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THE STABILITY OF MOVING CLUSTERS

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Introduction. — In 1922 Jeans published an important paper on the dynamics of moving clusters,[1] in which he estimated the probable shapes and rates of expansion for several galactic clusters. There are, however, reasons for repeating and extending Jeans' investigation at this time. A star inside a moving cluster moves under the influence of *a*, the forces exerted by the cluster as a whole on each of its members, and *b*, the forces due to our galactic system. Jeans was able to estimate the influence of the galactic forces on the basis of Kapteyn's picture of our galaxy. At present we are in doubt about many of the essential structural features of our own galaxy, but we are at least certain that Kapteyn's picture does not represent actual conditions. We have now abundant evidence that the motions of the stars in the neighborhood of our sun are largely determined by the effects of one — or perhaps more — distant galactic condensations; the tidal forces due to those star clouds will tend to disrupt at least the less dense of the galactic clusters. Jeans and ten Bruggencate[2] investigated the effects of the tidal forces upon the stability of globular clusters and it is the purpose of the present paper to study these effects for moving clusters.

We shall develop in Section 2 a method for studying the influence of the galactic tidal forces upon a cluster within a small distance of the galactic plane, and moving in a circular orbit around a distant galactic nucleus. For simplicity we shall suppose that the galactic components of the force of attraction of the galactic system depend only on the distance from the galactic center. Granting these assumptions it becomes possible to express the influence of the galactic forces through two constants n and α (see Section 1) in the equations of motion; n is inversely proportional to the period of rotation of the galaxy and α is a measure for the tidal forces. The constant α measures also the degree of stability of a given cluster; it is largely determined by the values assumed for the mass and distance of the galactic nucleus.

The Oort-Lindblad theory of galactic rotation offers a picture of the structure of our galaxy that represents the available observational data very well. Plaskett and Pearce's investigations on the velocities of early type stars have yielded much observational evidence in favor of this theory and quite recently Joy has shown that the radial velocities of distant Cepheids give a beautiful confirmation of the theory of galactic rotation. Local irregularities may be more important than this theory suggests, but even if this were so, we might expect to obtain a good first approximation by neglecting their influence over the long intervals of time that should be considered in the development of a cluster. The Oort-Lindblad theory has the additional advantage of analytic simplicity and gives well determined values for n and α.

Our conceptions of the structure of our own galaxy may change considerably within the next few years and our numerical estimates may be subject to considerable modifications, but it does not seem probable that disruptive forces of the type discussed here will go altogether out of the picture. Observations of nearby galaxies prove that tidal forces are at work in most of them, and there can be no doubt that a discussion like that of Jeans with $\alpha = 0$ leaves out one of the most essential features of the problem. We hope that it will be possible to adapt our method of attack to future theories of galactic structure.

The problem that we have set ourselves accordingly has been to make a study of the dynamics of a moving cluster with special reference to questions of stability. We have tried to predict the probable size and shape that a moving cluster will attain under the combined influence of

a. the attraction of the cluster on its members

b. the tidal forces of the galaxy as a whole

c. the disrupting effects of chance encounters.

As far as forces *a* and *b* are concerned this problem has much in common with that of the stability of comets and meteor swarms, which has been studied extensively during the last decade of the nineteenth century. Our method of analysis follows very closely that developed in Picart's doctor's thesis published at Paris in 1892 under the title: Sur la désagrégation des essaims météoriques. A concise and elegant summary of Picart's work has been given by Tisserand in Chapter XVI of volume IV of his Mécanique Céleste.

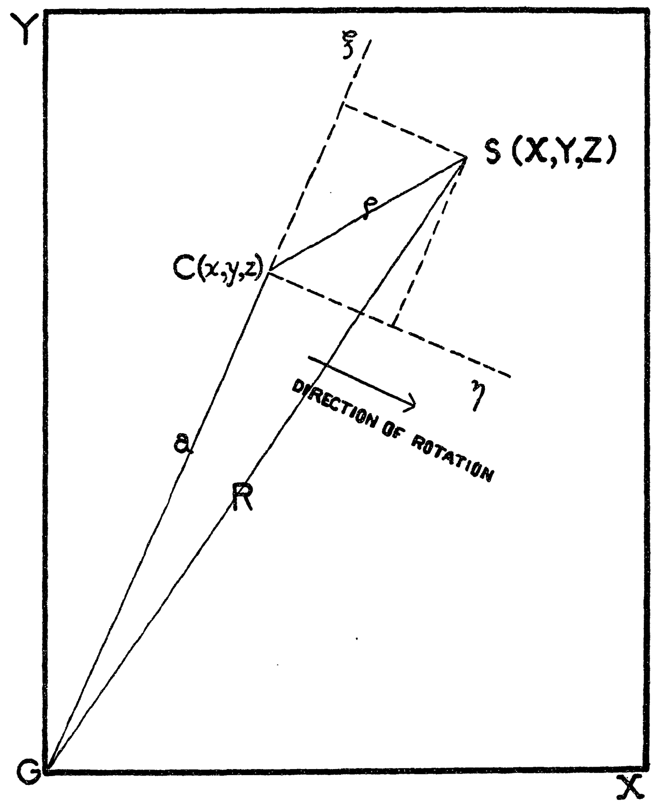
We shall attempt in the course of our investigation to distinguish between stable and unstable clusters. We should, however, be careful not to use these words too literally. All dense clusters that we have classified as "stable" must be loosening up gradually, but very slowly, through the effect of random encounters with passing stars. The rate of expansion of these clusters is however of an order of magnitude entirely different from that of the rapidly expanding clusters of low mean density disrupted by the tidal forces inside the galactic system.

The cluster problem resembles in many respects the satellite problem in our planetary system, but there are certain differences that it may be well to point out before entering into the detailed treatment of the problem.

a. Resonance phenomena give many complications in the planetary case, but should hardly be of importance for the galactic problem. Resonance arises in a planetary system, because we find there certain definite periods that remain constant over a large number of revolutions. The characteristic periods in our galaxy will, however, be changing continuously. We may take the period of revolution or oscillation of a star inside a cluster as an example. This period will depend largely upon the mean density in the cluster, and as this density decreases gradually during the development of the cluster the period will increase correspondingly.

b. If we compare a single star in the cluster with the satellite, the cluster as a whole with the planet, and our galaxy with the sun, we find that the fields of force in the cluster problem differ from those found from the inverse square law in our planetary system. Observation shows that the concentration of mass toward the nucleus is small in most galactic clusters and therefore that the attraction of the cluster as a whole upon its members should be more nearly directly proportional to the distance of the star from the center of the cluster. This simplifies the cluster problem considerably and makes it much easier to handle than the planetary problem.

c. Each planet has made millions of revolutions around the sun, and the problems of stability in our solar system are complicated on account of this large number of revolutions. Everything happens on a much more leisurely scale in our galaxy, where the forces are smaller and the distances larger. During the past three years it has become increasingly more probable that approximately 3×10^9 years ago some catastrophe must have occurred in our galaxy. The number of revolutions around the galactic nucleus that a cluster as a whole can have made since the occurrence of this catastrophe can hardly have been more than twenty. And even if we think in terms of a time scale of for instance 3×10^{12} years, the number of revolutions will not run into the millions.



1. *Form of the Equations of Motion.* — The diagram shows the projection on the galactic (X Y) plane. The symbols used have the following meanings:

G = galactic center taken as the origin of the x, y, z system of coordinates
C = center of cluster; coord., x, y, z; origin of the revolving system of coordinates ξ, η, ζ
S = star inside, or just outside the cluster; coord. X, Y, Z or ξ, η, ζ
ξ axis = prolongation of G C (positive outwards)
η axis = perpendicular to G C in direction of rotation of the galaxy
G C = a; C S = ρ; G S = R; $a^2 = x^2 + y^2 + z^2$

$$R^2 = X^2 + Y^2 + Z^2$$
$$\rho^2 = \xi^2 + \eta^2 + \zeta^2$$

We consider only clusters at a large distance from the galactic center and therefore only the case: $\rho << a$

$f_1(a)$ = the component parallel to the galactic plane of the attraction of the galaxy as a whole on a unit mass in C
$U(\rho)$ = attraction along CS (spherical cluster) of the cluster as a whole upon a unit mass in S.

We have not included the z-coordinate in the diagram. The ζ axis is parallel to the

Z axis, but has its origin in C. We have the relation: $\zeta = Z - z$. We have assumed that the cluster moves within a distance of 200 parsecs from the galactic plane; z and Z are therefore small compared with x, y, X and Y. Oort has shown [3] that for small values of z and Z, the Z-component of the attraction of the galaxy as a whole upon a unit mass placed in S is proportional to Z; the formulae take an elegant and symmetrical form, if we call this component: $\frac{Z}{a} f_2(a)$.

The equations of motion for the center of the cluster are:

$$
\begin{aligned}
\frac{d^2x}{dt^2} + f_1(a)\frac{x}{a} &= 0 \\
\frac{d^2y}{dt^2} + f_1(a)\frac{y}{a} &= 0 \qquad (1) \\
\frac{d^2z}{dt^2} + f_2(a)\frac{z}{a} &= 0
\end{aligned}
$$

The equations of motion for S are:

$$
\begin{aligned}
\frac{d^2X}{dt^2} + f_1(R)\frac{X}{R} + \frac{X - x}{\rho}U(\rho) &= 0 \\
\frac{d^2Y}{dt^2} + f_1(R)\frac{Y}{R} + \frac{Y - y}{\rho}U(\rho) &= 0 \qquad (2) \\
\frac{d^2Z}{dt^2} + f_2(R)\frac{Z}{R} + \frac{Z - z}{\rho}U(\rho) &= 0
\end{aligned}
$$

Combining (1) and (2), we find:

$$
\begin{aligned}
\frac{d^2(X - x)}{dt^2} + f_1(a)\frac{X - x}{a} + f_1(R)\frac{X}{R} - f_1(a)\frac{X}{a} + \frac{X - x}{\rho}U(\rho) &= 0 \\
\frac{d^2(Y - y)}{dt^2} + f_1(a)\frac{Y - y}{a} + f_1(R)\frac{Y}{R} - f_1(a)\frac{Y}{a} + \frac{Y - y}{\rho}U(\rho) &= 0 \qquad (3) \\
\frac{d^2(Z - z)}{dt^2} + f_2(a)\frac{Z - z}{a} + f_2(R)\frac{Z}{R} - f_2(a)\frac{Z}{a} + \frac{Z - z}{\rho}U(\rho) &= 0
\end{aligned}
$$

On the theory of galactic rotation, C moves in a circle around G (at least to a high degree of approximation) and hence:

$$x = a \cos nt \qquad y = -a \sin nt$$

Now $\rho << a$ and $z << a$ and neglecting quantities of the order $\frac{\rho^2}{a^2}$ and $\frac{z^2}{a^2}$ we have:

$$R = a\left[1 + \frac{\xi}{a}\right]$$

and as the transformation formulae are:

$$X = x + \xi \frac{x}{a} + \eta \frac{y}{a}$$

$$Y = y + \xi \frac{y}{a} - \eta \frac{x}{a}$$

$$Z = z + \zeta$$

we find, after some simple algebraic transformations, the formulae for the motion of S with respect to C in the system with rotating axis ξ, η, ζ to be:

$$\begin{aligned} &\frac{d^2\xi}{dt^2} - 2n\frac{d\eta}{dt} + \alpha\xi + \frac{\xi}{\rho}U = 0 \\ &\frac{d^2\eta}{dt^2} + 2n\frac{d\xi}{dt} + \frac{\eta}{\rho}U = 0 \\ &\frac{d^2\zeta}{dt^2} + n'^2\zeta + \frac{\zeta}{\rho}U = 0 \end{aligned} \tag{4}$$

where $n^2 = \frac{f_1(a)}{a}$, $n'^2 = \frac{f_2(a)}{a}$, $\alpha = \frac{df_1(a)}{da} - \frac{f_1(a)}{a}$.

