κ decreases ρ_{max} and T_{max}, the former being proportional to $1/\kappa'^3$ and the latter to $1/\kappa'$. From the result at the end of § 1 it seems likely that κ'/κ is fairly large and that the reduction of the upper limits will be considerable; but we are scarcely in a position to guess the extent of the reduction.

We see that with assumptions (*a*), (*b*), and (*c*) the values of ρ_{max} and T_{max} calculated in § 1 remain as extreme upper limits in all circumstances.

3. *Remarks.*—The strict upper limits naturally correspond to rather extravagant conditions, and are not likely to be attained. Besides the reduction due to incomplete degeneracy, there is a further reduction due to the fact that the density of a star is not uniform, so that P_{max} is not attained. To see the effect of this we calculate P for a polytrope with $\gamma = \frac{5}{3}$ instead of for a uniform sphere of the same central density. The result is

$$P = 0{\cdot}296(\tfrac{4}{3}\pi)^{\frac{1}{3}}GM^{\frac{2}{3}}\rho^{\frac{4}{3}}.$$

Comparing with (1), B is reduced approximately in the ratio 0·6, which leads to ρ_{max} being divided by 5 and T_{max} by 2. Thus for a distribution more or less of the character which might be expected in a white dwarf, the upper limits are only moderately reduced.

It is not supposed that the "reduced upper limits" (taking into account non-degeneracy and non-uniformity of density) give the actual temperature and density of a star. That depends on the radius, etc. But the star would probably attain these reduced limits at some stage during its collapse to the final state of extinction.

Upper limits to the temperature and density given by a universal formula must necessarily be rather high, since white dwarfs are included. Our result may, however, be of use in curbing riotous speculations which go beyond a temperature of 10^9 degrees and a density of 10^6.

The Dissociation Formula according to the Relativistic Statistics.

By S. Chandrasekhar.

(*Communicated by R. H. Fowler and E. A. Milne.*)

1. The temperatures conceived by Professor Milne * for the interiors of stars quite transcend all previous expectations and are precisely those temperatures at which the relativistic effect becomes important in statistical theory—at least for the electron-assembly, if not for the atomic assembly. It was thought worth while therefore to work out the dissociation formulæ on the relativistic statistics. The method followed in this paper will be the very elegant method developed by Milne in his recent paper.† Indeed the following will have to be considered as merely supplementing his results.

* *The Observatory*, **53**, 238, 1930.
† *M.N.*, **90**, 769, 1930.

2. *The Enunciation of the Dissociation Formulæ for Particles obeying Different Statistics.*—If we consider the case of a neutral atom dissociating into an ion and an electron, we know for certain that the electrons obey the Fermi-Dirac statistics, but the neutral atoms in general obey only the Einstein-Bose statistics. We have in fact the empirical rule that particles carrying a charge of an even or an odd multiple of the electronic charge obey the Einstein-Bose or the Fermi-Dirac statistics respectively. Hence in considering the dissociative equilibrium of different types of particles " chemically " reacting with each other, in the light of the new statistics we must, in general, consider the case of the different particles obeying different statistics.*

Let us for simplicity consider the case of an equilibrium characterised by the chemical equation

$$a\mathrm{A} + b\mathrm{B} = \mathrm{A}_a\mathrm{B}_b + \chi \quad . \quad . \quad . \quad (1)$$

representing the reaction in which a particles of type A react reversibly with b particles of type B to produce one particle of type $\mathrm{A}_a\mathrm{B}_b$ with an evolution of energy χ. Let the particles A and B obey the Fermi-Dirac statistics, and $\mathrm{A}_a\mathrm{B}_b$ the Einstein-Bose statistics,† which must happen when $a + b$ is even. Suppose that the total number of A's present in volume V be X, that of the B's Y. Let these occur in the assembly (in equilibrium) as L *free* particles of type A, and M *free* particles of type B. Then we have obviously

$$\mathrm{X} = \mathrm{L} + a\mathrm{N} \quad . \quad . \quad . \quad . \quad (2)$$

$$\mathrm{Y} = \mathrm{M} + b\mathrm{N} \quad . \quad . \quad . \quad . \quad (3)$$

