KINEMATICS AND WORLD-STRUCTURE

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ABSTRACT

The *idealized cosmological problem*, in which the nebulae are considered as particles in homogeneous flow, is analyzed from the standpoint of the operational methodology, allowing the fundamental observers the use only of clocks, theodolites, and light-signals. It is found, as an extension of the Helmholtz-Lie solution of the problem of physical space, that such a space-time necessarily admits the introduction of an *invariant Riemannian metric* of precisely the form and generality of that on which the general relativistic theory of cosmology is based, and in terms of which all given elements can be interpreted in the same way as in the relativistic theory.

INTRODUCTION

The increased knowledge of the structure, distribution, and radial motion of extra-galactic nebulae, which has been amassed during the past decade, has brought about a renewed interest in cosmological speculations. Perhaps the least *ad hoc* of these is that offshoot of Einstein's relativistic theory of gravitation most often referred to by the somewhat misleading designation "theory of the expanding universe." It seeks to develop from the general theory of relativity —which has other more firmly established successes to its credit in explaining phenomena within the solar system and within the galaxy —an idealized universe suitable for the gross description of the observed nebular phenomena. This it accomplishes with the aid of a general *a priori* uniformity postulate suggested by the observations themselves, which may or may not survive the test of further observation; the solution which it offers is by no means unique, nor does it seem possible to predict the extent to which the choice it offers will be restricted by further data.[1]

More recently E. A. Milne has put forward a theory, which is based on purely kinematical considerations, eschewing any and all theories of gravitation, although leaving open the possibility of the subsequent imposition of any gravitational theory consonant with the kinematics.[2] In the present form of this theory Milne attempts

[1] For a unified account of this theory see the writer's report, "Relativistic Cosmology," *Rev. Mod. Phys.*, **5**, 62–90, 1933.

[2] A complete account of the present stage of this theory is to be found in Milne's *Relativity, Gravitation and World-Structure* (Oxford, 1935); to be reviewed in an early issue of this *Journal*.

to go beyond his earlier considerations and to derive, in terms of the purely operational methodology, the most general kinematics suitable for a cosmological theory based on a uniformity postulate of the kind suggested by the observational material. In this idealization individual nebulae are to be replaced by "fundamental particle-observers" A, A′, ... , each equipped with a clock, a theodolite, and apparatus for sending and receiving light-signals—these latter considered as corpuscular impulses in order to avoid an indeterminacy foreign to the problem. Briefly stated, the operational viewpoint restricts the observations of each of these fundamental observers to such as can be made on events on his own world-line with the aid of these instruments. The uniformity postulate, which Milne fittingly calls "the cosmological principle," asserts that the description of the whole system, as given by A in terms of his immediate measurements, is to be identical with the description given by any other fundamental observer A′ in terms of his own measurements; as such it is equivalent to that uniformity requirement on which the general relativistic theory is based (although this is denied by Milne). Actually Milne imposes the further restriction that the world-lines of all fundamental observers concurred at a given event O ("creation"), which is then taken as the zero of all their clocks; although (as Milne himself recognizes) this is not a necessary condition, we shall adopt it temporarily in order to avoid circumlocution, and indicate in due time what change will result from its surrender.

Now in spite of the apparent similarity of initial viewpoint between this theory and the kinematical aspects of the general relativistic theory, their conclusions seem quite at variance. The relativistic theory finds that the cosmological speculations may, on assigning appropriate co-ordinates τ, η^α ($\alpha=1, 2, 3$), to each event E, be based on any Riemannian map whose invariant metric ds^2 is of the form

$$ds^2=d\tau^2-\frac{R^2(\tau)}{c^2}\,du^2\,, \tag{0.1}$$

where $R(\tau)/c$ is an arbitrary function of τ and

$$du^2=h_{\alpha\beta}(\eta^\gamma)d\eta^\alpha d\eta^\beta \tag{0.2}$$

defines any three-space of constant Riemannian curvature k (which may, without loss of generality, be restricted to the values $k=-1$, 0, $+1$). We thus have here a wide range of possibilities, corresponding to the three possible values of k and the choice of the arbitrary function $R(\tau)$—although the condition adopted temporarily above requires that $R(0)=0$. The world-lines of the fundamental observers are given by $\eta^a=$ const., and the relations between the co-ordinate (τ, η^a) of an event E, as obtained by different observers, are given by transformations of a six-parameter continuous group G_6 on the η^a which leave the auxiliary line-element (0.2) invariant in form as well as in fact, the value of τ being the same for all observers. This Riemannian map has, among others, the following properties of particular interest in the present investigation:

a) The world-line of each fundamental observer is a geodesic of the metric ds^2, and his clock-time along it is measured by this metric.

b) The world-line of every light-signal is a null-geodesic of the metric ds^2.

