# THE RECIPROCITY PRINCIPLE IN LUNAR PHOTOMETRY

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## ABSTRACT

In order to ascertain the exact meaning of the optical reciprocity principle, stated by Helmholtz, some simple idealized experiments are considered. By then applying this principle to the case of a scattering surface, we find a general reciprocity law which is independent of the nature of the surface considered. Definite relations must, therefore, exist between the brightness of two points on the moon, if the lunar surface at these points has identical properties; a photometric test for this identity is thus obtained. From the work of Öpik, Fessenkoff, and Bennett examples are selected in which similar lunar formations are found to have similar surface layers, and other examples in which the surfaces are found to be different. The reciprocity principle restricts the possible forms of the general law of illumination of the lunar surface. Some of the laws thus far proposed are incompatible with it.


The photometric properties of the average lunar surface are at present fairly well known. For more detailed information about the surface features of the moon the photometric investigation of individual points is now required. This work has been initiated by Pickering, Wislicenus-Wirtz, Goetz, Rosenberg, Öpik, Schoenberg, Barabascheff, Fessenkoff, Fessenkoff and Parenago, and Bennett.[1] The measurements by Öpik, by Fessenkoff, and by Bennett are the most important, both because of the precision of the methods used and because of the great number of points investigated.

A detailed photometric investigation of the moon requires (*a*) a comparison between the brightness of different points of the lunar surface at one definite moment and (*b*) a comparison between the same points on different days. Doubtless, observations of the second kind are more uncertain than those of the first, because changes in the atmospheric transmission or in the plate sensitivity may interfere to a certain extent, even if precautions are taken in order to eliminate these influences. This will have to be duly considered in a critical discussion of the results.

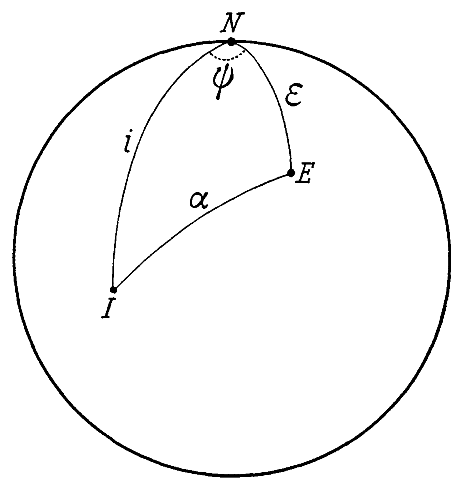


Fig. 1

The ultimate aim must be to determine the characteristic photometric function for each of the typical lunar formations. The surface brightness of a solid body, illuminated by parallel rays from a definite direction and observed from another direction, cannot be expressed by a simple physical law; the formulae thus far proposed are rather empirical and vary from one substance to another. In any case the brightness $(i, \epsilon, \psi)$ is a function $H(i, \epsilon, \psi)$ of the angle $i$ between the normal and the incident rays, the angle $\epsilon$ between the normal and the direction of observation, and the angle $\psi$ between the azimuth of the incident ray and of the direction of observation (Fig. 1). In most cases, the diffusing properties may be assumed to be symmetrical around the normal, otherwise the azimuth

[1] Pickering, *Selenograph. J.*, 1882; Wislicenus-Wirtz, *A.N.*, **201**, 289, 1915; P. Goetz, *Veröff. Sternwarte Oesterberg, Tübingen*, **1**, No. 2, 1919; H. Rosenberg, *A.N.*, **214**, 137, 1921; Öpik, *Pub. Tartu*, **26**, 3, 1924; E. Schoenberg, Acta Soc. Sc. Fennicae **50**, Nr. 9, 1925; Barabascheff, *Kharkov Pub.*, **1**, 35, 1927; B. Fessenkoff, *Pub. inst. ap. Russie*, **4**, 1, 1928. B. Fessenkoff and Parenago, *Russ. A.J.*, **6**, 279, 1929; A. L. Bennett, *Ap. J.* **88**, 1, 1938.

of the incident ray should be introduced as a fourth co-ordinate. If we consider the surface element under examination as the center of a sphere, the directions of the normal, of the incident light, and of the straight line toward the observer are represented by the points $N, I, E$; moreover, $i = \widehat{IN}$, $\epsilon = \widehat{EN}$, $\psi = \widehat{INE}$. These are the independent co-ordinates used, for example, by Bennett.

