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uncertainties involved. The actual error committed is only 0·5 per cent., a negligible quantity in *any* application of the formula, and less than the systematic errors likely to be present in both $\bar{\mu}$ and $\bar{R}$. The formula would then read

$$\bar{p} = 3\bar{\mu}/\bar{R}.$$

The formula has recently been applied by Mohr * to moving clusters, which were chosen because for them very accurate values of the "true" parallaxes can be calculated from the motion of the cluster as a whole. Unfortunately, it is clear from the derivation of the formula that this is perhaps the one case where it may most certainly *not* be applied, since its validity assumes a Maxwellian distribution of peculiar velocities, but also random distribution of the stars with respect to the apex of the Drift motion.

University of Minnesota,
Minneapolis:
1935 *March* 10.

RELATIVISTIC DEGENERACY.

Chr. Møller, D.Phil., and S. Chandrasekhar, Ph.D.

1. It has generally been accepted that the pressure P exerted by a gas is simply the mean rate of transfer of momentum across an ideal surface of unit area in the gas. From this definition it follows that

$$PV = \frac{1}{3}\int_0^\infty N(p) p v_p dp, \tag{1}$$

where $N(p)$ is the number of particles having momenta between p and $p+dp$, and v_p is the velocity associated with the momentum p. For an electron assembly which is completely degenerate ($T=0$) (1) reduces to

$$P = \frac{8\pi}{3h^3}\int_0^{p_0} p^3 \frac{dE}{dp} dp, \tag{2}$$

where E is the kinetic energy of the electron which has a momentum p. If we substitute in (2) the relation between E and p given by relativistic mechanics, we immediately get the relation between PV and N (the total number of electrons in the assembly) for completely degenerate matter.

2. However, Eddington has recently suggested † that (2) is based on a misunderstanding and that we cannot simply insert in (2) the relativistic relation between E and p. We are quite unable to follow his arguments in his

* *Astr. Journal*, 44, 10, 1934.
† *M.N.*, 95, 197, 1935.

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first paper, but in his second paper * he has made a more direct derivation of his result, using the energy-stress tensor $T_{\mu\nu}$. Since the use of the energy-stress tensor $T_{\mu\nu}$ in the quantum theory is not new it is now possible to see why Eddington obtains a result different from the usual treatment of a degenerate gas. Eddington defines the energy-stress tensor $T_{\mu\nu}$ as "the expectation value of the differential operator

$$T_{\mu\nu} = -\frac{1}{m}\frac{\partial^2}{\partial x_\mu \partial x_\nu}, \tag{3}$$

where the right-hand side is to be summed in an appropriate (invariant) way." Eddington does not justify his choice of (3) except by the statement "to satisfy tensor conditions." However, in relativistic quantum mechanics one defines the energy-stress tensor in a different way. In fact, the energy-stress tensor for one particle in free space is defined by (*cf.* W. Pauli, *Handbuch der Physik*, **24** (1), 255)

$$T_{\mu\nu} = \frac{c\hbar}{2i}\left(\psi^* \alpha^\nu \frac{\partial \psi}{\partial x_\mu} - \frac{\partial \psi^*}{\partial x_\mu}\alpha^\nu \psi\right), \tag{4}$$†

where α^1, α^2, α^3 are the Dirac matrices and α^4 is here defined as i times the unit matrix; also $x_4 = ict$. In (4) ψ is a solution of the Dirac equation and ψ^* is its conjugate complex.

To go over to the case of N electrons in a finite volume V which satisfy the exclusion principle *one has to consider ψ and ψ^* as non-commuting quantities satisfying the commutation rules established by Jordan and Wigner.*‡

Let $\psi_{\mathbf{p},s}$ be a suitably normalised eigen-solution satisfying the Dirac equation § and representing a plane wave with a definite value for the momentum $\mathbf{p}$ and spin s. Since the electrons are confined in a finite volume, $\mathbf{p}$ takes on only discrete values and, further, s can take two different values corresponding to the two different directions for the spin.

