following short table gives the values of the maximum and the zero, and the corresponding values of η :—

TABLE VIII

	Maximum value of ψ	First zero of ψ
$\omega_{\frac{3}{2}}=264\cdot7769$	$\psi=1\cdot15339$ at $\eta=1\cdot24248$	at $\eta=0\cdot356895$
$\omega_{\frac{3}{2}}=529\cdot5538$	$\psi=1\cdot69110$ at $\eta=1\cdot67456$	at $\eta=0\cdot706227$

The writer desires to acknowledge that these computations were carried out at the request of Professor E. A. Milne, with the aid of a grant made for the purpose by the Department of Scientific and Industrial Research.

THE MINIMUM PRESSURE OF A DEGENERATE ELECTRON GAS.

Edmund C. Stoner, Ph.D.

(Communicated by Sir A. S. Eddington)

A knowledge of the variation of pressure with density of matter at very high densities is essential in the investigation of stellar structure. Very high densities are known to occur in white dwarfs, and recent theoretical developments suggest that they may also occur in the central regions of more ordinary stars.* At these densities matter may be regarded as an aggregate of nuclei and electrons, the electrons being treated either as free, or as constituting, with the nuclei, a gigantic composite molecule. In either case, the material pressure is almost entirely due to the electrons, and calculations of the pressure may be made by treating the electrons as a gas subject to the Fermi-Dirac statistics. The pressure depends on the temperature, but for the range of electron concentrations under consideration, unless the temperature is very high, it does not differ greatly from the value at absolute zero. The pressures calculated for absolute zero temperature (corresponding to *complete* degeneracy of the electron gas) are, of course, minimum pressures. Provided that the electron concentration, n, is not too large, the pressure varies as $n^{5/3}$. This holds as long as the momenta are such that the relativity change of mass with velocity may be neglected. At higher concentrations, this relativity effect has to be taken into account. Eventually the pressure varies as $n^{4/3}$. Application has been made of the equations of state corresponding to these two limits,*† but concentrations which are of astrophysical interest (corresponding roughly to densities of the order 10^5 to 10^8) happen to fall in the range where neither equation can be strictly applied—possibly *because* this is a transition region. It has seemed worth while, therefore, to investigate how the pressure varies over the whole range from ordinary to

* E. A. Milne, *M.N.*, **91**, 4, 1930.
† S. Chandrasekhar, *ibid.*, **91**, 456, 1931.

relativistic degeneracy. In a previous paper * on the limiting density of white dwarfs this transition from ordinary to relativistic degeneracy was quantitatively discussed, but no explicit calculations of the pressures were made; though certain functions closely connected with the pressure were computed with a view to the particular application. Reference may be made to this paper for a fuller discussion of the astrophysical application, and to two other papers in the same series; † also to another paper by Chandrasekhar, ‡ as well as to the papers mentioned above. The present paper is concerned simply with the computation and the results for the minimum pressure and kinetic energy density of a degenerate electron gas; and an attempt is made to give the results in such a form that application of them can readily be made.

Derivation of Expressions for Pressure and Energy

The following symbols will be used :—

n = number of electrons per unit volume.
P = pressure (due to electrons).
E = kinetic energy of electrons per unit volume.
ϵ = kinetic energy of electron.
p = momentum of electron.
p_0 = maximum momentum.

In the completely degenerate state there are two electrons in each cell of the 6-dimensional phase space (of 6-volume h^3), 2 being the weight factor. The cells corresponding to the lowest possible energy will be filled, so that the total number of electrons, N, in a volume V is given by

$$N = (V/h^3)\int_0^{p_0} 2 \times 4\pi p^2 dp,$$

$$n = N/V = \frac{8\pi}{h^3}\int_0^{p_0} p^2 dp = \frac{8\pi p_0^{\,3}}{h^3}, \qquad (1)$$

$$p_0 = \left(\frac{3h^3 n}{8\pi}\right)^{1/3}. \qquad (1')$$

Ultimately values for E and P in terms of n are required; the simplest procedure seems to be to obtain expressions for E and P first in terms of p, and then, by integration, in terms of p_0.

$$\epsilon = (m - m_0)c^2$$

$$= m_0 c^2\left(\frac{1}{\sqrt{1-\beta^2}} - 1\right), \qquad (2)$$

$$p^2 = (m\beta c)^2 = m_0^{\,2} c^2\left(\frac{1}{1-\beta^2} - 1\right). \qquad (3)$$

* E. C. Stoner, *Phil. Mag.*, **9**, 944, 1930.
† E. C. Stoner, *ibid.*, **7**, 63, 1929; E. C. Stoner and F. Tyler, *ibid.*, **11**, 986, 1931.
‡ S. Chandrasekhar, *ibid.*, **11**, 592, 1931.