But evidently the spherical triangle $INE$ may also be defined by the co-ordinates $i$, $\epsilon$, and the phase angle $\alpha = IE$. The use of the angle $\alpha$ as a third co-ordinate instead of $\psi$ has practical advantages: (1) When one of the angles, $i$ or $\epsilon$, is small, a minute change in the position of $N$, $I$, or $E$, leaving the brightness nearly unchanged, often corresponds to a shift of $\psi$ over nearly the entire range from $-180°$ to $+180°$; this is not the case for the angle $\alpha$; (2) For all points of a lunar photograph $\alpha$ has the same value, while $\psi$ varies from point to point. The majority of the authors have, indeed, used the co-ordinate $\alpha$.

The complete photometric investigation of one determinate surface element of the moon is made impossible because of two limiting conditions: (*a*) the point $I$, representing the direction of the incident rays, is always on the same great circle—the "intensity-equator"—and (*b*) the point $E$, corresponding to the direction of observation, has a fixed position at the center of the lunar disk ($\epsilon = 0$). The effect of the nutations is so small that it may be practically neglected for our purpose. However, it is possible to select a number of points on the surface of the moon, having the same morphological character, of which it may be assumed a priori that they have identical surface layers. From all these points the complete photometric function for this special material may then be derived. There remains only the uncertainty as to whether the points selected have really the same photometric properties.

We shall now show that the limitations due to our position at the surface of the earth may be partly overcome by the application of an optical theorem, namely, the Helmholtz principle of reciprocity.

## THE RECIPROCITY PRINCIPLE

Helmholtz has formulated the following theorem:

Vom Punkte $A$ gehe das Einheitsquantum an Licht von bestimmter Farbe, polarisirt nach einer bestimmten Richtung $\alpha$, in einer solchen Strahlrichtung aus, dass nach einer Reihe von Spiegelungen und Brechungen schliesslich in $B$ das Quantum $x$ ankomme, und zwar nach einer Richtung $\beta$ polarisiert. Lassen wir nun von diesem Ziel rückwärts in der umgekehrten Richtung des Endstrahls das Einheitsquantum nach $\beta$ polarisirten Lichtes, von derselben Farbe ausgehen, so kommt nach allen jenen reciproken und reversibelen Vorgängen, die das Licht erleidet, dasselbe Quantum $x$ also derselbe Bruchtheil nach $\alpha$ polarisirten Lichtes am Ausgangsorte an.[2]

This principle is widely applicable; it applies to reflecting, refracting, absorbing, and scattering media, for each separate wave length and for each plane of polarization. It fails only in the cases of fluorescence or of magnetic rotation. In practice one is immediately confronted with the question whether the principle applies to the total intensity, $J$, of the light pencil, measured by the energy passing through the entire cross-section, or to the specific intensity, $H$, passing through each square centimeter of the cross-section.[3] In the case of a radiating surface element, the specific intensity emitted in a given direction is what we call the brightness of this surface element. We shall try to answer this question by imagining some very simple idealized experiments (Fig. 2). We consider an optical system of arbitrary shape, composed of reflecting, refracting, scattering, and absorbing material, inclosed in a hollow vessel which is opaque to radiation except for two minute apertures. These are so small that all the surface elements within the aperture may be considered to emit the same radiation. We imagine one of the holes at

[2] *Theorie der Wärme*, 1, 3, § 42.

[3] In both cases and in all following considerations the energy measurement refers to the radiation contained within the unit of solid angle.

the left side, the other at the right, and we shall write correspondingly at the left or at the right side the intensities relating to each of them. For each experiment and its reciprocal complement both the specific and the total intensities are given in Table 1.