The formula for ζ requires some explanation. For small distances from the galactic plane the value of $f_2(a)$ is determined by the density in the galactic plane at a distance a from the center of the galaxy. Local variations in density may affect $f_2(a)$ considerably, but in our present state of knowledge we have to put $\frac{df_2}{da} = 0$.

The cluster itself has been taken as spherical. The equations would have taken a slightly more complicated form, if we had assumed an ellipsoidal cluster; we shall treat the case of such a cluster in Section 4.

α and n are two constants closely related to Oort's constants A and B of galactic rotation. It is important, however, to notice that the orders of magnitude of α and n are determined by the values assumed for the central mass of the galactic nucleus and the distance to the galactic center. If M is the mass of the galactic nucleus and a the distance to the galactic center, we have:

$$f_1 = \frac{k^2M}{a^2}$$

(k^2 = gravitational constant) and hence:

$$\alpha = \frac{-3k^2M}{a^3} \quad \text{and} \quad n^2 = \frac{k^2M}{a^3} \tag{5}$$

Our galaxy can roughly be described as consisting of a nucleus with a mass of the order of 10^{11} solar masses at a distance of 10,000 parsecs from our sun. We have then approximately:

$$\alpha = -1.5 \times 10^{-30}$$
$$n^2 = 0.5 \times 10^{-30}$$

Formula (5) helps to indicate in what direction subsequent changes in our conceptions of galactic structure might alter our results. Oort's value of 10^{11} solar masses for the mass of the galactic nucleus is generally considered to be high, and if it had to be reduced, α and, as we shall see in the next section, the limiting density in a cluster beyond which instability sets in, should be reduced correspondingly. But the value of α is also very sensitive to the value assumed for a, the distance to the center of the galaxy, because α is inversely proportional to the third power of a. Formula (5) may be helpful in a study of the importance of tidal effects in galaxies outside our own, for which the central mass and linear scale are known from observations of rotation and a determination of the parallax.

At this moment we can obtain the most accurate value for α by noticing the simple relation that exists between α and Oort's constant A of galactic rotation. We have in our notation:

$$A = \frac{a}{4V'}\left[\frac{f_1}{a} - \frac{df_1}{da}\right]$$

where V' is the velocity of rotation of our galaxy at a distance, a, from its center. We have therefore:

$$\alpha = -4\frac{AV'}{a} \qquad n = \frac{V'}{a} \tag{6}$$

and it is a simple matter to determine both α and n, assuming the most probable values for A, V', and a.

All our computations will be carried through in c.g.s. units. The values of α and n used throughout the present paper have been determined by using the following values for the constants involved in (6):

$$\begin{aligned} A &= +0.0155 \text{ km/sec/pc} &&= +5.0 \times 10^{-16} \text{ c.g.s.} \\ V' &= 270 \text{ km/sec} &&= 2.7 \times 10^{7} \\ a &= 10{,}000 \text{ pcs} &&= 3.1 \times 10^{22} \end{aligned}$$

and hence:

$$\alpha = -1.74 \times 10^{-30} \quad \text{and} \quad n = 0.87 \times 10^{-15}$$

The corresponding value for the period of rotation of our sun around the galactic center is:

$$T = 2.3 \times 10^8 \text{ yrs.}$$

and the mass of the galactic nucleus appears to be of the order of 10^{11} solar masses. n' can be found from Oort's investigation in B.A.N., **238.** Using the values of K(z) given in the seventh column of Table 14 of Oort's paper, we find:

$$n' = 2.4 \times 10^{-15},$$

the period of one oscillation in z being equal to:

$$T' = 0.84 \times 10^8 \text{ years.}$$

2. *The Criterion of Stability.* — Hill [4] discussed in his well known memoir on "Researches in the Lunar Theory" the stability of the lunar orbit by using Jacobi's velocity integral. Picart [5] applied Hill's method of analysis for the case of a meteor swarm in the solar system and with slight modifications it holds also for our present problem.

Equations (4) are the basis of our discussion. Multiply the first equation by $\frac{2d\xi}{dt}$, the second by $\frac{2d\eta}{dt}$, the third by $\frac{2d\zeta}{dt}$, and add:

$$\frac{d}{dt}\left[\left(\frac{d\xi}{dt}\right)^2 + \left(\frac{d\eta}{dt}\right)^2 + \left(\frac{d\zeta}{dt}\right)^2\right] + \alpha\frac{d}{dt}(\xi^2) + n'^2\frac{d}{dt}(\zeta^2) + \frac{U}{\rho}\frac{d}{dt}(\rho^2) = 0$$

Upon integration we obtain the relation:

$$V^2 + \alpha\xi^2 + n'^2\zeta^2 + 2\int_r^\rho U(\rho)\,d\rho = C$$

where V is the linear velocity and r the initial value of ρ. We can write this relation more conveniently in the form:

$$V^2 = V_r^2 - \alpha(\xi^2 - \xi_r^2) - n'^2(\zeta^2 - \zeta_r^2) - 2\int_r^\rho U(\rho)\,d\rho \tag{7}$$

Hill considered the form of the surface

$$V^2 = 0$$

This surface will in general divide space into two parts, one in which V^2 will be positive and the velocity real, and a second part in which V^2 will be negative and the velocity imaginary. We know that $V^2 = V_r^2$ for $\xi = \xi_r$, $\eta = \eta_r$, $\zeta = \zeta_r$ and hence the star will be initially at the positive side of the surface $V^2 = 0$. The star may describe in time a very complicated orbit, but we can be certain that the square of the velocity will not become negative under any circumstances. It is therefore impossible for the star ever to pass through the surface $V^2 = 0$. If the surface $V^2 = 0$ is closed around the origin the star will continue to belong to the cluster, because it will be unable to pass beyond a certain distance from its center. If on the other hand the surface $V^2 = 0$ is open, the star may move away from the center of the cluster and cease to be a member. A study of the surface $V^2 = 0$ can give us some information about the stability of the cluster.

In considering (7) we have to bear in mind that the quantity α is essentially negative. The term $-\alpha(\xi^2 - \xi_r^2)$, which will be positive for all values $\xi > \xi_r$, represents the disruptive action of the tidal forces in our galaxy. It is now easy to see why the problem of stability in this form did not enter into Jeans' investigation. In the Kapteyn universe we can assume that for points close to the center $f_1(a) = Ka$, where K is a constant and hence $\alpha = 0$. But it is also evident that this is a special limiting case and that α will certainly be different from zero and negative in our galaxy.

The term $-n'^2(\zeta^2 - \zeta_r^2)$ is of minor importance in most of our discussions, because n'^2 is positive. It shows how the attraction of the galactic layer tends to keep the cluster together in the z-direction. If we put $\alpha = n'^2 = 0$ in (7), the equation has the form:

$$V^2 = V_r^2 - 2\int_r^\rho U(\rho)\, d\rho$$

which is the usual equation for the velocity in a radial field of force. A cluster will therefore be stable if the attraction of the cluster on its members is sufficient to counterbalance the disrupting effects due to: *a*, the tidal effects of the galactic nucleus and *b*, the random velocities of the individual stars in the cluster. If the mean density of the cluster is high, the second and third terms in the right hand side of (7) will be unimportant, but they should certainly be considered, if the mean density of the cluster becomes small.

Before entering into a detailed discussion of the form of the surface $V^2 = 0$, it is of interest to note that a cluster on the verge of instability will first become unstable in the direction of the ξ axis. It is convenient to introduce polar coordinates in this discussion. We take the $\xi\eta$ plane as equatorial plane, call the galactic longitude of the star as it is observed from the center of the cluster l, and its latitude b.

Putting the zero point, from which to measure the longitude, in the direction of the positive ξ axis, we find that (7) takes the form

$$V^2 = C - \alpha\rho^2 \cos^2 l \cos^2 b - n'^2\rho^2 \sin^2 b - 2\int_r^\rho U(\rho)\, d\rho.$$

We consider a star with a definite value of C (depending upon the amount of the velocity V_r at the initial point ξ_r, η_r, ζ_r), and compute the value of V^2 at a fixed distance ρ_0 from the center of the cluster, but for different values of l and b. Before instability has set in, it will always be possible to give a certain value $\rho = \rho_0$ for which V^2 will be negative, whatever l and b may be. The maximum value of V^2 on the sphere with $\rho = \rho_0$ will be found for $l = 0°$ or $180°$, $b = 0°$, or in other words the smallest negative values will be found at the points where the ξ axis passes through the sphere.

Let us now keep C constant, consider a cluster which is stable at the beginning and see what happens if we decrease the density of this cluster gradually. It will not be difficult to find in the beginning a value $\rho = \rho_0$ that defines a sphere upon whose surface the velocity will everywhere be negative. In the beginning we can choose for ρ_0 any value that will fall within a certain definite range in ρ, but we shall find that the interval from which we have to choose our value of ρ_0 will become smaller as the density decreases. Finally we shall reach a certain critical density which is so small that it becomes impossible to find a suitable value for ρ_0. It is especially interesting to see what happens at the moment that the average density will be just so small as to make the cluster unstable. We can then choose a value $\rho = \rho_0$ as the radius for a sphere upon which V^2 will be negative for all points with the exception of the two points where the ξ axis passes through the sphere; V^2 will be equal to zero at these two points.

It is now easy to see what happens to the surface $V^2 = 0$ as we pass from a stable cluster to an unstable one. We found that this surface was closed around the origin for the stable cluster. At the moment that instability sets in, the value of V^2 will be zero at two points of the ξ axis and positive for all other points along this axis. The surface $V^2 = 0$ is about to open; if we take the mean density of the cluster still smaller, it will definitely do so, and the cluster will then be unstable. Our analysis indicates that the first chances of escape will be for stars moving in paths that pass very close to the ξ axis through either of the open "necks" of the surface $V^2 = 0$. We can therefore say that instability will first occur in the direction of the ξ axis. The "necks" of the surface $V^2 = 0$ will open more and more as the average density of the cluster continues to decrease.

If we wish to find out first what clusters are just on the verge of becoming unstable, it will be sufficient to compute the values of V^2 along the ξ axis for each specific cluster and assumed value for V_r. If it appears that the value of V^2 will not be negative for any point on the ξ axis, the cluster will be unstable; it will on the other hand be stable, if we find negative values for V^2 along part of the ξ axis.

We attack the problem in two steps. The first one is to put $V_r = 0$ and investigate just where we should draw the line between stable and unstable clusters. We can be certain that any cluster, which is found to be unstable on the assumption $V_r = 0$, will certainly be so if V_r differs from zero. The next step will be to investigate the stable clusters further and see for what values of V_r a cluster, that was found stable for $V_r = 0$, will become unstable.