The expressions are simplified by using x and y, defined by

$$\left.\begin{aligned} y &= \frac{p}{m_0 c} \\ x &= \frac{p_0}{m_0 c} \end{aligned}\right\}. \tag{4}$$

From (1′) and (4),

$$\left.\begin{aligned} x &= \frac{h}{m_0 c}\left(\frac{3}{8\pi}\right)^{1/3} n^{1/3} = \alpha n^{1/3} \\ n &= \left(\frac{m_0 c}{h}\right)^3 \left(\frac{8\pi}{3}\right) x^3 = \frac{1}{\alpha^3} x^3 \end{aligned}\right\}. \tag{5}$$

(3) may be written

$$y^2 = \frac{1}{1-\beta^2} - 1,$$

giving

$$\beta = \left(\frac{y^2}{1+y^2}\right)^{1/2}. \tag{6}$$

Substituting in (2),

$$\epsilon = m_0 c^2 \{(1+y^2)^{1/2} - 1\}. \tag{7}$$

The energy per unit volume,

$$\begin{aligned} E &= \frac{8\pi}{h^3}\int \epsilon p^2 dp \\ &= 8\pi\left(\frac{m_0 c}{h}\right)^3 m_0 c^2 \int_0^x y^2\{(1+y^2)^{1/2} - 1\} dy. \end{aligned}$$

Let

$$A = \frac{8\pi (m_0 c)^5}{3 m_0 h^3}. \tag{8}$$

Then

$$E = A \times 3\int_0^x y^2\{(1+y^2)^{1/2} - 1\} dy. \tag{9}$$

An expression will now be derived for the pressure. The ordinary transport of momentum result, $P = \frac{1}{3} \times v \times p \times n$, may be generalised to

$$\begin{aligned} P &= \tfrac{1}{3}\int \beta c p \, dn \\ &= \tfrac{1}{3}\int_0^{p_0} \beta c p \frac{8\pi p^2}{h^3} dp \\ &= \frac{8\pi (m_0 c)^5}{3 m_0 h^3}\int_0^x y^4 (1+y^2)^{-1/2} dy \\ &= A\int_0^x y^4 (1+y^2)^{-1/2} dy. \end{aligned} \tag{10}$$

Evaluation of Integrals

$$E/A = f_E(x) = 3\int_0^x y^2(1+y^2)^{1/2}dy - x^3. \tag{11}$$

Since

$$\frac{d}{dy}\{y(1+y^2)^{3/2}\} = 3y^2(1+y^2)^{1/2} + (1+y^2)^{3/2}$$
$$= 4y^2(1+y^2)^{1/2} + (1+y^2)^{1/2};$$

$$f_E(x) = \tfrac{3}{4}\left[x(1+x^2)^{3/2} - \int_0^x (1+y^2)^{1/2}dy\right] - x^3$$
$$= \tfrac{3}{8}[x(1+x^2)^{1/2}(1+2x^2) - \log\{x+(1+x^2)^{1/2}\}] - x^3.$$

Let

$$f_1(x) = x(1+x^2)^{1/2}(1+2x^2) - \log\{x+(1+x^2)^{1/2}\}. \tag{12}$$

Then

$$f_E(x) = \tfrac{3}{8}f_1(x) - x^3. \tag{13}$$

$$P/A = f_P(x) = \int_0^x y^4(1+y^2)^{-1/2}dy. \tag{14}$$

Since

$$\frac{d}{dy}\{y^3(1+y^2)^{1/2}\} = y^4(1+y^2)^{-1/2} + 3y^2(1+y^2)^{1/2},$$

$$f_P(x) = x^3(1+x^2)^{1/2} - \tfrac{3}{8}f_1(x). \tag{15}$$

Expansions for x *Small and Large*

For x small, a series expansion for $f_E(x)$ and $f_P(x)$ may be obtained by noting that the log term is equal to $\int(1+x^2)^{-1/2}dx$. For x large, the log term may be neglected.