If, further, $\mathrm{T_A}$, $\mathrm{T_B}$, and $\mathrm{T_{AB}}$ are kinetic energies of the respective particles, the total energy U is given by

$$\mathrm{U} = \mathrm{T_A} + \mathrm{T_B} + \mathrm{T_{AB}} - \mathrm{N}\chi \quad . \quad . \quad . \quad (4)$$

Let x_s, y_s, z_s represent the numbers of cells associated with the energy-range E_s to $\mathrm{E}_s + d\mathrm{E}_s$ for the particles A, B, and $\mathrm{A}_a\mathrm{B}_b$ respectively. The entropy of the whole assembly by the Boltzmann principle is

$$\mathrm{S} = k \log \Pi_s \frac{x_s!}{(x_s - p_s')!\, p_s'!}\, \Pi_s \frac{y_s!}{(y_s - q_s')!\, q_s'!}\, \Pi_s \frac{(z_s + r_s')!}{z_s!\, r_s'!} \quad . \quad (5)$$

where each separate factor corresponds to the total number of " complexions " obtained by arranging p_s' particles of type A into x_s cells, q_s' particles of type B into y_s cells, and r_s' particles of type $\mathrm{A}_a\mathrm{B}_b$ into z_s cells with the conditions

$$\sum p_s' = \mathrm{L}, \quad \sum q_s' = \mathrm{M}, \quad \sum r_s' = \mathrm{M} \quad . \quad . \quad (6)$$

and that the particles $\mathrm{A}_a\mathrm{B}_b$ obey the Einstein-Bose statistics, while the

* I am indebted to Mr. Fowler for a full appreciation of this point.

† If $a=b=1$, this would correspond to a neutral atom dissociating into an ion and an electron.

other two types of particles obey the Fermi-Dirac statistics. It is convenient to introduce

$$\left.\begin{aligned} p_s^0 &= x_s - p_s' \\ q_s^0 &= y_s - q_s' \\ r_s^0 &= z_s + r_s' \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (7)$$

The expression for the entropy (5) then reduces to

$$S = k \log \Pi \frac{x_s!}{p_s^0!\, p_s'!} \Pi \frac{y_s!}{q_s^0!\, q_s'!} \Pi \frac{r_s^0!}{z_s!\, r_s'!} \quad . \quad . \quad (5')$$

Now the entropy will have to be a maximum subject to the conditions (2), (3), (6), (7), and the further condition

$$U = \sum p_s' E_s^A + \sum q_s' E_s^B + \sum r_s' (E_s^{AB} - \chi) \quad . \quad (8)$$

We apply, as usual in such problems, the Lagrange method.

Applying Stirling's theorem in the form

$$\log n! \sim n \log n - n,$$

and effecting the variation in the p_s', etc. (with x_s, y_s, z_s constant), we find

$$0 = -\sum \delta p_s^0 (1 + \log p_s^0) - \sum \delta p_s' (1 + \log p_s') - (\text{similar terms in } q)$$
$$+ \sum \delta r_s^0 (1 + \log r_s^0) - \sum \delta r_s' (1 + \log r_s') \quad . \quad . \quad . \quad (9)$$
$$0 = \delta p_s^0 + \delta p_s' \quad . \quad . \quad . \quad . \quad (10)$$
$$0 = \delta q_s^0 + \delta q_s' \quad . \quad . \quad . \quad . \quad (10')$$
$$0 = \delta r_s^0 - \delta r_s' \quad . \quad . \quad . \quad . \quad (10'')$$
$$0 = \sum \delta p_s' + a \sum \delta r_s' \quad . \quad . \quad . \quad (11)$$
$$0 = \sum \delta q_s' + b \sum \delta r_s' \quad . \quad . \quad . \quad (11')$$
$$0 = \sum E_s^A \delta p_s' + \sum E_s^B \delta q_s' + \sum (E_s^{AB} - \chi) \delta r_s' \quad . \quad (12)$$