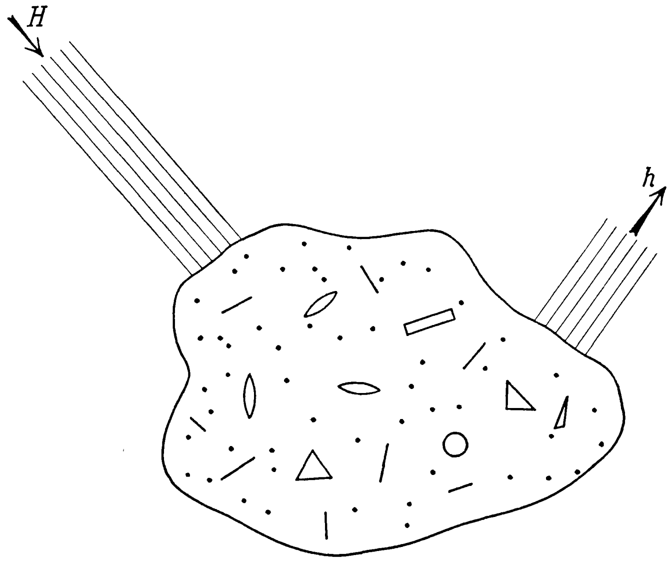


FIG. 2.

TABLE 1

IDEALIZED EXPERIMENTS WITH A DIFFUSING OPTICAL SYSTEM

| Experiment | Incident⟶Escaping | | Escaping⟵Incident | |
|---|---|---|---|---|
| 1......... | $H$<br>$J$ | $h$<br>$j$ | $h$<br>$j$ | $H$<br>$J$ |
| 2......... | $2H$<br>$2J$ | $2h$<br>$2j$ | $2h$<br>$2j$ | $2H$<br>$2J$ |
| 3......... | $H$<br>$2J$ | $2h$<br>$2j$ | $h$<br>$2j$ | $H$<br>$J$ |
| 4......... | $H$<br>$2J$ | $2h$<br>$2j$ | $2h$<br>$4j$ | $2H$<br>$2J$ |
| 5......... | $H$<br>$J$ | $h$<br>$j \cos \vartheta$ | $h \cos \vartheta$<br>$j \cos \vartheta$ | $H$<br>$J \cos \vartheta$ |
| 6......... | $H$<br>$J$ | $h$<br>$j \cos \vartheta$ | $h$<br>$j$ | $H \sec \vartheta$<br>$J$ |

*Experiment 1.*—Both the holes have an area of 1 $cm^2$. In this case the specific and the total intensities are equal, and there is no difficulty.

*Experiment 2.*—Here $H$, $J$, $h$, and $j$ have the same numerical value as in experiment 1; the result is obvious.

*Experiment 3.*—Now let the left hole have an area of 2 $cm^2$, the right one an area of 1 $cm^2$. Moreover, assume that the same specific intensity which first was admitted through the left hole is afterward admitted through the right one. Already, in this very simple case we find that the reciprocity principle is, strictly speaking, not verified, either when it is applied to the specific intensities (upper line) or when it is applied to the total intensities (lower line). It is valid only when we compare the incident specific intensity with the observed total intensity; or when we compare the incident total intensity with the observed specific intensity.

*Experiment 4.*—Let the left hole again have an area of 2 $cm^2$, the right one of 1 $cm^2$. We now assume that the same total intensity which first was admitted through the left hole is afterward admitted through the right hole. The conclusion to be drawn from this experiment is the same as for experiment 3.

*Experiment 5.*—Let us assume that the radiation leaving one of the holes is distributed over the different directions according to the simplest law, i.e., so that the specific intensity is the same in all directions. Let both holes have an area of 1 $cm^2$. Let the incident light at the left side be perpendicular to the hole, and let it be observed at the right side under the angle $\vartheta$ with the normal. In reversing the experiment, send back the same specific intensity. The conclusion of experiment 3 is again confirmed.