$x << 1$

$$f_E(x) = \tfrac{3}{10}x^5 - \tfrac{3}{56}x^7 \cdots \tag{13'}$$

$$f_P(x) = \tfrac{1}{5}x^5 - \tfrac{1}{14}x^7 \cdots \tag{15'}$$

These expressions give values correct to within 0·5 per cent. for $x \lesseqqgtr 0{\cdot}4$.

$x >> 1$

$$f_E(x) = \tfrac{3}{4}x^2(x^2+1) - x^3 \cdots \tag{13''}$$

$$f_P(x) = \tfrac{1}{4}x^2(x^2-1) \cdots \tag{15''}$$

These expressions give values correct to within 0·1 per cent. for $x \gtreqqless 10$.

It may be noticed that for $x << 1$, since $P/E = f_P(x)/f_E(x)$,

$$P = \tfrac{2}{3}E, \tag{16}$$

as for an ordinary gas ; while for $x >> 1$,

$$P = \tfrac{1}{3}E, \tag{16'}$$

the relativity effect then being predominant, and the same relation holding as for radiation.

Computation

The values of $f_P(x)$ and $f_E(x)$ have been computed by means of the sets of expressions (13) and (15). An accuracy within 0·5 per cent., which is probably amply sufficient for any applications that are likely to be made, could be easily attained with 4-figure logs, except in the range $x = 0{\cdot}4 - 0{\cdot}7$, for which 7-figure logs were used.

Numerical Values

The results for the energy and pressure are

$$E = Af_E(x), \tag{17}$$

$$P = Af_P(x), \tag{18}$$

where

$$x = \frac{h}{m_0 c}\left(\frac{3}{8\pi}\right)^{1/3} n^{1/3} = a n^{1/3}, \tag{5}$$

$$A = \frac{8\pi(m_0 c)^5}{3 m_0 h^3}. \tag{8}$$

The following values were used, obtained from Birge's table :—

$$h = 6{\cdot}547 \times 10^{-27},$$
$$c = 2{\cdot}998 \times 10^{10},$$
$$m_0 = 9{\cdot}035 \times 10^{-28}.$$

The main uncertainty is in connection with m_0. The spectroscopic value has been used, as the conditions when there are electrons at high concentration resemble those in atoms rather than those under which the deflection experiments are carried out. (The latter lead to $8{\cdot}994 \times 10^{-28}$ as the most probable value of m_0.) The above values lead to the following numerical relations between x and n:—

$$\left.\begin{aligned} x &= 1{\cdot}1901 \times 10^{-10} n^{1/3} \\ \log x &= \tfrac{1}{3}\log n - 10 + 0{\cdot}07558 \\ n &= 5{\cdot}932 \times 10^{29} x^3 \\ \log n &= 3 \log x + 29{\cdot}7732 \\ a &= 1{\cdot}1901 \times 10^{-10} \\ \log a &= \overline{10}{\cdot}07558 \end{aligned}\right\}. \tag{19}$$

[With $m_0 = 8{\cdot}994 \times 10^{-28}$ the results are

$$\left.\begin{aligned} x &= 1{\cdot}195 \times 10^{-10} n^{1/3} \\ \log x &= \tfrac{1}{3}\log n - 10 + 0{\cdot}0775 \\ n &= 5{\cdot}854 \times 10^{29} x^3 \\ \log n &= 3 \log x + 29{\cdot}7675 \end{aligned}\right\}. \tag{19'}$$

In previous papers m_0 was taken as $9{\cdot}01 \times 10^{-28}$, giving $n = 5{\cdot}88 \times 10^{29} x^3$.]

The conditions approximate to ordinary degeneracy for $x << 1$, or $n << n'$, and to relativistic degeneracy for $x >> 1$, or $n >> n'$, where $n' = 5\cdot932 \times 10^{29}$.