We shall on occasion have need of a more specific form for the auxiliary metric du^2; we may then, without serious limitation of the astronomical applicability, choose co-ordinates $\eta^a=(\eta, \theta, \varphi)$ in terms of which

$$du^2=d\eta^2+\sigma^2(\eta)[d\theta^2+\sin^2\theta d\varphi^2]\ , \tag{0.3}$$

where

$$\sigma(\eta)=\left\{\begin{array}{c}\sinh\eta,\ (0\leq\eta),\\ \eta,\ (0\leq\eta),\\ \sin\eta,\ \left(0\leq\eta\leq\frac{\pi}{2}\right)\ ,\end{array}\right\}\ \text{for } k=\left\{\begin{array}{c}-1\\ 0\\ +1\end{array}\right\} \tag{0.4}$$

In each case $0\leq\theta\leq\pi$, $0\leq\varphi<2\pi$.

Milne, on the other hand, obtains a solution of the full three-dimensional problem (i.e., with three spatial dimensions) only for the case interpreted by him to mean that the fundamental observers are relatively unaccelerated. His results may be expressed in terms of the Minkowski map, whose metric

$$dS^2=dT^2-\frac{dX^2+dY^2+dZ^2}{c^2} \tag{0.5}$$

satisfies the conditions (*a*) and (*b*) above. The relations between privileged observers are here given by that six-parameter group L_6 of Lorentz transformations which have the unique event O as fixed point and which preserve the direction of time. We remark in passing that the quantity

$$\mathrm{T}^2 = T^2 - \frac{X^2+Y^2+Z^2}{c^2} \tag{0.6}$$

is invariant under this group, and that on performing the transformation

$$\left.\begin{aligned} T &= \mathrm{T}\cosh\eta\ , & X &= c\mathrm{T}\sinh\eta\sin\theta\cos\varphi\ , \\ Y &= c\mathrm{T}\sinh\eta\sin\theta\sin\varphi\ , & Z &= c\mathrm{T}\sinh\eta\cos\theta \end{aligned}\right\} \tag{0.7}$$

the map (0.5) is seen to be contained in (0.1) as the special case $R(\mathrm{T}) = c\mathrm{T}$, $k = -1$. Only on restricting himself to "collinear" events and observers does Milne's attack on the problem of relatively accelerated observers yield some prospect of success; in this case he finds that their mutual relations must be given by transformations of the form

$$T' - \frac{X'}{c} = p\left(T - \frac{X}{c}\right)\ , \qquad T' + \frac{X'}{c} = p^{-1}\left(T + \frac{X}{c}\right)\ , \tag{0.8}$$

where p^{-1} is the inverse of the arbitrary function p. A discrete linear system of fundamental observers satisfying the cosmological principle is in fact obtained on requiring that the transformation between each two consecutive observers of the set be given by the same functions p, p^{-1}, but Milne does not extend this result to the case in which there exists a one-parameter family of such collinear observers—the only one which could offer a starting-point for the solution of the full three-dimensional problem of accelerated observers.[3]

We propose here to analyze the general problem *ab initio*, using the operational methodology throughout and avoiding what Milne chooses to call the "conceptual terms" of the general theory of relativity. We shall be led to the conclusion that, although none of the postulates characteristic of this latter theory are introduced, any

[3] Cf. *ibid.*, p. 357, n. 8.

such attack necessarily leads to the existence of a quadratic differential metric of precisely the form and generality of (0.1), and which furthermore satisfies the conditions (*a*) and (*b*) imposed in general relativity.

I. ONE-DIMENSIONAL KINEMATICS

We begin, then, with an analysis and a completion of Milne's solution of the problem of collinear observers and events. We adopt Milne's tacit assumptions that if two events are connected by a light-path, that light-path is unique, and that the contracting and opening light-cones with vertices at any event other than O each cut the world-line of each fundamental observer in one, and only one, point.[4] A light-signal sent out by a given observer A at time t_1 by his clock is received by a second observer A′ at time t_1' by his, where t_1' is assumed to be a continuous function of t_1. The locus of these light-paths for all $t_1 \geq 0$, extended backward from A as far as possible and forward from A′, is a certain surface which we denote by AA′. Now if the cosmological principle is to be satisfied, this surface AA′ must coincide with the locus A′A of light-paths leaving A′ at times t_2' and arriving at A at times t_2, as otherwise there would exist a one-parameter family of light-paths from any event on the world-line of A to a corresponding event on that of A′—as may be seen by considering the situation as observed by some appropriate intermediate observer A″. Similarly, if any point (other than the common event O) on the world-line of another fundamental observer A″ is on the surface AA′, then his entire world-line is also, and the three observers A, A′, A″ are said to be "collinear."