In applying the Lagrange method multiply the equations (10), (10'), (10''), (11), (11'), and (12) by $F_s + 1$, $G_s + 1$, $H_s - 1$, α, β, and γ respectively. Adding the complete set of equations and equating to zero the coefficients of $\delta p_s^0 \ldots \delta r_s'$, we have easily

$$-\log p_s^0 + F_s = 0, \qquad -\log q_s^0 + G_s = 0, \qquad \log r_s^0 + H_s = 0.$$
$$-\log p_s' + F_s - \alpha - \gamma E_s^A = 0,$$
$$-\log q_s' + G_s - \beta - \gamma E_s^B = 0,$$
$$-\log r_s' - H_s - a\alpha - b\beta - \gamma (E_s^{AB} - \chi) = 0.$$

Altering the constants F_s, G_s, H_s to new constants C_s^{AB}, etc., such that we have $r_s^0 = C_s^{AB}$, etc.,

$$r_s' = C_s^{AB} e^{-a\alpha - b\beta - \gamma (E_s^{AB} - \chi)},$$

or $$z_s = r_s^0 - r_s' = C_s^{AB}(1 - e^{-a\alpha - b\beta - \gamma(E_s^{AB} - \chi)}),$$

giving us $$r_s' = \frac{z_s}{e^{a\alpha + b\beta + \gamma(E_s^{AB} - \chi)} - 1}.$$

Similarly

$$p_s' = \frac{x_s}{e^{\alpha + \gamma E_s^A} + 1}, \qquad q_s' = \frac{y_s}{e^{\beta + \gamma E_s^B} + 1}.$$

γ is easily identified to be $\frac{1}{kT}$. The values of L, M, and N follow from (6). Replacing the sums by integrals and writing $kT\theta$ for E, we have

$$L = kT\int_0^\infty \frac{x_s d\theta}{e^{\alpha+\theta} + 1},$$

$$M = kT\int_0^\infty \frac{y_s d\theta}{e^{\beta+\theta} + 1},$$

$$N = kT\int_0^\infty \frac{z_s d\theta}{e^{\left(a\alpha + b\beta - \frac{\chi}{kT}\right)+\theta} - 1}.$$

If we write in the integral for N

$$\alpha' = a\alpha + b\beta - \frac{\chi}{kT}$$

we have its equivalent

$$a\alpha + b\beta = \frac{\chi}{kT} + \alpha'. \qquad . \quad . \quad . \quad (13)$$

This is for the simple reaction (1). This result is easily generalised for more complicated reactions, and we can enunciate the following generalisation of the theorem proved by Milne (*loc. cit.*) :—

Let us consider a gaseous mixture in which the constituents C_1, C_2 . . . C_r . . . react reversibly according to the chemical equation

$$\sum_r n_r C_r = \sum_{r'} n_{r'} C_{r'} + \chi$$

where χ is energy (expressed in ergs) evolved when n_r molecules of C_r, etc. react to form $n_{r'}$ molecules of $C_{r'}$, etc. Let the gaseous mixture be contained in a volume V at temperature T and consist of N_r molecules of C_r, etc., $N_{r'}$ molecules of $C_{r'}$, etc. The condition for dissociative equilibrium is obtained by eliminating α_1, . . . α_r ; $\alpha_{1'}$ . . . $\alpha_{r'}$ from the set of equations

$$N_r = kT\int_0^\infty \frac{x_r d\theta}{e^{a_r+\theta+\epsilon_r}}; \text{ etc.} \qquad . \quad . \quad . \quad (14)$$

$$N_{r'} = kT\int_0^\infty \frac{x_{r'} d\theta}{e^{a_{r'}+\theta} + \epsilon_{r'}}; \text{ etc.} \qquad . \quad . \quad . \quad (15)$$

and

$$\sum_r n_r a_r = \sum_{r'} n_{r'} a_{r'} + \frac{\chi}{kT} \qquad . \quad . \quad . \quad (16)$$

where x_r, etc., $x_{r'}$, etc., are the numbers of cells in a given energy range E to E + dE for the particles of the type C_r, etc., $C_{r'}$, etc., respectively, and ϵ_r equals + 1 or − 1 according as the particles "C_r" obey the Einstein-Bose or the Fermi-Dirac statistics.