Inserting numerical values,

$$\left.\begin{aligned} P &= Af_P(x) = 4\cdot818 \times 10^{23} f_P(x) \\ \log A &= 23\cdot6828 \end{aligned}\right\}. \tag{20}$$

To obtain the value of P (or E) for a given value of n, the simplest general procedure is to calculate the corresponding value of x, and then to find the value of $f_P(x)$ (or $f_E(x)$) by interpolation from the tables. It is, however, frequently convenient to use expressions in forms which show more clearly the relation to the limiting forms for low and high concentrations.

For

$$x << 1, \qquad f_P(x) = \tfrac{1}{5}x^5,$$
$$x >> 1, \qquad f_P(x) = \tfrac{1}{4}x^4,$$

$$P = A\{f_P(x)/(\tfrac{1}{5}x^5)\}(\tfrac{1}{5}x^5) = A\{f_P(x)/(\tfrac{1}{4}x^4)\}(\tfrac{1}{4}x^4), \tag{21}$$

$$= (\tfrac{1}{5}A\alpha^5)\{f_P(x)/(\tfrac{1}{5}x^5)\}n^{5/3} = (\tfrac{1}{4}A\alpha^4)\{f_P(x)/(\tfrac{1}{4}x^4)\}n^{4/3}, \tag{22}$$

$$= \kappa_1' n^{5/3} \times \{f_P(x)/(\tfrac{1}{5}x^5)\} = \kappa_2' n^{4/3}\{f_P(x)/(\tfrac{1}{4}x^4)\}, \tag{23}$$

where

$$\left.\begin{aligned} \kappa_1' &= 2\cdot300 \times 10^{-27} \\ \log \kappa_1' &= \overline{27}\cdot3617 \\ \kappa_2' &= 2\cdot415 \times 10^{-17} \\ \log \kappa_2' &= \overline{17}\cdot3830 \end{aligned}\right\}. \tag{24}$$

Thus for n small and large, the results are

$$n << 5\cdot932 \times 10^{29}, \qquad P = 2\cdot300 \times 10^{-27} n^{5/3}, \tag{25}$$

$$n >> 5\cdot932 \times 10^{29}, \qquad P = 2\cdot415 \times 10^{-17} n^{4/3}. \tag{26}$$

The values of $f_P(x)/\tfrac{1}{5}x^5$ are tabulated over the range of x for which it is greater than $0\cdot5$; and the values of $f_P(x)/\tfrac{1}{4}x^4$ over the whole range, as this function has other applications.

Expressions for E similar to those for P may readily be obtained if required. For low and high concentrations the results are

$$n << 5\cdot932 \times 10^{29}, \qquad E = 3\cdot450 \times 10^{-27} n^{5/3}, \tag{27}$$

$$n >> 5\cdot932 \times 10^{29}, \qquad E = 7\cdot245 \times 10^{-17} n^{4/3}. \tag{28}$$

Pressure as a Function of Density

In many astrophysical applications it is convenient to express the pressure as a function of the density. Let ρ be the density and μ the mean molecular weight per electron. (This differs slightly from the usual use of μ, but is more convenient in the present connection.) Let c_A be the fractional

TABLE I

Functions for Calculating Kinetic Energy per Unit Volume, E, *and Pressure*, P, *in Terms of Electron Concentration*, n

$$n = 5{\cdot}932 \times 10^{29} x^3,$$
$$P = 4{\cdot}818 \times 10^{23} f_P(x) = 2{\cdot}300 \times 10^{-27} n^{5/3} f_P(x) / (\tfrac{1}{5}x^5)$$
$$= 2{\cdot}415 \times 10^{-17} n^{4/3} f_P(x) / (\tfrac{1}{4}x^4),$$
$$E = 4{\cdot}818 \times 10^{23} f_E(x).$$