Consider now the traces on AA′ of all light-cones with vertices at all events E on AA′ (and contained within the light-cone opening out from O, which is in fact the entire universe in Milne's interpretation); these traces, which are possible light-paths, fall into two families of non-intersecting lines such that through each event E there passes one, and only one, line of each family. Let each line

[4] At least within a sufficiently large domain containing the events in question. We concern ourselves mainly with differential geometrical properties, supplemented, as occasion demands, with remarks on the situation arising in case any of the spaces met later are or may be finite in extent.

of one of these families, say that representing light advancing in the direction $A \rightarrow A'$, be assigned the parameter t_1 by A, where, as above, t_1 is the time, as measured by his clock, at which the path in question cuts his world-line, and let him similarly assign the parameter t_2 to those of the other family. A′ will then assign to any two such lines of parameters t_1, t_2 (as given by A) *his* clock-times t'_1, t'_2, respectively, at which they cut his world-line. A may now assign the *co-ordinates* (t_1, t_2) to any event E, where t_1 and t_2 are the parameters of the two lines of the first and second families, respectively, which pass through E; similarly A′ will assign to the same event the co-ordinates (t'_1, t'_2) obtained from his own clock readings, as illustrated in Figure 1. In terms of these co-ordinates, the unique event O is (0, 0), the world-line of A is given by the line $t_1 = t_2$, that of A′ by $t'_1 = t'_2$, the light-paths of the first family by $t_1 =$ const. or $t'_1 =$ const., and those of the second family by $t_2 =$ const. or $t'_2 =$ const.; all light-paths on AA′ are thus given in terms of differentials by the vanishing of either of the quadratic differential forms $dt_1 dt_2$ or $dt'_1 dt'_2$.

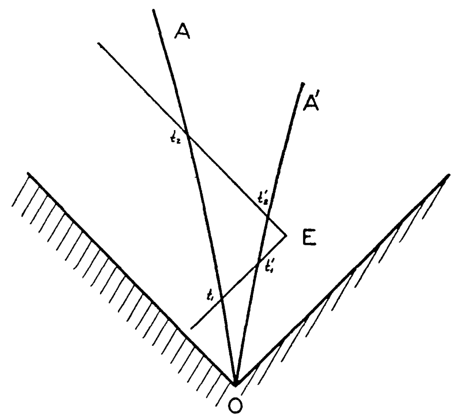

FIG. 1.—*A* assigns to *E* the co-ordinates (t_1, t_2); *A′* the co-ordinates (t'_1, t'_2).

The transformation $t_1, t_2 \rightarrow t'_1, t'_2$ from the co-ordinates assigned E by A to those assigned by A′ must clearly consist in a reparameterization of the two families of light-paths, i.e., it must be of the form $t'_1 = p(t_1)$, $t'_2 = q(t_2)$, where p and q are some two (continuous) functions such that $p(0) = 0$, $q(0) = 0$. Further, if the cosmological principle is to be satisfied, this transformation must be such that on interchanging the observers A, A′ and at the same time the two families of light-paths it is left completely unaltered—the interchange of the two families being occasioned by the fact that the parameter t_1 represents a signal advancing in the direction $A \rightarrow A'$, and hence must be replaced by a signal t'_2 advancing in the direction $A' \rightarrow A$. But this means that t_2, t_1 must be obtained from t'_2, t'_1 by exactly the same transformation as t_1', t_2' are obtained from t_1, t_2, i.e., $t_2=p(t_2')$, $t_1=q(t_1')$. Hence the transformation in question must be of the form

$$t_1'=p(t_1)\ , \qquad t_2'=p^{-1}(t_2)\ , \tag{1.1}$$

where p^{-1} is the inverse of the arbitrary function p and $p(0)=0$, $p^{-1}(0)=0$; this accomplishes at the same time the standardization of clocks. This result is completely equivalent to that of Milne cited above.

We now complete this solution for the case in which there exists a one-parameter family of possible collinear observers. The transformation (1.1) between any two observers of the family must then be a member of a one-parameter continuous group; hence the transformation p of t_1 must be a member u of a one-parameter continuous group G_1, and t_2 must transform according to its inverse $\breve{u}$, which is also a member of G_1:

$$t_1'=f(t_1;\, u)\ , \qquad t_2'=f(t_2;\, \breve{u})\ . \tag{1.2}$$

We may without restriction take $u=0$ as defining the identical transformation $t_1'=t_1$, $t_2'=t_2$. Now, it is well known that in order for such transformations to form a continuous group, t_1' and t_2', considered as functions of the parameter u, must satisfy a certain set of differential equations;[5] these equations may here, on suitable choice of the parameter u, be taken in the form

$$\frac{dt_1'}{du}=\xi(t_1')\ , \qquad \frac{dt_2'}{du}=-\xi(t_2')\ , \tag{1.3}$$

where $\xi(t_1)$ is the generator of the infinitesimal transformation δu of G_1:

$$t_1'=t_1+\xi(t_1)\delta u \text{ (and } t_2'=t_2-\xi(t_2)\delta u)\ . \tag{1.4}$$