Instability will set in first along the ξ axis and, in our problem with $V_r = 0$, we shall therefore have to study the function $V^2 = f(\xi)$, where

$$f(\xi) = -\alpha(\xi^2 - \xi_r^2) - 2\int_{\xi_r}^{\xi} U(\xi)\, d\xi \tag{8}$$

Upon differentiation of (8) we find:

$$\frac{df}{d\xi} = -2[\alpha\xi + U]$$

If we assume the density of the cluster either to remain constant, or to decrease outwardly beyond $\xi = \xi_r$, it can readily be proved that the cluster will be unstable if $\left(\frac{df}{d\xi}\right)_r$ = positive. We can write for $\xi \geqq \xi_r$

$$U(\xi) \leqq \tfrac{4}{3}\pi k^2 \Delta_r \xi,$$

where k^2 is the constant of gravitation and Δ_r is the average density of the part of the cluster within a distance ξ_r from its center. U is positive, α negative, and using our formula for $U(\xi)$, we find for $\xi > \xi_r$

$$\alpha\xi + U \leqq \frac{\xi}{\xi_r}(\alpha\xi_r + \tfrac{4}{3}\pi k^2 \Delta_r \xi_r)$$

If $\left(\frac{df}{d\xi}\right)_r$ is positive, $(\alpha\xi_r + \frac{4}{3}\pi k^2 \Delta_r \xi_r)$ must necessarily be negative, and we find therefore that $\frac{df}{d\xi}$ will be positive for all values of $\xi \geqq \xi_r$. Formula (8) shows that $f(\xi_r) = 0$, and hence we have for $\xi > \xi_r$:

$$f(\xi) = \text{positive.}$$

This proves the instability of the cluster. In the same way we can prove that the cluster will be stable if $\left(\frac{df}{d\xi}\right)_r$ is negative.

The case where $\left(\frac{df}{d\xi}\right)_r = 0$ marks the transition from a stable to an unstable cluster. If $\alpha\xi_r + U(\xi_r)$ becomes equal to zero we have evidently:

$$\alpha + \tfrac{4}{3}\pi k^2 \Delta_r = 0$$

and we find therefore that instability sets in if the average density within a distance r from the center becomes equal to:

$$\Delta_r = -\frac{3\alpha}{4\pi k^2} \tag{9}$$

The cluster will be unstable if Δ_r is smaller than this value, and stable if it exceeds it; instability will also occur if Δ_r is just equal to the critical density.

In the investigation of the stability of a cluster we proceed in the following way. Compute Δ_r for different values of r, Δ_r being the average density within a sphere having a radius r (it is *not* the mean density at a distance r from the center!). The cluster will certainly be unstable if this average density becomes nowhere as high as the critical value given by (9), and rapid disintegration of such a cluster cannot be avoided. If the average density exceeds the critical value for the core of the cluster, but becomes negative for points in the outer shells, the cluster may be partly stable. The stars in the outer shells will be lost as members of the cluster, but the core may be retained. All our computations have been based on the assumption $V_r = 0$, and we should remember that the velocity dispersion of the stars in the core may cause instability. A cluster for which Δ_r exceeds the critical value throughout will be stable, unless instability arises on account of the dispersion in velocity among the cluster members.

The critical density Δ_r for clusters in the neighborhood of our sun can be computed from the value of α found at the end of Section 1, the result being:

$$\Delta_r = 6.2 \times 10^{-24}\ \text{g/cm}^3$$

or

$$\Delta_r = 0.093 \text{ solar masses per cubic parsec}$$

We shall postpone the detailed discussion of this result to Section 5, but it seems well to note the following points before continuing our analysis: 1. The mean density in most of the conspicuous galactic clusters exceeds considerably the critical value of one tenth of a solar mass per cubic parsec; the Pleiades, Praesepe, and h and χ Persei are examples of such apparently stable clusters. 2. The mean densities in the Hyades and in the nucleus of the Ursa Major cluster should be close to the critical value, but accurate estimates are difficult on account of the lack of observational data. 3. The mean densities in the extended Ursa Major cluster, in Eddington's moving cluster in Perseus, and in the Scorpio-Centaurus cluster are probably much less than the critical density, and these clusters will therefore probably be unstable. 4. If a local cluster exists at all, the average density near its center will probably be of the same order of magnitude as our critical density, making it rather unstable.

We know now which clusters should certainly be classified as unstable, but we cannot guarantee stability for the clusters whose mean density exceeds the critical density, because our computations were based on the assumption $V_r = 0$. If we consider a cluster with a well defined radius, and with sufficient density to be classified as stable, the outward velocities of the stars near the outer boundary will naturally be very small. But on account of the dispersion in velocity among the members of the cluster, it may well be possible to find stars with an outward velocity which is sufficiently large to make escape from the cluster possible. Again, escape will occur most readily along the ξ axis and we shall therefore compute what value V_r should have at the outer boundary of the cluster to make escape possible.

The formula for the velocity again takes the form:

$$f(\xi) = V_r^2 - \alpha(\xi^2 - \xi_r^2) - 2\int_{\xi_r}^{\xi} U(\xi)\, d\xi,$$

where $U(\xi) = \dfrac{k^2 M}{\xi^2}$, M being the total mass of the cluster and $\xi \geqq \xi_r$. We introduce now for convenience the average density in the cluster expressed in terms of the critical density Δ_r as a unit. Calling the ratio of both densities P, we have:

$$M = \frac{4\pi}{3} P \Delta_r \xi_r^3 = -\frac{P\alpha}{k^2}\xi_r^3,$$

and hence:

$$f(\xi) = V_r^2 - \alpha(\xi^2 - \xi_r^2) + 2P\alpha\xi_r^3\left[\frac{1}{\xi_r} - \frac{1}{\xi}\right]$$

The cluster will be on the verge of becoming unstable if we find:

$$f(\xi') = 0$$

for the point where:

$$\left(\frac{df}{d\xi}\right)_{\xi=\xi'} = 0$$

We have upon differentiation

$$\xi' = P^{\frac{1}{3}}\xi_r$$

and after substitution in the formula for $f(\xi)$ and putting $f(\xi') = 0$:

$$V_r^2 = -\,a\xi_r^2[2P - 3P^{\frac{2}{3}} + 1] \tag{10}$$

If we had neglected the action of the galactic forces upon the cluster the formula for the velocity of escape for a star in the outer parts of the cluster would have been:

$$V_r^2 = -2Pa\xi_r^2 \tag{11}$$

and a comparison of (10) and (11) tells us how much the velocity of escape from the cluster is reduced through the galactic tidal effects.

Formula (10) applies only to clusters with an average density exceeding the critical density and for which $P \geqq 1$. If we compute the ratio between the values of V_r^2 computed from (10) and (11), we find this ratio to be very small for values of P slightly exceeding 1, showing that galactic tidal effects increase the probability of escape considerably. The ratio is equal to 0.20 for $P = 5$, equal to 0.35 for $P = 10$, and slowly converges toward unity for larger values of P. Our results indicate that the galactic tidal effects, which are responsible for the instability of clusters with $P \leqq 1$, will have an appreciable influence on the motion of stars in a cluster for which P lies between one and three, but they can be considered as causing only a minor perturbation in the state of motion in the more massive clusters.

3. *Disintegration of a Cluster with Negligible Mean Density.* — The problem that we set out to solve was the determination of the probable size and shape that a moving cluster will attain under the combined influence of:

a. The attraction of the cluster on its members

b. The tidal forces of the galaxy as a whole

c. The disrupting effects of random encounters.

The analysis of the preceding section gave us a general criterion for the stability of a cluster, but, as its results were altogether based upon one single integral of the equations of motion, it left the bulk of the problem untouched. Since it would be confusing to try to solve the whole problem at once, we shall study first a simplified case that shows some essential features of the larger problem. We neglect in this section the attraction of the cluster upon its members. We mentioned before that the mean density in some of the most conspicuous galactic clusters is considerably less than the critical density found from formula (9), and the solution of the present section should apply to a certain extent to those cases. We shall investigate the influence of forces, a, further in Section 4.

Let us consider a cluster, the members of which form a definite limited group in space, but placed at such large distances apart that their mutual gravitational action is negligible. Such a cluster will be unstable; and the rate of disintegration will depend upon the distribution of the initial velocities in the cluster. We shall study however the least unstable cluster of this type, one in which the initial velocities and coordinates have been so balanced that the cluster would have held together in the absence of chance encounters with passing stars. We shall see that it is quite possible to choose the initial velocities of the cluster members in such a way as to make the periods of revolution around the center of our galaxy the same for all. Such a cluster will however be broken up gradually through the influence of random encounters with passing stars. Small additional velocity components will be acquired in this way and the galactic tidal effects will do the rest. It may be well to point out that any deviations from the initial conditions used here will result in a more rapid disintegration of the cluster.

Equations (4) will now take the form:

$$\begin{aligned} \frac{d^2\xi}{dt^2} - 2n\frac{d\eta}{dt} + a\xi &= 0 \\ \frac{d^2\eta}{dt^2} + 2n\frac{d\xi}{dt} &= 0 \\ \frac{d^2\zeta}{dt^2} + n'^2\zeta &= 0 \end{aligned} \tag{12}$$

We solve these equations for the motion of a star in the cluster assuming that for $t = 0$, we have

$$\xi = \xi_0, \quad \eta = \eta_0, \quad \zeta = \zeta_0 \quad \text{and} \quad \left(\frac{d\xi}{dt}\right)_0 = V_\xi, \quad \left(\frac{d\eta}{dt}\right)_0 = V_\eta, \quad \left(\frac{d\zeta}{dt}\right)_0 = V_\zeta.$$

The equation in ζ can be solved immediately, the solution having the form:

$$\zeta = A \cos n't + B \sin n't$$

Integration of the second equation of (12) gives:

$$\frac{d\eta}{dt} = -2n\xi + C$$

and upon substitution of this in the equation for $\frac{d^2\xi}{dt^2}$, we find:

$$\frac{d^2\xi}{dt^2} + p^2\xi - 2nC = 0,$$

where $p^2 = 4n^2 + \alpha$. The solution of this equation is:

$$\xi = A\cos pt + B\sin pt + \frac{2n}{p^2}C$$

We substitute this value for ξ in the equation for $\frac{d\eta}{dt}$, integrate once more, and obtain as our final solution, making use of the initial conditions:

$$\begin{aligned}
\xi &= \left[\frac{\alpha}{p^2}\xi_o - \frac{2n}{p^2}V_\eta\right]\cos pt + \frac{1}{p}V_\xi \sin pt + \frac{2n}{p^2}[V_\eta + 2n\xi_o] \\
\eta &= \frac{2n}{p^2}V_\xi \cos pt - \left[\frac{2n\alpha}{p^3}\xi_o - \frac{4n^2}{p^3}V_\eta\right]\sin pt + \eta_o - \frac{2n}{p^2}V_\xi + \frac{\alpha}{p^2}[V_\eta + 2n\xi_o]t \qquad (13) \\
\zeta &= \zeta_o \cos n't + \frac{1}{n'}V_\zeta \sin n't
\end{aligned}$$

For brevity we shall write (13) in the form

$$\xi = \xi_1(t) \qquad \eta = \eta_1(t) \qquad \zeta = \zeta_1(t),$$

where ξ_1, η_1, and ζ_1 represent the right-hand members of (13). We imagine now that the star, whose motion is represented by (13), meets at a time $t = t'$ another star not belonging to the cluster, the result of the passage being that the cluster star acquires at that moment additional velocity components Δ_ξ, Δ_η, and Δ_ζ. Jeans has already shown that the time during which the velocity transfer takes place is small compared with the period $\frac{2\pi}{p}$ of the periodical terms in (13). The solution for $t > t'$ is therefore:

$$\begin{aligned}
\xi &= \xi_1(t) + \frac{2n}{p^2}\Delta_\eta[1 - \cos p(t - t')] + \frac{1}{p}\Delta_\xi \sin p(t - t') \\
\eta &= \eta_1(t) + \frac{\alpha}{p^2}\Delta_\eta(t - t') + \frac{4n^2}{p^3}\Delta_\eta \sin p(t - t') - \frac{2n}{p^2}\Delta_\xi[1 - \cos p(t - t')] \\
\zeta &= \zeta_1(t) + \frac{1}{\eta'}\Delta_\zeta \sin n'(t - t')
\end{aligned}$$