*Experiment 6.*—Experiment 5 is repeated, but now in the reciprocal experiment the same total energy is sent back. The result is the same. We remark that in experiments 5 and 6 a uniform distribution of the specific intensity over the different directions has been assumed only in order to find out for a very simple case how the reciprocity principle has to be formulated. The applicability of the principle, however, remains quite general.

The curious form of the reciprocity principle which we have found for optical systems presents an interesting analogy with what is known about the principle of Le Chatelier–van 't Hoff in thermodynamics.[4] Here also a reciprocity theorem is valid only when an "intensity parameter" ($P$, $T$) is combined with a "quantity parameter" ($V$, $Q$). In order to apply the general principle to lunar photometry we shall have to consider the following case. Let radiation of specific intensity $H$ fall upon the left hole under an angle $\theta$ with the normal and observe the radiation escaping through the right hole under an angle $\vartheta$. Now try the reciprocal experiment. From the reciprocity principle we conclude that the total intensities, $j, j'$, observed in these two cases will be equal; the observed specific intensities are $h = j/\cos\vartheta$ for the first experiment, $h' = j'/\cos\vartheta'$ for the second. Consequently, $h/h' = \cos\theta/\cos\vartheta$. In these reciprocal experiments, the observed brightnesses are proportional to the cosines of the angles of incidence. We shall give for this theorem an alternative, more direct, derivation which reveals at once its connection with the second principle of thermodynamics.

*Experiment 7.*—The incident light forms and angle $\theta$ with the normal to the diffusing surface, the observed light an angle $\vartheta$; let the two holes have an area of 1 $cm.^2$ Consider now the surface element of unit area $A$, sending the radiation $J$ toward the diffusing system, and the surface element of unit area $B$, receiving the diffused radiation $i$; let both of them be perpendicular to the rays of light. In reciprocal experiments we will have: $i/J = i'/J'$. For otherwise, if the two surfaces $A, B$ have initially the same temperature, one of them would receive more heat than it would lose, the case for the other one being reversed. This would contradict the second principle of thermodynamics. In reciprocal experiments the incident radiations must be equal, thus $J = J'$ and $i = i'$. Now the brightness of the diffusing surface is defined as the diffused radiation received by $B$, divided by the projected area of the diffusing surface; i.e., $h = i/\cos\vartheta$. In the reciprocal experiment the brightness will be $h' = i'/\cos\theta$. Therefore

$$\frac{h}{h'} = \frac{i}{i'} \frac{\cos\theta}{\cos\vartheta} = \frac{\cos\theta}{\cos\vartheta} .$$

This is precisely the result already obtained.

[4] P. Ehrenfest, *Zs. f. phys. Chem.*, **77**, 227, 1911.

### RECIPROCAL EXPERIMENTS WITH DIFFUSING SURFACES

A solid substance of which a diffusing-surface element is investigated may be considered as an optical system, inclosed between opaque walls, in which two coincident holes have been made. The following theorem now is a direct consequence of the reciprocity principle: Let $H$ be the specific intensity of the light, incident under an angle $\theta$, and let $h$ be the brightness of the surface element, observed under an angle $\vartheta$; now try the reciprocal experiment, and let $h'$ again be the brightness observed (the phase angle between the rays of light being kept the same). Then $h/h' = \cos\theta/\cos\vartheta$. This fundamental principle is independent of any assumption about the law according to which the substance diffuses the light; the scattering properties may even be quite asymmetrical with respect to the normal, for example when the surface is covered by minute parallel furrows. It is therefore important for the photometric investigation of any diffusing surface, be it opal glass, the surface layer of a road, or the plains of the moon.

