negative}\,. \qquad (54)$$

VI. THE VELOCITY OF THE MERIDIONAL CURRENT

The velocity with which the stationary current circulates in the meridional plane is primarily a question of the thermal stability of the star, since it is the nonuniform heating of the material which keeps the motion going.[13] However, assuming the currents to be present, we can also study their velocity from the point of view of dynamical equilibrium, since the transfer of momentum by the current must be adjusted so as to balance the effect of viscosity. Since we are considering here only the equations of motion, we shall apply the last procedure.

We turn back to the equation (16), which has the form

$$\rho V \nabla \Omega = \mu \operatorname{div} (R^2 \nabla W)\,, \qquad (55)$$

or, introducing Ω also on the right-hand side,

$$V\nabla\Omega = \eta\left(\nabla^2\Omega - 2R^{-1}\frac{\partial\Omega}{\partial R}\right), \tag{56}$$

where $\eta = \mu/\rho$ is the kinetic coefficient of viscosity. We at once notice an interesting property of equation (56). Because the operator ∇ has no φ-component, only the meridional components of V appear in the product $V\nabla\Omega$. As a result equation (56) is homogenous in Ω, which means that we can multiply Ω, or W, by a constant and still have the same meridional velocity V_z, V_R. The physical reason for this is obvious. Multiplying W by a large factor means increasing the differential rotation, and consequently increasing the viscous forces by the same factor. The meridional current will, however, now pass through a correspondingly increased change of W per unit length, and transport of momentum by the current will, therefore, increase in the same ratio, so that the balance is still there.

The expression in the bracket in equation (56) cannot vanish over the whole region because then V would be parallel to the lines of constant Ω, or, in other words, the streamlines would follow the Ω-lines and could not be closed (according to theorem [*a*] of sec. V).

If we know the distribution of angular velocity it is seen that equation (55) together with the equation of continuity (17) is sufficient to determine the two quantities ρV_z and ρV_R. In fact, the formal solution of this problem is very simple.[10] However, without knowing W or ρ we can obtain a general idea about the order of magnitude of the meridional velocity most easily by equation (56), according to which the meridional velocity V_m must be of the same order of magnitude as η/R, or:

$$|V_m| \approx \frac{\eta}{R}. \tag{57}$$

As has already been shown, this result is independent of the magnitude of the angular velocity. In order to find the order of magnitude of the period of the meridional circulation, we have to divide the stellar radius by the velocity V_m. Hence, for the period P we have

$$P \approx \frac{R^2}{\eta}. \tag{58}$$

The question now is which value to assume for the kinetic viscosity. If the motion is laminar we have to deal with the usual gaseous coefficient of viscosity for stellar temperatures. This coefficient will probably have the order of magnitude unity.[6] Using the radius of the sun as a measure of the linear dimensions, we obtain by equation (58)

$$P \approx 10^{13} \text{ years}. \tag{59}$$

If the motion is turbulent, we have to use for η the so-called eddy-diffusivity[15] K, given by the product $\bar{\omega} \cdot l$, where $\bar{\omega}$ is the mean turbulent velocity and l the distance passed by a turbulence element during its lifetime,

$$K = \bar{\omega} \cdot l. \tag{60}$$

[15] D. Brunt, *Physical and Dynamical Meteorology*, Cambridge University Press, 1939.

The eddy-diffusivity is obviously not a property of the stellar material alone, as the coefficient of viscosity, but depends upon the degree of turbulence. In order to obtain a value for the period of the meridional current in this case, we must, therefore, have some knowledge about the turbulence in the interior of the star. As an illustration we shall consider the case when the turbulence is supposed to be similar to that observed on the surface of the sun. From observations of the solar granulation we estimate the turbulent velocity $\bar{\omega}$ to be as to order of magnitude 1 km/sec, and the lifetime of a turbulence element 10^2 sec. This gives for K

$$K \approx 10^5 \cdot 10^7 = 10^{12} \text{ cm}^2/\text{sec} . \tag{61}$$

With this value introduced for η and with the solar radius for R, equation (58) gives, for the period of the meridional current,

$$P \approx 10 \text{ years} . \tag{62}$$

Not much confidence should be placed in this result, however, as the surface turbulence of the sun is probably a very poor indicator of the interior conditions. The fact that any amount of turbulence will change the period of the steady current enormously seems, however, to be beyond doubt. The value of K given in equation (61) may be an enormous overestimate, but it is known from terrestrial observations that the increase of viscosity by turbulence may very well amount to a factor of 10^6–10^7. As will be seen in section VII there are reasons against too high values of the eddy-diffusivity.

