

THE EMPIRICAL MASS-LUMINOSITY RELATION

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ABSTRACT

After a brief historical introduction the problem is subdivided into two main parts. The first, on the temperature scale and bolometric corrections, is treated in the preceding article. The second problem, the derivation of the empirical mass-luminosity relation, is treated in three sections in this article: (1) the visual binaries, (2) some selected spectroscopic binaries, and (3) Trumpler's massive stars in clusters. An attempt has been made to obtain the most accurate observational data for all quantities entering into the discussion, including magnitudes and spectral types.

Tables 1 and 7 give the visual binaries used at present. Table 5 shows the derived quantities for the stars of Table 1, and Table 6 shows the quantities of theoretical interest for the stars of Table 5 for which the accuracy is sufficiently great. Visual binaries in the same class as those of Table 1, but for which the data are still incomplete, are collected in Table 8.

The problem of the spectroscopic binaries is only partly treated in this paper. Only some representative objects are discussed; they are found in Table 12. Theoretical values of the ellipticity and reflection constants are used in the discussion of the observations. Trumpler's massive stars are discussed in Table 13.

In all three sections the results of the preceding article have been used. The data are shown graphically in Figures 1 and 2.

INTRODUCTION

The discovery of the mass-luminosity relation has been of great importance to the progress of astronomy. The relation has been used in statistical astronomy and in double-star astronomy and has been a central problem of theoretical astrophysics. Since for most stars no direct determination of the mass can be made, the use of the mass-luminosity relation is the only method of estimating the total mass of the known stars per volume of space—an important dynamical quantity.¹ In binary-star statistics the observable Δm and a rough knowledge of the absolute magnitude can now be used in obtaining a statistically useful determination of the mass ratio, which is of great cosmogenetic interest.² Investigations, such as a recent one on ϵ Aurigae³ show the need of having the relation well established. But particularly the theoretical importance, both in ab-

¹ E.g., Oort, *B.A.N.*, 6, 285, Table 34, 1932.

² *Pub. A.S.P.*, 47, 17 ff., 143 ff., 1935.

³ *A.p. J.*, 86, 574, 1937.

stract form and in numerical form for particular stars, is clear from Eddington's work and from more recent developments.^{4, 5}

1. *Historical*.—The idea that the mass and the luminosity are correlated seems to have developed gradually as the knowledge of parallaxes increased. Even today the knowledge of masses is to a large extent limited by the knowledge of accurate parallaxes. The fact that one of the best-known stars, Sirius B, does not follow the general mass-luminosity relation must have delayed the discovery of that relation.

Halm was probably the first to state explicitly the existence of a statistical relation between intrinsic brightness and mass.⁶ His conclusion was essentially based on the correlation of mass with spectral type and of spectral type with luminosity. The first relation was established mainly from double-line spectroscopic binaries. The result was therefore partly accidental, because double-line spectroscopic binaries are very rare among giants, so that Halm was essentially dealing with main-sequence stars only. In fact, the relation was not recognized in Russell's paper of 1913,⁷ although in 1914 Russell found evidence⁸ for a definite correlation, which was obtained by comparing absolute magnitudes derived from hypothetical parallaxes with absolute magnitudes derived by methods of stellar statistics.

Hertzsprung, in 1918, gave the relation⁹

$$\log m = -0.06(M_v - 5),$$

in fair agreement with modern data. He also gave the corresponding formula for the mass ratio of a binary derived from Δm , and the first formula for dynamical (not hypothetical) parallaxes. Van Maanen shortly afterward emphasized that the mass-luminosity relation is independent of spectral type.¹⁰

⁴ B. Strömberg, *Erg. d. exact. Naturwiss.*, **16**, 497, 513, 1937.

⁵ S. Chandrasekhar, *Introduction to the Study of Stellar Structures*, "Astrophysical Monograph," chaps. vii and viii, 1938. (In press.)

⁶ *M.N.*, **71**, 638, 1911. ⁷ *Observatory*, **36**, 327, 1913. ⁸ *Pop. Astr.*, **23**, 340, 1914.

⁹ *A.N.*, **208**, 96, 1919. Already in 1915 Hertzsprung had given the same relation in a less explicit form (*Aph. J.*, **42**, 115, n. 2).

¹⁰ *Pub. A.S.P.*, **31**, 231, 1919.

Many other investigations followed, most of which have been reviewed by Lundmark.¹¹ Of all these, Eddington's results are undoubtedly the best known, particularly because Eddington used the relation in connection with his well-known theoretical investigations.

The most recent study of the subject is that by P. P. Parenago,¹² which was received after most of the present investigation was completed. In some ways the two investigations run parallel: for instance, in the use of bolometric corrections derived from radiometric observations and the estimation of weights of the individual mass determinations. But the present study uses several improved orbits, mass ratios, and Δm values and does not use spectroscopic parallaxes. Minor discrepancies will be found by comparing Parenago's Table 2 with our Tables 1 and 5. The treatment of the spectroscopic binaries is entirely different in the two papers: Parenago uses spectroscopic absolute magnitudes whereas we use the stellar temperature scale. A number of objects are used here which are not considered by Parenago; among these are Trumpler's stars.

2. *The present investigation* is limited to those individual stars for which the three fundamental quantities—the mass m , the radius R , and the luminosity L —may be obtained from the observations with fair precision. The derivation of the radius, in addition to the other two quantities, does not require any extra information, since radius and luminosity are simply related by the effective temperature, which is needed in any case before the eclipsing binaries can be used. In view of the fact that all three parameters are required in theoretical work, we shall give them explicitly as the final result of the investigation.

Since all the emphasis is laid on accurate data for individual stars (as required in theoretical work), no attempt is made here to supplement the results by a statistical treatment of other data of smaller individual accuracy. Such a statistical treatment would be useful for determining the trend of the mass-luminosity relation for the giants and the O stars, where few data of high individual accuracy are available.

A close co-operation between theory and observation will certainly

¹¹ *Handbuch der Astrophysik*, 5, Part II, 1933.

¹² *A.J. Sov. Union*, 14, 33, 1937.

be the quickest and least wasteful method of solving the fundamental problems of stellar structure and evolution.⁵ In this connection it would seem that the visual binaries of Tables 1, 7, and 8 deserve preference in double-star observations and in determinations of parallax and mass ratio. At least four to six determinations for each object will be needed in order to make the mass determinations sufficiently accurate. Photometric and spectroscopic measures of double-line spectroscopic binaries that are also eclipsing variables are obviously the second group of observations especially needed. Finally, investigations on the stellar temperature scale are required for the computation of luminosities from radii and vice versa.

Many important direct or indirect contributions to the knowledge of stellar masses have recently become available: Schlesinger's *Catalogue of Parallaxes*; Boss's *General Catalogue of 33342 Stars*, containing many mass ratios for visual binaries, based on all the available meridian positions; many new orbits, of which we want to mention especially those derived with the help of photographic measures (Strand, van den Bos, Hertzsprung); and several new mass ratios by van de Kamp. For the discussion of the eclipsing binaries, of special importance is Pannekoek's rediscussion of the temperatures of O and B stars, derived from maxima of spectral series. Data for very interesting systems, such as ζ Aurigae and VV Cephei, have recently been published. Last, but not least, Trumpler's discovery of very massive stars in galactic clusters should be mentioned. These many advances make it promising to assemble and discuss the data now available.

We have already referred to the stellar temperature scale as an integral part of the subject under discussion, being needed in the conversion of radii into luminosities and vice versa, according to the formula

$$\frac{L}{L_{\odot}} = \left(\frac{R}{R_{\odot}}\right)^2 \cdot \left(\frac{T_e}{T_{e\odot}}\right)^4. \quad (1)$$

Before this formula can be used, the relation between M_{vis} and L has to be known. This relation is given by the bolometric correction.

Hence, the subject falls into two parts: (1) the discussion of the stellar T_e scale and the derivation of the bolometric corrections; and

(2) the discussion of the binary systems, with suitable observational data, and of Trumpler's stars. The first problem is treated in the preceding article; the second naturally falls into three parts: the visual binaries, the eclipsing binaries, and Trumpler's massive stars.

THE VISUAL BINARIES

If the orbital elements and the parallax of a visual binary are known, Kepler's third law

$$3 \log a'' - 2 \log P - 3 \log \pi'' = \log \Sigma m \quad (2)$$

will give the total mass, Σm , of the system. If the orbit is well determined, and hence the error in $(3 \log a'' - 2 \log P)$ is small, the error in Σm will depend mainly on the error in the parallax; we then have $3\Delta(\log \pi'') = -\Delta(\log \Sigma m)$. If the probable error of the parallax equals one-tenth of the parallax itself, the probable error in $\log \Sigma m$ will be ± 0.13 . This value corresponds to about ± 1.4 absolute magnitude in the region of the mass-luminosity relation covered by the visual binaries of our list. The computed absolute magnitude is also affected by an error in the parallax, and a certain compensation of errors occurs when the data are used in a diagram correlating mass and luminosity; the value ± 1.4 mag. is then reduced to ± 1.2 mag. As true deviations from the mean mass-luminosity curve larger than 2 mag. are very rare indeed, it is obvious that data with probable errors larger than ± 1.2 mag. are of no individual value. Hence, all systems for which the probable error of the parallax exceeds 10 per cent are excluded from our discussion.¹³

The remaining binaries are collected in Table 1. It would be of great importance for the knowledge of stellar masses if parallax observers would pay special attention to these objects, particularly to those for which the probable error of the parallax still exceeds $\pm 0''.004$. Of not much less importance are the stars in Table 8, which may be used in the near future.

The first three columns of Table 1 need no explanation. The fourth column gives, on the international photovisual system, the

¹³ We have, furthermore, excluded stars with parallaxes smaller than $0''.045$. No attempt is made here to use averages for binaries of different spectral types—a problem considered recently by Professor Russell.

apparent magnitude of the combined light of the binary. Most of these data are based on Zinner's catalogue (which is on the Potsdam