3. *The Equations of Relativistic Statistics.*—When the relativistic effect is taken into account, the usual formula for the number of cells associated with a given energy range E_s to $E_s + dE_s$ (*e.g.*)

$$Z_s dE_s = \frac{2\pi V q}{h^3}(2m)^{\frac{3}{2}}E_s^{\frac{1}{2}}dE_s \quad . \quad . \quad . \quad (17)$$

(q = the statistical weight) has to be replaced by

$$Z_s dE_s = \frac{2\pi V q}{h^3}(2m)^{\frac{3}{2}}E_s^{\frac{1}{2}}\left(1+\frac{E_s}{2mc^2}\right)^{\frac{1}{2}}\left(1+\frac{E_s}{mc^2}\right)dE_s \quad . \quad (17')$$

If the relativistic effect becomes predominant (we state the physical conditions for this later), we shall have to work with

$$Z_s dE_s = \frac{4\pi V q}{h^3 c^3}E_s^2 dE_s. \quad . \quad . \quad . \quad (18)$$

To work out the dissociation formula when certain types of particles have a predominant relativistic effect while others have not, we have simply to use for the corresponding x_r or $x_{r'}$ the expression (17) or (18) in equation (14) or (15). But we take this opportunity to write down the equations of the relativistic statistics for the *Fermi-Dirac* case in a form convenient for application. Working with (18), we can derive

$$\left.\begin{aligned} N &= 8\pi V q\left(\frac{kT}{hc}\right)^3 U_2 \\ E &= 24\pi V q kT\left(\frac{kT}{hc}\right)^3 U_3 \\ pV &= \tfrac{1}{3}E \\ S &= nk\left[4\frac{U_3}{U_2} - \log A\right] \end{aligned}\right\} \quad . \quad . \quad . \quad (18')$$

where

$$U_\rho = \frac{1}{\Gamma(\rho+1)}\int_0^\infty \frac{\theta^\rho d\theta}{\frac{1}{A}e^\theta + 1}.$$

In making the transition from the degenerate to the non-degenerate case we simply pass from one limiting value for U_ρ characterised by large A to the other limiting value characterised by small A. It can easily be shown that the condition for degeneracy in this case is different from the Sommerfeld criterion for the non-relativistic case. If we should regard an assembly as degenerate when the relativistic effect is predominant, we must have the inequality

$$n >> \left(\frac{kT}{hc}\right)^3 \frac{4\pi q}{3} \left(= 2{\cdot}86T^3\right)$$

satisfied. The assembly must satisfy also another condition, namely, that which secures that the relativistic effect is predominant.

For convenience of reference we collect below the conditions which an assembly must satisfy in order that we can regard a system as degenerate or non-degenerate in the two distinct cases—where the relativistic effect is negligible and where it is highly predominant:

Magnitude of Relativistic Effect.	State of the Assembly.	Conditions. (The Numerical Values are those for an *Electron*-Assembly.)
Second order effect	(i) Degenerate	$n << \frac{8\pi m^3c^3}{3h^3} = 5{\cdot}88 \times 10^{29}$
		$n >> \frac{q}{h^3}(2\pi mk\mathrm{T})^{\frac{3}{2}} = 4{\cdot}87 \times 10^{15} \times \mathrm{T}^{\frac{3}{2}}$
	(ii) Non-degenerate	$n << \frac{q}{h^3}(2\pi mk\mathrm{T})^{\frac{3}{2}} = 4{\cdot}87 \times 10^{15} \times \mathrm{T}^{\frac{3}{2}}$
		$\mathrm{T} << \frac{mc^2}{k} = 5{\cdot}9 \times 10^9.$
Predominant	(iii) Degenerate	$n >> \left(\frac{k\mathrm{T}}{hc}\right)^3 \frac{4\pi q}{3} = 2{\cdot}86\mathrm{T}^3$
		$n >> \frac{8\pi m^3c^3}{3h^3} = 5{\cdot}88 \times 10^{29}$
	(iv) Non-degenerate	$n << \left(\frac{k\mathrm{T}}{hc}\right)^3 \frac{4\pi q}{3} = 2{\cdot}86\mathrm{T}^3$
		$\mathrm{T} >> \frac{mc^2}{k} = 5{\cdot}9 \times 10^9.$

