

enabled me to prepare this paper. Also to Professor E. A. Milne, F.R.S., who suggested the outlines, as well as methods to overcome difficulties in the actual numerical integration.

Newport, Mon. :
1930 Nov. 11.

The Solutions of Emden's and Similar Differential Equations.

By R. H. Fowler, F.R.S.

§ 1. *Introduction. Physical Bearings of the Problem.*—A sphere of perfect gas, in equilibrium under gravitation, gas-pressure and radiation-pressure, generating energy at a certain rate per unit mass, has been shown by Eddington on certain natural simplifications to have a structure controlled by the equation

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\theta}{d\xi} \right) + \theta^3 = 0 \quad . \quad . \quad . \quad (1)$$

In this equation ξ represents a "reduced" radius, and θ a "reduced" temperature. Other physical quantities follow at once when θ is known as a function of ξ . The equation is a member of the family of equations shown by Emden to govern the polytropic * equilibrium of a gravitating gas sphere, namely,

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\theta}{d\xi} \right) + \theta^n = 0 \quad . \quad . \quad . \quad (2)$$

Equation (2) for $n = \frac{3}{2}$ is allied to that governing the Thomas-Fermi distribution of electrons in an atom,† differing only in the sign of $\theta^{\frac{3}{2}}$.

The application of equation (1) and similar equations to astrophysical problems has recently been re-examined by Milne, to see whether it may be that any oversight has occurred such as might affect the conclusions drawn. Milne's investigations will be found on page 4 of this number, and it is not within my province to comment on them here, beyond what is essential to explain the origin of my own studies. In the great papers in which Eddington initiated the present theory of the internal structure of stars, Eddington bases his discussion on the properties of Emden's solution of the equation (1) which satisfies the central conditions

$$\theta = 1, \quad \frac{d\theta}{d\xi} = 0 \quad (\xi = 0) \quad . \quad . \quad . \quad (3)$$

But $\xi = 0$ is a singularity of the equation, and it may not be safe without further examination to base any discussion on the properties of such a particular solution, unless it is known *a priori* that this is the solution required. The reader of Milne's paper will see that it

* Polytropic equilibrium of a gas denotes a state of the gas in which by hypothesis $p = \varphi \rho^\gamma$; γ, φ constant.

† L. H. Thomas, *Proc. Camb. Phil. Soc.*, **23**, 542 (1926).

follows at once from Milne's arguments that for just this reason Eddington's theory may benefit by revision, or at least by reconsideration. In view, therefore, of the great importance of Milne's contentions, and so of the general character of *all* the solutions of Emden's and similar equations, it seems desirable to re-examine the character of the solutions of equation (1) or (2) as an exercise in pure mathematics. This I attempt to do in this paper, stimulated thereto, as is obvious, by many discussions with Professor Milne.

In undertaking this discussion it serves no useful purpose to restrict it unduly to equation (1), or even (2). The more general equation

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\theta}{d\xi} \right) + \xi^\sigma \theta^n = 0 \quad (n > 1) \quad . \quad . \quad . \quad (4)$$

might also be of value, and is about as easy to discuss.

Emden himself was well aware that the solution which he constructed numerically was a particular one, and that there were others with quite different properties as $\xi \rightarrow 0$, but he has not determined the properties of these other solutions—at least not in a sufficiently usable form. A rigorous though incomplete study of equations such as (4) has been given before, but has unfortunately remained unknown to astrophysicists. The study was started by Horn and Kneser, arranged as a discussion of the form as $x \rightarrow +\infty$ of real continuous solutions of equations such as

$$\frac{d}{dx} \left(x^\sigma \frac{dy}{dx} \right) + bx^\sigma y^n = 0 \quad . \quad . \quad . \quad (5)$$

and similar equations of higher order. It was restarted on rather different lines by Hardy* for the more tractable first-order equations

$$P(x, y, y') = 0 \quad . \quad . \quad . \quad . \quad (5a)$$

where P is a polynomial in x, y, y' . Finally, perhaps the most nearly complete existing discussion of equation (5) is my own,† at least so far as I am aware. Like everyone else I had entirely overlooked the fact that these discussions had astrophysical applications. The form of the more general solutions of equation (1) has, of course, been rediscovered, and still more their importance recognised for the first time by Milne. The values of n , considered in equation (5), have in all previous studies except Emden's been positive integers. It is comparatively simple to extend all the theorems to rational values of n greater than unity. The assumption that n is a positive integer is only relevant in controlling the behaviour of y^n when y changes sign. None of these studies tells the astrophysicist quite all he wants to know, and it is my object here to simplify and fill in the gaps, so that the behaviour of any solution having an astrophysical interest may be rigorously established.