VII. FIRST-ORDER APPROXIMATION OF THE EQUATIONS OF MOTION

For stars possessing any appreciable rotation it is obvious from the foregoing that the velocity of the meridional current is vanishingly small as compared to the rotational velocity in the main parts of the star. In the equations of motion (10), (11), and (12), of which we thus far have considered only (11), we may, therefore, safely neglect second and higher orders of the meridional components of the velocity. In so doing, however, we must, according to equation (57), also neglect squares of the quantity η/R or products like $(\eta/R)V_z$, $\eta(\partial V_z/\partial R^2)$, etc., in order to obtain a consistent approximation. Correct to the first order, the equations of motions (sec. II) then take the form

$$-\rho \frac{V_\varphi^2}{R} = -\rho \frac{\partial \Phi}{\partial R} - \frac{\partial p}{\partial R} , \tag{63}$$

$$0 = -\rho \frac{\partial \Phi}{\partial z} - \frac{\partial p}{\partial z} , \tag{64}$$

$$\rho V \nabla (W R^2) = \mu \operatorname{div} (R^2 \nabla W) . \tag{65}$$

The last of these equations is the familiar φ-equation in the form (16). The R- and z-components, equations (63) and (64), are seen to reduce to the same form as the equations of motion for purely axial differential rotation of a nonviscous star, as considered by Rosseland[16, 17, 18]. In his work Rosseland finds that the expression for the angular velocity always contains an arbitrary function $F(R)$. His results are based upon the assumption of constant conductivity and energy production all through the star. The

[16] *Ap. J.*, **63**, 342, 1926.

[17] *Astroph. Norvegica*, **2**, 173, 1936.

[18] *Ibid.*, p. 249, 1937.

appearance of the arbitrary function of R is, however, a quite general phenomenon, independent of any special assumption about conductivity or energy production. It follows from the form of the equations (63) and (64) in the following way: Multiplying equation (63) by dR, equation (64) by dz and adding gives

$$-\rho W^2 R dR = -\rho d\Phi - dp \,, \tag{66}$$

or

$$W^2 R dR = d\Phi + \rho^{-1} dp \,. \tag{67}$$

Now p will normally not be a function of Φ in a rotating star, and we may, therefore, consider the density ρ as a function of p and Φ through the star. We can then multiply equation (67) by a factor $\tau(p, \Phi)$, which makes the right-hand side a complete differential dQ:

$$\tau(p, \Phi) W^2 R dR = dQ \,. \tag{68}$$

From equation (68) it follows that

$$Q = F(R) \,, \tag{69}$$

where $F(R)$ is an arbitrary function of R, while

$$\tau(p, \Phi) W^2 R = \frac{\partial F(R)}{\partial R} \,. \tag{70}$$

Consequently,

$$W^2 = 2\tau^{-1}(p, \Phi) \frac{\partial F(R)}{\partial (R^2)} \,, \tag{71}$$

which is a generalization of Rosseland's relation. As an illustration consider the simple case of negligible radiation pressure, constant conductivity K and energy production ϵ. Then, with certain assumptions about the boundary conditions, Rosseland obtains the relation

$$\Phi + \frac{GK}{\epsilon} T = \alpha \,; \qquad \alpha = \text{const} \,. \tag{72}$$

Eliminating T by the gas equation

$$p = k\rho T \,,$$

equation (72) becomes

$$\rho = \frac{GK}{\epsilon k} \frac{p}{\alpha - \Phi} \,. \tag{73}$$

Introducing ρ from equation (73) into equation (67), we see that the integrating factor is

$$\tau = (\alpha - \Phi)^{-1} = \left(\frac{GK}{\epsilon} T\right)^{-1} \,. \tag{74}$$

Hence, finally,

$$W^2 = \frac{2GK}{\epsilon} T \frac{\partial F(R)}{\partial (R^2)} , \tag{75}$$

which is Rosseland's result. If we were considering pure axial rotation of a nonviscous star, there would not seem to be any way of determining $F(R)$. In our case, however, we have the boundary conditions for the viscous material to take care of, and it is easily seen that $F(R)$ is no longer arbitrary. In fact, differentiating equation (75) logarithmically, we have

$$2 \frac{\nabla W}{W} = \frac{\nabla T}{T} + \frac{\nabla\left(\frac{1}{R} \frac{\partial F}{\partial R}\right)}{\frac{1}{R} \frac{\partial F}{\partial R}} . \tag{76}$$

The function $F(R)$ now has to be such that ∇W is parallel to the surface along the whole of the outer boundary. If the distribution of temperature in the star is known, this condition determines $F(R)$ apart from a constant factor. If in solutions of Rosseland's type we adjust $F(R)$ so as to satisfy the above condition, we will, however, not generally obtain a first-order approximation for the distribution of W in a star with slow meridional currents. This is because, in addition to the above boundary condition, we have also the kinematic boundary condition $A = 0$ along the boundary. Since A is determined by the distribution through the star of W^{10}, as was mentioned earlier, this gives a further condition to be satisfied by W.