x	log n	$f_E(x)$	$f_P(x)$	$f_P(x)/(\frac{1}{5}x^5)$	$f_P(x)/(\frac{1}{4}x^4)$
·1	26·7732	$2{\cdot}99 \times 10^{-6}$	$1{\cdot}99 \times 10^{-6}$	·996	·0796
·2	27·6762	$9{\cdot}53 \times 10^{-5}$	$6{\cdot}31 \times 10^{-5}$	·984	·158
·3	28·2045	$7{\cdot}17 \times 10^{-4}$	$4{\cdot}70 \times 10^{-4}$	·967	·232
·4	·5795	$2{\cdot}99 \times 10^{-3}$	$1{\cdot}94 \times 10^{-3}$	·947	·303
·5	·8702	$8{\cdot}99 \times 10^{-3}$	$5{\cdot}76 \times 10^{-3}$	·921	·369
·6	29·1078	$2{\cdot}20 \times 10^{-2}$	$1{\cdot}39 \times 10^{-2}$	·894	·429
·7	·3085	4·67	2·90	·863	·483
·8	·4825	8·92	5·47	·835	·534
·9	·6358	$1{\cdot}57 \times 10^{-1}$	9·44	·800	·575
1·0	·7732	2·60	$1{\cdot}54 \times 10^{-1}$	·770	·616
1·2	30·0108	6·18	3·53	·709	·682
1·4	·2115	$1{\cdot}27 \times 1$	7·04	·652	·733
1·6	·3855	2·37	$1{\cdot}26 \times 1$	·601	·774
1·8	·5391	4·06	2·12	·561	·807
2·0	·6762	6·55	3·33	·520	·832
3	31·2045	$3{\cdot}99 \times 10$	$1{\cdot}84 \times 10$	...	·908
4	·5795	$1{\cdot}39 \times 10^{2}$	6·07	...	·948
5	·8702	3·58	$1{\cdot}51 \times 10^{2}$	...	·964
6	32·1078	7·82	3·16	...	·975
7	·3085	$1{\cdot}49 \times 10^{3}$	5·89	...	·981
8	·4825	2·61	$1{\cdot}01 \times 10^{3}$	...	·985
9	·6358	4·25	1·62	...	·988
10	·7732	6·57	2·47	...	·990
20	33·6762	$1{\cdot}12 \times 10^{5}$	$3{\cdot}99 \times 10^{4}$	...	·997
30	34·2045	5·81	$2{\cdot}02 \times 10^{5}$	...	·999
40	·5795	$1{\cdot}86 \times 10^{6}$	6·40	...	1·000
50	·8702	4·56	$1{\cdot}56 \times 10^{6}$	...	1·000
60	35·1078	9·50	3·24	...	1·000
70	·3085	$1{\cdot}77 \times 10^{7}$	6·00	...	1·000
80	·4825	3·02	$1{\cdot}02 \times 10^{7}$	...	1·000
90	·6358	4·85	1·64	...	1·000
100	·7732	7·40	2·50	...	1·000

concentration (by mass) of atoms of atomic weight A, and atomic number Z, so that $\Sigma c_A = 1$. For completely ionised matter,

$$\mu = 1 \bigg/ \left(\Sigma c_A \frac{Z}{A}\right). \tag{29}$$

(With this definition $\mu = 1$ for hydrogen, 2 for helium, 2·53 for lead, 1·43 for a mixture of 50 per cent. hydrogen and 50 per cent. lead.)

The density and electron concentration are then related by

$$\left.\begin{aligned} \rho &= \mu m_H n \\ &= 1{\cdot}6618 \times 10^{-24} \mu n \\ \log \rho &= \overline{24}{\cdot}22055 + \log \mu + \log n \\ n &= 6{\cdot}017 \times 10^{23} (\rho/\mu) \\ \log n &= 23{\cdot}77945 + \log \rho - \log \mu \end{aligned}\right\}. \tag{30}$$

As in dealing with pressure as a function of electron concentration, the simplest general procedure for finding the pressure for a given density is to calculate first the corresponding value of x, from which the value of $f_P(x)$ may be found from the tables, the pressure then being given by (20). The relation between ρ and x, using (19) and (30), is given by

$$\left.\begin{aligned} \rho &= 9{\cdot}856 \times 10^5 \mu x^3 \\ \log \rho &= 3 \log x + \log \mu + 5{\cdot}9937 \\ x &= 1{\cdot}005 \times 10^{-2} (\rho/\mu)^{1/3} \\ \log x &= \tfrac{1}{3} \log \rho - \tfrac{1}{3} \log \mu - 2 + 0{\cdot}0021 \end{aligned}\right\}. \tag{31}$$