* Hardy, *Proc. Lond. Math. Soc.*, **10**, 451 (1911).

† Fowler, *Quart. J. of Math.*, **45**, 289 (1914). Fairly full earlier references are given in these papers and will not be repeated here.

As I have said, all discussions have been aimed at obtaining the form of the solutions of the proposed differential equation as $x \rightarrow +\infty$, and it is frequently convenient to adhere to this practice. In astrophysical problems it is the form of the solutions as $\xi \rightarrow 0$ which is in doubt and of supreme importance to us. On writing

$$\xi = 1/x$$

equation (4) becomes

$$\theta'' + x^{-(4+\sigma)}\theta^n = 0.$$

We shall therefore here frequently discuss rather the equation

$$\theta'' + x^\sigma \theta^n = 0 \quad . \quad . \quad . \quad (6)$$

remembering that for astrophysical purposes $\sigma = -4 - \sigma'$ and that usually $\sigma = -4$.

We encounter certain special cases, of which one at least is more important than the general case, for it covers equation (1). This case is

$$\sigma + n + 1 = 0 \quad . \quad . \quad . \quad (7)$$

For example, equation (1) with $\sigma = -4$, $n = 3$ satisfies (7). We can then write

$$\theta = xy \quad x = e^t \quad (t \rightarrow \infty),$$

for the equation is homogeneous, and obtain

$$y'' + y' + y^n = 0 \quad . \quad . \quad . \quad (8)$$

Another reduction is possible when

$$\sigma + 2 = 0 \quad . \quad . \quad . \quad (9)$$

On putting

$$x = e^t \quad (t \rightarrow \infty),$$

we have

$$\theta'' - \theta' + \theta^n = 0 \quad . \quad . \quad . \quad (10)$$

This equation is not of much interest to astrophysicists, but the interest of the special case (7) is extreme. We shall therefore start with its discussion.

We close this introduction with one remark on notation. Since it is convenient to pass freely to and fro between the independent variables ξ , x , t , and the dependent variables θ , y , we standardize the notation

$$\xi = 1/x = e^{-t} \quad (\xi \rightarrow 0, t, x \rightarrow +\infty) \quad . \quad . \quad (11)$$

$$\theta = \frac{1}{\xi}y = xy = e^t y \quad . \quad . \quad . \quad (12)$$

In writing differential coefficients we use y' , etc., indiscriminately for dy/dX , where X is whichever of the variables ξ , x , t we are working with at the moment.

§ 1.1. *Summary for Astrophysicists.*—For the sake of such astrophysicists as may be interested in the results of this paper, rather than in their derivation, I close this introduction with a short statement of the

results proved to hold for the two equations which are used extensively in Milne's work—namely,

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\theta}{d\xi} \right) + \theta^3 = 0 \quad . \quad . \quad . \quad (I.)$$

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\theta}{d\xi} \right) + \theta^{\frac{3}{2}} = 0 \quad . \quad . \quad . \quad (II.)$$

Equations I. and II.—(1) As $\xi \rightarrow 0$ all the solutions are ultimately monotonic functions of ξ and steadily approach either a finite limit or $\pm \infty$.

(2) Consider the group of all the solutions which start inwards from a given point (ξ_0, θ_0) , $\theta_0 > 0$, with various values of the initial slope such that θ initially increases inwards. Starting-points ξ_0, θ_0 are divided into two classes by a simple *critical curve* of the form $\theta = \alpha \xi^{\varpi}$ (α a positive constant, $\varpi = -1$ for equation I., $\varpi = -4$ for equation II.).