TABLE 1
VISUAL BINARIES FOR THE MASS-LUMINOSITY RELATION

Name	a 1900	δ 1900	m_{AB}	Spec.	Δm	P	a''	Ref., Q	π'' (Trig.)
HR 159	0 ^h 32 ^m 2	-25° 19'	5.61	G7	0.3	25.00	0.670	2, A	0.072 ± 6
η Cas	0 43.0	+57 17	3.57	G0, K5+	3.74	526.0	12.534	3, B	.182 ± 5
ρ Eri	1 36.0	-56 42	5.21	G5	0.06	251	8.31	4, C	.161 ± 7
α_2 Eri BC	4 10.7	-7 49	9.37	B0, M5e	1.48	247.9	6.894	1, AB	.202 ± 3
α Aur	5 09.3	+45 54	0.19	G4, F4	0.15	104.02d	0.0536	1, A	.0632
α CMa	6 40.7	-16 35	-1.52	A0, A5	10.06	49.94	7.62	1, A	.376 ± 2.6
α Gem	7 28.2	+32 06	1.59	A0, A1	0.86	340	5.84	1, B	.073 ± 3
α CMi	7 34.1	+5 29	0.45	F3, ...	10.3	40.23	4.26	1, A	.291 ± 4
ρ Pup	7 47.1	-13 38	5.27	G2	0.7	23.34	0.69	1, A	.061 ± 4
ξ UMa	11 12.9	+32 06	3.77	F0, F9	0.43	59.86	2.5355	1, A	.138 ± 6
γ Vir	12 36.6	-0 54	2.98	F0, F0	0.03	171.37	3.746	3, A	.089 ± 7
ζ UMa A	13 19.9	+55 27	2.43	A2, A2	0.0 ±	20.54d	0.0115	1, A	.045
α Cen	14 32.8	-60 25	-0.20	G4, K1	1.37	80.09	17.593	5, A	.756 ± 7
ξ Boo	14 46.8	+19 31	4.51	G8, K5	1.96	149.95	4.884	3, A	.147 ± 6
44 Boo	15 00.5	+48 03	4.79	G1, G5	(0.82)	219.5	3.609	3, B	.079 ± 5
ADS 10075	16 24.5	+18 37	6.82	K2	0.01	217.1	2.198	1, AB	.059 ± 6
ζ Her	16 37.5	+31 47	2.93	G1, ...	2.80	34.42	1.349	1, A	.110 ± 5
-8 ⁴³⁵²	16 50.1	-8 09	9.20	M4	0.1	1.72	0.19	6, AB	.148 ± 4
HR 6416	17 11.5	-46 32	5.54	K0, ...	2.6	242	4.94	7, BC	.132 ± 6
HR 6426	17 12.2	-34 53	5.82	K5, ...	1.2	42.2	1.83	1, A	.147 ± 6
HR 6516	17 25.2	-0 59	5.22	G6	0.14	46.0	1.06	1, A	.050 ± 4
26 Dra	17 34.0	+61 57	5.22	G1, ...	2.61	76.06	1.55	8, B	.066 ± 5
μ Her BC	17 42.5	+27 47	9.80	M4	0.1	43.02	1.287	1, A	.109 ± 6
70 Oph	18 00.4	+2 31	3.97	K1, K4	1.68	87.85	4.558	3, A	.196 ± 4
99 Her	18 03.2	+30 33	5.13	F5, ...	3.40	54.7	1.13	9, A	.049 ± 5
HR 7162	18 53.3	+32 46	5.17	G0, ...	1.78	61.8	1.24	10, A	.058 ± 5
δ Equ	21 09.6	+9 36	4.50	F6	0.0	5.70	0.26	4, A	.060 ± 4
τ Cyg	21 10.8	+37 37	3.73	F0, ...	2.82	49.8	0.94	11, A	.046 ± 4
Kr. 60	22 24.4	+57 12	9.64	M4+, M6	1.5	44.52	2.362	1, A	.256 ± 4
85 Peg	23 56.9	+26 33	5.80	G2, ...	3.5:	26.46	0.82	4, AB	0.090 ± 6

EXPLANATION OF NINTH COLUMN (REF.)

1. W. S. Finsen, "Catalogue of Visual Binary Star Orbits," *Circ. Union Obs.*, 91, 24, 1934. In most cases the same orbit is given in R. G. Aitken's recent book, *The Binary Stars*, p. 284, Table 1. This last table is complete to September, 1933. Baize's catalogue (*Bull. Astr.*, 10, 273, 1937) was received after this section was written.
2. W. H. van den Bos, *Circ. Union Obs.*, No. 98, 1938.
3. K. A. Strand, *Annals Leiden Obs.*, 18, Part II, 1937.
4. W. J. Luyten, "Investigations of Binary Stars," *Pub. Minnesota*, 2, Part I, 1934.
5. E. Hertzsprung, *B.A.N.*, 7, 330, 1936.
6. G. P. Kuiper, unpublished. The orbit is nearly circular in appearance; $i = 0^\circ$, $e = 0.04$.
7. H. N. Russell, *A.J.*, 45, 95, 1936; also W. S. Finsen, *Circ. Union Obs.*, No. 95, 230, 1936. Finsen's orbit is a parabola: $n = 0.0707$; $q = 2.175$. Russell's orbit represents the measures better than does Finsen's. In computing the mass, Russell's orbit has been given double weight.
8. P. Rudnick, *A.J.*, 43, 164, 1934.
9. W. J. Luyten and E. Ebbighausen, *ibid.*, 45, 54, 1936.
10. R. C. Huffer, *A.p. J.*, 80, 270, 1934.
11. W. J. Luyten and E. Ebbighausen, *A.J.*, 44, 119, 1935.

system),¹⁴ corrected by the amount $-0.22 + 0.11C$.¹⁵ Other magnitude data were first reduced to the PD system with the aid of Zin-

¹⁴ *Veröff. Bamberg*, 2, 1926.

¹⁵ F. W. Seares, *Trans. I.A.U.*, 4, 142, 1932; C. Payne Gaposchkin, *Harvard Ann.*, 89, 105-107, 1935.

ner's tables, and then to the IP_v system.¹⁶ The magnitudes of α_2 Eridani BC, BD-8°4352, and Krüger 60 are my own determinations. For μ Herculis BC, the reduced Harvard magnitude is 9.79 mag., and my own measures give 9.82 mag.

For convenience, the correction $IP_v - PD = -0.22 + 0.11C$, which seems to be well established, is tabulated against spectral type in Table 2. The color index C is defined as $IP_g - IP_v$, which makes $C = 1.08$ for gK₀ if $C = 0$ for A₀.

TABLE 2

Spec.	IP _v -PD	Spec.	IP _v -PD	Spec.	IP _v -PD
O.....	-0.24	dGo.....	-0.17	gGo.....	-0.15
B0.....	.24	G5.....	.16	G5.....	.13
B8.....	.23	K0.....	.14	K0.....	.10
A0.....	.22	K2.....	.12	K2.....	.09
A2.....	.21	K5.....	.09	K5.....	.05
A5.....	.20	dMo.....	-0.05	gMo.....	-0.03
F0.....	.20				
F5.....	-0.18				

The fifth column of Table 1 gives the spectral type either of the two components separately or of the combined light. For A stars the HD classes were used, and for most of the other stars spectral types were obtained by Dr. Morgan or the writer.¹⁷ In particular, the writer wishes to acknowledge Dr. Morgan's classification of the spectra of the components of Capella. Three-prism spectrograms were used which were compared with similar spectrograms of other F and G giants. Dr. Morgan estimates $\Delta m(\text{pg}) = 0.0$, which is in good agreement with the small visual difference found interferometrically.

For α Centauri B the published spectral type is K₅; but since this value led to an abnormally large radius for this star, the writer has asked Dr. Shapley's permission to compare the Harvard spectra of this star with those of 61 Cygni (dK₅), ϵ and 40 Eridani (dK₂),

¹⁶ The valuable compilation by C. Payne Gaposchkin (*Harvard Obs. Mimeograms*, 3, 1938), was received after this paper was practically completed. It was used for Table 8 (except γ CrA, which seems to be in error).

¹⁷ Cf. pp. 451 and 463 of the preceding article.

and τ Ceti and ξ Bootis (dG8). The result, K_I , is quite well determined and makes the star a normal dwarf.¹⁸ The writer wishes to thank Dr. Shapley and Miss Hoffleit for their kind co-operation.

The sixth column of Table 1 contains the magnitude differences between the components. The majority of the determinations are my own, but ten Harvard values were available and these were included.

The seventh, eighth, and ninth columns give the relevant orbital data, including the references explained at the bottom of the table, and the quality Q of the orbit. $Q = A$ indicates an orbit as good or nearly as good as may be obtained for the pair in question; B means a more uncertain orbit, usually due to the fact that the measures cover the orbit only partly; and C means that the orbit is quite uncertain.

The list of references illustrates the activity in orbital determinations since the appearance of the "Catalogue of Orbits," by Finsen in 1934. Of particular interest is Strand's publication on six classical binaries (five of which are used here), in which many photographic measures are incorporated.

The last column gives the trigonometric parallax, in all but a few cases the same as in Schlesinger's *Catalogue* of 1935. The few more recent determinations available were included.

The data on mass ratios for the binaries of Table 1 are collected in Table 3. A most important source of information is the new *Boss General Catalogue*, which contains the mass ratios that can be derived from meridian observations. In addition, van de Kamp has recently determined several mass ratios photographically, each based on at least fifty plates.

One word should be said about mass ratios determined for binaries that are not separated on the plates used, or, if observed visually, are not resolved in the meridian circle. These binaries are ζ Herculis, 99 Herculis, HR 7162, τ Cygni, and 85 Pegasi. The result of the finite size of the image (caused by imperfect seeing and optical effects) will be that in such cases not the brighter star is observed but the center of light. The position of this center may be computed if

¹⁸The result is in agreement with that just published by Miss Hoffleit (*Harvard Ann.*, 105, 57, 1937).

TABLE 3

Star	$\frac{m^2}{m^1+m^2}$	Authority	Ref.	Star	$\frac{m^2}{m^1+m^2}$	Authority	Ref.
η Cas...	0.405	Boss GC	1	ζ Her...	0.372	Boss GC	1
	.385	Strand	2		.382	Van de Kamp	7
α_2 Eri BC	.31	Van den Bos	3	26 Dra...	.38	Boss GC	1
α Aur...	.443 \pm	Reese, Sanford	4	70 Oph..	.478	Boss GC	1
α CMa..	.282	Boss GC	1		.416	Comstock	8
	.326	Van de Kamp	5		.427	Strand	9
α Gem..	.64	Boss GC	1	99 Her...	.265:	Boss GC	1
α CMi..	.235	Boss GC	1	HR 7162..	.45	Van de Kamp	10
γ Vir...	.508	Boss GC	1	τ Cyg...	.328	Van de Kamp	11
ζ UMa.	.501	Hadley	6	Kr. 60...	.359	Van de Kamp	12
α Cen...	.446	Boss GC	1	85 Peg...	.52	Boss GC	1
ξ Boo...	.461	Boss GC	1		0.51	Van de Kamp	13
44 Boo ..	.791	Boss GC	1				

EXPLANATION OF FOURTH COLUMN (REF.)

1. B. Boss, *General Catalogue of 33342 Stars*, 1, Appen. II, 1937. ζ Her: B (observed) = 0.313; $\Delta m = 3.00$; B (corrected) = 0.372. 26 Dra: since the separation reaches 1".7, the meridian positions must refer to the primary, and B needs no correction. 99 Her: B (obs.) = 0.223; $\Delta m = 3.40$; B (corr.) = 0.265. According to a personal communication by Dr. Jenkins, the weight of this determination is small. 85 Peg: B (obs.) = 0.48; $\Delta m = 3.5$; B (corr.) = 0.52.

2. *Ann. Leiden Obs.*, 18, Part II, 75, 1937; the probable error is ± 0.019 .

3. *B.A.N.*, 3, 132, 1926.

4. *Lick Obs. Bull.*, 1, 35, 1901; *Pub. A.S.P.*, 34, 179, 1922. Value is approximate.

5. *A.J.*, 45, 124, 1936; 59 plates were used, dating from 1920 to 1933.

6. *Pub. Michigan Obs.*, 2, 101, 1915.

7. *A.J.*, 44, 83, 1935. From 50 plates between 1919 and 1934, $B = 0.323 \pm 0.015$ was found, which, corrected for $\Delta m = 3.00$, gives 0.382.

8. *Ibid.*, 32, 157, 1920. Based on relative micrometer positions of three field stars, separated by the present writer from Comstock's values obtained from meridian positions. The probable error is ± 0.025 .

9. *Ann. Leiden Obs.*, 18, Part II, 135, 1937. Value derived from radial-velocity observations published by Berman. The probable error is ± 0.028 .