The relation may be expressed as

$$\left.\begin{aligned} x &= c_1 \rho^{1/3} \\ \rho &= c_2 x^3 \end{aligned}\right\}, \tag{32}$$

c_1 and c_2 depending on μ as shown by (31), so that

$$\left.\begin{aligned} \log c_1 &= \overline{2}{\cdot}0021 - \tfrac{1}{3} \log \mu \\ \log c_2 &= 5{\cdot}9937 + \log \mu \end{aligned}\right\}. \tag{32'}$$

It may be noted that, for $x = 1$,

$$\rho = c_2 = \rho', \tag{33}$$

where ρ' corresponds to $x = 1$; that is, the conditions correspond to ordinary or to relativistic degeneracy according as $\rho << \rho'$ or $\rho >> \rho'$.

The equations corresponding to (23) are

$$\left.\begin{aligned} P &= \kappa_1 \rho^{5/3} \times \{f_P(x)/(\tfrac{1}{5}x^5)\} \\ P &= \kappa_2 \rho^{4/3} \times \{f_P(x)/(\tfrac{1}{4}x^4)\} \end{aligned}\right\}. \tag{34}$$

κ_1 and κ_2 depend on μ, and are related to κ_1' and κ_2' of (24) as follows:—

$$\left.\begin{aligned} \log \kappa_1 &= \log \kappa_1' - \tfrac{5}{3} \log m_H - \tfrac{5}{3} \log \mu \\ &= 12{\cdot}9941 - \tfrac{5}{3} \log \mu \\ \log \kappa_2 &= \log \kappa_2' - \tfrac{4}{3} \log m_H - \tfrac{4}{3} \log \mu \\ &= 15{\cdot}0889 - \tfrac{4}{3} \log \mu \end{aligned}\right\}. \tag{35}$$

The values of κ_1, κ_2, c_1 and c_2 (together with the logarithms, in brackets) are given in the following table for four values of μ:—

TABLE II

Values of Constants in the Formulæ

$$P = \kappa_1 \rho^{5/3} \text{ for } \rho << \rho',$$
$$P = \kappa_2 \rho^{4/3} \text{ for } \rho >> \rho',$$
and
$$x = c_1 \rho^{1/3}.$$

(Logarithms are given in brackets)

μ	κ_1	$c_2 = \rho'$	κ_2	c_1
1·0	9.865×10^{12} (12·9941)	9.856×10^{5} (5·9937)	1.227×10^{15} (15·0889)	1.005×10^{-2} ($\bar{2}$·0021)
1·5	5.019×10^{12} (12·7006)	1.479×10^{6} (6·1698)	7.147×10^{14} (14·8541)	8.778×10^{-3} ($\bar{3}$·9434)
2·0	3.109×10^{12} (12·4926)	1.971×10^{6} (6·2947)	4.871×10^{14} (14·6876)	7.977×10^{-3} ($\bar{3}$·9018)
2·5	2.143×10^{12} (12·3310)	2.463×10^{6} (6·3916)	3.617×10^{14} (14·5584)	7.404×10^{-3} ($\bar{3}$·8695)

General Character of Results

With the aid of Tables I and II and the relevant equations, calculations of the pressure and electronic kinetic energy density corresponding to any value of the electron concentration, or of the material density (for $\mu = 1$ to $\mu = 2.5$), may be made without much difficulty to a degree of accuracy which is probably amply sufficient for most purposes.

In order to show more immediately the general orders of the magnitudes involved, Table III has been drawn up. This shows the variation with n and ρ (for $\mu = 2$) of the pressure in the transition range. As will be obvious

TABLE III

Subsidiary Table, showing Values of Pressure (P) *for a Series of Values of the Electron Concentration* (n) *in the Transition Range; with some Values obtained from the Simple Formulæ for Comparison*