The conditions (ii) and (iv) for the case of non-degeneracy hold good for the Einstein-Bose case also. This is clear when we note that the condition for the rapid convergence of the series expansion of (14) for large α_r, when ϵ_r equals either $+1$ or -1, is just the same (and the non-degeneracy condition is merely the analytical expression of this). In the sequel we do *not* consider the case of degeneracy for the atomic assembly (some constituents of which will obey the Einstein-Bose statistics), for, first, such cases are of no practical value, and secondly, the question of degeneracy for the particles obeying the Einstein-Bose statistics is closely bound up with the difficulty pointed out by Uhlenbeck.* In fact, in the degenerate case the particles can *never* have a predominant relativistic effect because all the particles go down to the zero state. (In the Fermi-Dirac statistics this is prevented by Pauli's principle.)

4. *The Dissociation Formulæ for the Relativistic Case.*—(*a*) Let all the constituents be non-degenerate. We consider the case of the ionisation of an atom A capable of being in a state of lowest energy content $\mathrm{A}^{(1)}$, its ionised form being $\mathrm{A}_+^{(1)}$. Let χ be the energy of ionisation. If $\mathrm{C}^{(1)}$ and $\mathrm{C}_+^{(1)}$ are the atomic concentrations per unit

* G. E. Uhlenbeck, *Over Statistische Methoden in die Theorie der Quanta*, Thesis, Leiden (1927), p. 70.

volume we have equations corresponding to (14) and (15) reducing, on using for the number of cells, the value given by (18), to *

$$C^{(1)} = 8\pi q^{(1)}\left(\frac{kT}{hc}\right)^3 e^{-\alpha^{(1)}} \quad . \quad . \quad . \quad (19)$$

$$C_+^{(1)} = 8\pi q_+^{(1)}\left(\frac{kT}{hc}\right)^3 e^{-\alpha+^{(1)}} \quad . \quad . \quad . \quad (20)$$

$q^{(1)}$ and $q_+^{(1)}$ being the statistical weights. For C_e, the electron-concentration, we have

$$C_e = 8\pi q_e\left(\frac{kT}{hc}\right)^3 e^{-\alpha_e} \quad . \quad . \quad . \quad (21)$$

and as the chemical equation in this case is

$$A_+^{(1)} + e = A^{(1)} + \chi,$$

we have for our fourth equation, in addition to (19), (20), and (21), according to (16),

$$\alpha_+^{(1)} + \alpha^{(e)} = \alpha^{(1)} + \frac{\chi}{kT} \quad . \quad . \quad . \quad (22)$$

Hence our dissociation formula in this case is

$$\frac{C_+^{(1)} \cdot C_e}{C^{(1)}} = \frac{q_+^{(1)} q^{(e)}}{q^{(1)}} 8\pi\left(\frac{kT}{hc}\right)^3 e^{-\frac{\chi}{kT}} \quad . \quad . \quad (23)$$

The above equation is analogous to the Saha equation for the non-relativistic case.

(*b*) Let us now consider the case where the electrons are degenerate but the atoms and the ionised atoms are non-degenerate. Then we have in this case for the electron-concentration (because it obeys the Fermi-Dirac statistics)

$$C_e = \frac{4\pi}{3} q_e\left(\frac{kT}{hc}\right)^3 (-\alpha_e)^3 \text{ approximately} \quad . \quad . \quad (24)$$

giving

$$-\alpha_e = \left(\frac{3C_e}{4\pi q_e}\right)^{\frac{1}{3}} \frac{hc}{kT} \quad . \quad . \quad . \quad . \quad (24')$$

We have therefore for our eliminant

$$\log\left(\frac{C^{(1)}}{C_+^{(1)}} \frac{q_+^{(1)}}{q^{(1)}}\right) = \frac{\chi}{kT} + \frac{hc}{kT}\left(\frac{3C_e}{4\pi q_e}\right)^{\frac{1}{3}} \quad . \quad . \quad (25)$$

* (19) and (20) are true, independent of what statistics they obey, but (19) would in general correspond to the Einstein-Bose case, and (20) to the Fermi-Dirac case.