10. *A.J.*, 46, 36, 1937. From 54 plates between 1919 and 1936, B (obs.) = 0.288 \pm 0.045 was found, which, corrected for $\Delta m = 1.78$, gives $B = 0.45$.

11. *Ibid.*, 45, 121, 1936. From 53 plates between 1916 and 1935, B (obs.) = 0.257 \pm 0.024. Corrected for $\Delta m = 2.82$, $B = 0.328$.

12. *Ibid.*, 47, 1, 1938. From 57 plates between 1916 and 1936, $B = 0.359 \pm 0.007$. (The difference between the x - and y -solutions suggests a somewhat larger uncertainty.)

13. Personal communication. From measures in the x -direction B (obs.) = 0.476 \pm 0.012; from the y -direction B (obs.) = 0.419 \pm 0.034, weighted mean B (obs.) = 0.470 \pm 0.012; $\Delta m = 3.5$; B (corr.) = 0.51.

the magnitude difference, Δm , is known. The center will be on the line connecting the components, at a distance $[L_2/(L_1 + L_2)]s$ from the brighter star, where L_1 and L_2 are the luminosities of the components, measured in the wave length considered, and s is their separation. It follows that the observed (or apparent) values of $B = m_2/(m_1 + m_2)$ will have to be increased by $L_2/(L_1 + L_2)$, in order to give the true values of B .¹⁹ This correction has been applied in Table 3, with the details given in the footnotes. In Table 4 the correction ($B - B_{\text{obs}}$) is tabulated against Δm . It appears that Δm , except if larger than 3 mag., must be known quite accurately in order that significant mass ratios may be obtained. If Δm is well

TABLE 4

Δm	$B - B_{\text{obs}}$	Δm	$B - B_{\text{obs}}$
0.0.....	0.500	2.5.....	0.091
0.5.....	.387	3.0.....	.059
1.0.....	.285	3.5.....	.038
1.5.....	.201	4.0.....	.025
2.0.....	0.137	5.0.....	0.010

determined, there is no difficulty in obtaining mass ratios even for close pairs with small Δm .

We have not attempted to include in Table 3 all mass ratios that have been published, but only those that seemed most reliable. Several extensive investigations based on meridian positions have now been replaced by the GC mass ratios. Particularly when the whole orbit has not yet been observed, as in the case of η Cassiopeiae, α Geminorum, γ Virginis, and ξ Bootis, the meridian observations should give a more reliable determination of the mass ratio than even the best photographic measures extended over a period of less than twenty years. This is obvious from the fact that in such cases the accuracy involves the square of the time interval used.

The binaries for which the mass ratios are known yield *two* absolute mass determinations each, free from assumptions. The determination of mass ratios is therefore most important.

¹⁹ Cf. E. Silbernegel, *A.N.*, 233, 168, 1928; P. van de Kamp, *A.J.*, 46, 37, 1937.

If the mass ratio is not known, and the components have nearly equal magnitudes, we shall assign half the total mass to the average of the two magnitudes. This has been done for nine pairs in Table 1. In these cases each binary yields only one absolute mass determination.

Only two pairs, HR 6416 and 6426, are left for which Δm is not small, and no mass ratio is available from observations. Here we shall use the mass ratio corresponding to the observed Δm , adopting the slope of the mass-luminosity relation found in this paper. We shall use the primaries of these two systems only.

For the systems in Table 3 with more than one determination of the mass ratio, relative weights must be assigned to these determinations. The weights of the two values of η Cassiopeiae are probably in ratio about 1:2; hence, $B = 0.392$ has been adopted. For Sirius, van de Kamp's photographic determination extends over thirteen years near apastron, where the curvature of the relative orbit is slightly over $1''$. The size of the whole orbit, $14''$, is covered by the meridian observations (for simplicity, the comparison is made in the *relative* orbit). The value $B = 0.295$ has been adopted, which gives weight 0.7 to the GC determination.

For ζ Herculis the agreement is better than could be expected. In time the photographic determination will be very accurate.

Three determinations are available for γ Ophiuchi, with approximate weights 2:1:1, making the average 0.450; this value is very well established.

Table 3 contains two anomalous values: for 44 Bootis and 85 Pegasi. Taking the duplicity of the companion into account, the slope of the mass-luminosity relation would give $B = 0.61$ for 44 Bootis. In view of the relatively small part of the orbit covered by the observations, the observed value of B can probably not be considered contradictory to the value $B = 0.61$.

The case of 85 Pegasi is perhaps more puzzling. The GC value of the mass ratio is based on 0.400 from the x -direction (weight 0.71) and 0.669 from the y -direction (weight 0.29). Van de Kamp's determination was described in note 13 of Table 3, and is of high weight. The observations as they stand would indicate that the primary is of abnormally low mass and the secondary of abnormally

high mass; but if the parallax of the system should be too large by twice its probable error, the primary would be nearly normal and the large mass of the secondary might be explained by assuming it to be double.

The data of interest in connection with the mass-luminosity relation are found in Table 5. Explanation of the columns is given beneath the table.

We have not attempted to give numerical estimates of the uncertainties of the mass ratios used. Qualitative data are given of pages 480 and 482. The weights in column 8 will be upper limits for the masses in the tenth column; the secondaries will be of lower relative accuracy than the primaries unless the components are nearly equal. For the nine objects where the average component is used in column 10, the weight remains identically the same as given in column 8, of course.

Table 6 gives the quantities of theoretical interest for the stars of Table 5 having weight 3 or more; α Geminorum and ξ Ursae Majoris are omitted because the components are spectroscopic binaries, and HR 6426 is omitted because the spectral type refers to a blend of the two components. Furthermore, faint and close companions, such as that of Procyon, cannot be used because no spectral type is known, whereas for Krüger 60 B no reliable estimate of T_e can be made. 70 Ophiuchi has occasionally been suspected to be a triple system, but the evidence has never been entirely convincing, and Strand's discussion of accurate photographic measures gives no indication of the presence of a third body.

The masses of Table 6 are taken directly from Table 5; the luminosities follow from the bolometric magnitudes of Table 5, in connection with the bolometric magnitude of the sun, 4.62.²⁰ The value $\Delta \log T_e = \log T_e / \odot$ follows from the temperature scale of the preceding article,²¹ except for dwarfs later than M2, for which no radiometric measures are available. For these stars we shall adopt the temperature scale for the giants, on the basis of the following considerations.

As we have seen, the heat indices (or bolometric corrections) are nearly the same for gM2 and dM2; the color indices are also nearly

²⁰ Cf. p. 438 of the preceding paper.

²¹ Table 13, p. 464.

TABLE 5
VISUAL BINARIES FOR THE MASS-LUMINOSITY RELATION

(1) Star	(2) M_{pv}	(3) M_{bol}	(4) $\log \Sigma m$	(5) p.e. (π)	(6) p.e. (Orb.)	(7) p.e.	(8) Wt.	(9) $\frac{m_2}{m_1+m_2}$	(10) $\log m$
HR 159.....	5.65	5.55	+0.11	±0.09	±0.03	±0.10	1	av.	-0.19
η Cas.....	4.90	4.84	+0.07	.03	.015	.03	9	0.392	-0.14 -0.33
ρ Eri.....	6.99	6.89	+0.34	.05	.08	.09	1	av.	+0.04
α_2 Eri BC.....	11.14	10.26	-0.10	.018	.015	.02	18	.31	.35 - .70
α Aur.....	0.13	0.58	+0.8801	.01	100	.443	.02 + .52
α CMa.....	1.36	0.64	+0.523005	.009	120	.295	.37 + .01
α Gem.....	1.32	1.08	+0.65	.04	.04	.00	3	.64	+ .35
α CMi.....	2.77	2.71 <	+0.29	.016	.015	.02	21	.235	.17 - .34
9 Pup.....	4.95	4.88	+0.42	.06	.04	.08	2	av.	+ .12
ξ UMa.....	5.24	5.18	+0.24	.05	.02	.05	4	av.	- .06
γ Vir.....	3.47	3.37	+0.40	.09	.005	.09	1	.508	.10 + .11
ζ UMa A.....	1.45	0.93	+0.7204	.04	6	.501	.41 + .41
α Cen.....	4.40	4.37	+0.29	.010	.002	.01	100	.446	.04 - .06
ξ Boo.....	5.52	5.42	+0.21	.04	.005	.04	5	.461	.06 - .12
44 Boo.....	5.21	5.14	+0.30	.07	.03	.08	2	.7	- .2
ADS 10075.....	6.42	6.27	+0.04	.11	.03	.12	1	av.	- .26
ζ Her.....	3.22	3.15	+0.19	.05	.01	.05	4	.38	- .02 - .23
-8 ^h 4352.....	10.80	8.1	-0.15	.03	.04	.05	4	av.	- .45
HR 6416.....	6.23	6.12	-0.01	.05	.04	.06	2	(.366)	- .21
HR 6426.....	6.97	6.44	+0.04	.04	.03	.05	3	(.438)	- .22
HR 6516.....	4.46	4.36	+0.65	.09	.03	.09	1	av.	+ .35
26 Dra.....	4.41	4.34	+0.35	.08	.05	.10	1	.38	+ .14
μ Her BC.....	10.74	8.0	-0.05	.06	.015	.06	3	av.	- .35
70 Oph.....	5.64	5.56	+0.21	.024	.002	.02	17	.450	.05 - .13
99 Her.....	3.63	3.59	+0.61	.11	.03	.12	1	.264	.48
HR 7162.....	4.18	4.12	+0.41	.10	.02	.10	1	.45	.15 + .06
δ Equ.....	4.14	4.10	+0.40	.07	.02	.08	2	av.	+0.10
τ Cyg.....	2.12	2.02	+0.54	.10	.015	.10	1	.328	.36 + .05
Kr. 60.....	11.04	9.0	-0.40	.016	.012	.02	25	.359	- .60 - .85
85 Peg.....	5.61	5.54	+0.03	±0.07	±0.03	±0.08	2	0.51	-0.28 -0.20

EXPLANATION OF TABLE 5

1. The name of the star.
2. The absolute visual magnitude, on the IPv scale. Either both components are given or the average, depending on the data in the last column.
3. The bolometric magnitude, computed from column 2, Tables 6, 9, and 10 of the preceding paper, and the spectral type from Table 1. If radiometric observations were available, they were also used; they received double weight, compared with the values derived from the visual magnitudes.
4. The logarithm (base is 10) of the sum of the masses, computed from P, a'' , and π'' of Table 1; the unit of m is the solar mass.
5. The probable error in $\log \Sigma m$, resulting only from the probable error of π'' . For this purpose the probable error of π'' was computed to one more decimal.
6. The probable error in $\log \Sigma m$, resulting only from the uncertainty of the orbital elements used. This value is an estimate, of course; but considerable time was spent in making the estimate as close as possible. For Capella and Mizar, the uncertainty in the masses depends mainly on the uncertainty in i , which was determined from interferometer observations. The estimated probable errors correspond to a probable error of ± 0.4 in i for Capella and of $\pm 3^\circ$ for Mizar.
7. The probable error, resulting from the combination of the probable errors in the two preceding columns. This combination can be made in the usual manner, since the two sources of error are independent and since the small compensation of errors mentioned on page 476 of this paper was removed before the values in column 5 were computed. Hence, columns 5, 6, and 7 all refer to the uncertainty parallel to the ($\log m$) axis of a mass-luminosity diagram, the uncertainty in the other co-ordinate having been eliminated.
8. The weight of the determination of $\log m$, corresponding to the probable error in column 7. The unit corresponds to a probable error of ± 0.10 in $\log m$.
9. The adopted value of the mass ratio, based on Table 3 and subsequent discussion in the text.
10. The logarithm of the masses of the components or the average component based on columns 4 and 9.

the same.²² Hence, the effective temperatures are probably nearly the same, as we have assumed in the preceding paper.