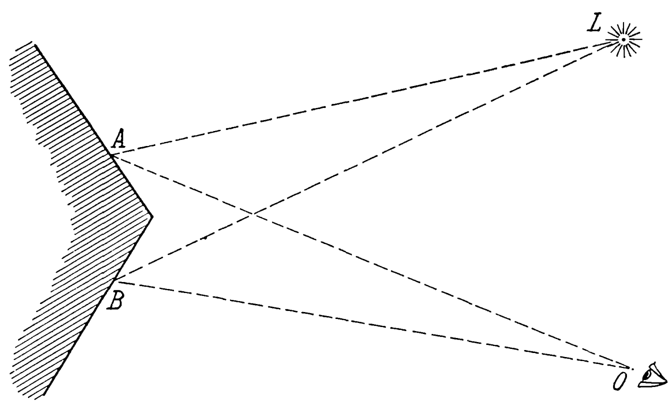


FIG. 3

A very simple way to test the reciprocity principle is indicated in Figure 3. A prism composed of any diffusing substance is illuminated by the lamp $L$ and observed from $O$. It is now stated that the brightness ratio between $A$ and $B$ is the same for all substances and that it is equal to $\cos\theta/\cos\vartheta$. This principle could be used for photometric standardization just as well as the inverse-square law.

### PHOTOMETRIC COMPARISON BETWEEN POINTS OF THE LUNAR SURFACE BY MEANS OF THE RECIPROCITY PRINCIPLE

As a direct consequence of the reciprocity principle it is possible to calculate very simply how the brightness of the individual points of the lunar surface would be altered if the positions of the sun and of the earth were interchanged. Thus, it is possible to "observe" (so to speak) the full moon from all directions on the intensity equator. The calculation to be performed is independent of the physical nature of the surface.

The full importance of the reciprocity principle for the photometry of the moon becomes clear when it is combined with the assumption of symmetry in the scattering properties around the normal. This makes possible an exact photometric comparison between two points of the lunar surface, under definite conditions of illumination and observation. When this comparison shows that the brightness ratio is not as expected, we may conclude with certainty that in the two points investigated the surface properties are not the same. If the ratio is as expected, it may be that the surface material is the same, but certainty can be obtained only in special cases when a series of photometric comparisons under varying conditions is possible. We shall now enumerate the possibilities for photometric comparison, assuming symmetry of the scattering properties around the normal and distinguishing between a simultaneous and a successive comparison (Fig. 4). The effect of the nutations will be neglected.

1. Two arbitrary points of the same latitude may be simultaneously compared for one determinate position of the sun. The comparison between the points $P_1$ and $P_2$ is possible when they are situated symmetrically with respect to the earth, $E$, and the sun, $S$. We then have:

$$\widehat{EP_1} = \widehat{SP_2}\ ; \qquad \widehat{SP_1} = \widehat{EP_2}\ ; \qquad \widehat{EP_1S} = \widehat{EP_2S}\ .$$

Thus,

$$\frac{h_1}{h_2} = \frac{\cos \widehat{SP_1}}{\cos \widehat{SP_2}} = \frac{\cos i_1}{\cos i_2}\ .$$

A connection between the several parallels is possible by means of the following consideration.

2. At full moon the points on any circle, concentric with the moon's circumference, may be simultaneously compared. Those points where the surface conditions are identi-

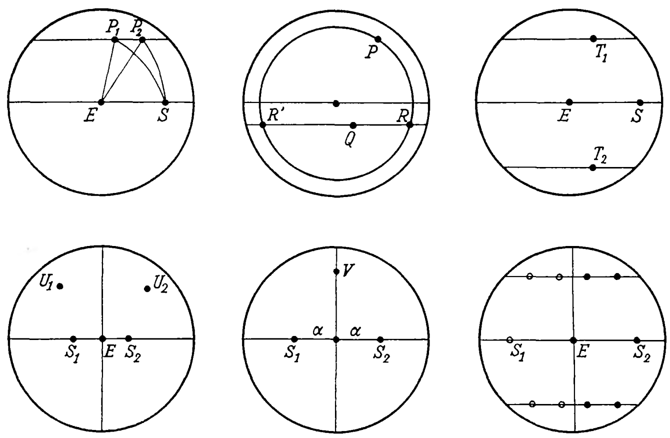


FIG. 4

cal must then be equally bright. By means of the comparisons (1) and (2), two arbitrary points $P$ and $Q$ of the lunar surface may be compared, either directly or in two steps, introducing a third point $R$ or $R'$ (either two or four points $R$ are always available).