Instead of considering the effect of one single encounter with a passing star, we should consider the cumulative effect of a great many passages that must have occurred since $t = 0$. The period of one oscillation of a star in the cluster is of the order of 2×10^8 years, and it is therefore probable that each cluster has gone through at least ten oscillations. Following Jeans [1] we can again compute the probable mean values $\overline{\xi^2}$, $\overline{\eta^2}$, $\overline{\zeta^2}$ for all stars in the cluster. Neglecting all terms that will be small compared with those given here, we find:

$$\overline{\xi^2} = \overline{\xi_1^2(t)} + \frac{4n^2}{p^4}\overline{[1 - \cos p\,(t - t')]^2}\sum \Delta_\eta^2 + \frac{1}{p^2}\overline{\sin^2 p\,(t - t')}\sum \Delta_\xi^2$$

$$\overline{\eta^2} = \overline{\eta_1^2(t)} + \frac{\alpha^2}{p^4}\sum \Delta_\eta^2 (t - t')^2 + \frac{16n^4}{p^6}\overline{\sin^2 p\,(t - t')}\sum \Delta_\eta^2 + \frac{4n^2}{p^4}\overline{[1 - \cos p\,(t - t')]^2}\sum \Delta_\xi^2$$

$$\overline{\zeta^2} = \overline{\zeta_1^2(t)} + \frac{1}{n'^2}\overline{\sin^2 n'\,(t - t')}\sum \Delta_\zeta^2.$$

We stated in the beginning of this section that we would study the least unstable of all clusters and we can now say what relation should exist between the initial velocities and coordinates. If we put

$$V_\eta + 2n\,\xi_o = 0$$

for all stars in the cluster, the linear term in t in the formula for $\eta_1(t)$ will disappear, and the formula for ξ_1, η_1, ζ_1 will contain only periodical terms. We make this assumption and can then be certain that the mean values $\overline{\xi_1^2(t)}$, $\overline{\eta_1^2(t)}$ and $\overline{\zeta_1^2(t)}$ will be constant throughout the development of the cluster. It is clear that a cluster, for which the value of

$$V_\eta + 2n\,\xi_o$$

is not the same for the different members, will disintegrate very rapidly in the η direction, but that our cluster would not do so in the absence of random encounters.

The cluster must have passed through several oscillations and we can therefore take mean values for the trigonometric terms. We find:

$$\overline{\xi^2} = \overline{\xi_1^2} + \frac{6n^2}{p^4}\sum \Delta_\eta^2 + \frac{1}{2p^2}\sum \Delta_\xi^2$$

$$\overline{\eta^2} = \overline{\eta_1^2} + \frac{\alpha^2}{p^4}\sum \Delta_\eta^2 (t - t')^2 + \frac{8n^4}{p^6}\sum \Delta_\eta^2 + \frac{6n^2}{p^4}\sum \Delta_\xi^2$$

$$\overline{\zeta^2} = \overline{\zeta_1^2} + \frac{1}{2n'^2}\sum \Delta_\zeta^2.$$

Our problem will therefore be solved, if we are able to find the values of:

$\sum \Delta_\xi^2$, $\sum \Delta_\eta^2$, $\sum \Delta_\zeta^2$ and $\sum \Delta_\eta^2 (t - t')^2$ as a function of t.

The effect of random encounters upon the members of a moving cluster has been studied by Jeans, Eddington, and Schwarzschild, and while the results are naturally uncertain, because of our lack of knowledge of "the past" of a cluster, we can make reasonable estimates of the order of magnitude of the resulting changes in the velocities of the members of a cluster.

The transverse velocity imparted in an encounter of two stars is equal to:

$$\Delta V = \frac{2k^2M}{VD} \tag{14}$$

where $k^2 = 6.67 \times 10^{-8}$
M = mass of disturbing star
V = initial relative velocity of the stars
D = minimum distance at which the stars would have passed if no gravitational action had occurred.

It is worth noticing: *a*, that ΔV is independent of the mass of the star receiving a transverse component ΔV; and *b*, that ΔV is inversely proportional to V, and hence that encounters with low relative velocities are most effective.

Strictly speaking ΔV is the transverse velocity acquired by a star in the cluster in a direction perpendicular to V_o, where V_o is the velocity of the cluster star measured with respect to the center of gravity of both passing stars. If the velocity of the cluster as a whole is large with respect to the neighboring stars, ΔV will practically always have a direction perpendicular to the direction of motion of the cluster; if on the other hand the cluster is practically at rest, the direction of ΔV will depend largely on the direction of motion of the passing star.

Formula (14) will hold if:

$$V^2D >> k^2(M_1 + M_2)$$

where $M_1 + M_2$ is the sum of the masses of both stars. Jeans and Eddington have shown that single large deflections are uncommon and that they probably play no rôle of importance during the time that our galaxy has existed in its present form. Formula (14) will therefore be valid for all practical purposes.

In addition to (14) we need to know the frequency of encounters for different values of D. If N is the number of stars per unit volume, the number of encounters between distances D and D + dD occurring during a time t is equal to:

$$2\pi VND\,dD\,t$$

We assume a certain mean value for V, and find for the transverse component V′ acquired after a time t the value given by the formula:

$$V'^2 = \frac{8\pi k^4 (NM) M}{V} t \ln \frac{D_2}{D_1}$$

where D_1 and D_2 are the effective upper and lower limits of D between which the essential encounters occur. The present method of analysis is due to Jeans.

We have substituted the following numerical values in the formula for V'^2: $NM = 2 \times 10^{-24}$ (Oort in B.A.N., **238**), $M = 4 \times 10^{33}$, $V = 1.5 \times 10^6$, $D_1 = 7.5 \times 10^{15}$ (= 500 astron. units), $D_2 = 1.5 \times 10^{19}$ (= 5 parsecs), all values being given in c.g.s. units. D_1 was taken equal to 500 a.u., because closer passages are too infrequent; D_2 was taken as low as 5 parsecs because passages at a larger distance will tend to change the path of a cluster as a whole rather than that of an individual star. Upon substitution we find:

$$V'^2 = 4.5 \times 10^{-9} \times t \qquad \text{(c.g.s. units)}$$

In clusters with low density, which are of special interest for our investigation, the average distance between two cluster members is at least of the order of 2 parsecs. The problem of computing accurately the effect of a passage at a distance of 5 parsecs from the center of a cluster upon its members is quite involved, but we can readily show that our simple procedure gives the right order of magnitude. Assuming the average distance between the cluster members to be of the order of 2 parsecs, no serious complications would arise for passages within 1 parsec. If we had taken $D_2 = 3 \times 10^{18}$ (= 1 parsec) instead of the value of 5 parsecs used, we would have found:

$$V'^2 = 3.5 \times 10^{-9} \times t.$$

In this case we would have neglected the influence of the more distant passages completely, and we see that the numerical result differs only slightly from that obtained before. The choice of the upper limit for D is of secondary importance and we shall obtain the right order of magnitude if we take, for any cluster, D_2 equal to two or three times the average distance between the cluster members.

Jeans' method of analysis gives admittedly only a very first approximation to the problem of computing the influence of random encounters. We have preferred its use to that of a more refined analysis, like, for instance, the method developed by Schwarzschild in the Seeliger Festschrift, because there did not seem to be much point in making a detailed, and necessarily complicated, study of certain parts of the problem. The problem of estimating the influence of such encounters is very complex, and we can say *a priori* that a statistical treatment will never give us more than the order of magnitude of the effect. The total number of revolutions that our galaxy has made is so small that for this reason alone we can say that a statistical treatment can give only approximate results. In addition it may be that the cluster has spent some time in the past in regions where the star density differed considerably from that now observed in the neighborhood of our sun. Occasional encounters with massive clusters, or dense nebulae, may raise havoc with the results of any statistical treatment. A more refined analysis should also take into account the influence of encounters among the cluster members; our computed rate of expansion must therefore be somewhat too slow. We believe however that our approximate treatment should be sufficiently accurate to serve as a basis for the discussion of the quite general questions concerning the structure of moving clusters that we are attempting to solve in the present paper. These questions are: 1. Why do very loose moving clusters exist at all? 2. Why do the observed extended moving clusters show no trace of the elongation along the η axis that we would expect to find on account of the shearing effect of the rotation of our galaxy? and 3. Should these extended clusters not be much more flattened with respect to the galactic plane than they are observed to be?

Jeans found that a fast-moving cluster should be flattened in the direction of its motion. I doubt whether this conclusion can be maintained, if we accept the Oort-Lindblad theory of galactic rotation as correct. Let us consider for a moment a cluster that always remains in the plane of the galaxy. If the cluster as a whole has an initial velocity that differs only slightly in amount or direction from the velocity of rotation at its starting point in the galaxy, it will describe an ellipse with respect to a suitably chosen rotating system of coordinates ξ', η', having its origin within a few hundred parsecs from the center of the cluster. This can easily be seen by studying equations (12) and is part of Lindblad's explanation of the star streams as a consequence of galactic rotation.[6] The direction of motion of the cluster with respect to the neighboring stars will therefore change completely during one revolution of the cluster. Part of the time it will move toward or away from the center of the galaxy, and for the rest it will move more or less in a direction perpendicular to this. If the cluster moves in the galactic plane we can as a first approximation assume that the cluster will be moving half of the time in the ξ direction and half of the time in the η direction. We would then have:

$$\sum \Delta_\xi^2 = \sum \Delta_\eta^2 = 1.1 \times 10^{-9} \times t$$

$$\sum \Delta_\zeta^2 = 2.3 \times 10^{-9} \times t$$

Most clusters have however a motion of their own in the ζ direction. Also the direction of the additional velocity components acquired in encounters will not always be perpendicular to the direction of the motion of the star in the ξ, η, ζ system. It seems therefore correct to put for our problem:

$$\sum \Delta_\xi^2 = \sum \Delta_\eta^2 = \sum \Delta_\zeta^2 = 1.5 \times 10^{-9} \times t \tag{15}$$

and keep in mind that there may actually be slight differences between the three sums. It is not difficult to find the value of $\sum \Delta_\eta^2 (t - t')^2$. Integrating from $t = 0$ until the present time we have:

$$\sum \Delta_\eta^2 (t - t')^2 = \tfrac{1}{3} t^2 \sum \Delta_\eta^2$$

or inserting numerical values:

$$\sum \Delta_\eta^2 (t - t')^2 = 0.5 \times 10^{-9} \times t^3 \tag{15a}$$

Substitution in the equations for $\overline{\xi^2}$, $\overline{\eta^2}$ and $\overline{\zeta^2}$ gives:

$$\begin{aligned} \overline{\xi^2} &= \overline{\xi_1^2} + 4.6 \times 10^{21} \times t \\ \overline{\eta^2} &= \overline{\eta_1^2} + 9 \times 10^{-10} \times t^3 + 7.0 \times 10^{21} \times t \\ \overline{\zeta^2} &= \overline{\zeta_1^2} + 1.3 \times 10^{20} \times t \end{aligned} \tag{16}$$

c.g.s.