We give below the corresponding equation for the non-relativistic case for comparison : *

$$\log\left(\frac{C^{(1)}}{C_+^{(1)}}\frac{q_+^{(1)}}{q^{(1)}}\right) = \frac{\chi}{kT} + \frac{h^2}{2m_e kT}\left(\frac{3C_e}{4\pi q_e}\right)^{\frac{2}{3}} \quad . \quad . \quad (26)$$

5. *Other Possible Cases.*—We now pass on to the consideration of cases intermediate to the relativistic and the non-relativistic cases. As our table shows, it is quite possible for the electron-assembly to be governed by the relativistic equations, while the atomic assembly obeys reasonably well the non-relativistic equations. We assume that the atomic system is non-degenerate. We have then two cases corresponding to the two possible behaviours of the electron-assembly.

(i) *Non-degenerate Electrons.*—In the equations corresponding to (14) and (15) for $C^{(1)}$ and $C_+^{(1)}$ we have to use for the x_r's the expression (17) and for C_e the expression (18). We have then

$$C^{(1)} = \frac{q^{(1)}(2\pi m^{(1)}kT)^{\frac{3}{2}}}{h^3}e^{-\alpha^{(1)}} \quad . \quad . \quad . \quad (27)$$

$$C_+^{(1)} = \frac{q_+^{(1)}(2\pi m_+^{(1)}kT)^{\frac{3}{2}}}{h^3}e^{-\alpha+^{(1)}} \quad . \quad . \quad . \quad (28)$$

But we have for the electron-concentration

$$C_e = \frac{8\pi q_e}{h^3c^3}(kT)^3 e^{-\alpha_e} \quad . \quad . \quad . \quad . \quad (29)$$

We have therefore for our dissociation formula, remembering (22),

$$\frac{C_+^{(1)}C_e}{C^{(1)}} = \frac{q_+^{(1)}\cdot q_e}{q^{(1)}}\left(\frac{m_+^{(1)}}{m^{(1)}}\right)^{\frac{3}{2}} 8\pi\left(\frac{kT}{hc}\right)^3 e^{-\chi/kT} \quad . \quad . \quad (30)$$

which, neglecting the trivial factor $m_+^{(1)}/m^{(1)}$ is exactly the same as (23) derived for the case when the relativistic-non-degenerate equations were used for the atomic assembly.

(ii) *Degenerate Electrons.*—In this case, instead of (29) we have to use (24), and our dissociation formula is again

$$\log\left[\frac{C^{(1)}}{C_+^{(1)}}\frac{q_+^{(1)}}{q^{(1)}}\left(\frac{m_+^{(1)}}{m^{(1)}}\right)^{\frac{3}{2}}\right] = \frac{\chi}{kT} + \frac{hc}{kT}\left(\frac{3C_e}{4\pi q_e}\right)^{\frac{1}{3}} \quad . \quad . \quad (31)$$

which is exactly the same as (25).

Thus we have the result that the dissociation formula is independent of whether the atomic assembly is governed by the relativistic or the non-relativistic equations in so far as it is postulated that it is in any case non-degenerate. It is the behaviour of the electron-assembly (*i.e.* whether it is degenerate or non-degenerate) that matters. Since this is so in the two extreme cases, we can reasonably generalise and assume that the equations (23) and (25) hold for all intermediate

* General formulæ for dissociative equilibrium were first given by R. H. Fowler, *Proc. Roy. Soc.*, A, **118**, 432, 1926. Explicit forms were given by S. Chandrasekhar, *Phil. Mag.*, **9**, 292, 1930, and also by E. A. Milne (*loc. cit.*).

behaviour of the atomic assembly. It is unlikely that the atomic assembly can be degenerate at those high temperatures (of the order of 10^{10}) when the concentration would have to be greater than 5×10^{38}. (The electron-assembly is degenerate at very much higher temperatures and much lower concentration obviously because of its very small mass.) So far as the atomic assembly is concerned, therefore, the only question is whether the relativistic effect is large or small for the non-degenerate case. This question, as we have already seen, does not affect our dissociation formulæ. If we could be sure about the behaviour of the electron-assembly, we could immediately choose our dissociation formula independent of the behaviour of the atomic assembly.