3. Two points, $T_1$, $T_2$, symmetrical with respect to the equator may be simultaneously compared at all phase angles.

4. Two points, $U_1$, $U_2$, symmetrical with respect to the central meridian may be successively compared at all phase angles; if one point is observed at a phase angle $+\alpha$, the other one must be measured at a phase angle $-\alpha$.

5. Points $V$ on the central meridian must have the same brightness at phase angles $+\alpha$ and $-\alpha$. This gives a connection between the photometry of the moon at these two moments, if the scattering properties are assumed to be symmetrical around the normal. Conversely, if the photometric comparison is trustworthy, a test of the symmetry assumption is obtained.

It will be seen that on a lunar photograph any particular point may be directly compared to three other points. If a photograph at phase angle $+\alpha$ and another at phase angle $-\alpha$ are combined, groups of eight points are intercomparable.

## APPLICATION OF THE RECIPROCITY PRINCIPLE TO PREVIOUS PHOTOMETRIC MEASUREMENTS

The principles enumerated give a criterion which must be applied before points of the lunar surface can be assumed as identical in surface composition. The use of this criterion will be illustrated by some examples taken from previous work. For reasons given before, we shall make use of simultaneous comparisons only. A systematic survey of all published data is being attempted and will be communicated later if possible.

The points for which measurements are available have not always the exact position required for the application of the reciprocity principle. In order to estimate the influence of these small deviations, the brightness $J$ has been reduced to the value $J_0$ which would have been found if the angle of incidence had been $i_0$ instead of $i$. This reduction has been made according to the illumination law derived by Öpik. From this law, we find, by taking the logarithms and by differentiating,

$$\frac{dJ}{J} = -k \tan i \, di$$

where $k$ is a function of the phase angle, varying from 0 to 0.83 for the continents, from 0 to 0.88 for the Maria.

We shall give examples of good agreement with the reciprocity principle as well as examples of poor agreement; in this last case we must conclude that the points compared have not the same surface composition.

### CONSEQUENCES OF THE RECIPROCITY PRINCIPLE FOR THE GENERAL FORM OF THE LAW OF ILLUMINATION OF THE LUNAR SURFACE

Any law giving an exact description of the photometric properties of a solid substance must conform to the reciprocity principle. The laws of Lambert and Lommel[5] satisfy this criterion.

The law of Seeliger,

$$\frac{\cos i}{\mu \cos i + \cos \epsilon} ,$$

conforms only if the constant $\mu$ is equal to 1. For in reciprocal experiments the brightnesses observed will be

$$h = \frac{\cos i}{\mu \cos i + \cos \epsilon} \quad \text{and} \quad h' = \frac{\cos \epsilon}{\mu \cos \epsilon + \cos i} .$$

According to the reciprocity principle, $h/\cos i = h'/\cos \epsilon$. This implies that $\mu = 1$. The law derived by Fessenkoff for the continents is an extension of Seeliger's law and thus cannot be considered as exact, though it may be sufficiently accurate in a restricted region.

Nor is Öpik's law satisfactory. From his paper the impression is obtained that the dependence on $i$ and $\alpha$ has been well determined, while the functional dependence on $\epsilon$ is less well established. However, this may be easily determined from the reciprocity principle. For assuming $J = c(\cos i)^k \cdot f(\epsilon)$, we have to satisfy the condition

$$\frac{c(\cos i)^k f(\epsilon)}{\cos i} = \frac{c(\cos \epsilon)^k f(i)}{\cos \epsilon} ,$$

which is possible only if $f(\epsilon) = (\cos \epsilon)^{k-1}$; so that

$$J = c(\cos i)^k (\cos \epsilon)^{k-1} .$$

This law seems to represent the measurements at least as well as Öpik's formula. In

[5] *Sitzungsber. Akad. München, Math.-phys. Class.*, p. 95, 1887.

particular, the requirement is fulfilled that the brightness should increase with $\epsilon$; for $k$ is always $< 1$, and therefore $k - 1 < 0$.