Formula (16) yields the desired results: it gives us the rate of growth of a cluster that has throughout its development a negligibly small mean density. The coefficient of the term in t^3 in the formula for $\overline{\eta^2}$ is small, but a very marked elongation in the direction of the η axis will set in soon after t becomes sufficiently large to make this term comparable with the linear terms. The term in t^3 arises from the fact that a linear term in t occurs in (13), and it is natural to inquire if this "secular term" is not due to some form of approximation and is, in reality, perhaps a periodical term with a long period. This is not so. A linear term in (13) will occur for any star whose period of revolution around the galactic center differs slightly from that for the origin of the ξ, η, ζ system of coordinates. We assume that for $t = 0$ the coefficient of the linear term in t in (13) should be equal to zero for all members of the cluster, or, in other words, we assume that all cluster stars had initially the same period of revolution around the galactic center. Random encounters with passing stars will affect these periods differently, depending upon the circumstances of the encounter, and the increasing dispersion in the periods of revolution for the original members of the cluster introduces the term in t^3 in (16).

We have tabulated in Table I the values of $\sqrt{\overline{\xi^2} - \overline{\xi_1^2}}$, $\sqrt{\overline{\eta^2} - \overline{\eta_1^2}}$ and $\sqrt{\overline{\zeta^2} - \overline{\zeta_1^2}}$ (in parsecs) for different values of t (in years) as computed from (16).

TABLE I

t (in years)	$\sqrt{\overline{\xi^2} - \overline{\xi_1^2}}$	$\sqrt{\overline{\eta^2} - \overline{\eta_1^2}}$ (in parsecs)	$\sqrt{\overline{\zeta^2} - \overline{\zeta_1^2}}$
10^8	1.3	2.3	0.2
2×10^8	1.7	5.2	0.3
5×10^8	2.8	20	0.5
10^9	4.0	55	0.7
2×10^9	5.5	150	1.0
5×10^9	8	610	1.5
10^{10}	13	1800	2.1
2×10^{10}	17	5000	2.9
5×10^{10}	28	20000	4.8
10^{11}	40	50000	6.5

This table shows clearly what the shearing effect of galactic rotation can do to a cluster of very low density. We shall consider in the next section to what extent these results are changed, when the cluster's attraction upon its members is taken into account. It is better to postpone a discussion of the significance of the results of Table I for our galaxy until we have finished the investigation of the influence of the forces arising within the cluster itself.

4. *Disintegration of a Cluster With Appreciable Mean Density.* — The results of Section 3 tell us something about the way in which a cluster, whose mean density is so small that it can be neglected, will disintegrate. It is, however, difficult to interpret the results obtained, if we do not know to what extent these may have been affected by our not taking the forces of attraction due to the cluster itself into consideration. It is also impossible to deduce from Section 3 anything as to the rate of disintegration of more massive clusters like the Hyades, with a mean density slightly exceeding the critical one, and we shall therefore discuss in the present section the effect of the cluster itself upon the motion of its members. In doing so we shall, for purposes of analysis, replace our real cluster, consisting of a definite and often very small number of distinct stars, by a smooth distribution of mass throughout the volume of the cluster. We try to choose the density distribution inside this volume so that the actual path described by a star in the cluster would agree closely with that which we would find for the same initial conditions on the assumption of a continuous distribution of density.

We wish to apply our analysis mostly to galactic clusters that show little or no central condensation, and it is sufficient for our purpose to assume a constant uniform density throughout the volume of the cluster. It did not seem correct to neglect the marked deviations from a spherical shape that occur in nature; we have therefore assumed the cluster to have the shape of an ellipsoid with, in general, three different axes. We have, however, avoided additional mathematical complications by having the three axes of the ellipsoid coincide with the direction of the ξ, η, and ζ axes. Such a model probably represents actual conditions sufficiently and makes it possible for us to understand what goes on in a cluster without leading us into too many mathematical complexities. We shall assume toward the end of this section that a cluster can be replaced at each stage of its evolution by an ellipsoid of uniform density. It may be possible to show that a cluster can never in the process of disintegration pass through such a series of homogeneous ellipsoids, but this will not affect the validity of our conclusions. The problem that we wish to solve is to determine the approximate rate of disintegration and the probable changes in shape at different stages of a cluster's development, and we can hardly doubt that our simplified model will enable us to make correct estimates. Our analysis does not apply in its present form to globular clusters, where a strong central concentration of mass is always present.

Formula (4) has to be changed slightly, if we wish to apply it to ellipsoidal clusters. Again we call the force of attraction, due to the cluster as a whole, "U", but this vector will no longer point toward the origin. Instead we have now the following formulae for the ξ, η, and ζ components of U for a point inside the cluster:

$$U_\xi = \beta_1 \xi; \quad U_\eta = \beta_2 \eta; \quad U_\zeta = \beta_3 \zeta; \quad \lambda^2 = \frac{b^2 - a^2}{a^2}; \quad \lambda'^2 = \frac{c^2 - a^2}{a^2}$$

$$\beta_1 = \frac{3k^2M}{a^3} \int_0^1 \frac{u^2\,du}{\sqrt{(1+\lambda^2u^2)(1+\lambda'^2u^2)}}$$

$$\beta_2 = \frac{3k^2M}{a^3} \int_0^1 \frac{u^2\,du}{(1+\lambda^2u^2)\sqrt{(1+\lambda^2u^2)(1+\lambda'^2u^2)}} \tag{17}$$

$$\beta_3 = \frac{3k^2M}{a^3} \int_0^1 \frac{u^2\,du}{(1+\lambda'^2u^2)\sqrt{(1+\lambda^2u^2)(1+\lambda'^2u^2)}}$$

where a, b and c are the semi axes of the ellipsoid in the direction of the ξ, η, and ζ axes and M is the total mass of the cluster.[7]

We have therefore instead of (4) the following three equations for the motion of a star inside the cluster:

$$\begin{aligned}
\frac{d^2\xi}{dt^2} - 2n\frac{d\eta}{dt} + (\alpha + \beta_1)\xi &= 0 \\
\frac{d^2\eta}{dt^2} + 2n\frac{d\xi}{dt} + \beta_2\eta &= 0 \\
\frac{d^2\zeta}{dt^2} + (n'^2 + \beta_3)\zeta &= 0
\end{aligned} \tag{18}$$

The third equation can be integrated without difficulty and the solution has the form:

$$\zeta = A' \cos p_3 t + B' \sin p_3 t, \qquad \text{where} \quad p_3^2 = n'^2 + \beta_3.$$

The first and second equation should be considered together; their solution is well known.[5]

We distinguish between two separate cases:

1. If $\alpha + \beta_1 > 0$ the solution has the form:

$$\begin{aligned}
\xi &= A_1 \cos p_1 t + B_1 \sin p_1 t + A_2 \cos p_2 t + B_2 \sin p_2 t \\
\eta &= \lambda_1 A_1 \sin p_1 t - \lambda_1 B_1 \cos p_1 t + \lambda_2 A_2 \sin p_2 t - \lambda_2 B_2 \cos p_2 t
\end{aligned} \tag{19}$$

where:

$$\begin{aligned}
p_1^2 &= +\tfrac{1}{2}[4n^2 + \alpha + \beta_1 + \beta_2] + \tfrac{1}{2}\sqrt{[4n^2 + \alpha + \beta_1 + \beta_2]^2 - 4\beta_2[\alpha + \beta_1]} \\
p_2^2 &= +\tfrac{1}{2}[4n^2 + \alpha + \beta_1 + \beta_2] - \tfrac{1}{2}\sqrt{[4n^2 + \alpha + \beta_1 + \beta_2]^2 - 4\beta_2[\alpha + \beta_1]}
\end{aligned}$$

and

$$\lambda_1 = -\frac{2np_1}{p_1^2 - \beta_2}; \qquad \lambda_2 = -\frac{2np_2}{p_2^2 - \beta_2}.$$

The A's and B's are the constants of integration and therefore related to the initial values of ξ, η, $\frac{d\xi}{dt}$ and $\frac{d\eta}{dt}$.

2. If $\alpha + \beta_1 < 0$ the solution has the form:

$$\begin{aligned}
\xi &= A_1 \cos p_1 t + B_1 \sin p_1 t + A_2 e^{p_2 t} + B_2 e^{-p_2 t} \\
\eta &= \lambda_1 A_1 \sin p_1 t - \lambda_1 B_1 \cos p_1 t + \lambda_2 A_2 e^{p_2 t} - \lambda_2 B_2 e^{-p_2 t}
\end{aligned} \tag{20}$$

where:

$$\begin{aligned}
p_1^2 &= +\tfrac{1}{2}[4n^2 + \alpha + \beta_1 + \beta_2] + \tfrac{1}{2}\sqrt{[4n^2 + \alpha + \beta_1 + \beta_2]^2 - 4\beta_2[\alpha + \beta_1]} \\
p_2^2 &= -\tfrac{1}{2}[4n^2 + \alpha + \beta_1 + \beta_2] + \tfrac{1}{2}\sqrt{[4n^2 + \alpha + \beta_1 + \beta_2]^2 - 4\beta_2[\alpha + \beta_1]}
\end{aligned}$$

and

$$\lambda_1 = -\frac{2np_1}{p_1^2 - \beta_2}; \qquad \lambda_2 = -\frac{2np_2}{p_2^2 + \beta_2},$$

the A's and B's being again the constants of integration.

The solution for $\alpha + \beta_1 > 0$ contains only periodical terms and the corresponding clusters will therefore be stable. If $\alpha + \beta_1$ becomes negative, exponential terms appear and these clusters will certainly be unstable. Apparently the transition from the stable to the unstable clusters occurs if:

$$\alpha + \beta_1 = 0 \tag{21}$$

We consider further the clusters that are on the verge of instability. It is obvious that condition (21) gives the same results as criterion (9) for spherical clusters. But most clusters are more or less flattened with respect to the galactic plane and (21) makes it possible to determine the limiting density for which instability occurs in flattened clusters. As an illustration we have computed the critical density for a number of clusters, having the shapes of ellipsoids of revolution (a = b), for different values of the ratio $\frac{a}{c}$, the minor axis being assumed to point toward the galactic pole. Table II shows to what extent flattened clusters are more unstable than the spherical.

TABLE II

a =	Critical Density
1.0 c	0.09 ⊙/pc³
1.2	.10
1.5	.11
2.0	.13
2.5	.15
3.0	.17

The solutions that we have given for (18) will both break down for $\alpha + \beta_1 = 0$, and it is well to consider this special case for a moment. The first two equations in (18) are now:

$$\begin{aligned} \frac{d^2\xi}{dt^2} - 2n\frac{d\eta}{dt} \qquad\quad &= 0 \\ \frac{d^2\eta}{dt^2} + 2n\frac{d\xi}{dt} + \beta_2\eta &= 0 \end{aligned} \tag{22}$$

It is interesting to compare (22) with (12), the equations for a cluster with all β's equal to zero. The solution of (22) resembles (13), with the important difference that *the linear term in t will now occur in the formula for ξ instead of that for η.* This suggests that a cluster that is just becoming unstable will start disintegrating along the ξ axis, or in directions toward and away from the galactic center. This result comes more or less as a surprise, because we think as a rule that the shearing effect of galactic rotation will tend to lengthen out the cluster along the η axis. We should, however, remember that we have already had an indication of our present results on page 10, where we found that for a spherical cluster "instability will first occur

TABLE III

p_1, p_2, λ_1, and λ_2 are the constants used in formulae (19) and (20).