$\log n$	$\log \rho$ ($\mu = 2$)	a	P/a	$\kappa_1' n^{5/3}/a$	$\kappa_2' n^{4/3}/a$
27	0·5215 + 3	10^{18}	2·284	2·300	24·15
28	4	10^{20}	1·077		
29	5	10^{21}	4·490		
30	6	10^{23}	1·638	2·300	2·415
31	7	10^{24}	4·571		
32	8	10^{26}	1·089		
33	9	10^{27}	2·400	23·00	2·415

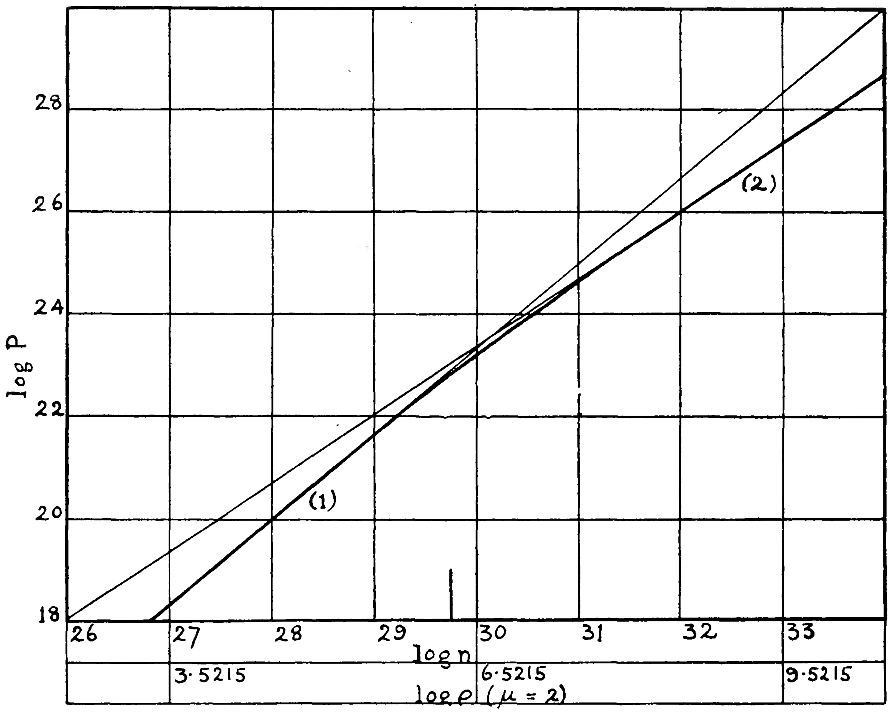

FIG. 1.—*Graph showing Relation between Pressure,* P, *and Electron Concentration,* n.

The curve gives log P as a function of log n.

The straight lines are log curves corresponding to

$$(1)\ P=\kappa_1' n^{5/3}=2{\cdot}300\times10^{-27}n^{5/3},$$
$$(2)\ P=\kappa_2' n^{4/3}=2{\cdot}415\times10^{-17}n^{4/3}.$$

The log n axis is also calibrated in terms of ρ for $\mu=2$.

The point marked is that corresponding to $n=5{\cdot}932\times10^{29}$, for which $x=1$.