When a spectrum (extending from 4000 Å to 6600 Å) of BD $-8^{\circ}4352$ (dM₄) was compared with spectra of M₄ giants, it was found that although the red *TiO* bands correspond (as they should, since the classification is based on them), the *TiO* bands in the yellow, green, and blue are considerably weaker. Dr. Morgan has found the

TABLE 6

Star	log <i>m</i>	log <i>L</i>	Δ log <i>T_e</i>	log <i>R</i>	log ρ	log <i>g</i>	<i>E</i> km/sec
η Cas A.....	-0.14	-0.09	+0.021	-0.09	+0.27	4.47	0.56
B.....	-0.33	-1.16	-0.166	-0.25	+0.56	4.60	0.53
ο ₂ Eri B.....	-0.35	-2.26	+0.303	-1.74	+5.01	7.56	15.5
C.....	-0.70	-1.96	-0.305	-0.37	+0.56	4.48	0.30
α Aur A.....	+0.62	+2.08	-0.082	+1.20	-2.84	2.65	0.17
B.....	+0.52	+1.90	+0.066	+0.82	-1.79	3.32	0.32
α CMa A.....	+0.37	+1.59	+0.273	+0.25	-0.23	4.31	0.84
B.....	-0.01	-2.59	+0.175	-1.65	+5.08	7.72	27.5
α CMi A.....	+0.17	+0.76	+0.077	+0.23	-0.36	4.16	0.56
ζ UMa A.....	+0.41	+1.48	+0.229	+0.28	-0.28	4.29	0.86
α Cen A.....	+0.04	+0.10	-0.018	+0.09	-0.07	4.30	0.57
B.....	-0.06	-0.43	-0.078	-0.06	+0.27	4.50	0.64
ξ Boo A.....	-0.06	-0.32	-0.051	-0.06	+0.25	4.49	0.63
B.....	-0.12	-0.83	-0.157	-0.10	+0.34	4.52	0.61
ζ Her A.....	-0.02	+0.59	+0.010	+0.29	-0.74	3.84	0.31
-8°4352 AB.....	-0.45	-1.40	-0.290	-0.12	+0.06	4.23	0.30
μ Her BC.....	-0.35	-1.37	-0.290	-0.10	+0.12	4.30	0.36
70 Oph A.....	-0.05	-0.38	-0.078	-0.03	+0.20	4.46	0.61
B.....	-0.13	-0.86	-0.136	-0.16	+0.49	4.62	0.68
Kr. 60 A.....	-0.60	-1.77	-0.298	-0.29	+0.42	4.42	0.31
Sun.....	0.00	0.00	0.000	0.00	+0.15	4.44	0.64

same phenomenon even more strikingly in Barnard's star (dM₆), as compared with M₅-M₆ giants. Another difference between the spectra of giants and dwarfs of type M is, of course, the broad absorption feature near *Ca* 4227, present in dwarfs but absent in giants. If the effective temperatures of M giants and M dwarfs were about the same, the spectral differences mentioned would tend to give the dwarfs the larger color index (for two reasons) but the smaller bolometric correction. The bolometric corrections are not known for the later M dwarfs, but the color indices are, approxi-

²² Seares, *Pub. Nat. Acad. Sci.*, 5, 232, 1919.

mately, from Seares's determination for Barnard's star²³ (dM6, $C = 1.76$) and from the approximate value for Wolf 359 (dM8, $C = 2.0$).²⁴ The M giants are known to become slightly bluer (as measured by the ordinary color index) with advancing type.²² The color difference is, therefore, in the direction to be expected from the appearance of the spectra.²⁵ Whether or not the full difference would be accounted for (which would mean that the effective temperatures of giants and dwarfs are equal) would be difficult to determine. Another approach to the temperatures of the M dwarfs would be to estimate the distortion of the color index of Barnard's star by the *TiO* absorption. Dr. Morgan estimates that the corrected value may be about 2.0 mag., with a considerable uncertainty.

With this (uncertain) value, the effective temperature in our system may be computed if color indices for two other spectral types are adopted. We take $C = 0.00$ for A₀, and 1.22 for dK5+ (being the mean of Seares's average for the components of 61 Cygni, 1.18²², and W. Becker's photoelectric color index reduced to the international system, 1.25). With the aid of Table 13²¹ we then find $c_2/T = 5.19$, which happens to be very near to the value 5.22 for gM6 given by Table 13. Much significance cannot be attached to this result; only radiometric observations of Barnard's star will solve the problem of the effective temperatures of the later M dwarfs.

Since the temperatures for giants and dwarfs seem to be nearly the same for M6 as well as for M2, the temperature scale of the giants will be used also for the dwarfs.

The mean densities, ρ , and the values of the surface gravity, g , given in Table 6 are in c.g.s. units. The variation of g along the main sequence is seen to be quite small. The mean densities vary from 0.5 to 0.6 for Mizar A and Sirius A to about 3 for K and M dwarfs.

The last column gives the Einstein gravitational red shift. It ap-

²³ Seares, *Pub. A.S.P.*, 28, 281, 1916.

²⁴ *Contr. Mt. W. Obs.*, No. 356, 3, 1928.

²⁵ For the reasons given, the treatment of the M dwarfs in papers by Öpik and Gabovits, who assume equal *TiO* absorption in the visual region for giants and dwarfs, and who adopt a constant difference between the color indices (0.3 mag., the dwarfs being bluer), is not correct, although the errors in the two assumptions tend to compensate. Cf. *Pub. Tartu*, 28, Nos. 3 and 5, 1935-1936; 30, No. 1, 1938.

pears to diminish slightly along the main sequence. Only for the white dwarfs is the red shift appreciable, although in statistical studies dealing with the F, G, and K stars it should not be overlooked that the standard radial velocity system is corrected by an amount of 0.64 km/sec, the red shift of the sun, in order that the sun would show zero radial velocity. The giants of these spectral types should, therefore, show a small negative K term.

The two white dwarfs of Table 6 will be discussed, together with similar objects, in a forthcoming paper.

A theoretical discussion of the masses, radii, and luminosities derived here will not be given, since Drs. Chandrasekhar and Ström-

TABLE 7

ADS	M_{pv}	Sp.	M_{bol}	$\log L$	$\Delta \log T_e$	$\log R$	$\log m$	Wt.
3264.....	3.02	F0	2.90	+0.69	+0.118	+0.11	+0.07	1½
3483.....	3.66	F5	3.62	+ .40	+ .054	+ .09	+ .04	1
3135.....	4.48	F6	4.44	+ .07	+ .046	- .06	- .195	3
3169.....	4.52	F8	4.47	+ .06	+ .032	- .03	- .21	½
3475.....	4.59	F7	4.54	+ .03	+ .039	- .06	- .19	3
3210.....	5.05	G5	5.55	-0.37	-0.028	-0.13	-0.44	2

gren intend to undertake such a discussion with the methods they have developed.

The values themselves are shown graphically in Figures 1 and 2. Figure 1 contains also objects of lower weight, taken from Table 5, and the binaries in the Hyades, given in Table 7. The latter table is a revision of a table published previously;²⁶ for ADS 3135 a new orbit was published by van den Bos,²⁷ who used measures up to date. As mentioned in the previous paper, the mean of the six objects in the Hyades should give a reliable determination of the hydrogen content, but deviations from the mean are not considered significant. In due time the pair 68 Tauri ($M_{pv} = +1.4$, $M_{bol} = +0.9$); discovered recently by the writer,²⁸ will be valuable in settling the question of possible variations of composition with absolute magnitude.

The companion to Procyon was entered in Figure 1 as a probable

²⁶ *Ap. J.*, **86**, 176, Tables 4 and 5, 1937.

²⁷ *Circ. Union Obs.*, No. 98, 1937.

²⁸ *Pub. A.S.P.*, **49**, 341, 1937.

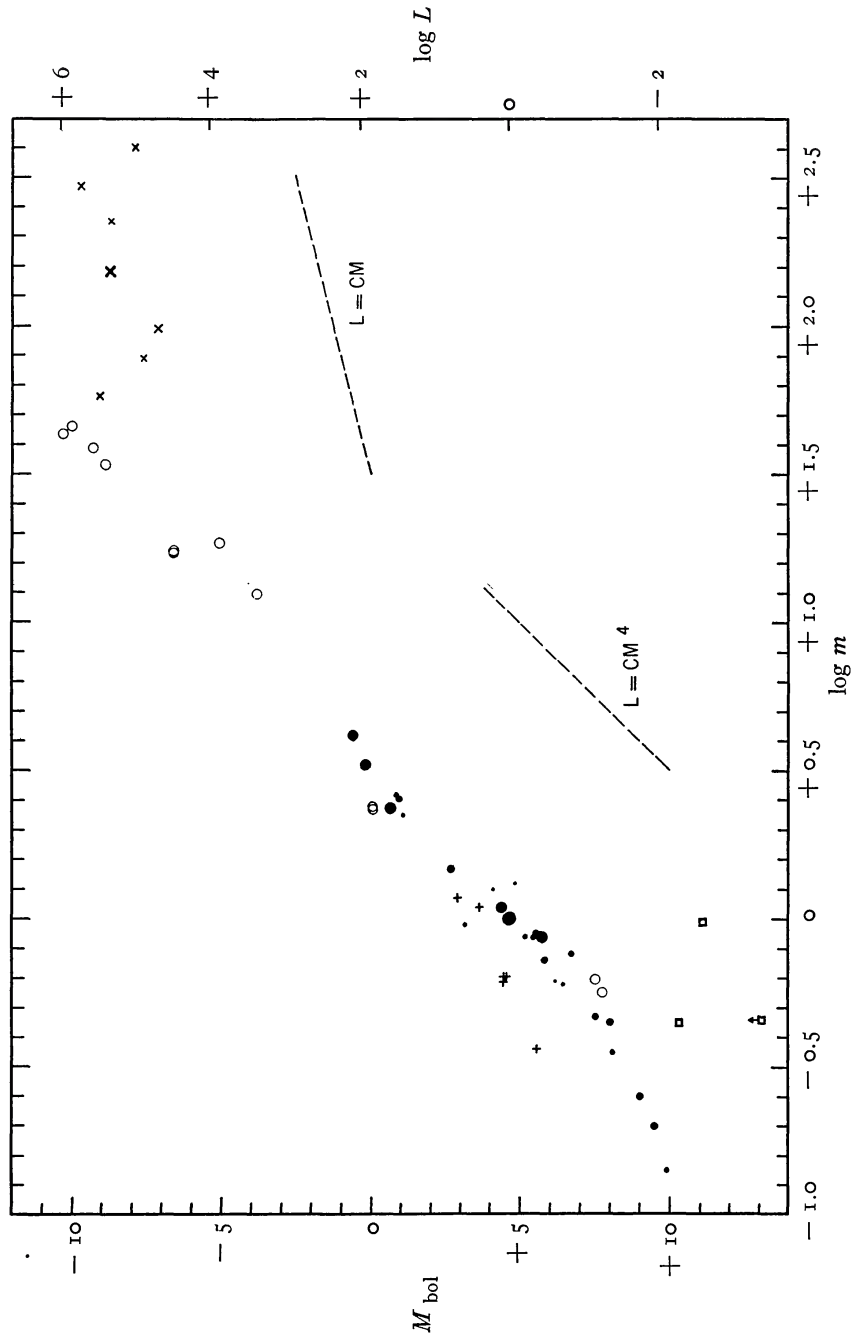


FIG. 1.—The empirical mass-luminosity relation. Closed dots: visual binaries; open circles: spectroscopic binaries; vertical crosses: Hyades; diagonal crosses: Trumpler's stars; square's: white dwarfs.

white dwarf for the following reason. Observations show that the correlation between absolute visual magnitude and spectral type is close for the lower part of the main sequence. The deviation of Procyon B from the empirical mass-luminosity relation may therefore be computed under the assumption that the star is a normal dwarf. The deviation is in that case 2 mag., or over 0.4 in $\log m$. Until other large deviations are found in this region of the mass-luminosity diagram, we may conclude that Procyon B is probably not a normal red dwarf.