From equations (63) and (64) we can obtain another relation, which has often been used in earlier work on stellar rotation. After dividing both equations by ρ, we differentiate equation (64) with respect to R, equation (63) with respect to z, and subtract. The result is

$$R \frac{\partial (W^2)}{\partial z} = \left[\nabla \frac{1}{\rho} \times \nabla p\right] . \tag{77}$$

This relation shows that if the density is constant, or is a function of the pressure, the angular velocity is a function of R only, and it is impossible to satisfy the condition that ∇W is parallel to the surface at the surface of a star. Independent variation of ρ and p is necessary. Consequently, we cannot find solutions of the kind we are discussing in a star in convective equilibrium (because of the adiabatic relation between p and ρ). This fact is of some interest in connection with the possibility of extremely high eddy-viscosity as was discussed in section VI. The transfer of momentum by eddies is, of course, accompanied by transfer of heat as well. An extremely high momentum-diffusivity will be accompanied by an equally high heat-diffusivity. The convective transfer of energy in this case will probably take over the whole transport of energy, and the star would be in convective equilibrium, with an approximately adiabatic temperature gradient. The central regions, where the heat content per gram is greatest, would be the first place for this to happen. In the atmosphere the transport by radiation would still be important. The eddy-viscosity, necessary to regulate the exchange of momentum in a meridional current of appreciable velocity, therefore itself sets an upper limit to the velocity by producing a state of convective equilibrium, in which the steady current cannot exist.

We can introduce the temperature T instead of ρ in equation (77) by the relation

$$p = k\rho T + aT^4 . \tag{78}$$

We have by equation (78)

$$\rho = k^{-1}\left(\frac{p}{T} - aT^3\right) , \tag{79}$$

which, introduced into equation (77), gives

$$R\,\frac{\partial(W^2)}{\partial z} = -\frac{1}{\rho^2}\,[\nabla\rho \times \nabla p] = \frac{1}{k\rho^2}\left(\frac{p}{T^2} + 3aT^2\right)[\nabla T \times \nabla p] \,. \tag{80}$$

With negative $\partial W/\partial z$ (sec. V) we can conclude from equation (80) that ∇T is to the right of ∇p (see definition of cross-product, eq. [24]), considered in the upper right quadrant. Hence the temperature will increase toward the pole as we follow an isobar, and the polar regions will be hotter than the equatorial regions. Under the same conditions we also see that $\nabla\rho$ must point to the left of ∇p. Hence the density decreases toward the pole along an isobar. Consequently, the density is lower in the polar regions than in the equatorial region. This will have the effect of decentralizing further the streamline pattern (Fig. 2), which for geometrical reasons must already, by constant density, be displaced toward the equatorial region, to satisfy the equation of continuity. In a simple meridional circulation we shall, therefore, expect to find the center of revolution in the outer, equatorial region.

VIII. NONVISCOUS STARS

The question still remains to be answered whether it is possible to have steady meridional currents in a rotating nonviscous star. It is easy to see that a motion of this kind would in any case be dynamically unstable. The reason for this is the following: If we put the coefficient of viscosity equal to zero in the familiar φ-equation

$$\rho V \nabla\Omega = \mu \text{ div } (R^2 \nabla W) , \tag{81}$$

we see that the meridional current must follow the level lines of Ω. Since the steady current has closed streamlines, Ω must have a maximum. (It should be noticed that the theorem [*a*] in section V is not valid for nonviscous fluids.) This means that there are regions where Ω decreases outward. Such a region is, however, dynamically unstable,[17] and a small perturbation of the steady current will have a tendency to increase and will break down the steady motion completely if there is not a strong thermal stability to counteract the dynamical instability.

There is also a further difficulty connected with the case of a nonviscous star. Because of the assumed equatorial symmetry the meridional current cannot cross through the equatorial plane. Consequently, the equatorial radius in a meridional section is a line of constant Ω. Since the radius goes right to the center of the star, the constant value of Ω is zero. The continuation of this Ω-line at the outer end is the surface itself, since the current follows the surface. We must, therefore, conclude that $\Omega = 0$ along the surface of the star, which means that the outside does not rotate at all. Considering the two difficulties which we have mentioned, we must conclude that steady meridional currents cannot exist in a rotating, nonviscous star.

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