[The discreteness is obtained as usual by imposing the following periodicity condition:—

$$\psi_{\mathbf{p},s}(x + l_x, y + l_y, z + l_z) = \psi_{\mathbf{p},s}(x, y, z), \tag{i}$$

where l_x, l_y, l_z define the sides of a rectangular "box" and the volume V is clearly given by

$$V = l_x l_y l_z.$$

From the above condition (i) we find that the eigen values of the components of momentum are given by

$$p_x = \frac{2\pi n_x \hbar}{l_x}; \qquad p_y = \frac{2\pi n_y \hbar}{l_y}; \qquad p_z = \frac{2\pi n_z \hbar}{l_z}, \tag{ii}$$

where n_x, n_y, n_z are arbitrary integers.]

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* We are indebted to Sir A. S. Eddington for allowing us to see a manuscript copy of his paper.

† The tensor $T_{\mu\nu}$ was introduced by Tetrode, *Z. f. Physik*, **49**, 858, 1928.

‡ P. Jordan and E. Wigner, *Z. f. Physik*, **47**, 631, 1928.

§ For the explicit expressions see C. G. Darwin, *P.R.S.*, **118**, 654, 1928.

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We now expand ψ and its conjugate complex ψ^* in terms of $\psi_{\mathrm{p},s}$ and $\psi^*_{\mathrm{p},s}$ respectively :

$$\psi = \sum_{\mathrm{p},s} a_{\mathrm{p},s}\,\psi_{\mathrm{p},s}, \tag{5}$$

$$\psi^* = \sum_{\mathrm{p},s} a^*_{\mathrm{p},s}\,\psi^*_{\mathrm{p},s}. \tag{6}$$

In (5) and (6) $a_{\mathrm{p},s}$ and $a^*_{\mathrm{p},s}$ are q-numbers satisfying the commutability relations (*cf.* Jordan and Wigner, *loc. cit.*).

$$\left.\begin{aligned} a^*_{\mathrm{p},s}\,a_{\mathrm{p}',s'} + a_{\mathrm{p}',s'}\,a^*_{\mathrm{p},s} &= \delta_{\mathrm{p},\mathrm{p}'}\,\delta_{s,s'} \\ a_{\mathrm{p},s}\,a_{\mathrm{p}',s'} + a_{\mathrm{p}',s'}\,a_{\mathrm{p},s} &= 0 \\ a^*_{\mathrm{p},s}\,a^*_{\mathrm{p}',s'} + a^*_{\mathrm{p}',s'}\,a_{\mathrm{p},s} &= 0 \end{aligned}\right\}. \tag{7}$$

From (4) we now have

$$S = \tfrac{1}{3}(T_{11}+T_{22}+T_{33})$$

$$= \frac{c\hbar}{6i}\sum_{\mathrm{p},s}\sum_{\mathrm{p}',s}\psi^*_{\mathrm{p},s}\left\{a^*_{\mathrm{p},s}\left(\sum_{\nu=1}^{3}\alpha^\nu\frac{i}{\hbar}p_\nu'\right)a_{\mathrm{p}',s'} + a^*_{\mathrm{p},s}\left(\sum_{\nu=1}^{3}\alpha^\nu\frac{i}{\hbar}p_\nu\right)a_{\mathrm{p}',s'}\right\}\psi_{\mathrm{p}',s} \tag{8}$$

or

$$S = \frac{c}{6}\sum_{\mathrm{p},s}\sum_{\mathrm{p}',s} a^*_{\mathrm{p},s}\,a_{\mathrm{p}',s'}\sum_{\nu=1}^{3}(p_\nu+p_\nu')\psi^*_{\mathrm{p},s}\,\alpha^\nu\psi_{\mathrm{p}',s}. \tag{9}$$

We shall denote the expectation value of an operator by putting a bar over it. Thus the pressure P is given by the expectation value $\bar{S}$ of S for a state of the whole assembly in which the N lowest states of the particles are occupied. It now follows from the commutation rules that

$$\overline{a^*_{\mathrm{p},s}\,a_{\mathrm{p}',s'}} = N_{\mathrm{p},s}\,\delta_{\mathrm{p},\mathrm{p}'}\,\delta_{s,s'}, \tag{10}$$*

where $N_{\mathrm{p},s}=1$ if the state (p,s) is occupied and zero otherwise. From (9) we now have

$$P = \frac{c}{3}\sum_{\mathrm{p},s}N_{\mathrm{p},s}\sum_{\nu=1}^{3}\psi^*_{\mathrm{p},s}\,\alpha^\nu\psi_{\mathrm{p},s}\,p_\nu. \tag{11}$$