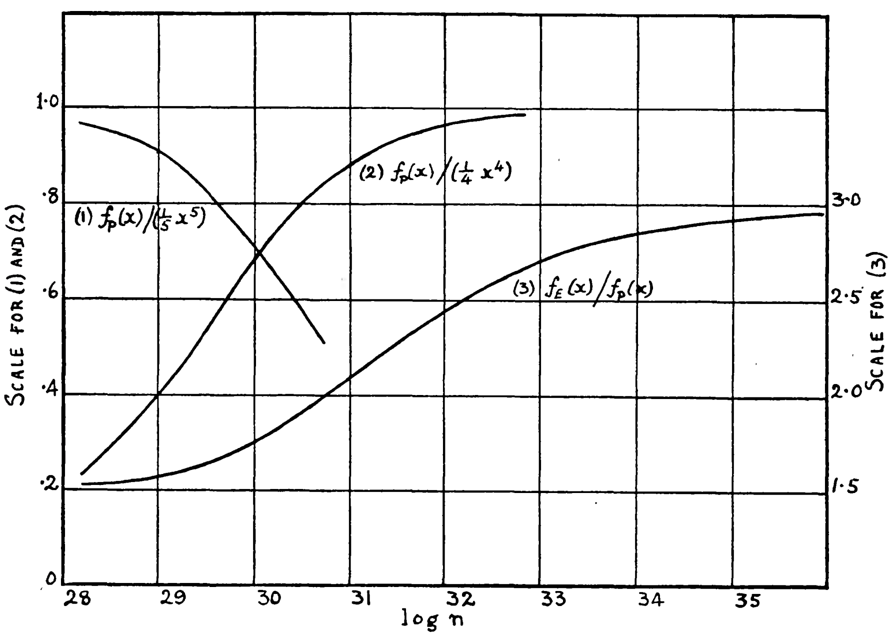

FIG. 2.—*Curves showing Relation between log* n *and*

$(1)\ f_P(x)/(\frac{1}{5}x^5)$, $(2)\ f_P(x)/(\frac{1}{4}x^4)$, $(3)\ f_E(x)/f_P(x)=E/P$.

For (1) and (2) let g_1, g_2 be the ordinates. Then

$$P=g_1\kappa_1' n^{5/3}=g_2\kappa_2' n^{4/3}.$$

from the comparative values given, below $n=10^{27}$ the formula $P=\kappa_1'n^{5/3}$ (or $P=\kappa_1\rho^{5/3}$) is adequate; and above $n=10^{33}$, the formula $P=\kappa_2'n^{4/3}$. The pressures given by the simple formulæ are always too great; roughly, using the formula $P=\kappa_1'n^{5/3}$ *below* $n=10^{30}$ increases the error to about 50 per cent. as n increases, while using $P=\kappa_2'n^{4/3}$ *above* $n=10^{30}$ increases the error to about 50 per cent. as n decreases.

In fig. 1 a graph of log P against log n is shown; the slope of the initial part being 5/3 and that of the final part, 4/3.

In fig. 2 the functions $f_P(x)/(\frac{1}{5}x^5)$ and $f_P(x)/(\frac{1}{4}x^4)$ are plotted. The curves show at once the extent of the deviation from the simple formulæ. A curve is also given showing the manner in which the pressure changes, owing to relativistic effect, from a value equal to two-thirds of the kinetic energy density at low concentrations to one-third at high concentrations.

Summary

In the investigation of the structure of white dwarfs and of the central regions of ordinary stars a knowledge of the pressure of matter at high densities is necessary. The pressure will be mainly due to the pressure of the " electron gas " subject to the Fermi-Dirac statistics. The minimum pressure is obtained by treating the gas as completely degenerate. Provided that the electron concentration, n, is not too large, the relativity change of mass with velocity may be neglected. Under these conditions, $P=\kappa_1'n^{5/3}$. When the relativity effect is predominant, $P=\kappa_2'n^{4/3}$. As the transition region corresponds to densities of astrophysical interest, the variation of P with n over the whole range from ordinary to relativistic degeneracy has been investigated.

A general expression for P (and also for E, the electronic kinetic energy density) is derived as an explicit function of x, where $x=\frac{h}{m_0c}\left(\frac{3}{8\pi}\right)^{1/3}n^{1/3}$. The value of this function is tabulated for $x=0{\cdot}1$ to $x=100$ (Table I). This enables P to be calculated to within 1 per cent. for any value of n. For n small and large the results reduce to

$$P=2{\cdot}300\times10^{-27}n^{5/3} \text{ for } n<<n' \quad\text{and}\quad P=2{\cdot}415\times10^{-17}n^{4/3} \text{ for } n>>n',$$

where

$$n'=5{\cdot}932\times10^{29}.$$

For a mean molecular weight (per electron) μ, the corresponding results in terms of density are $P=\kappa_1\rho^{5/3}$ or $P=\kappa_2\rho^{4/3}$, according as $\rho<<\rho'$ or $\rho>>\rho'$. The values of κ_1, κ_2 and ρ' are calculated for $\mu=1{\cdot}0$, $1{\cdot}5$, $2{\cdot}0$ and $2{\cdot}5$ (Table II).

The main features of the results are shown in two figures, one giving log P, the other the ratio of the pressure to the values calculated by the simple formulæ, and also E/P, as functions of log n.

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1932 *April.*