The dots in Figure 1 define an average empirical mass-luminosity relation which is useful in the discussion of such problems as that presented by ϵ Aurigae. Since, however, different selection factors operate for the different groups of stars used, it is doubtful whether this mean empirical relation has any definite physical significance; no interpolation-curve was therefore drawn. Some stars, such as the Hyades and ζ Herculis, are seen to be brighter than the average star of the same mass; these differences are considered real. Other differences, particularly among the stars of low accuracy, are probably spurious. The conclusion, therefore, is that we are not dealing with only *one* mass-luminosity relation but with many such relations, involving still a parameter different from the mass. Strömberg has identified this parameter with the hydrogen (or hydrogen and helium) content. This conclusion is based on another result, by Strömberg and Chandrasekhar, that the stars of mass smaller than five to ten times the sun are built essentially on the same model, so that changes in model cannot be held responsible for the differences in luminosity for stars of the same mass. For the more massive stars, however, Chandrasekhar has found very striking differences in model, which must be largely responsible for the scatter in the mass-luminosity relation in that region.

For the stars with masses smaller than $0.6\odot$ the slope of the mass-luminosity relation changes;²⁹ the exact slope will depend, of course, on the bolometric corrections used, but there is no doubt that the change is real. The explanation may be simply that the stars Krüger 60 and α_2 Eridani C have a small hydrogen content, as would follow

²⁹ The change found now is better determined than in Fig. 1 of *Ap. J.*, **86**, 194, 1937; cf. also Parenago, *loc. cit.*, Gabovits, *Pub. Tartu*, **30**, No. 1, 1938 (see n. 25).

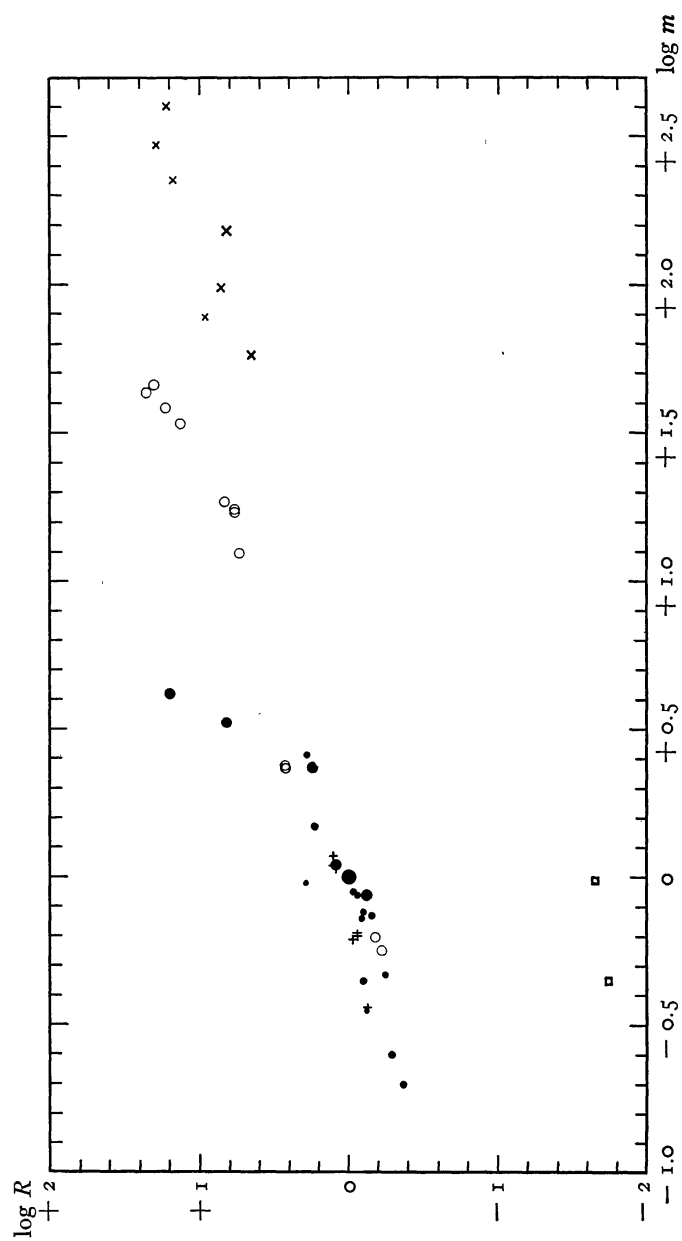


FIG. 2—Mass-radius diagram. Symbols as in Fig. 1

from an extrapolation of Strömgen's tables. A more detailed theoretical investigation of these low-temperature stars will be needed before this conclusion can be accepted. If it holds, it may be expected that stars will be found which deviate strongly from the mean relation in Figure 1, toward the fainter side. It is of great interest that the white dwarf σ_2 Eridani B (also) has a very small hydrogen content, which follows from the data in Table 6 and Chandrasekhar's theoretical relations. Castor C, connected with the A-type star Castor AB, has probably a fairly high hydrogen content. Furthermore, the brighter component of μ Herculis (A) lies well above the main sequence and may therefore be expected to be relatively poor in hydrogen. It would, therefore, not be surprising to find BC somewhat bright for its mass. It is also noteworthy that both components of η Cassiopeiae deviate in the same sense from the mean relation of Figure 1.

These remarks are all of a provisional nature, but they indicate the direction in which future research may be continued. The accuracy of the observations (parallaxes, temperatures) will set the limits to which discussion is profitable.

In conclusion, in Table 8 the visual binaries are collected which promise to be of future interest in the study of the empirical mass-luminosity relation. They are of special importance in parallax and double-star observations. Spectral types marked with an asterisk are my own determinations. Of particular importance is Wolf 424, which will provide a significant extension of the mass-luminosity relation beyond Krüger 60 B. The value of the spectral parallax of this star has been unusually uncertain because of the great steepness of the main sequence in this region. A change of M8 to M7 corresponds to a change of 1.7 in M_v or a factor 2.2 in the spectral parallax. That means that even estimates of spectral type with a probable error of only one-half of one-tenth will still give quite uncertain results for types later than dM6.³⁰

The limits set in the compilation of Table 8 are necessarily arbitrary. If the evidence available indicated that the parallax is (or may be) in excess of $0''.050$, the pair was included, provided the projected separation between the components did not exceed about

³⁰ *Ap. J.*, **87**, 593, Fig. 1, 1938.

sixty times the parallax (this limit will already include many pairs of periods over five hundred years). For stars of very large parallax, the limit was taken somewhat wider, since a much greater relative accuracy may be obtained for such pairs (on account of the larger angular scale of the orbit and the applicability of the photographic method of observation; further, mass ratios may be determined for such pairs from relatively short arcs). Boundary-line cases are entered in parentheses.

The magnitudes, m_{AB} , are reduced to the IPv system wherever possible; values depending on estimates are indicated by a colon. Of the magnitude differences, Δm , however, many values depend on estimates. Two systems, ADS 1322 and 9476, were excluded as probably optical, the spectral parallaxes derived by the author being very much smaller than the computed dynamical parallaxes.

Not mentioned in the table is Ross 614, found by Reuyl to have a variable motion; if the pair can be resolved at maximum separation, it would be a most interesting addition to Table 8. Other stars of particular interest are ADS 9655 and 16644. All these objects deserve a prominent place in parallax and double-star observations.

Other objects not mentioned are those for which interferometer observations could be combined with the spectrographic observations. They will be few in number but of great importance individually (luminous stars).

Finally, Ross 492³¹ and a pair announced by Luyten,³² may appear to have parallaxes in excess of $0''.050$.

SOME SELECTED SPECTROSCOPIC BINARIES

1. After a discussion of the visual binaries it is natural to collect all the data on spectroscopic binaries which have a bearing on the problem of the mass-luminosity relation. These data fall into two groups: (1) double-line spectroscopic binaries that are also eclipsing variables (for these objects the masses and radii may be found and the luminosities computed by means of the stellar temperature scale); (2) double-line spectroscopic binaries for which the mass ratio may be measured and the magnitude difference found from photometry in the spectrum.

³¹ *Pub. A.S.P.*, 47, 282, 1935.

³² *Harvard Bull.*, No. 906, 1937.