(3) If the starting-point lies below the critical curve there pass through the starting-point two, and only two, solutions of the group which remain positive in the range $0, \xi_0$ and tend to finite limits as $\xi \rightarrow 0$. These solutions are both simple transformations of Emden's solution, and are called E-solutions. If the starting-point lies above the critical curve no solution of the group is an E-solution, and if the starting-point lies on the curve there is just one member of the group which is an E-solution touching the critical curve at the starting-point. [The critical curve is the envelope of the E-solutions.]

(4) If the starting-point lies below the critical curve and the starting-slope lies between the slopes of the two E-solutions through the starting-point, then any such solution (called an M-solution) remains always positive as $\xi \rightarrow 0$, and in fact tends steadily to positive infinity with the asymptotic form

$$\theta \sim \frac{1}{\xi} \left[\frac{1}{2 \log (C/\xi)} \right]^{\frac{1}{2}} \quad . \quad . \quad . \quad (\text{Equation I.})$$

$$\theta \sim \frac{C}{\xi} \quad . \quad . \quad . \quad (\text{Equation II.})$$

where C is some positive constant. It remains always below the critical curve.

(5) If the starting-point lies below the critical curve and the starting-slope is *less* than the *lesser* of the slopes of the two E-solutions, then there exists a number ξ_1 , such that $0 < \xi_1 < \xi_0$, and $\theta(\xi_1) = 0$. The solution lies entirely below both E-solutions in the interval ξ_0, ξ_1 .

(6) If the starting-point lies below the critical curve and the starting-slope is *greater* than the *greater* of the slopes of the two E-solutions, then the solution must pass above the critical curve.

(7) If the starting-point lies above the critical curve or if the solution ever rises above the critical curve as in (6), then there again exists a number ξ_1 , such that $0 < \xi_1 < \xi_0$, $\theta(\xi_1) = 0$.

Only E-solutions and M-solutions are permissible solutions in any astrophysical applications over a range which reaches to $\xi = 0$, as all other solutions have illegitimate zeros. The establishment of the

asymptotic forms and relative arrangements of the E-solutions and the M-solutions constitutes the main result of this paper.

§ 2. *Study of the Equation* * ($n = \frac{p}{q} > 1$; p, q odd)

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\theta}{d\xi} \right) + \xi^{n-3} \theta^n = 0, \quad y'' + \frac{1}{\xi^2} y^n = 0, \\ \theta'' + x^{-n-1} \theta^n = 0, \quad y'' + y' + y^n = 0. \quad (A)$$

Part I. Possible forms as $t \rightarrow \infty$ of the solutions of $y'' + y' + y^n = 0$.—This equation is one of those not completely discussed before. Part I. of the discussion can be finished off by the proper appeal to Hardy's work, and is given here in a simplified form *de novo*.† After the results of Part I. are obtained, the complete properties of the solutions of (A) can be established by bringing in a different type of argument.

All variables are to be real throughout. We start by confining attention to solutions $y = y(t)$, which are such that $y(t), y'(t)$ exist and are real, finite, and continuous for all finite $t (> t_0)$, without inquiring more closely whether any such solutions actually exist. Such solutions of course also have continuous $y''(t)$ and differential coefficients of all higher orders for all $t (> t_0)$, except perhaps at $y = 0$. They will be called *proper solutions* when a name is required. We interpret

$$(-y)^n = -(y^n).$$

We prove the following theorem:—

Theorem I.‡—Any proper solution $y = y(t)$ of (A) is such that $y(t) \rightarrow 0$ with or without an infinity of zeros for $t > t_0$.

For any positive k , however small, and any proper solution $y = y(t)$ there are three *a priori* possibilities:

- (i) There exists a t_0 such that $y(t) > k$ ($t > t_0$).
- (ii) There exists a t_0 such that $y(t) < k$ ($t > t_0$).
- (iii) $y = y(t)$ cuts $y = k$ in a sequence of points $\{t_s\}$ such that $t_s \rightarrow \infty$.

We shall, for the sake of shortness, call any such sequence simply “a $\{t_s\}$ ”.