Since $c\alpha^\nu$ has the meaning of velocity in Dirac's theory we clearly have

$$c\psi^*_{\mathrm{p},s}\,\alpha^\nu\psi_{\mathrm{p},s} = \frac{p_\nu c^2}{EV} = \frac{v_\nu}{V}, \tag{12}$$

where

$$E = c\sqrt{m^2c^2+\sum_{\nu=1}^{3}p_\nu^2}. \tag{13}$$

Relations (12) and (13) are easily verified using the explicit expressions for $\psi_{\mathrm{p},s}$ as given by Darwin (*loc. cit.*).

From (11) and (12) we now have

$$PV = \frac{1}{3}\sum_{\mathrm{p},s}N_{\mathrm{p},s}\frac{|\,\mathrm{p}\,|^2c^2}{E} \tag{14}$$

or

$$PV = \frac{1}{3}\sum_{\mathrm{p},s}N_{\mathrm{p},s}(\mathrm{p},\mathrm{v}), \tag{15}$$

* This is clear from the matrix representations for the $a_{\mathrm{p},s}$ (*cf.*, for instance, H. Weyl, *Gruppen Theorie und Quanten Mechanik*, pp. 223, 224).

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which is precisely our earlier expression (2) and therefore leads to the same final result.

3. Since Eddington prefers to work with standing waves, it may be mentioned here that we could just as well have taken for the set of orthogonal functions the following set of standing waves instead of our $\psi_{p,s}$:—

$$\left.\begin{aligned}\phi_{p,s}&=\frac{1}{\sqrt{2}}(\psi_{p,s}+\psi_{-p,s})\\ \phi_{-p,s}&=\frac{1}{\sqrt{2}}(\psi_{p,s}-\psi_{-p,s})\end{aligned}\right\}. \tag{16}$$

It is easily verified that (12) continues to be true if the $\psi_{p,s}$'s are replaced by the $\phi_{p,s}$'s. Thus (15) continues to be true even if the fundamental set of orthogonal functions were taken to correspond to standing waves.

4. In conclusion we wish to state that we do not intend this note as a reply in any sense to Eddington's papers. We thought it of interest, however, to point out that, starting with the energy-stress tensor as is defined in *relativistic* quantum mechanics and following Eddington's own procedure for calculating the pressure, we are simply led back to the relation between P and N one had earlier derived from (2) by directly inserting in it the relation between E and p given by relativistic mechanics.

Cambridge:
1935 *June* 7.

STELLAR CONFIGURATIONS WITH DEGENERATE CORES. (SECOND PAPER.)

S. Chandrasekhar, Ph.D.

1. In a previous communication * the general problems of stellar structure as they present themselves on the standard model were rediscussed, using the exact relativistic equation of state to describe degenerate matter.† The method developed in I is, however, quite general and consists essentially in relating the completely degenerate gas spheres governed by the differential equation

$$\frac{1}{\eta^2}\frac{d}{d\eta}\left(\eta^2\frac{d\phi}{d\eta}\right)=-\left(\phi^2-\frac{1}{y_0^2}\right)^{3/2}, \tag{1}$$ ‡

* *M.N.*, **95**, 226–260, 1935. This paper will be referred to as I.

† In a recent paper, *M.N.*, **95**, 297, 1935, Eddington has questioned the validity of the relativistic equation of state for degenerate matter which is still generally accepted. There are, however, grounds for not abandoning the accepted form of the equation of state—the arguments are presented in the preceding paper by Dr. Christian Møller and the writer.

‡ This equation was established in the author's paper, *M.N.*, **95**, 207–226, 1935. This paper will be referred to as H.C. II. The earlier paper, *M.N.*, **91**, 456, 1931, will be referred to as H.C. I.

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