The parameter u is hereby determined only to within an arbitrary constant factor C and the generator ξ to within the factor $1/C$; for convenience we limit the choice of this factor by the condition that

[5] Lie's "first fundamental theorem"; cf. S. Lie and F. Engel, *Theorie der Transformationsgruppen*, **3**, Part VI, Leipzig, 1893, to which we shall have occasion to refer in the sequel.

$\xi(t)$ be positive for $t>0$ (or in the range between $t=0$, at which $\xi=0$, and its next zero).

The integration of the fundamental equations (1.3) yields the finite equations (1.2) of the group. Defining

$$F(z)=\int_a^z \frac{dz}{\xi(z)}, \tag{1.5}$$

where a is any suitably chosen lower limit >0 (which never actually enters into the analysis), the finite equations of the group are immediately found to be

$$F(t_1')=F(t_1)+u\ , \qquad F(t_2')=F(t_2)-u\ . \tag{1.6}$$

Each observer A may now characterize each other privileged observer A′ in the set by a fixed value of the continuous parameter u, i.e., by that value of the parameter which defines the transformation sending his measurements into those of A′; he of course assigns to himself the parameter $u=0$, defining the identity. The solution obtained by Milne for a discrete set of such observers is contained herein as the infinite cyclic subgroup $u=n\kappa$, $n=0, \pm 1, \pm 2, \ldots$, where κ is a fixed constant (which may, on suitable normalization, be taken as unity).[6] Note that the group (1.6) admits as its only fundamental invariant $F(t_1)+F(t_2)$; for convenience we choose instead of this quantity itself that solution τ of the equation

$$2F(\tau)=F(t_1)+F(t_2) \tag{1.7}$$

which reduces to $\tau=t_1=t_2$ for events on the world-line of A. All first-order differential invariants are expressible in terms of τ, $dF(t_1)=dt_1/\xi(t_1)$, and $dF(t_2)=dt_2/\xi(t_2)$.

These formal considerations must now be supplemented by an

[6] We have here tacitly assumed, with Milne, that there exists an infinite set of distinct observers, i.e., that they may be thought of as equidistant points along an open straight. If, however, there are but a finite number N of distinct observers, as in the case of N points at equal distances on the circumference of a circle, this subgroup is a finite group of order N, and u must be taken modulo $N\kappa$. From the standpoint of the continuous group, this situation arises if there exists a value of u other than zero which also yields the identity; the significance of this alternative topology for the continuous group will become apparent in the following discussion.

appeal to the empirical in order to determine the hitherto arbitrary function $\xi(t)$ from the given motions. To accomplish this we fix attention on two particular observers A, A′, and complete the normalization of u and $\xi(t)$ by assigning to the transformation $A \rightarrow A'$ whichever of the parameter values $u = \pm 1$ leads to positive ξ, as agreed above. A signal sent out by A′ at time $t'_1 = t'_2$ is received by A at the time t_2 defined by the equation $F(t_2) = F(t'_2) \pm 1$ obtained from (1.6). We may now suppose that A′ informs A of the time t'_2 at which the signal was sent, so that t_2 is in principle an empirically determinable function of t'_2; the foregoing equation then determines implicitly the function $F(t)$ and hence $\xi(t) = 1/F'(t)$. We come much closer to actual practice, however, by relaxing our too stringent hypothesis concerning the nature of light-signals, and allowing A to determine the "Doppler shift-ratio" $s(t_2) = dt_2/dt'_2 = \nu/\nu + \Delta\nu$ directly with the aid of a spectroscope;[7] for that observer A′ characterized by general u this Doppler shift-ratio is given by

$$s(t_2) = \frac{dt_2}{dt'_2} = \frac{\xi(t_2)}{\xi(t'_2)}, \tag{1.8}$$

where $F(t_2) = F(t'_2) + u$. Note that for the infinitesimal transformation (1.4) the foregoing formula becomes $s = 1 + \xi'(t_2)\delta u$ and would be interpreted by an observer using the traditional formulae for Doppler effect[8] as due to a radial velocity

$$v = c\xi'(t_2)\delta u . \tag{1.9}$$

Finally, we may say that, in principle at least, the whole development, including the determination of the generator ξ, is expressible in terms of the operational methodology, although we have, of course, not hesitated to use such purely mathematical tools as seemed most appropriate.