$f = \frac{\beta}{-\alpha}$	p_1	p_2	λ_1	λ_2	
1000	42.6×10^{-15}	40.8×10^{-15}	-1.00	$+0.99$	
100	14.1	12.3	-1.02	$+.98$	
10	5.0	3.3	-1.11	$+.86$	
5.0	3.8	2.0	-1.14	$+.78$	
3.0	3.1	1.4	-1.17	$+.69$	Stable
2.0	2.7	0.9	-1.20	$+.60$	
1.5	2.5	.61	-1.21	$+.48$	
1.2	2.3	.36	-1.24	$+.32$	
1.1	2.2	.26	-1.25	$+.25$	
1.05	2.2	.17	-1.25	$+.17$	
.95	2.2	.17	-1.25	$-.17$	
.90	2.1	.24	-1.26	$-.25$	
.80	2.0	.35	-1.27	$-.40$	
.60	1.9	.45	-1.30	$-.63$	
.25	1.6	.49	-1.38	-1.25	Unstable
.10	1.3	.40	-1.43	-2.1	
.05	1.2	.30	-1.47	-2.9	
.01	1.2	.14	-1.49	$-6.$	
.001	1.1	.05	-1.53	$-16.$	

along the ξ axis." There is a second important difference between (13) and the solution of (22). We have in (13):

$$p^2 = 4n^2 + \alpha$$

and in the solution of (22)

$$p^2 = 4n^2 + \beta_2.$$

Now α is essentially negative and β_2 always positive, and the value of p in the solution of (22) will be considerably larger than the p in (13).

It is possible to see without elaborate computations why a cluster that begins to disintegrate along the ξ axis will finally end by disintegrating along the η axis. We compute to this end for a spherical cluster the values of p_1, p_2, λ_1 and λ_2 for different values of the ratio:

$$f = \frac{\beta}{-\alpha}$$

During the gradual disintegration of the cluster, f will decrease continuously and the dimensions and shapes of the orbits of the stars inside the cluster will change correspondingly. The results are given in Table III.

The clusters with $f > 1$ are all stable. Table III indicates that galactic rotation cannot have very much effect upon clusters whose mean density is at least twice that at which instability occurs; Jeans' analysis of 1922 is essentially correct for clusters with $f > 2$. Clusters with f values between 1 and 2 already begin to show the effects due to the rotation of our galaxy and should be slightly elongated along the ξ axis; we shall find below (see Table IV) that this elongation will be hardly perceptible until disintegration sets in.

The most interesting part of Table III is that where $f \leqq 1$. The table shows how the transition from $f = 1$ to $f = 0$ takes place. If f is practically equal to 1, λ_2 is a very small quantity and the exponential terms in (20) will amount to much more in the formula for ξ than in that for η; such a cluster will therefore tend to disintegrate along the ξ axis. The density of the cluster, and also f, will decrease continuously during the disintegration and the absolute value of λ_2 will increase rapidly; λ_2 becomes equal to -1.0 for $f = 0.4$ and the tendency for disintegration along the ξ axis will now disappear. The well known shearing effect due to the rotation of the galaxy begins to show and we find already, for $f = 0.1$, $\lambda_2 = -2.1$. It is worth noticing that p_2 does not vary greatly between $f = 0.95$ and $f = 0.01$.

We are now in a position to attack the problem of the disintegration of a fairly massive cluster, and we shall carry through the computations for a cluster that closely resembles the Taurus cluster. We start with the Hyades as we know them at present and will attempt to predict the future development of this cluster.

The Taurus cluster has roughly the shape of an ellipsoid of revolution with $a = b = 6.0$ parsecs and $c = 4.0$ parsecs. We assume its total mass to be equal to 150 $\odot$, an estimate which is naturally quite uncertain. But if we study in detail the rate of disintegration of this cluster it will afterward be an easy matter to predict the probable rates of disintegration for clusters having a somewhat different mass or different dimensions. We assume, as we have mentioned above, that the cluster can be represented at each particular stage of its development by an ellipsoid of uniform density. At the beginning $\alpha + \beta_1$ will be positive and the cluster therefore stable. The velocities of the stars in the cluster will increase gradually through the influence of random encounters with passing stars. The sizes of the orbits that these stars describe in the cluster will also increase gradually and so will the volume occupied by the cluster. It is clear that the β's will decrease gradually. Let us consider the influence of random encounters over an interval of 5×10^8 years, which is so short that we can consider the β's as constants throughout this interval without committing any serious errors. The relations between the initial values $\overline{\xi_o^2}$, $\overline{\eta_o^2}$ and $\overline{\zeta_o^2}$ and the values of $\overline{\xi^2}$, $\overline{\eta^2}$ and $\overline{\zeta^2}$ after an interval t has elapsed are given by the equations:

$$\overline{\xi^2} = \overline{\xi_o^2} + \left[\frac{1}{(\lambda_1 p_1 - \lambda_2 p_2)^2} + \frac{\lambda_1^2 + \lambda_2^2}{2(\lambda_2 p_1 - \lambda_1 p_2)^2}\right] \times At$$

$$\overline{\eta^2} = \overline{\eta_o^2}\left[\frac{\lambda_1^2 + \lambda_2^2}{2(\lambda_1 p_1 - \lambda_2 p_2)^2} + \frac{\lambda_1^2\lambda_2^2}{(\lambda_2 p_1 - \lambda_1 p_2)^2}\right] \times At \tag{23}$$

$$\overline{\zeta^2} = \overline{\zeta_o^2} + \frac{A}{2p_3^2}t$$

where the λ's and p's have been computed according to (19) from the initial values of β_1, β_2, and β_3. The method of derivation for (23) agrees closely with that given in the preceding section and there seems no point in giving it here in detail; we have, according to formula (15), $A = 1.5 \times 10^{-9}$.

The interval t must be chosen so small that the same values of β can be used throughout the first step, but after an interval t has elapsed we must compute new values for β_1, β_2, and β_3. It is convenient for our numerical analysis to disregard all changes in the β's during the first interval of time and have them change abruptly to the new values at the end of this interval. We can then compute new values for the p's and λ's to be used for the second interval.

In reality the changes in the p's and λ's will be slow and gradual, and our cluster can to a certain extent be compared with a harmonic oscillator whose binding is gradually being loosened and for which we wish to compute the change in amplitude over a large number of oscillations. Our problem is a bit more complicated because the amplitudes of our oscillators determine the strength of the binding. The simple oscillator problem has been solved in physics by making use of adiabatic invariants.[8] I believe that Rosseland [9] has been the first to point out how convenient it is to study the adiabatic changes in our galaxy by methods similar to those that have been developed with special reference to the quantum theory. We can use the principle of adiabatic invariance for computing the changes in the dimensions of the cluster at the conclusion of each step in our analysis. But the difficulty is that we cannot make use of this principle after the cluster has become unstable, as it applies only to strictly periodical solutions. We shall have to consider unstable clusters as well as stable ones and we therefore suggest the following method that gives for a stable cluster results that agree closely with those computed on the basis of the principle of adiabatic invariance and that can also be used after the cluster has become unstable.

We shall call the new values for the p's and λ's, computed from the values of β_1, β_2 and β_3 at the beginning of the second interval, π_1, π_2, π_3, l_1, and l_2. We imagine a star in the cluster, moving, shortly before the end of the first interval of time, in an orbit characterized by the initial values p_1, p_2, p_3, λ_1, λ_2. At the beginning of the second interval the constants of the orbits are supposed to change abruptly to π_1, π_2, π_3, l_1, and l_2, and we must now compute the change in the dimensions of the orbits, assuming that at the epoch t the positions and velocities in both orbits are identical. If we indicate the values of $\overline{\xi^2}$, $\overline{\eta^2}$ and $\overline{\zeta^2}$, computed from (23) at the end of the interval of time t, by $\overline{\xi'^2}$, $\overline{\eta'^2}$ and $\overline{\zeta'^2}$, we find the actual values $\overline{\xi_1^2}$, $\overline{\eta_1^2}$ and $\overline{\zeta_1^2}$, computed after applying the "adiabatic correction" from the formulae:

$$\begin{aligned}
\overline{\xi_1^2} &= P_1[\overline{\eta'^2} - \lambda_2^2\overline{\xi'^2}] + P_2[\lambda_1^2\overline{\xi'^2} - \overline{\eta'^2}] \\
\overline{\eta_1^2} &= l_1^2P_1[\overline{\eta'^2} - \lambda_2^2\overline{\xi'^2}] + l_2^2P_2[\lambda_1^2\overline{\xi'^2} - \overline{\eta'^2}] \\
\overline{\zeta_1^2} &= \frac{\pi_3^2 + p_3^2}{2\pi_3^2}\overline{\zeta'^2}
\end{aligned} \tag{24}$$

where P_1 and P_2 are constants that depend on p_1, p_2, λ_1, λ_2, π_1, π_2, l_1, and l_2. As a check upon the last formula we put $\pi_3 = (1 - \epsilon)p_3$ and take ϵ to be a small quantity. The formula reduces to:

$$\overline{\zeta_1^2} = (1 + \epsilon)\overline{\zeta'^2}$$

which agrees with what we would have found if we had made use of the principle of adiabatic invariance.

$\overline{\xi_1^2}$, $\overline{\eta_1^2}$, and $\overline{\zeta_1^2}$ having been obtained, we are ready for the next step. We compute new values for a, b, and c, and the corresponding β's, and from (19) new values for the p's and λ's. Formula (23) enables us to compute the effect of random encounters over the next interval of 5×10^8 years; we find again new values for $\overline{\xi'^2}$, $\overline{\eta'^2}$, and $\overline{\zeta'^2}$ and can now compute from (24) what the probable values of $\overline{\xi_2^2}$, $\overline{\eta_2^2}$, and $\overline{\zeta_2^2}$ will be at the end of the second interval of 5×10^8 years. We have checked each successive step by using different values for the interval of time, so that no errors of any importance have been introduced by our numerical method of analysis.

We follow this procedure until we reach the point where $\alpha + \beta_1$ equals zero and is finally negative. The problem now becomes more complicated, and by applying again the method used in the derivation of (24) we can compute the changes in size and shape of the cluster in three or four steps, over perhaps a total interval of 4×10^8 years. The computations become however more and more approximate, because we now find appreciable changes in the values of the β's during one "oscillation" of the cluster. The process of averaging, used in the derivation of (23) and (24), loses its meaning. The data in Table IV have been given only up to the point where we can be certain that they are essentially correct. We know that the cluster will continue to disintegrate rapidly beyond this point, but accurate analysis is not easily possible.

It appears that random encounters play only a minor rôle after the cluster becomes unstable. This is not surprising; for the stars moving in an unstable cluster will after a fairly short interval of time move further and further away from the center of the cluster. But this will mean again an increase in the total volume and a corresponding rapid decrease in the β's. Random encounters serve to loosen the cluster while it is still stable, and are of the utmost importance until $\alpha + \beta_1$ becomes equal to zero; they bring the cluster to the point where its mean density falls below the critical density. But from there on, the cluster proceeds independently to disintegrate at a rapid rate; computation shows that random encounters are thereafter unimportant.

The results of the computations for the Hyades are given in a concise form in Table IV, where we have tabulated the values of a, b, and c, and the corresponding values for β_1, β_2, and β_3, and the average density, in solar masses per cubic parsec, for several epochs in the coming 3×10^9 years.