TABLE 8*

STAR	R.A.	DECL.	m_{AB}	Δm	SPEC.	d''	P	1000 PARALLAX			SOURCE $\pi(d)$
								$tr.$	$dy.$	$sp.$	
ADS 48.....	0 ^h 00 ^m 4	+45° 16'	8.1	0.1	*K6, K6	5''	89±5	120	79	930
(ADS 61).....	0 01.0	+57 53	6.0	0.8	G3	106.	14±8	53	39	930, 91
PASP 47,282.....	0 07.7	+68 47	12.2	0.3	*M5	0.7	56±7
ADS 433.....	0 26.0	+66 42	10.2	1.9	*M3	3	103±7	69
ADS 490.....	0 30.1	- 4 09	5.2	0.6	F7	7.	52±8	(51)	46	930
-69°44.....	1 11.6	-69 21	7.4	0.6	G5	128.	52	98
-30°529 AB.....	1 30.4	-30 25	7.2	0.2	G5	4.6	36±9	56	91
AB, C.....	4.9	1.5	59	93
ADS 1733.....	2 11.1	-18 42	7.8	1.0	*K3	2.	74	(42)	930
ADS 1865.....	2 22.5	+ 3 59	8.6	0.2	*K6	25.	61	98
α For.....	3 07.8	-29 23	4.2	3.5	F5	1.	72±5	55	87	93
+68°278 BC.....	3 38.2	+68 21	10.8	0.0	*M2+	0.4	54±9	60
(ADS 2995).....	4 00.9	+37 49	7.1	1.3	*K2	1.	36±6	58	44	930
68 Tau.....	4 19.7	+17 42	4.4	4.0	A2	1.5	Hyad	25
ADS 3248.....	4 23.3	+15 56	6.6	1.0	F8	40.	Hyad	28	98
ADS 3321 CD.....	4 30.2	+16 19	11.2	2.4	*K6	1.7	Hyad
HR 1504.....	4 38.6	-59 08	6.4	0.2	G0	1.	50	93
104 Tau.....	5 01.5	+18 31	5.1	0.0	G1	0.1	2±?	58±8	46
α Aur Hh.....	5 10.0	+45 44	10.1	3.5	*M1+	2.	73±3	Intf = 63
(ADS 3900).....	5 14.1	- 3 11	7.9	2.8	*K4	4.	66±5	110	58	930
-51°1540.....	5 31.2	-51 08	9.3	0.2	K0	1.	110	93
BDS 3112 AB.....	5 56.6	-31 03	7.6	0.4	*K5+	< 1.	53±11	79	(78)	93
AB, C.....	0.1	2.	51	93
HR 2162.....	6 02.2	-48 27	6.5	0.4	G5	2.	52	93
-46°3046.....	7 14.6	-46 49	7.6	1.1	K0	1.	73±9	14	93
ADS 6487.....	7 52.9	- 0 33	8.3	5.0	*K4	3.	81	(69)	485
HR 3430.....	8 34.8	-22 19	5.1	1.8	G6	1.	62±6	36	76	93
ADS 7067.....	8 46.0	+71 11	8.6	0.2	*K6, K6	3.	93±4	90	66	930
ϵ UMa BC.....	8 52.4	+48 26	10.2	0.0	M1	35±	66±6	30
A, BC.....	3.2	7.1	A5	6.	91	63	930
10 UMa.....	8 54.2	+42 11	4.0	2.0	F2	0.6	72±6	66
(σ_2 UMa).....	9 01.6	+67 32	4.9	3.3	F4	1.5	53±6	40	50	91
(ADS 7251).....	9 07.7	+53 07	7.2	0.0	*K6, K6	19.	162±3	(240)	(152)	930
ADS 7284.....	9 12.0	+29 00	7.2	0.0	*K3	34.	60±8	57	48	930, 91
ψ Arg.....	9 26.8	-40 02	3.7	1.2	F0	35.	70±9	61	55	930, 91
11 LMi.....	9 29.7	+36 16	5.4	7.5	K0	2.	95±5	91
-60°2911.....	11 20.3	-61 06	7.4	1.2	K5	234.	67±13	104	91
Wolf 424.....	12 28.4	+ 9 34	11.9	0.0	*M7(+)	1.	(220)	(320)	91
78 UMa.....	12 56.4	+56 54	5.0	2.4	F0	90.	UMa(?)	47	24	BAN 305
42 Com.....	13 05.1	+18 04	4.7	0.0	F4	26.	57±7	56	43	930, 91
(ADS 8841).....	13 11.9	+17 34	6.4	2.7	K3	5.	68±7	94	79	930
ADS 8887.....	13 18.0	+29 45	8.9	0.4	*K6	0.6	67±11	19	46	930†
ADS 9031.....	13 44.5	+27 29	7.0	0.4	K6, K6	156.	60±9	79	66	91
ADS 9090.....	13 58.5	+46 49	9.3	0.1	*M1, M1	3.	79±6	95
(ADS 9446).....	14 51.6	-20 58	5.4	2.2	K5, M2	19.	172±4	(200)	(174)	93
ADS 9544.....	15 08.8	- 0 58	6.5	0.0	G8	0.1	2±?	60±4	42
η CrB.....	15 19.1	+30 39	5.1	0.3	F9	42.	68±8	(57)	44	930
ADS 9716.....	15 32.5	+40 08	6.7	0.3	*K3	57.	46±9	47	54	§
ψ Ser.....	15 39.0	+ 2 50	5.9	6.3	G5	3.	46±10	89	52	485

* Spectral types marked by an asterisk (*) are my own determinations. The last column gives the references for the dynamical parallaxes; 930=A.J., No. 930; 91, 93, 95, and 98 are the numbers of the *Union Observatory Circulars*; 451 and 485 are the numbers of the *Lick Observatory Bulletins*.

† Two good spectra available for this star give M7+ and M7, respectively. The star is, therefore, definitely somewhat earlier than Wolf 359.

‡ The spectral parallax is about 0.06. It seems unlikely that the parallax should be much less than 0.05 or 0.04, as the dynamical parallax would indicate. The value of the latter was confirmed by the inclusion of recent measures. Hence, the orbital inclination must be high (in which case the binary should soon close in rapidly), or the mass of the binary is unusually small. Additional trigonometric determinations will be valuable.

§ Derived from Pitman's orbit, *Pub. A.S.P.*, 46, 196, 1934.

TABLE 8—Continued

STAR	R.A.	DECL.	m_{AB}	Δm	SPEC.	d''	P	1000 PARALLAX			SOURCE $\pi(d)$
								$tr.$	$dy.$	$sp.$	
+45°2505.....	17 ^h 09 ^m 2	+45° 50'	9.4	0.4	*M3+	< 1.	143±6	115
36 Oph.....	17 09.2	-26 27	4.4	0.0	K1, K3	4.	179±5	52	174	93
ADS 11632.....	18 41.7	+59 29	8.5	0.8	*M4, M4	16.	282±4	(240)	(282)	930
γ CrA.....	18 59.7	-37 12	4.2	0.0	F7, F7	119.	62	51	91
17 Lyr C.....	19 03.6	+32 21	11.2	0.4	*M5	0.2	123±6
Ross 165.....	19 41.7	+26 55	12.2	0.0	*M5+	0.7	116±11
ADS 12889.....	19 41.8	+33 22	7.7	0.2	*K3	1.	48±5	57	44	930¶
HR 7703.....	20 04.6	-36 21	5.2	5.4	K4	8.	177±9	191
+4°4510.....	20 34.6	+ 4 37	7.9	1.0	*K5	0.7	57±4	87
Furuh 54.....	20 56.2	+39 41	10.0	1.9	*M2+	1.	97±8	87
(61 Cyg).....	21 02.4	+38 15	4.9	0.7	*K5, K6	26.	299±3	(360)	(276)	930
-58°7803.....	21 39.4	-58 08	8.6	0.1	Ma	6.8	49±5	54	95
(μ Cyg).....	21 39.7	+28 17	4.5	1.4	F6, F3	< 1.	46±8	66	44	930**
HR 8501.....	22 11.7	-54 06	5.5	5.1	Go	3.	80±7

|| Another case (n. †, above) where the dynamic parallax is much too small.

¶ Strand (*B.A.N.*, 8, 206, 1937) has computed a provisional orbit for this star and found the dynamical parallax 0".117. It is almost certain that this value is too large, probably as a result of the fact that the orbital inclination found by Strand is too close to 90°. The trigonometric parallax, the Mount Wilson spectroscopic parallax, and my spectral parallax (0".050) agree very well. Also Strand's doubt whether this binary is physically connected with 17 Cygni seems unjustified. Proper motion, radial velocity, and parallax suggest that the connection is real.

The nature of this case is apparently the opposite of cases † and ||, above.

** The secondary was classified at Mount Wilson earlier than the primary.

The first group gives fundamental data for the problem of stellar structure, and the second group is useful in a study of the scatter of the mass-luminosity relation³³ and in giving data that may be used statistically in a determination of the mean position of the mass-luminosity relation (the mean absolute magnitude may be determined from the spectral types; the mean mass, by substitution of the mean value of $\sin^3 i$ into the mean value of $m \cdot \sin^3 i$).

A review of the available data showed that for many close pairs contradictions seem to be present;³⁴ it was clear that for many objects new observations would be required in order to find the interpretation of such discrepancies. Particularly close pairs with very unequal components are likely to present apparent anomalies; the systems with nearly identical components, however, are more easily interpreted. In the present paper we shall discuss only a few objects of the latter class; the discussion of additional spectroscopic binaries will be made later by Dr. Morgan and the present writer.

³³ Cf. *Pub. A.S.P.*, 47, 144, 1935; the scatter found in this manner will be a minimum value, since the components of binaries seem, at least statistically, to have common properties (hydrogen content). Cf. *Harvard Bull.*, No. 903, 1936.

³⁴ E.g., Wyse, *Bull. Lick Obs.*, No. 464, 1934.

2. It is well known that in many classes of light-curves the ratio of the radii cannot be found with any great precision. It is therefore necessary to determine the ratio of the total brightnesses from the relative strength of the two superimposed spectra; in combination with the ratio of the surface brightnesses determined from the light-curve the ratio of the radii is found. All our discussions of the radii will be made in this manner. Fortunately, the only cases which are of interest here are binaries which show two spectra.

3. A second question in connection with the determination of the radii arises when the pair is very close, so that the components are ellipsoidal in shape. We have a good theoretical evidence that the stars possess strong central concentrations; only certain classes of very massive stars may be exceptions to this rule.³⁵ Hence the Roche, or point-mass, model should give a close approximation in the determination of the equipotential surfaces; the boundaries of the components will, in general, be two of these surfaces. Since for the present we shall limit ourselves to pairs of nearly identical stars, the model considered will have two equal components with envelopes of negligible mass, in which the periods of rotation are equal to the orbital period. In this case the equipotential surfaces are given by the formula³⁶

$$\Omega = -GM \left[\frac{1}{r_1} + \frac{1}{r_2} + \frac{x^2 + y^2}{a^3} \right]. \quad (3)$$

The symbols have the following meaning: Ω is the potential for which a constant value is adopted; G is the gravitational constant; and M is the mass of each component. The orbit is supposed to be circular, with radius a ; the xy -plane is assumed to coincide with the orbital plane, and $x = y = 0$ with the center of gravity; r_1 and r_2 are the distances of any point (x, y, z) from the centers of the two stars; x, y, z refer to a rotating co-ordinate system.

With the aid of this formula the writer has computed a family of curves, $\Omega = C$, both for the xy -plane and for a plane parallel to the yz -plane through the center of one of the components. The results were plotted on large-scale diagrams from which the ratio of the

³⁵ Chandrasekhar, *op. cit.*, chap. viii.

³⁶ E.g., Jeans, *Problems of Cosmogony*, p. 162, 1919.

axes, a_1 , b_1 , and c_1 , were read off for a given a_1/a (if a is the distance between the components). A certain amount of rounding-off was applied to the pointed sides of the stars facing each other, since theoretically the intensity of these points (apart from reflection effects) should be low. We are interested in knowing the mean stellar radius, $R = \sqrt[3]{a_1 b_1 c_1}$, whereas the light-curve gives a_1 , or a_1 and b_1 . We shall therefore tabulate R/a_1 against a_1/a , so that the former quantity may be found from the latter, which is observed. The quantity b_1/a_1 to be expected from the Roche model is also given, since this

TABLE 9

a_1/a	R/a_1	b_1/a_1	a_1/a	R/a_1	b_1/a_1
0.10.....	1.00	1.00	0.35.....	0.945	0.935
.15.....	0.995	0.995	.40.....	.915	.90
.20.....	0.99	0.99	.45.....	.885	.86
.25.....	0.98	0.98	0.48.....	0.88	0.855
0.30.....	0.965	0.96			

ratio may be obtained observationally. Table 9 contains the results, rounded off to the nearest half of 0.01. It should be noted that for $a_1/a = 0.43$, the two components just touch each other; for larger values of a_1/a there will be a bridge between them, which, as mentioned above, may be expected to have a low intensity (apart from reflection effects), so that in the photometric solution the components will still appear separated.

Before applying the tabular values to actual stars, we may consider the results of Wesselink's discussion of SZ Camelopardalis.³⁷ From an unusually accurate light-curve he finds $b_1/a_1 = 0.9323 \pm 0.0020$ (m.e.) in the uniform solution, for which $a_1/a = 0.38$, $a_2/a = 0.27$, and $\Delta m = 1.0$. The mean value of a_i/a (weighed according to intensity) is 0.35. For the darkened solution he found $b_1/a_1 = 0.9579 \pm 0.0013$. The theoretical darkening coefficient for this star is about 0.1, and we may adopt the empirical value $b_1/a_1 = 0.935 \pm 0.002$ (m.e.). This happens to agree precisely with the tabular value. (Wesselink had already pointed out this agreement with the Roche model.)