We now ask whether in this general collinear case we can map space-time with the aid of an invariant Riemannian metric ds^2, satisfying, if possible, the conditions (a) and (b) discussed in the introduction above; this is what is accomplished for the unaccelerated case $\xi(t) = t$ by the Minkowski map. Condition (b) requires

[7] Cf. Milne, *op. cit.*, p. 35. [8] Cf. *ibid.*, p. 37.

that such a metric be of the form $f(t_1, t_2)dt_1dt_2$,[9] and the invariance requirement that it be reducible to the form $f(\tau)dF(t_1)dF(t_2)$. For an interval on the world-line of A ($t_1=t_2=\tau$), this latter expression becomes $f(\tau)[d\tau/\xi(\tau)]^2$, and if ($a$) is to be satisfied, it must be exactly $(d\tau)^2$, i.e., we must have $f(\tau)\equiv[\xi(\tau)]^2$. Hence a metric having the required properties must, if it exists, be of the form

$$ds^2=\frac{\xi^2(\tau)}{\xi(t_1)\xi(t_2)}\,dt_1dt_2\,, \qquad (1.10)$$

i.e., it is uniquely determined by the function ξ characterizing the given motions. That the world-line of A, and therefore of any fundamental observer, is a geodesic of this metric is apparent from the symmetry of the latter in t_1 and t_2 (or from the alternative form [1.12] derived below); hence both conditions (a) and (b) are in fact satisfied.

Milne has defined two quantities, the epoch T and the distance X of any event E(t_1, t_2) relative to A, in terms of the quantities t_1, t_2, which are the results of immediate judgments by A;[10] we propose

[9] Thus, in the first instance the fact that the metric is to be quadratic is but an expression of the existence of exactly two families of light-paths; in general, as has been remarked by H. Bateman and by R. Courant in discussing the foundations of the general theory of relativity, a quartic or higher-order metric would imply double refraction or worse (or the introduction of imaginary elements foreign to the problem). This argument, arising from the consideration of light-paths alone, is but supplementary to those adduced by Helmholtz and Lie for the case in which space is considered as given a priori and by Weyl for the case in which it is considered as contingent (cf. references in sec. 2 below); the present investigations are based on an interfusion of these two standpoints. Altogether these arguments should allay the querulous skepticism expressed by Milne (*op. cit.*), p. 342.

[10] *Ibid.*, p. 29; explicitly, $T=(t_2+t_1)/2$, $X=c(t_2-t_1)/2$. Note that our τ satisfies all Milne's requirements for an "epoch" except the quite inappropriate one that on adding a constant to the graduations of the clock carried by A, the epoch be increased by the same amount—inappropriate, because we are here dealing with a situation in which, as previously agreed by Milne (p. 26), the unique event O offers a natural origin for the measurement of time. Note also that for the case of unaccelerated motion our epoch $\tau=(t_1 t_2)^{\frac{1}{2}}$, i.e., we have merely used the *geometrical* in place of Milne's *arithmetical* mean—surely no great deviation from his methodology! It is hoped that these remarks will serve to dispel the prejudice against "cosmic" time τ, as opposed to "the time [T] used in timing a race, stating athletic records, or arranging a railway time-table" (*ibid.*, p. 44)—incidentally, over here the officials at such experiments usually carry their own watches!

similarly to define two quantities τ, η, which will also serve as the co-ordinates of any event E, but we shall be more noncommittal in naming the second. First, the epoch τ of $E(t_1, t_2)$ is defined by (1.7); it does coincide with the clock time of that fundamental observer A′ whose world-line passes through E, but A has no need of calling upon A′ to determine it. Second, we use in place of distance the u-interval $|\eta|$ between A and the event E in question; η itself is the value of the parameter u of that transformation (1.6) which sends the measurements obtained by A into those obtained by that unique observer A′ whose world-line contains the event E, and is accordingly defined in terms of t_1 and t_2 by the equation

$$2\eta = F(t_2) - F(t_1) \tag{1.11}$$

obtained from (1.6) upon setting $t_1' = t_2' = \tau$ and subtracting. In terms of these co-ordinates (τ, η) the invariant metric (1.10) of the space-time map is readily found to be

$$ds^2 = d\tau^2 - \xi^2(\tau)d\eta^2 , \tag{1.12}$$

and upon setting $\xi(\tau) = R(\tau)/c$, this is seen to be identical with the "one-dimensional" form of the line-element (0.1) upon which the relativistic theory of cosmology is based. We have precisely as much or as little right to assert that its measurement $R(\tau)\eta$ of the "distance" between "simultaneous" events coincides with that obtained with rigid rods as Milne has to identify his co-ordinate X with such measurements in the unaccelerated case. Note that in case we do so, the admittedly conventional assignment (1.9) of a velocity v to the observed Doppler effect in light from that fundamental particle of constant u-interval η would, understandably enough, be consistent with this interpretation only in the first approximation. We choose, however, not to introduce this additional hypothesis or "law," in order to retain the kinematical and general character of the analysis and to avoid premature resort to the empirical.