Case (i) is impossible, for the hypothesis implies

$$y'' + y' < -k' \quad (t > t_0), \\ \frac{d}{dt}(e^t y') < -k' e^t, \\ e^t y' < -k' e^t + C, \\ &< -k'' e^t \quad (t > t_0'), \\ y' &< -k'', \\ y &< -k'' t + C', \\ &< k \quad (t > t_0''),$$

* It is, of course, intended that p and q are positive integers with no common factor.

† For the sake of completeness we include steps in the argument which really duplicate former work; for example, Fowler, *loc. cit.*

‡ This theorem is part of a rather more general theorem whose proof has been given by the author, *loc. cit.* See p. 343, Table V.

contrary to hypothesis. Similarly, $y(t) < -k$ ($t > t_0$) is also impossible.

We can now re-enunciate the remaining possibilities as follows:—

(a) For any positive k there exists a t_0 such that $y(t) > 0$ (< 0) for $t > t_0$ and $y = y(t)$ cuts $y = k$ ($-k$) in a $\{t_s\}$.

(b) $y = y(t)$ oscillates finitely or infinitely about $y = 0$ cutting $y = \pm k$ and $y = 0$ in $\{t_s\}$'s.

(c) $y = y(t) \rightarrow 0$ as $t \rightarrow \infty$ with or without cutting $y = 0$ in a $\{t_s\}$.

Case (a) is impossible; for let $y(t) > 0$ and let us start at t_0' , a member of $\{t_s\}$, such that $t_0' > t_0$, $y_0 = y(t_0') = k$, $y_0' = y'(t_0') = -\delta$, $\delta > 0$. Then

$$\begin{aligned} y'' + y' &< 0 & (t > t_0'), \\ y' + y &< y_0' + y_0 = k - \delta, \\ e^t y - e^{t_0'} k &< (k - \delta)(e^t - e^{t_0'}), \\ y &< k - \delta(1 - e^{(t_0' - t)}), \\ &< k, \end{aligned}$$

which contradicts the hypothesis.

Case (b) is also impossible. For consider any one of the assumed infinity of oscillations about $y = 0$. Let τ_s be the abscissa of the

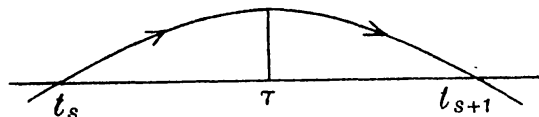


FIG. 1.

maximum of $y(t)$, which is easily proved to be the sole stationary value of $y(t)$ in the interval $t_s t_{s+1}$. On the upward arc $t_s \tau_s$, $y'(t) \geq 0$. Therefore

$$\begin{aligned} -y'' &\geq y^n, \\ -y''y' &\geq y^n y', \end{aligned}$$

$$\frac{1}{2}[y'(t_s)]^2 > \frac{1}{n+1}[y(\tau_s)]^{n+1}.$$

Similarly, from the downward arc $\tau_s t_{s+1}$,

$$\frac{1}{2}[y'(t_{s+1})]^2 < \frac{1}{n+1}[y(\tau_s)]^{n+1}.$$

Hence

$$|y'(t_s)| > \left(\frac{2}{n+1}\right)^{\frac{1}{2}} |y(\tau_s)|^{\frac{n+1}{2}} > |y'(t_{s+1})| > \left(\frac{2}{n+1}\right)^{\frac{1}{2}} |y(\tau_{s+1})|^{\frac{n+1}{2}} \dots \quad (13)$$

These inequalities state that both $|y'(t_s)|$ and $|y(\tau_s)|$ form decreasing monotonic sequences as $s \rightarrow \infty$. Hence either both $|y'(t_s)| \rightarrow 0$, $|y(\tau_s)| \rightarrow 0$ as $s \rightarrow \infty$, or else

$$|y'(t_s)| \rightarrow l, \quad |y(\tau_s)| \rightarrow m, \quad \left(l = \left(\frac{2}{n+1}\right)^{\frac{1}{2}} m^{\frac{n+1}{2}}\right).$$

Suppose now that $|y'(t_s)| \rightarrow l$, and consider the foregoing inequalities more exactly. We have on the upward arc $t_s\tau_s$

$$-y''y' = y^n y' + y'^2,$$

and therefore, in particular,

$$\frac{1}{2}[y'(t_s)]^2 = \frac{1}{n+1}[y(\tau_s)]^{n+1} + \int_0^{y(\tau_s)} |y'| dy.$$