³⁷ Dissertation, Leiden, 1938.

4. Finally, the *reflection effect* should be taken into account in the determination of the masses. Since the centers of light on the two disks do not project themselves precisely on the centers of mass, but are displaced toward the center of gravity, the amplitudes measured are found too small even if the components are of similar brightness. The reflection effect will increase the bolometric brightness on certain areas of the stars; in order to find the photographic increase of

TABLE 10
CORRECTION OF MASSES FOR REFLECTION EFFECT

a_1/a	$\Delta \log m$	a_1/a	$\Delta \log m$
0.10.....	0.000	0.35.....	0.021
.15.....	.001	.40.....	.032
.20.....	.003	.45.....	.046
.25.....	.007	(0.50).....	(0.066)
0.30.....	0.013		

TABLE 11
FACTOR TO APPLY TO THE CORRECTION OF TABLE 10

Spec.	Factor	Spec.	Factor
O7.....	0.45	F0.....	0.95
B0.....	.45	dF5.....	1.15
B5.....	.5	dG0.....	1.4
A0.....	.65	dK0.....	1.5
A5.....	0.8	dK6.....	2.5

intensity, the ratio of photographic increase to bolometric increase has to be known, which will depend on the spectral type. The simplest procedure is to tabulate the correction to $\log m$ as a function of a_1/a for bolometric intensities; this correction will then have to be multiplied by a factor giving the rate of increase of photographic intensity for a given increase of bolometric intensity. This factor is readily found from Table 3, or from Tables 9 and 13, of the preceding paper, and a table of color indices; the result is found in Table 11. The correction to $\log m$ for bolometric intensities is given in Table 10; Eddington's formula for the reflection effect³⁸ was used.

³⁸ *The Internal Constitution of the Stars*, p. 212, formula 144.3, 1926.

The reflection has, of course, also its effect on the determination of $a \sin i$, and hence on R (involving a) and L (involving a^2), as well as on m (involving a^3). The corrections are of less importance than those for the masses, but may be taken from the same tables (10 and 11), after multiplication by $1/3$ for $\log R$, and by $2/3$ for $\log L$.

After these preparations, some close binaries with nearly equal components will be discussed.

5. *Castor C*.—From the discussion in the preceding article³⁹ we have $\log \bar{R} = -0.198$. The ratio of the radii follows from the ratio of the intensities of the two spectra, 5:4, and the equality of the surface brightnesses. We find $\log R_1 = -0.18$ and $\log R_2 = -0.22$.

The direct determination of the effective temperature gave $\log T_e = 3.550 \pm 0.014$;³⁹ from the spectral type K6+ and the stellar temperature scale⁴⁰ we have $\log T_e = 3.562$. We adopt the mean, 3.556. Hence,⁴¹ $\log L_1 = -1.16$, $\log L_2 = -1.24$; $M_{\text{bol}}(1) = +7.53$, $M_{\text{bol}}(2) = +7.73$. With $i = 86^\circ$ the masses are given by⁴² $\log m_1 = -0.204$; $\log m_2 = -0.250$. With $R/a = 0.16$, the correction for the reflection effect is about $+0.003$, making $\log m = -0.201$ and -0.247 , respectively.

6. β *Aurigae*.—The photographic magnitudes are nearly equal,⁴³ and also the surface brightnesses.⁴⁴ Hence, the radii are nearly equal: $\log R_1 = \log R_2 = 0.431$.⁴⁵ The direct determination of T_e gave $\log T_e = 3.99 \pm 0.04$; from the spectral type A1 we have $\log T_e = 4.01$; we adopt 4.00. Then $\log L_1 = \log L_2 = 1.83$; $M_{\text{bol}(1,2)} = +0.04$.

With $\cos i = 0.223$ ⁴⁵ and Baker's orbit the masses are found: $\log m = 0.376$ and 0.370 , respectively. The reflection effect found from $a_1/a = 0.15$ adds less than 0.001 to $\log m$.

7. μ_1 *Scorpii*.—From our former discussion⁴⁶ we take $\log R_1 = \log R_2 = 0.73$. The direct determination of T_e was $\log T_e = 4.20 \pm 0.05$. From the spectral type B3 of the average component

³⁹ P. 458.

⁴⁰ Table 13, p. 34, of the preceding paper.

⁴¹ Formula 1, p. 475.

⁴² Joy and Sanford, *Ap. J.*, **68**, 253, 1926. Computed from K , P , and i .

⁴³ Baker, *Pub. Allegheny Obs.*, **1**, 173, 1910.

⁴⁵ P. 442 of the preceding paper.

⁴⁴ Stebbins, *Ap. J.*, **34**, 126, 1911.

⁴⁶ P. 443 of the preceding paper.

we have $\log T_e = 4.27$. We adopt 4.23. Hence $\log L = 3.35$ and $M_{\text{bol}} = -3.76$; this value applies to the average of the components. Since the mass ratio for this binary is not known, we shall use the average mass of the components in connection with the average $\log L$. With $i = 62^\circ$, the average uncorrected mass is $\log m = 1.082$; the correction for reflection ($a_1/a = 0.370$) is 0.012, and $\log m = 1.094$.

8. *V Puppis*.—The mass ratio is not known; therefore we shall use the average component. The strength of the companion is about 0.7 of that of the primary.⁴⁷ Roberts' photometric observations being visual, the amount of darkening toward the limb must be negligible; hence Shapley's uniform solution⁴⁸ will be used. Then $J_b/J_f = 1.27 = 1/0.79$. This value in connection with the ratio of the total brightnesses indicates substantially equal radii, as Shapley has remarked. Shapley finds the mean of a_i/a to be 0.438, and $b_i/a_i = 0.813$. The Roche model gives $b_i/a_i = 0.87$. Since the light-curve near the minimum depends much more on b_i than on a_i , we shall use the value of $(3b_i + a_i)/4$ as found by Shapley, and compute R by means of the Roche model. The result is $\bar{R}/a = 0.373$. With Shapley's value of $\cos i$ (0.219) and Maury's value of $a \sin i$ we find $\log \bar{R} = 0.821$. The mean spectral type being B2 (Maury), $\log T_e = 4.31$, and the mean $\log L$ is 3.85. From P , K , e , and i , we further find $\log \bar{m} = 1.249$. From the computed value $a_i/a = 0.409$ we find $\Delta \log m = 0.016$ and $\Delta \log L = 0.011$; the corrected values for the mean component are $\log m = 1.265$, $\log R = 0.83$, and $\log L = 3.86$ ($M_{\text{bol}} = -5.04$).

9. *Y Cygni*.—As Redman⁴⁹ and Dugan⁵⁰ have pointed out, the observations for this binary are not all satisfactorily explained (possibility of a third body, etc.), so that some caution should be exercised in using the results. Yet it is doubtful whether the chief quantities, R and m , are much affected by these difficulties.

The spectra are of nearly equal intensity (Plaskett, Redman) and the minima are of equal depth (Dugan); hence the radii are nearly

⁴⁷ A. C. Maury, *Harvard Ann.*, **84**, 173, 1920.

⁴⁸ *Princeton Pub.*, No. 3, 1915.

⁴⁹ *M.N.*, **90**, 754, 1930; *Pub. Dom. Ap. Obs. Victoria*, **4**, 341, 1931.

⁵⁰ *Pub. Princeton Obs.*, No. 12, 1931.

equal. Dugan found $R_1 = R_2 = 0.206a = 5.86\odot$ (uniform solution, which will be used since the darkening must be small). The ellipticity effect is negligible because the width of the minimum determines b_i , which is nearly equal to R if $R/a < 0.25$.

The writer is under obligation to Dr. Morgan, who obtained an accurate spectral class for the star from the plates available at the Yerkes Observatory. This was of importance, in view of the discordant values published: B2 in the *Henry Draper Catalogue*, B2 by J. S. Plaskett,⁵¹ O9.5 by Pearce, and O9 in the Victoria catalogue of radial velocities of O and B stars. Dr. Morgan finds the type to be certainly as early as O9, but probably not much earlier than O9. He suggests the value O9 as the best obtainable from the data at hand.

From Dugan's dimensions we now have: $\log R = 0.768$; $\log L_1 = \log L_2 = 4.51$; $M_{\text{bol}(1,2)} = -6.6$. The masses are 17.1 and 17.3 (Dugan, Redman), $\log m = 1.233$ and 1.238; the correction for reflection to $\log m$ is only $+0.002$.

10. *AO Cassiopeiae*.—Pearce has made four photometric solutions, corresponding to $\Delta m = 0.25, 0.50, 0.80$, and 1.11 mag.⁵² He estimates the ratio of the intensities from the spectrum to be 2.6:1, but this may perhaps be an upper limit and we shall adopt the solution for $\Delta m = 0.80$ mag. The mean value of a_i/a is then 0.500; of b_i/a , 0.478. This is not possible for completely separated stars built on the Roche model, for which $b_i/a < 0.38$. If the photometric solution is correct, it is probable that the components have a common envelope; and since the stars are not identical, it is then further probable that the system is not in equilibrium. This possibility should be borne in mind if comparisons with other stars are made.

It is in this case uncertain what values for the radii should be used; since, according to Table 9, b_i and R are never very different, we shall use the values of b_i : $R_1/a = 0.549$ and $R_2/a = 0.406$.

In the case of unequal components, the effect of reflection on the constants of the system is slightly more complicated than in the case of identical components. With the ratio of the radii given and with equal surface temperatures (both spectra are O8.5), Δm is

⁵¹ *Pub. Dom. Ap. Obs. Victoria*, 1, 213, 1920.

⁵² *Ibid.*, 3, 297, 1926. Cf. also: M. Güssow, *A.N.*, 207, 321, 1930.

found to be 0.65. In order to make $\Delta m(\text{pg}) = 0.80$, we shall have to make the temperatures slightly different: $\Delta \log T_e = 0.03$, which makes $\Delta m(\text{bol}) = 1.0$.

The reflection effect on the amplitudes, K , is one-third of that on the masses, which is found from Tables 10 and 11. For equal components and $a_i/a = 0.50$, the effect on $\log K$ is $\frac{1}{3}(0.066) \cdot (0.45) = 0.010$. With the intensity ratio $2\frac{1}{2}:1$, the effect on $\log K_1$ will be about $0.4 \times 0.010 = 0.004$, and on $\log K_2$, $2\frac{1}{2} \times 0.010 = 0.025$. Applying these corrections to the observed values, we have $K_1 = 220$; $K_2 = 248$; $m_2/m_1 = 0.887$; $\log a \odot / = 1.625$; $\log R_1 = 1.365$; $\log R_2 = 1.234$; $\log L_1 = 5.97$, $\log L_2 = 5.58$; $M_{\text{bol}(1)} = -10.3$, $M_{\text{bol}(2)} = -9.3$; $\log m_1 = 1.634$, $\log m_2 = 1.582$; $m_1 = 43.0 \odot$; $m_2 = 38.2 \odot$.

In allowing for the reflection effect we have increased the masses by a fair amount and have made them more unequal, which is more favorable to the difference in dimensions found for the two components (since they are probably in contact) and in better agreement with the difference in brightness.