2. THREE-DIMENSIONAL KINEMATICS

In order to extend these results to the three-dimensional case, consider any pair B, B′ of fundamental observers in the given three-parameter family, and let BB′ denote the surface containing all events and observers collinear with them. The cosmological principle requires that all that can be said concerning the family AA′ of collinear observers can also be said of the family BB′; thus, in particular, the relations between the various observers in BB′ are given by the same transformation group (1.2), and its infinitesimal generator is therefore the same function $\xi(t)$ to within a constant factor $1/C$. We now normalize the parameter u for this new family by requiring that this factor C be unity; there then exists a unique nonnegative u-interval $|u|$ between each pair B, B′ of fundamental observers, which can be determined operationally as described in section 1 above and which vanishes only if B and B′ coincide. Hence there exists a three-dimensional positive-definite metric "u-space" of observers (not to be confused with "physical space") which, in virtue of the cosmological principle, must be homogeneous and isotropic. Now the fundamental Helmholtz-Lie investigations concerning such a space show that *this u-space must admit a positive-definite quadratic metric du^2 of constant Riemannian curvature.*[11] The curvature k of this three-space may, without loss of generality, be restricted to one of the three distinct possibilities $k=-1, 0, +1$ by renormalizing the parameter u (in case $k\neq 0$) with the aid of the constant factor $C=|k|^{\frac{1}{2}}$. On introducing co-ordinates $\eta^\alpha(\alpha=1, 2, 3)$, the metric du^2 of this auxiliary u-space of observers assumes the form (0.2), and each event E in the complete space-time map may be assigned the co-ordinates $(\tau, \eta^1, \eta^2, \eta^3)$. Furthermore, the considerations at the end of the last section and the beginning of this show that we may associate with each pair of "neighboring" events (τ, η^α), $(\tau+d\tau, \eta^\alpha+d\eta^\alpha)$ the invariant interval

$$ds^2=d\tau^2-\xi^2(\tau)du^2\;; \tag{2.1}$$

[11] Lie-Engel, *op. cit.*, 3, Part V. For a more direct account of this remarkable theorem (and for further more general results on the uniqueness of the quadratic form in physical geometry) see H. Weyl's Barcelona lectures, *Mathematische Analyse des Raumproblems* (Berlin, 1923). The group of congruences in terms of which the uniformity is expressed (Weyl, *op. cit.*, p. 30) consists here of the transformations from the measurements of any observer A, with arbitrarily oriented theodolite, to those of any other such observer A′, and is described in more detail below.

hence our kinematical space-time map admits an invariant metric of the same form and generality as that which is at the basis of the general relativistic treatment of the cosmological problem. The relations between the fundamental observers A (η^α=const.) are given by the group G_6 of automorphisms of du^2, and τ is invariant. The world-lines of these observers A are clearly geodesics, and the light-paths are null-geodesics, as shown below; hence the conditions (*a*) and (*b*) are again fully satisfied.

We return to the operational definition of the co-ordinates η^α and of the group relating the observations of different fundamental observers, the definition of τ having been discussed in section 1 above; with this in mind, we consider first the light-paths in terms of the metric (2.1). A signal sent out by an observer A, whose world-line is given by η_1^α const., and received by another observer A′, η_2^α const., traces out a certain path in the auxiliary u-space between the points η_1^α and η_2^α. Now, by the cosmological principle and the assumption that the light-path between two events is unique, this projection on the u-space must be a geodesic of the auxiliary metric du^2, as otherwise there would exist a one-parameter family of such paths; the considerations of the previous section show that the light-path itself must be a null-line of the metric ds^2. It is readily shown that these two facts together imply that the light-path in the space-time map is necessarily a *null-geodesic* of the metric ds^2,[12] thus proving for the present case the condition (*b*) which is one of the cardinal assumptions of the general theory of relativity. Each observer A may now supplement the epoch τ and the u-interval $\eta^1=\eta(\geq 0)$ of a distant event E by two further co-ordinates $\eta^2=\theta$, $\eta^3=\varphi$ defined as follows. Let him set up his theodolite, once and for all, in any arbitrary way, i.e., by choosing a zenith and a base meridian. He then assigns to E the co-ordinates θ, φ obtained immediately from the altitude $\pi/2-\theta$ and the azimuth φ of the theodolite when it is set in position to receive light from E. These operationally defined geodesic polar co-ordinates (η, θ, φ) are in fact those employed in the canonical form (0.3) of du^2, and enable A to obtain from immediate judgments alone

[12] This fact may be established by a reversal of the analysis employed in the relativistic theory, for which see Appendix E, p. 87, of the report referred to in footnote 1 above.

the co-ordinates of any event E in his observable universe.[13] Any two of the six-parameter family of co-ordinate systems thus obtained—the choice of an origin (three parameters), of an equatorial plane-element (two), and of a meridian (one)—are related by a transformation of the fundamental group G_6, which is hereby brought into canonical form.