Similarly, from the downward arc

$$\frac{1}{2}[y'(t_{s+1})]^2 = \frac{1}{n+1}[y(\tau_s)]^{n+1} - \int_0^{y(\tau_s)} |y'| dy.$$

Hence

$$\frac{1}{2}[y'(t_s)]^2 - \frac{1}{2}[y'(t_{s+1})]^2 > \int_0^{y(\tau_s)} |y'|_{\text{down. arc}} dy.$$

It is therefore necessary for consistency that

$$\int_0^{y(\tau_s)} |y'|_{\text{down. arc}} dy \rightarrow 0.$$

But as we trace y backwards from t_{s+1} up the downward arc $|y'|$ increases until $y^n = |y'|$, i.e. at least as far as $y^n = l$. Hence

$$\int_0^{y(\tau_s)} |y'|_{\text{down. arc}} dy > \int_0^{l^{\frac{n}{n+1}}} l dy = l^{\frac{n+1}{n}},$$

and the condition fails. It is therefore impossible that $l \neq 0$, and Case (b) itself is impossible. The proof of Theorem I. is completed.

The next stage depends upon an application of Hardy's results for equation (5a). Since t does not occur explicitly in equation (A) we can convert our equation into an equation of the first order in y and v , where $v = y'$, namely,

$$v \frac{dv}{dy} + v + y^n = 0 \quad (14)$$

Since by Theorem I. $y \rightarrow 0$ as $t \rightarrow \infty$ we are required to determine the asymptotic nature of the solutions of (14) as $y \rightarrow 0$, but it may be that $y = 0$, and therefore $y \rightarrow 0$ without $t \rightarrow \infty$. Equation (14) can be transformed by the substitution $y = 1/\eta$ into

$$-v\eta^2 \frac{dv}{d\eta} + v + \frac{1}{\eta^n} = 0 \quad (\eta \rightarrow \infty) \quad (15)$$

and then still further by the substitution $\eta = \zeta^a$ into an equation of form (5a). Thus to equation (15) and therefore to (14) we may apply Hardy's theorem that *there is an asymptotic relation of one of the forms*

$$a \succ b, \quad a \sim -b, \quad a \prec b \quad (16)$$

*between any two terms a and b as $\eta \rightarrow \infty$ ($y \rightarrow 0$).**

* The notation is that of Hardy, *Orders of Infinity*, Cambridge Tracts, No. 12. In exceptional cases, more than two terms being of equal highest order, we may have $a \sim ab$, a a constant (in Hardy's notation, $a \equiv b$) instead of $a \sim -b$.

We observe, further, that if any solution starts from given t_0 with finite values of y and y' it can only fail to be a *proper* solution by y or y' (y or v) becoming infinite as $t \rightarrow t_1$ ($> t_0$), and such an event can also be studied by means of Hardy's theorem. We shall thus prove Theorem II. :

Theorem II.—(1) *All solutions of (A) are proper solutions.* (2) *Solutions of (A) which never vanish for $t > t_0$ are such that either*

$$y = Ce^{-t} - \frac{\alpha C^n e^{-nt}}{n(n-1)} \quad (\alpha \sim 1), \quad . \quad . \quad [\text{E-solutions},$$

or

$$\pm y \sim \left[\frac{1}{(n-1)(t+C)} \right]^{\frac{1}{n-1}} . \quad . \quad [\text{M-solutions},$$

(1) Suppose $y \rightarrow \infty$ as $t \rightarrow t_1$. Then the only possibilities are

(a) $v \sim -y^n$, $v' \not\sim 1$. The latter implies $v \not\sim y$, which is a contradiction; or

(b) $v' \sim -1$, $vv' \not\sim y^n$. The former implies $v \sim -y$, $vv' \sim -y$, which is a contradiction; or

(c) $vv' \sim -y^n \not\sim v$. The former implies $\frac{1}{2}v^2 \sim -y^{n+1}/(n+1)$, which is consistent with the second relation, but is itself impossible owing to the essential difference of sign. There exist, therefore, no real solutions of (14) as $y \rightarrow \infty$.