What has been said regarding the dissociation formula with the relativistic equations is equally true for the electron-assembly with the non-relativistic equations, as can easily be verified. But a state of affairs where the electron-assembly is governed by non-relativistic equations while the atomic assembly obeys the relativistic equations is physically inconceivable. We have therefore to deal with four kinds of dissociation formulæ—namely, for the two possible behaviours of the electron-assembly for both the relativistic and the non-relativistic cases.

6. *Numerical Calculations.*—(i) According to Milne,* the central density, and the central and the effective temperatures of his highly collapsed configurations (which he identifies with the white dwarfs), are respectively for $M = \cdot 85\ \odot$:

$$\left.\begin{aligned} \rho_c &= 2\cdot 87 \times 10^6 \beta^3 \\ T_c &= 2\cdot 92 \times 10^9 \beta (1-\beta)^{\frac{1}{4}} \\ T_e &= 11{,}300 \beta^{\frac{1}{2}} \end{aligned}\right\} \quad . \quad . \quad . \quad (32)$$

where

$$\beta = 1 - \frac{\kappa L}{4\pi c G M} \qquad (\kappa = \text{opacity}).$$

If the effective temperature should correspond to, say, 10000°, β must be about 0·9. This gives us immediately

$$\left.\begin{aligned} C_e &= 7 \times 10^{29} \text{ per cm.}^3 \\ T_c &= 1\cdot 4 \times 10^9 \text{ degrees} \end{aligned}\right\}.$$

The electron-assembly therefore just satisfies those conditions necessary for the system to be degenerate and at the same time have the relativistic effect predominant. Hence the dissociation formula we ought to use is (25), which reduces on using numerical values (x = degree of dissociation) to

$$\log_e \frac{1-x}{x} = \frac{\chi}{kT} + \cdot 7045 \frac{C_e^{\frac{1}{3}}}{T}$$

$$= \frac{\chi}{kT} + 4\cdot 483 \quad . \quad . \quad . \quad (33)$$

* E. A. Milne, *M.N.*, **91**, 4, 1930.

For temperatures of the order of 10^9 the first term in (33) is negligible compared with the second, even when χ corresponds to the K-ionisation potentials for the heaviest elements. The degree of dissociation given by (34) is therefore practically zero, and independent of the ionisation potential. This at first sight would appear to contradict the common-sense view that at these extreme conditions of temperature and pressure matter must necessarily be ionised down to the bare nuclei. At the same time we have to recognise that at such high concentrations every electron must be under the influence of one atomic nucleus or another all the time. The whole system is, in fact, "a gigantic molecule in the lowest quantum state." * But "statistically" they must all be free in the sense that they travel the whole of the phase space. Our analysis giving 0 per cent. ionisation will have to be interpreted this way. Thus, whenever we get the calculated degree of dissociation as being practically zero and independent to a large extent of our choice of the ionisation potential, we can then safely conclude that matter has reached such extreme conditions that we must regard (statistically) all the atoms as being "ionised" down to their bare nuclei, and, further, the electron-assembly as being highly degenerate. We can use this result as a "degeneracy-condition." †