TABLE IV

Disintegration of the Taurus Cluster

t (in years)	a	b (in parsecs)	c	β_1	β_2 (c. g. s.)	β_3	Average density (in $\odot$/pc^3)
0.0×10^9....	6.0	6.0	4.0	3.92×10^{-30}	3.92×10^{-30}	6.20×10^{-30}	0.25
0.5×10^9....	6.3	6.2	4.1	3.50	3.57	5.76	.22
1.0×10^9....	6.6	6.4	4.2	3.08	3.21	5.23	.20
1.5×10^9....	7.0	6.6	4.3	2.66	2.86	4.70	.18
2.0×10^9....	7.5	6.9	4.4	2.25	2.52	4.16	.16
2.4×10^9....	8.7	7.1	4.5	1.64	2.15	3.56	.13
2.5×10^9....	9.9	7.2	4.5	1.25	1.83	3.31	.11
2.6×10^9....	21.8	11.2	4.6	0.18	0.46	1.21	.03

Table IV shows that the cluster will only change slightly during the next 2×10^9 years, but that it will become unstable shortly after this. It passes the point where the mean density becomes equal to the critical density after 2.3×10^9 years. The dimensions of the cluster will change rapidly from then on, and after 3×10^9 years the stars will have spread over such a large volume of space that the cluster as such will no longer exist. We have not carried through the computations beyond $t = 2.6 \times 10^9$ years, as the taking of averages becomes meaningless when the dimensions of the cluster change considerably in a short interval of time. We can, however, predict that a and b will continue to grow rapidly and that c will remain practically constant during the next few intervals of 10^8 years. λ_2 will have larger and larger negative values, but it is important to notice that there will be no marked elongation in the direction of the η axis (therefore no marked influence of the shearing effect due to the rotation of the galaxy) until the galactic dimensions of the cluster exceed 100 parsecs.

The value of p_2 determines approximately the rate of growth of the cluster. We find $p_2 = 0.5 \times 10^{-15}$ at the end of the last step in our computations, which means that the cluster will again double its galactic dimensions in less than 10^8 years. It is of interest to compute the average velocity of recession of the cluster members, or, as we may call it, the average galactic K-effect, that would be measured by an observer situated at the center of the exploding cluster. A rate of growth of 15 parsecs in 10^8 years corresponds to a linear velocity of recession of only 0.15 km/sec, and this K-effect would at no point in the development of the cluster exceed a few tenths of a kilometer per second.

We can summarize the results of the computations leading to Table IV as follows:

1. A cluster with a radius of the order of 6 parsecs and a mean density equal to twice the critical density will disintegrate in 3×10^9 years;

2. The cluster will become highly flattened with respect to the plane of the galaxy in the process of disintegration;

3. The cluster should be slightly elongated in the direction of the ξ axis shortly after it has become unstable, and some elongation along the η axis will occur in the last stages of disintegration. We would hardly say that the cluster would still exist by the time that a marked elongation along the η axis becomes observable;

4. The rate of expansion of the cluster will not be more than a few tenths of a kilometer per second at any stage of its development.

5. *The Moving Clusters in Our Galaxy.* — We know of several moving clusters in the neighborhood of our sun and we shall now discuss their shapes and dimensions in the light of the analysis of the preceding sections.

a. THE PLEIADES: Hertzsprung's [10] investigations of the Pleiades have made this cluster one of the best known in our galaxy. The total mass of the cluster is probably equal to 250 solar masses and assuming its distance to be 150 parsecs, the cluster has a radius of approximately 3 parsecs. The average density of $2.2\ \odot/\mathrm{pc}^3$ is equal to 24 times the critical density below which instability sets in. We considered in the preceding section the rate of disintegration for a loose cluster similar to the Taurus cluster and it seems of interest to repeat these computations for the Pleiades. The rate of development is much slower: 10^{10} years from now, the radius of the cluster will have increased to 4.0 parsecs and its average density will then be only 0.94 $\odot/pc^3$. After 2×10^{10} years the radius will be equal to 5.8 parsecs and the density equal to 0.31 $\odot/pc^3$. The cluster is then still sufficiently stable to last for another 5×10^9 years, but it will be disintegrated completely within 3×10^{10} years. Galactic forces will have hardly any influence on the shape of clusters like the Pleiades, as we know them at present, and any deviations from a spherical shape will probably be due to a rotation of the cluster around its own axis.

b. PRAESEPE AND h AND χ PERSEI: The total mass of Praesepe is probably of the order of 400 solar masses (Klein Wassink[11] has found 200 members of the cluster) and its radius equal to 5 parsecs. The average density should therefore be equal to 0.75 $\odot/pc^3$, which is $8 \times$ the critical density. Much less is known about the twin clusters h and χ Persei. Their distance is probably of the order of 1500 parsecs and the corresponding linear diameters equal to 16 parsecs each. We know very little about their total mass, but 1500 solar masses should be a fair estimate for each cluster. If these guesses are approximately correct, the average density will be equal to 0.70 $\odot/pc^3$. The density in all three clusters is well above the critical value and their complete disintegration will probably be a matter of 1.5×10^{10} years.

c. THE TAURUS CLUSTER: The future development of the Hyades has been investigated in detail in the preceding section and Table IV gives the results of the analysis. We shall discuss the outlying members under (e) and shall limit ourselves now to the restricted cluster. It is probable that it is slightly larger and more massive than we have previously assumed. In a recent investigation Wilson[12] finds 136 members inside an ellipsoid with a principal axis of 9 parsecs. He shows that the cluster is flattened with respect to the galaxy, while the principal axis points very nearly toward the galactic center. It is tempting to conclude from this that the average density of the cluster is close to the critical density, but we must keep in mind that Wilson does not consider his results on the shape of the cluster as definite. A still unpublished investigation of the Kapteyn Astronomical Laboratory shows that the number of faint cluster members is rather small and our estimated density of 0.25 $\odot/pc^3$ should be about right. The Taurus cluster is the best example of a cluster that will have disappeared as such in the fairly short interval of 3×10^9 years.

d. The nucleus of the Ursa Major cluster: An inspection of Rasmuson's space diagram for the Ursa Major cluster [13] shows that we have to distinguish between the eight or nine stars that form a dense nucleus and the more scattered members of the cluster. I doubt if it means anything to speak about the "shape" of a cluster of nine stars and all we can do is to represent the cluster by a sphere with a radius of 3 parsecs and assume its total mass to be equal to 20 solar masses. The average density of the cluster should then be equal to 0.18 $\odot/pc^3$, indicating that the nucleus of the Ursa Major cluster will probably disintegrate in less than 2×10^9 years.

e. The extended moving clusters and star streams: Apart from the outlying members of the Taurus and Ursa Major cluster we have evidence of the existence of extended moving clusters in our galactic system in Eddington's moving cluster in Perseus and in the Scorpio-Centaurus and Orion clusters discovered by Kapteyn. Recent investigations by Gyllenberg [14] and Wilson [12] indicate the existence of a considerable excess of fairly bright stars, chiefly in the northern hemisphere, with space motions agreeing closely, both in direction and in amount, with the absolute motion of the Taurus cluster. Hertzsprung [15] was the first to point out the existence of a number of bright stars, including Sirius, that appear to share the motion of the Ursa Major cluster. In 1923 Strömberg [16] pointed out that the space motions of the A stars indicate the existence of extended Ursa Major and Taurus streams. Strömberg's suggestion of the existence of these streams has recently been taken up by several astronomers, notably by Raymond and Wilson [17] and by Mohr.[18] The known members of the Perseus, Scorpio-Centaurus, and Orion clusters have all been found among the bright stars of early type, but it seems probable that fainter stars of later types will be added when additional proper motion and radial velocity data become available. It is of interest that Pannekoek [19] shows in his investigations based on the Henry Draper Catalogue that concentrations of A stars are present near the centers of the Perseus and Orion clusters and possibly in the Scorpio-Centaurus cluster.

It is a difficult matter to decide whether the available observational evidence is sufficient to prove the existence of extended moving clusters in our galaxy. The observed velocities are affected by large errors of observation, and we cannot see clearly for how many stars the motions will agree closely with those of the members of the cluster. A concentration of the apices of the velocities for stars outside the cluster around the velocity apex of the cluster itself can be interpreted in two ways. It may mean that the scattered stars are outlying members of the cluster; but we can just as well consider it as indicating that the cluster is only a minor unit in a giant star stream. The scattered stars, with velocities approximately equal in amount and direction to those of the stars belonging to the cluster, are members of the same stream that happen to be within a hundred parsecs of the center of the cluster. The lack of concentration of the group members around the center of the cluster supports the latter point of view.

Further difficulties arise on account of the effects of observational selection. Sufficiently accurate data on stellar motions are only available for stars brighter than the sixth magnitude, and it may very well be that the published dimensions of some of the extended clusters have no real meaning and that they simply indicate the distance for which our catalogues are reasonably complete. It is for instance very suspicious that the published values for the dimensions of the extended clusters indicate that they would all be more or less spherical. Wilson finds that the 200 stars of the Taurus group are largely within a sphere with a radius of approximately sixty parsecs. Bertaud [20] has recently shown that the extended Ursa Major cluster with eighty members is also spherical and that it has a radius of seventy five parsecs. Other clusters, like the Scorpio-Centaurus and Perseus clusters, are slightly flattened, but the diameter perpendicular to the galactic plane is in no case less than half the galactic diameter. The dimensions of these clusters come dangerously close to the values that we would expect to find if we had selected those members of much more extended star streams that could occur in our catalogues of radial velocities and proper motions.

We can look at the same problem from a different angle and notice how extremely improbable it is that we would find in our galaxy several, more or less spherical, extended moving clusters with a small average density and with a diameter not exceeding 150 parsecs. The computations of Section 4 have shown that we need not be surprised to find a very *flattened* cluster with more or less circular dimensions in a plane parallel to the galactic plane, and we have proved that this cluster should not necessarily exhibit the marked elongation along the η axis that we would have expected on account of the shearing effect of galactic rotation. Such a cluster could have evolved within the past 5×10^8 years from a more compact cluster with a density slightly exceeding the critical density, and should be disintegrating rapidly. The considerable degree of flattening indicated by the computations that lead to Table IV is however not observed in any of the extended moving clusters and we are now faced with the difficulty of explaining the absence of such flattening.

I doubt very much whether occasional close encounters with dense clusters or star clouds can explain the considerable thickness of these clusters. For, if this were so, the members of the original cluster should scatter rapidly through our galaxy, and the chances that rare and nearly spherical clusters would be observed seem very small. Local irregularities in our galaxy might introduce temporarily considerable deviations from the simple state of affairs where the ζ component of the galactic force is assumed to be proportional to ζ for points within 200 parsecs from the galactic plane. A few approximate calculations show that such deviations will affect the ζ dimensions of the cluster somewhat, because of the fact that the period of one single oscillation in ζ would no longer be the same for all members of the cluster. But again we would hardly expect that the observed clusters would show such a very slight degree of flattening.

We would probably have to revise our present conceptions of galactic structure considerably, if we were able to prove beyond doubt from observation that the extended moving clusters show only little flattening. I believe that their mere existence would then force us to accept one of two alternatives. Either we would have to consider it as a proof for what appears to be an impossibly short time-scale for the development of our galaxy, or we would have to discard *a priori* any theory of galactic structure that predicts the existence of a massive galactic nucleus. If tidal forces similar to those predicted by the theory of galactic rotation do exist, a very rare cluster with a radius of the order of 100 parsecs should disintegrate rapidly, and the five extended clusters that are now observed should scatter over a much larger volume in the coming 3×10^8 years. There would probably be no clusters to take their place at such short notice, and under such a theory a value of 3×10^9 years for the age of our galaxy would seem improbably large. We can, of course, avoid the taking place of such rapid developments if we exclude the presence of tidal forces due to the action of a galactic nucleus. This would practically mean a return to the Kapteyn Universe, for which Jeans found a very slow rate of expansion for extended moving clusters,[1] but I doubt if any astronomer of today would care to go this far.