However, perhaps the main uncertainty about this system is the value of the inclination, which enters seriously in the determination of the masses. This is particularly evident if a comparison is made with the ellipsoidal systems, π^5 Orionis, b Persei, and ζ Andromedae, discovered by Stebbins. A smaller inclination would obviously increase the derived ellipticity of the components, which is now found to be smaller than is possible for two stars which are just in contact. It will remain for a new photometric study of this system to settle this question.

The extremely high luminosity of this system is noteworthy (a smaller value of the inclination would even increase the luminosity). With the provisional values of the constants, it lies well above the mean empirical mass-luminosity relation; even a somewhat decreased inclination would not change that feature. This fact was interpreted by Chandrasekhar³⁵ to be a result of strong mass concentration, contrary to the massive stars found by Trumpler and even stars like V Puppis, which are less centrally condensed than Eddington's standard model. In that respect AO Cassiopeiae seems to resemble the stars of lower mass, whereas Trumpler's stars are

really the stars that deviate from a theoretical mass-luminosity relation based on the standard model. It will be of interest to discover which of the two classes of massive stars is the most frequent in nature.

On account of the uncertainties mentioned in connection with AO Cassiopeiae, it is very fortunate that a second system is known with similar properties. This system is 29 Canis Majoris.

11. *29 Canis Majoris*.—Spectroscopic orbits have been published by Harper,⁵³ Pearce,⁵⁴ and Luyten and Ebbighausen.⁵⁵ Pearce has measured both components and finds the spectral types both O7; the secondary is very faint. Furthermore, $K_1 = 217 \pm 5$ km/sec; $K_2 = 288 \pm 7$ km/sec; $P = 4.3935d$; and $e = 0.156 \pm 0.017$. Luyten and Ebbighausen found a smaller value of e , 0.077 ± 0.019 .

The star was found to be an eclipsing variable by S. Gaposchkin,⁵⁶ who derived photometric elements from a light-curve based on photographic estimates. Recently Elvey and Rudnick have observed the star photoelectrically at the McDonald Observatory. They have very kindly made their provisional results available to the writer. Two solutions were made, of which one, with $k = 0.8$, fitted the observations best. This solution, in which the orbital eccentricity was neglected, is

$$\begin{array}{ll} a \sin i = 3.01 \times 10^7 & a_2 = 18.1 \odot \\ m_1 \sin^3 i = 32.2 \odot & b_1 = 16.6 \odot \\ m_2 \sin^3 i = 24.3 \odot & b_2 = 13.2 \odot \\ k = 0.8 & b_i/a_i = 0.73 \\ i = 64^\circ & m_1 = 44.3 \odot \\ a_1 = 22.7 \odot & m_2 = 33.4 \odot \end{array}$$

It is seen that Gaposchkin's elements are in good general agreement with those derived by Elvey and Rudnick. The range being 0.4 mag., the star is photometrically a much better case than AO Cassiopeiae.

Dr. Morgan has very kindly examined the Yerkes plates of this star which were measured and discussed by Luyten and Ebbighausen, and he confirms the presence of the extremely faint com-

⁵³ *Pub. Dom. Obs. Ottawa*, 4, 115, 1917.

⁵⁴ *Pub. Dom. Ap. Obs. Victoria*, 6, 50, 1932.

⁵⁵ *Ap. J.*, 83, 246, 1935.

⁵⁶ *Bull. Harvard Obs.*, No. 902, 1936.

panion but finds the spectral class of the primary later than S Monocerotis (=O7 or O7.5) and perhaps a trace earlier than η Camelopardalis (O9). Morgan classifies it as O9–O8. In the Victoria catalogue of O and B stars, this star is given as O9; Pearce classifies it as O7. We shall adopt O8.5.

From Elvey and Rudnick's figures we find $a_1/a = 0.471$, $a_2/a = 0.376$. The average of a_i/a is 0.424; if we average according to brightness, we find 0.441. The theoretical values of b_i/a_i are, then, 0.877 and 0.863, respectively, compared with the empirical 0.73. We shall, as before, adjust the photometric solution by leaving $(3b_1 + a_1)/4$ the same and by using the theoretical ellipticity constant. We then find $a_1/a = 0.400$ and $a_2/a = 0.319$. Gaposchkin found $a_1/a = 0.44$ and $a_2/a = 0.35$; his ellipticity constant 0.9 is equal to the theoretical value. In view of the fact that Elvey and Rudnick's results are still provisional, we shall adopt the average values: $a_1/a = 0.415$ and $a_2/a = 0.33$. For the inclination Gaposchkin finds $68^\circ.3$; Elvey and Rudnick, 64° . We adopt 65° .

We have not yet used the condition that the total brightnesses of the components should correspond to the relative intensities of the spectra. It would seem that $\Delta m(\text{pg})$ is about 1.0 mag. Taking the small difference in surface brightness into account, we find that the ratio a_2/a_1 should be about 2/3. Leaving $a_1 + a_2$ unchanged, so that the representation of the light-curve will be little affected, we find $a_1/a = 0.45$; $a_2/a = 0.30$; hence $R_1/a = 0.42$ and $R_2/a = 0.28$.

Following the procedure of applying the correction for reflection used for AO Cassiopeiae, we have: correction to $\log K_1$, 0.002, to $\log K_2$, 0.010; $K_1 = 218$; $K_2 = 294$; $\log a/\odot = 1.686$; $\log R_1 = 1.309$, $\log R_2 = 1.133$; furthermore, allowing for the small difference in surface brightness, $\log L_1 = 5.84$, $\log L_2 = 5.39$; $M_{\text{bol}(1)} = -10.0$; $M_{\text{bol}(2)} = -8.9$; $\log m_1 = 1.66$, $\log m_2 = 1.53$; $m_1 = 45.7\odot$, $m_2 = 33.9\odot$.

The striking similarity between η Canis Majoris and AO Cassiopeiae has been already pointed out by Gaposchkin. But in η Canis Majoris the components appear to be separated; this fact and the deeper minima make this star of greater importance for the mass-luminosity relation.

In conclusion, the data on spectroscopic binaries thus far discussed are collected in Table 12.

The data are included in the mass-luminosity diagram (Fig. 1) and the mass-radius diagram (Fig. 2). The two binaries AO Cassiopeiae and 29 Canis Majoris stand out by their great luminosity; Y Cygni appears to be an intermediate case.

TABLE 12
SOME SELECTED SPECTROSCOPIC BINARIES

Star	$\log m$	$\log L$	$\log R$	M_{bol}
Castor C ₁	-0.201	-1.16	-0.18	+ 7.53
C ₂	-0.247	-1.24	-0.22	+ 7.73
β Aur A.....	+0.378	+1.83	+0.43	+ 0.04
B.....	+0.370	+1.83	+0.43	+ 0.04
μ_1 Sco \overline{AB}	+1.094	+3.35	+0.73	- 3.76
V Pup \overline{AB}	+1.265	+3.86	+0.83	- 5.04
Y Cyg A.....	+1.240	+4.51	+0.77	- 6.6
B.....	+1.235	+4.51	+0.77	- 6.6
AO Cas A.....	+1.634	+5.97	+1.36	-10.3
B.....	+1.582	+5.58	+1.23	- 9.3
29 CMa A.....	+1.66	+5.84	+1.31	-10.0
B.....	+1.53	+5.39	+1.13	- 8.9

TRUMPLER'S MASSIVE STARS

The method, used by Trumpler, of determining the masses of the very luminous O and B stars in galactic clusters from the Einstein gravitational red shift has, at least for main-sequence stars, extended the study of stellar masses to near the upper limit. For the most luminous of these objects are not appreciably exceeded in brightness (and probably not in mass) by any other known stars of classes O and B. Trumpler's method is, of course, capable of being extended to visual binaries, although the fact that only one comparison star can be used will be a disadvantage.

Table 13 gives the observational data for seven stars taken from Trumpler's paper.⁵⁷ The spectrum of the third star is classed O7 by Trumpler, O7 at Victoria, and O7.5 by E. G. Williams. We have used the average value. The temperatures and bolometric correc-

⁵⁷ *Pub. A.S.P.*, 37, 249, 1935.

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tions are taken from Table 3 of the preceding paper; the color index C was supposed to be -0.2 for all the stars. $\log L$ follows from M_{bol} and $M_{\text{bol}}(\text{sun}) = 4.62$; $\log R$ from $\log L$, $\log T_e$, and $\log T_e(\text{sun}) = 3.757$. It is seen that $\log m$ is roughly twice as large as $\log R$; this shows that the surface gravity is about the same as that of the sun, as was supposed in adopting the temperature scale for $\log g = 4.4$.

The last column gives the order of reliability of the measured red shifts, according to a private communication by Dr. Trumpler.

TABLE 13
TRUMPLER'S STARS

Star	E	$\log \frac{m}{R}$	M_{pg}	Sp.	$\log T_e$	B.C. - C	M_{bol}	$\log L$	$\log R$	$\log m$	Q
NGC 2244, 15	7.9	1.10	-4.4	O6	4.8:	-4.7	-9.1	5.49	0.66:	1.76:	3
..... 8	8.5	1.13	4.1	O9	4.50	3.0	7.1	4.69	0.86	1.99	2
NGC 2264, 60	14.6	1.36	4.7	O7(+)	4.68	4.0	8.7	5.33	0.82	2.18	1
NGC 2362, 1	9.9	1.19	6.4	O8.5	4.55	3.3	9.7	5.73	1.28	2.47	4
NGC 6871, 2	9.4	1.17	5.7	O9w	4.50	3.0	8.7	5.33	1.18	2.35	6
..... 5	15.2	1.38	5.4	B0	4.40	2.5	7.9	5.01	1.22	2.60	5
NGC 7380, 1	5.4	0.93	-4.6	O9	4.50	-3.0	-7.6	4.89	0.96	1.89	7

The data of the last columns are shown graphically in Figures 1 and 2. The general position of Trumpler's stars in these diagrams is explained by the observational selection for large values of m/R , in connection with an intrinsic scatter of the O and B stars in these diagrams.

The spectroscopic binaries Plaskett's star and HD 698, studied by Pearce, for which $m_1 \sin^3 i = 76 \odot$ and $113 \odot$, respectively, are probably objects similar to Trumpler's cluster stars, whereas AO Cas and 29 CMa are very luminous but of only medium large mass. The latter stars form a continuation of the less massive stars, both in stellar model (Chandrasekhar) and in the mass-luminosity relation, which for these stars is approximately

$$L = m^{3\frac{1}{2}} \quad \left(\frac{1}{3} < m < 40\right). \quad (4)$$

Considering the smaller masses only, the relation

$$L = m^{4\frac{1}{2}} \quad \left(\frac{1}{2} < m < 2\frac{1}{2}\right) \quad (5)$$

fits the data considerably better. This conclusion is consistent with the theoretical mass-luminosity relation

$$L = c. \frac{m^{5\frac{1}{2}}(\mu\beta)^{7\frac{1}{2}}}{\kappa_0 R^{\frac{1}{2}}} \quad (6)$$

in connection with the theoretical result $\beta \cong 1$, with the theoretical variation of κ_0 and the empirical variation of R , if we assume μ to be constant.

Formula (4) is only roughly correct, and its interpretation is less direct since the decrease of β with increasing m has also to be taken into account.

YERKES OBSERVATORY
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