(ii) We now use the *non*-relativistic dissociation formula (26). At the interface of the "composite configurations," where we pass from a solution of Emden's equation "$n = 3$" to another of Emden's, "$n = \frac{3}{2}$," the pressure and temperature are (Milne, *loc. cit.*, equations (82) and (83))

$$\rho' = 3{\cdot}66 \times 10^7 \frac{1-\beta}{\beta}\,;\quad T' = {\cdot}843 \times 10^{10}\left(\frac{1-\beta}{\beta}\right)^{\frac{2}{3}}$$

or

$$\frac{C_e'^{\frac{2}{3}}}{T'} = \frac{(\rho'/2m_H)^{\frac{2}{3}}}{T'} = 5{\cdot}9 \times 10^{10}\,. \qquad . \qquad . \qquad (34)$$

Introducing (34) in (26) we have simply

$$\log_e \frac{1-x}{x} = \frac{\chi}{kT} + 2{\cdot}482 \qquad . \qquad . \qquad . \qquad (35)$$

Comparing (35) with (33) we can deduce that matter must from now onwards be treated as "degenerate"—justifying therefore the transition to the degenerate Fermi-Dirac equation of state for the electrons.

In conclusion I wish to express my thanks to Mr. R. H. Fowler and Professor E. A. Milne for their advice and encouragement during the course of the work.

* R. H. Fowler, "On Dense Matter," *M.N.*, **87**, 114, 1926.

† In this connection see W. Anderson, *Physik. Zeits.*, **30**, 360, 1929, also S. Chandrasekhar (*loc. cit.*), § 6.

The Highly Collapsed Configurations of a Stellar Mass.

By S. Chandrasekhar.

(*Communicated by Professor E. A. Milne.*)

§ 1. Professor Milne in his recent paper * on "The Analysis of Stellar Structure" has put forward some essentially new considerations on the possible steady-state configurations of stellar aggregates of varying mass, luminosity, and opacity. One of the main consequences of the analysis is the explanation not only of the existence of white dwarfs—his collapsed configurations—but also of the principal physical characteristics of these configurations. The following is devoted to the development of Milne's theory of these collapsed configurations a stage further.

§ 2. Milne's estimates for the central density and temperature of these collapsed configurations indicate that in some cases we pass beyond the range of validity of the degenerate form of the Fermi-Dirac equation of state ($p = K\rho^{\frac{5}{3}}$). It can be shown that the pressure of an electron gas which is highly degenerate and which has a very highly predominant relativistic-mass variation effect, takes the limiting form †

$$p = \frac{n^{\frac{4}{3}}hc}{8}\left(\frac{3}{\pi}\right)^{\frac{1}{3}} \quad . \quad . \quad . \quad . \quad (1)$$

(c = velocity of light, h = Planck's constant) if the following two conditions are satisfied :—

$$n > \left(\frac{kT}{hc}\right)^3 \frac{4\pi G}{3} \equiv 2{\cdot}86T^3 \qquad (G = 2) \quad . \quad (i)$$

$$n > \frac{8\pi m^3 c^3}{3h^3} \equiv 5{\cdot}88 \times 10^{29} \quad . \quad . \quad . \quad (ii)\ ‡$$

(i) replaces Sommerfeld's degeneracy criterion, and (ii) ensures the predominance of the relativistic effect. As the condition that configurations are highly collapsed is precisely equivalent to the condition that in the central regions the electron assembly is degenerate,§ it would be sufficient to consider whether (ii) is satisfied in order to count the relativistic effect as highly predominant. Further, as L for these configurations is very small, T_c is normally of the order of 10^9,|| and (ii) then would automatically provide for (i). (See equation (19′)).

* *M.N.*, **91**, 4–55, 1930, referred to hereafter as *loc. cit.* For a general exposition of his main ideas see *Nature*, 1931 January 3, "Stellar Structure and the Origin of Stellar Energy."

† The corresponding expression for the energy E was obtained by E. C. Stoner, *Phil. Mag.*, **9**, 944, 1930. That $p = \frac{1}{3}E$ is generally true when the relativistic effect is highly predominant can easily be proved.

‡ (ii) is given in Stoner's paper; (i) now replaces Sommerfeld's criterion. These and other inequalities are briefly discussed in my paper on "The Dissociation Formula according to the Relativistic Statistics," *M.N.* (in course of publication).

§ Milne, *loc. cit.*, § 22.

|| Milne, *loc. cit.*, p. 39.