The most probable explanation of our difficulties lies however in the incorrectness of our initial assumption. The extended moving clusters have all approximately the same size and, as their galactic dimensions are equal to what we would expect from the unavoidable effects of observational selection, it seems natural to consider them as samples of much more extended streams. Lindblad has shown that the theory of galactic rotation is able to account for the general phenomena of star streaming,[6] but this theory gives, of course, only a very schematical picture of what actually happens in our complicated galaxy and purposely neglects all local irregularities. It is therefore not surprising that we find upon closer examination indications of several intermingling star streams. We may compare our own galaxy with some of the larger spiral nebulae. The changes in the radial velocity that are observed with increasing distance from the center of the nebula can readily be explained as being due to the action of a massive nucleus. The large scale photographs of these nebulae show that this "galactic rotation" can only give a rough first approximation to the state of motion in the nebulae and that local star streams are probably present within them.

If we accept as correct the explanation that the extended moving clusters are samples of much larger star streams, the connection between the observed dense nuclei and the stars occurring in the corresponding streams should be very loose; for these nuclei must then be considered as separate units partaking in the stream motion, that is, as minor "knots" in the "spiral arms" and not necessarily as the remnants of a nucleus from which the whole spiral arm has evolved.

If we consider this point of view correct, the five extended moving clusters fit in very well with the picture of galactic structure offered by the theory of galactic rotation. They do not, however, give a great deal of information as to the probable age of our galaxy and can hardly be expected to do so until more data about their true dimensions and dispersions in random velocities become available. At present we can only obtain an upper limit for the probable age of our galaxy. Formula (15) in Section 3 gave average values of the velocity components in the ξ, η, and ζ direction acquired by the members of the cluster through the influence of random encounters with passing stars. If we substitute for t a value corresponding to 10^{11} years the value of the square root out of $\sum \Delta_\xi^2$ corresponds to a velocity of 0.7 km/sec.*

The observed velocity dispersions are small. Miller[21] finds for instance from a recent discussion of the peculiar motions in the Scorpio-Centaurus cluster that the mean peculiar motion of its members does not exceed ± 1.0 km/sec. The smallness of the observed dispersion in velocity indicates that the extended star streams cannot have existed for more than 10^{11} years, and shows that an age of 10^{13} years is wholly out of the question. The results of Table I indicate however that this upper limit for the age can be considerably reduced. In spite of the fact that we have chosen our initial conditions so as to make the extended moving cluster, with low average density, as stable as it possibly can be, we see from Table I that this cluster

* It may seem surprising that this quantity is so much smaller than the one found by Jeans (M.N., **74**, 109, 1913). An average deflection of 1° in 3.2×10^9 years, for a cluster whose members have equal and parallel velocities of 40 km/sec, corresponds to a velocity of 0.7 km/sec. The reason for this apparent discrepancy lies largely in the fact that Jeans assumed a very high average density of slightly more than one solar mass per cubic parsec, whereas we have used a value of $0.03 \odot / pc^3$, which is slightly less than the density of the visible stars in the neighborhood of the sun found by Oort (B.A.N., **238**).

will disintegrate very rapidly. The galactic tidal forces will disperse the members of any loose group of parallel moving stars in an interval of time that is, cosmically speaking, very short, and even if we grant that the initial dimensions of the group may have been as large as one or two thousand parsecs, we shall find the stars scattered far apart in an interval of a few times 10^{10} years. The average density in the cluster should be so small by that time that we could never discover the group from a set of proper motions and radial velocities that would cover such a small volume of space as that covered by our present data. The existence of at least one or two extended moving clusters or star streams is established, and it seems safe to assert that their average densities will drop to at least one tenth of their present values in the next 2 or 3×10^{10} years; in fact the rate of development may be even more rapid. The presence of such loose clusters or streams in our galaxy is, by itself, one of the finest bits of evidence in favor of the short time-scale that we possess to date. Our knowledge concerning the true structure of these clusters is however very incomplete, and while we are able to show that they indicate an upper limit of 2×10^{10} years for the age of our galaxy, we should keep in mind that smaller values are acceptable.

6. *Miscellania.* — We shall discuss briefly a few problems of general interest that lie somewhat outside the main theme of the present paper and show how the type of analysis that we have worked out may contribute toward their solution.

a. The local cluster: The problem of the stability of our Local Cluster has, from the theoretical point of view, much in common with that of the stability of moving clusters with an average density equal to, or slightly exceeding, the critical density. The steep density gradient, which presumably exists in the Local Cluster, will make the mathematical treatment of the problem slightly more involved, but apart from this our approximate discussion should be sufficient, provided that the radius of the Local Cluster has a value not exceeding one tenth of its distance to the galactic center. Lack of observational data is again the chief stumbling block. We shall first have to establish the existence of the Local Cluster beyond doubt, but as data on the magnitudes and spectra of faint stars are now accumulating rapidly, it will presumably be a matter of not more than five or ten years before this point can be settled. A knowledge of its approximate total mass, dimensions, and shape will be a second prerequisite to a successful discussion of its stability, for it is obvious that the theory for a small definite cluster would differ considerably from that for a large elongated cluster that might be part of a spiral arm. In addition we shall need a better determination of the value of a. If a should prove to be equal to only one half of the value used here, the Local Cluster would at least be partly stable.

But if, on the other hand, we should have to multiply the assumed value of α by a factor two, the Local Cluster would be disintegrating rapidly.

We have not yet the observational foundation needed for the theoretical discussion, but even at this stage it may be of interest to point out that an observer near the center of a rapidly disintegrating Local Cluster would find a considerable K-effect. Imagine a Local Cluster with a radius of 300 parsecs, with an average density so far below the critical one that the value of p_2 in formula (20) of Section 4 would be 0.3×10^{-15}. The Local Cluster would then double its radius in less than 10^8 years and an average K-effect amounting to 3 km/sec would be observed. A purely dynamical interpretation of the K-effect should therefore not be rejected off hand. The fact that it is shown most conspicuously by the bright early type stars known to aggregate in loose, extended moving clusters suggests that these groups were originally held together in a much larger single cluster that must have split into several smaller ones in the past 5×10^8 years. Such a dynamical interpretation would explain immediately Plaskett's striking observational result that the apparently faint B stars show no evidence of a K-effect [22]; these faint stars should then be considered as field stars not belonging to the Local Cluster. Our interpretation of the K-effect should, however, be ruled out completely, if it appears that the average space density in low galactic latitudes decreases with increasing distance from the sun for stars which do not as a group show the K-effect.

We have neglected the influence of a Local Cluster completely in our analysis of known moving clusters. The Local Cluster, the Scutum Cloud, and similar galactic condensations may have some influence upon the rates of development and the changes in shape of clusters in their vicinity. We replaced the galaxy in our analysis by a simplified model that could be described completely by two constants, α and n. In addition to this we assumed that all stars and clusters were describing circular orbits around the center and that the time that has elapsed since the beginning of our galaxy, or at least since the occurrence of a catastrophe in its regular development, has been so short that we could neglect the slow but systematic changes in the values of α and n that must occur on account of an unavoidable slow expansion of our galaxy as a whole[23]. It should, however, be possible to adapt our type of analysis to our changing conceptions of galactic structure, and I hope that the methods of attack which I have outlined may be of more lasting value than the results obtained by applying them to the galaxy that represents modern observations.

b. Supergalaxies: Shapley has shown that the grouping of galaxies is one of the most striking features of the structure of our universe. It will be of interest to investigate how stable the observed configurations will probably be and whether it is possible that disintegration on a large scale may actually take place in the universe of galaxies. As an illustration we shall discuss the degree of stability of the Magellanic Clouds. Both clouds can be considered as satellites of our galaxy, as their distance from the galactic center — being of the order of 3×10^4 parsecs — is comparable with the dimensions of our galaxy. Oort [3] has recently estimated the total masses of both the Large and Small Clouds and found them equal to 2×10^{42} g and 6×10^{41} g, respectively. The radius of the Large Cloud is 1700 parsecs; that of the Small Cloud, 900 parsecs, and the corresponding densities are 3×10^{-24} g/cm^3 and 6×10^{-24} g/cm^3. The value of α for circular motion at the distance of the Magellanic Clouds is approximately equal to $\alpha' = -8 \times 10^{-32}$ c.g.s., and the corresponding critical density will only be 3×10^{-25} g/cm^3. The Clouds are therefore probably stable configurations and the forces due to our galaxy should play only a secondary rôle. Outlying objects, such as clusters, will only be retained as permanent members of one of the Clouds, if their distance from the center of each Cloud does not exceed 2.5 times the observed radius. It will be possible to discuss the stability of certain configurations in the Clouds, at least qualitatively, by similar methods.

The appearance of spiral structure in a great many external galaxies has lead to extensive dynamical studies and speculations, but as yet no satisfactory theory of the formation of spiral arms has been given. We have limited ourselves in our attack upon the problem of the development of galactic condensations to the special case where the dimensions of the condensation are small compared with the distance of the condensation from the galactic center. Difficulties of an entirely different order of magnitude arise if we wish to go beyond this limitation, and I doubt whether it will be possible to attack the problem of spiral structure by extending our analysis.

Summary. — We summarize briefly the contents of each section separately:

Introduction: The main problem of the present investigation is outlined, and reasons for repeating and extending Jeans' investigation of 1922 are given. The analogy between our problem and that of the disintegration of meteor swarms (Luc. Picart, 1892) is stressed.

Section 1: The equations of motion for a star moving inside a cluster are set up, and the probable numerical values for the constants involved are discussed.

Section 2: Following Hill and Picart, a general criterion for the stability of a cluster is found. Clusters in the neighborhood of the sun with an average density

smaller than 0.093 ☉/pc^3 will probably be unstable. A formula for the minimum outward velocity that a star should have to escape from a stable cluster is derived.

Section 3: The equations of motion are solved for the special case that the attraction of the cluster upon its members can be neglected; Table I shows what changes in the dimensions occur through the combined influence of random encounters and galactic tidal forces. If we accept the consequences of the theory of galactic rotation, Jeans' theorem that a fast moving cluster will be flattened in its direction of motion no longer holds.

Section 4: The equations of motion are solved for a massive cluster that can be considered as an ellipsoid of uniform density with three different axes. Table II gives the critical densities for clusters flattened with respect to the galactic plane. Disintegration along the ξ axis (in the direction toward and away from the galactic center) is indicated for a cluster with an average density equal to the critical one. Table III shows how the transition to the well known shearing effect, arising through the rotation of our galaxy, takes place.

The future development of the Taurus cluster is studied in detail, and Table IV shows the results of the computation. Disintegration will take place inside 3×10^9 years. The cluster will become highly flattened in a direction perpendicular to the galactic plane, but no pronounced elongation in the galactic plane will be observed. The rate of growth of the cluster does not exceed a few tenths of a kilometer per second in any stage of its development.

Section 5: The observed shapes and dimensions of several moving clusters are discussed. The Pleiades have an average density of twenty four times the critical density, and their disintegration will be completed in 3×10^{10} years. The Hyades and the nucleus of the Ursa Major cluster will continue to exist for at least 2×10^9 years.

The nature of the observed extended moving clusters or star streams in our galaxy is discussed, and a critical summary of the available data of observation is given. The analysis shows that it is impossible to account for the presence of very rare, more or less spherical, moving clusters unless we either accept an age of 3×10^8 years for these clusters or show that the massive galactic nucleus, indicated by the theory of galactic rotation, does not exist. The probable solution of the problem lies in the fact that we have only observed small samples of giant star streams through the influence of observational selection. 2×10^{10} years is indicated as a reasonable upper limit for the age of our galaxy.

Section 6: The problem of the stability of a possible Local Cluster is discussed, and a tentative explanation of the K-effect is given.

The stability of the Magellanic Clouds, considered as satellites of our galaxy, is investigated. Spiral structure is discussed, but we find that our analysis in its present form cannot contribute to the solution of the problem of the development of spiral arms.

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