TABLE 2

ÖPIK, CONTINENTS

| $a$ | Point No. | $i$ | $\epsilon$ | $J$ | $J_0$ | $\cos i_0$ | $\frac{J_0}{\cos i_0}$ | Agreement |
|---|---|---|---|---|---|---|---|---|
| 84° | 27 | 40° | 61° | 3.53 | 3.65 | 0.829 | 4.40 | good |
| | 37 | 64 | 34 | 2.02 | 2.11 | .485 | 4.35 | |
| 121 | 25 | 55 | 75 | 2.58 | 2.81 | .656 | 4.30 | good |
| | 35 | 80 | 49 | 0.94 | 1.22 | .259 | 4.72 | |
| 84 | 15 | 22 | 63 | 2.89 | 2.92 | .948 | 3.09 | poor |
| | 19 | 65 | 19 | 2.61 | 2.69 | .454 | 5.92 | |
| | 36 | 66 | 18 | 2.55 | 2.67 | 0.454 | 5.89 | |

BENNETT, CRATER BOTTOMS

| $a$ | Point No. | $i$ | $\epsilon$ | $J$ | $J_0$ | $\cos i_0$ | $\frac{J_0}{\cos i_0}$ | Agreement |
|---|---|---|---|---|---|---|---|---|
| 50° | 26*i* | 67° | 17° | 15 | 16.1 | 0.438 | 37 | good |
| | 44*i* | 15 | 64 | 39 | 38.7 | .956 | 41 | |
| 26 | 25*h* | 46 | 21 | 57 | 54.2 | .656 | 83 | poor |
| | 45*h* | 25 | 49 | 45 | 46.1 | 0.934 | 49 | |

The correction of $i$ to $i_0$ has been made according to Öpik's law for the continents.

FESSENKOFF, MARIA

| $a$ | $x$ | $y$ | $i$ | $\epsilon$ | $J$ | $J_0$ | $\cos i_0$ | $\frac{J_0}{\cos i_0}$ | |
|---|---|---|---|---|---|---|---|---|---|
| 64° | +690 | −046 | 21° | 44° | 5.27 | 5.27 | 0.934 | 5.65 | Mare Foecunditatis; mean: 6.20 |
| | +683 | −195 | 23 | 45 | 5.77 | 5.74 | .934 | 6.14 | |
| | +742 | −169 | 19 | 50 | 6.33 | 6.37 | .934 | 6.82 | |
| | +716 | −104 | 20 | 46 | 5.79 | 5.77 | .934 | 6.18 | |
| 64 | +342 | −055 | 45 | 20 | 4.86 | 4.86 | .707 | 6.88 | Mare Tranquillitatis; mean: 6.33 |
| | +343 | +031 | 45 | 20 | 4.28 | 4.28 | .707 | 6.05 | |
| | +314 | +090 | 47 | 19 | 4.08 | 4.14 | .707 | 5.85 | |
| | +286 | +149 | 48 | 19 | 4.21 | 4.31 | .707 | 6.10 | |
| | +345 | +177 | 45 | 23 | 4.36 | 4.36 | .707 | 6.17 | |
| | +316 | +235 | 47 | 23 | 4.83 | 4.91 | .707 | 6.95 | |
| 64 | +308 | −139 | 47 | 20 | 5.71 | 5.80 | .707 | 8.20 | Mare Nectaris; mean: 9.35 |
| | +334 | −202 | 46 | 23 | 7.36 | 7.42 | .707 | 10.50 | |

In deriving exact laws from a more extensive material, the data must be reduced in such a way that the condition of reciprocity is automatically fulfilled.

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August 1940