We have thus shown that any space-time satisfying the cosmological principle admits *ipso facto* an invariant quadratic metric ds^2 of the form (2.1) and having the properties (*a*) and (*b*); this result goes beyond the general relativistic treatment, for there a quadratic metric with these latter properties is assumed and is then shown to be necessarily of the form (2.1). No treatment of the cosmological problem, in which the uniformity principle is adopted, can be based on a more general kinematical background than that offered by the relativistic theory; on the other hand, any treatment in which no undue restrictions are imposed must lead to exactly the same general range of possible backgrounds. With regard to this last point, we remark that the assumption that the world-lines of all fundamental observers concurred at $\tau=0$ (i.e., $\xi(0)=0$) is clearly not a necessary restriction, and an examination of our analysis shows that it may be dropped, i.e., no restriction of this nature need be placed on $\xi(\tau)$. Further, while we have tacitly assumed "elliptical" rather than "spherical" space in introducing co-ordinates in the auxiliary u-space for the case $k=+1$, we are still free to restore this second possibility —but the decision on observational ground is presumably far beyond our present resources. Finally, we emphasize the fact that the geometry of the three-dimensional u-space is in no sense a matter of convention; whether it is one of (constant) negative, zero, or positive curvature is a purely empirical matter, as is the determination of $\xi(\tau)$.

[13] It follows from eq. (1.6) that light emitted at times τ_0 and reaching A at time τ traverses a u-interval $F(\tau)-F(\tau_0)$; hence A can at time τ have cognizance only of those fundamental particles whose u-interval does not exceed $[F(\tau)-F(0)]/2$, and he can never be aware of an event $E(\tau_0, u)$ whose u-interval exceeds $[F(\infty)-F(\tau_0)]/2$. Unless the integral $F(\sigma)$ is divergent, the first of these anomalies will always be found in the open models $k=-1$, 0, and in the closed ones $k=+1$ for sufficiently small τ; this has led Milne (*op. cit.*, chap. xvii) to reject, on methodological ground, all Friedmann universes, for in them $\xi(\tau)\sim\tau^{\frac{2}{3}}$.

3. INTERPRETATION OF MILNE'S SOLUTION

In our treatment no mention has been made of the "velocity of light"; much less has any question been raised as to its constancy. Indeed, we have not allowed ourselves to identify any property of the metric as giving rise to a "distance" as measured by "rigid rods," as such a procedure would necessitate another, for the present purpose unnecessary, appeal to the empirical. In Milne's attempt to deal with these points is to be found the reason for his failure to obtain the general solution of the problem. Milne has defined co-ordinates X, Y, Z in addition to a time T, has allowed his observers to employ Euclidean geometry with these co-ordinates as a rectangular Cartesian system, and has demanded

c) that the velocity of light c be constant and the same for all observers.

As a result of these assumptions he has of course obtained the well-known result that the differential form (0.5) must be invariant and that the relations between fundamental observers are accordingly given by Lorentz transformations, the linearity of which implies that the observers are in relatively unaccelerated motion; the further requirement that O be fixed restricts the full Lorentz group to that six-parameter subgroup L_6 mentioned in the introduction above, which leaves invariant the quantity T defined by (0.6).[14] He thus obtains the special case $\xi(\tau)=\tau$, $k=-1$ of (2.1) as the only metric satisfying the conditions (*a*) and (*b*) in addition to his condition (*c*), and is led to the conclusion that (*c*) must be altered.[15] Our treatment avoids this *impasse* by making no use of (*c*), but it is also a fact that we could adopt it in certain cases if we are willing to surrender the identification of T with clock time. This procedure, which is based

[14] For the requirement that the vanishing of dS^2 implies the vanishing of dS'^2 means that the transformations between fundamental observers are conformal, and the only conformal transformations in four dimensions are composed of (i) dilatations, (ii) translations, (iii) rotations (here including Lorentz transformations), and (iv) inversions, cf. Lie-Engel, *op. cit.*, **3**, Part IV, or G. Darboux, *Systemes orthogonaux* (Paris, 1910). The physical requirements throw out (i) and (iv), and the requirement that O be fixed leaves only rotations about this event.

[15] *Op. cit.*, p. 51.

on our remark made elsewhere that (0.1) is conformally flat,[16] can of course be expected to lead only to the open models without the introduction of imaginary quantities. It may be of some interest to carry through this interpretation for the case $k=-1$, $F(0)$ divergent, for here the group G_6 becomes exactly that group L_6 on which Milne's considerations are based; if desired, one could then retain for this case, with suitable reinterpretation, much of the formal structure developed in Parts II and III of his book in dealing with accelerated observers.

With this in view, we define a new time-like variable

$$\mathrm{T}=e^{F(\tau)} \tag{3.1}$$

in terms of cosmic time τ, where F is the integral (1.5). The line-element (2.1) for the hyperbolic models $k=-1$ then becomes

$$ds^2=f^2(\mathrm{T})dS^2\,, \tag{3.2}$$

where $f(\mathrm{T})=\xi(\tau)/\mathrm{T}$, upon subjecting the co-ordinates $(\mathrm{T}, \eta, \theta, \varphi)$ to the transformation (0.7) and defining dS^2 as in (0.5). For the cases mentioned above in which $F(0)$ diverges (therefore to $-\infty$), the singular event O becomes the origin of the Cartesian co-ordinates T, X, Y, Z, which is the fixed point O of the Lorentz group L_6. The condition (*b*) is satisfied by the subsidiary line-element dS^2 as well as by ds^2, for a simple calculation shows that the null-geodesics of one are the null-geodesics of the other; hence Milne's postulate (*c*) is formally satisfied in terms of the new space-time co-ordinates T, X, Y, Z. However, condition (*a*) is not fully met by dS^2, for although the world-line of the fundamental observer A is the straight $X=Y=Z=0$ radiating out from O toward the future, T is not a direct measure of clock time along it—except in the case $\xi(\tau)=\tau$ of unaccelerated motion, in which the conformality factor $f(\mathrm{T})$ may be taken as unity and all three conditions (*a*), (*b*), and (*c*) are met simultaneously.

[16] "On the Foundations of Relativistic Cosmology," *Proc. Nat. Acad. Sci.*, **15**, 825, 1929. Cf. also A. G. Walker, *M.N.*, **95**, 263, 1935.

The parabolic case $k=0$ lends itself to similar treatment. In this case it is only necessary to introduce Cartesian co-ordinates X, Y, Z in place of the polar co-ordinates η, θ, φ, for we may then write

$$ds^2 = \xi^2(T)dS^2 , \tag{3.3}$$

where $T=F(\tau)$. The solution is then reduced to one in which the fundamental group G_6 is the full group of Euclidean motions on X, Y, Z. In case $F(0)$ is divergent, the fixed event O recedes to $T=-\infty$, and we are dealing with the whole of the Newtonian universe—the co-ordinate velocity of light being constant merely because we are concerned only with fundamental observers whose mutual distances, as measured by the metric dS^2, are constant. The anomaly stressed by Milne[17] sets in whenever $F(0)$ converges, as in the so-called Einstein–De Sitter universe, for in the parabolic cases "creation" occurred at the finite "time" $T_0=F(0)$ in the past, and in the hyperbolic cases discussed above at "time" $\mathrm{T}_0=e^{F(0)}$.

CONCLUSION

We have examined, from the operational standpoint, the problem of determining the most general kinematical background suitable for an idealized universe in which the cosmological principle holds. Allowing the fundamental observers the use only of clocks and theodolites, and granting them the possibility of sending and receiving light-signals, we have shown that for each given mode of motion $\xi(\tau)$ there necessarily exists a quadratic line element (2.1) which is invariant, in form as well as in fact, under transformation from one fundamental observer to another. This metric is determined, apart from topological considerations, to within the sign of the Riemannian curvature of the "space" $\tau=\text{const.}$, the determination of which requires a second contact with the given and is in no sense conventional. This intrinsically unique metric has the property (*a*) of measuring observers' clock time along each member $\eta^a=\text{const.}$ of the three-parameter family of fundamental observers' world-lines, which are at the same time geodesics, and the property (*b*) of describing all light-paths as minimal geodesics. More it cannot do

[17] Cf. n. 13 in sec. 2 above.

without additional hypotheses, for we have with this accounted for all given elements—clocks, theodolites, all light-paths, and the world-lines of the given observers. Thus, although this metric map is identical with that to which the kinematical aspects of the general theory of relativity lead on imposing the cosmological principle, it cannot predict the motion of free particles other than the given fundamental particle-observers; it can only restrict their mode of motion by demanding consonance with the uniformity postulate. Any gravitational theory not in conflict with its basic assumptions may be imposed—but the existence of the dimensionless function $\xi(\tau)/\tau$ of τ would appear an insuperable obstacle, even in the idealization here considered, to the development of a purely kinematical-statistical theory of gravitation along the lines proposed by Milne.

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