Suppose, therefore, that $v = y' \rightarrow \infty$ as $y \rightarrow y_0$, $t \rightarrow t_1$. Then must $v' \sim -1$, $v \sim -y$, which contradicts $y \rightarrow y_0$. No solution, therefore, can cease to be a proper solution. This completes the proof of part (1).

(2) We have to determine possible forms of the proper solutions of equation (14), which are such that $y \rightarrow 0$, $t \rightarrow \infty$. By Hardy's theorem there are three possibilities.

Case (a). $vv' \sim -v \not\sim y^n$. This hypothesis yields

$$v \sim -y + C',$$

$$\frac{dy}{-y + C'} \sim dt,$$

which only allows $t \rightarrow \infty$ if $C' = 0$. In that case

$$y = Ce^{-t(1+\epsilon)} \quad (\epsilon \rightarrow 0)$$

consistently with our original hypothesis. If greater exactitude is required we may put $v = -y + z$ and repeat the argument. We find (the details are left to the reader) that we must have

$$v = -y + \frac{\alpha y^n}{n} \quad (\alpha \sim 1).$$

This equation is easily integrated approximately to give

$$y = Ce^{-t} - \frac{\alpha C^n e^{-nt}}{n(n-1)} \quad (\alpha \sim 1) \quad . \quad . \quad (17)$$

We shall see directly that such solutions actually exist and are in fact the solutions constructed by Emden and used by Eddington. A convenient name for them will therefore be E-solutions.

Case (b). $-v \sim y^n \succ vv'$. This hypothesis yields

$$\begin{aligned} -\frac{dy}{y^n} &\sim dt, \\ \frac{1}{y^{n-1}} &\sim (n-1)t + C, \\ \pm y &\sim \left[\frac{1}{(n-1)(t+C)} \right]^{\frac{1}{n-1}} \\ \pm \frac{dy}{dt} &\sim - \left[\frac{1}{(n-1)(t+C)} \right]^{\frac{n}{n-1}} \end{aligned} \quad (18)$$

which are consistent with $v' \prec 1$. We shall prove later that such solutions actually exist. They are, of course, those rediscovered by Milne, and it seems proper to refer to them as M-solutions.

Case (c). $vv' \sim -y^n \succ v$. This case is impossible, for it implies

$$\frac{1}{2}v^2 \sim C - \frac{y^{n+1}}{n+1}.$$

$C = 0$ does not allow of a real solution. $C \neq 0$ implies that $v^2 \sim C$ in contradiction with $y^n \succ v$, $y \rightarrow 0$. The proof of Theorem II. is thus completed. We can now proceed to Part II. of the discussion, in which we make a new start and obtain the rest of the information that is desired.

§ 3. *Study of Equation (A). Part II. The Actual Forms of the Solutions and their Arrangement.*—Emden has studied and actually constructed numerically a solution of the (θ, ξ) -equation (2) for various values of n , in particular $n = 3$ when it is equivalent to equations (A). Such methods of numerical integration are usually equivalent to an existence theorem. We may therefore assume that Emden's typical solution exists for $n = 3$, leaving open for the present for what other values of n similar solutions of equations (A) exist. Since $n = 3$ is the astrophysically important equation, this procedure is the most convenient. For completeness we give the necessary existence theorems themselves for $n > 1$ in § 6. We discuss therefore now odd values of $n > 1$ which have an Emden solution similar to that for $n = 3$. This solution has the following properties:—

$$\begin{aligned} \theta &= 1, \theta' = 0 \text{ at } \xi = 0, \\ [y &= 0, y' = 1, \text{ at } \xi = 0, y \sim \xi = e^{-t}]; \end{aligned}$$

$\theta(\xi)$ first vanishes for a certain value $\xi = \xi_0$, and $\theta'(\xi) < 0$ everywhere in the range $0 < \xi \leq \xi_0$. It is one of the E-solutions of Theorem II.

On referring to the (y, t) -equation, we see that it does not contain t explicitly, and therefore if $y = y(t)$ represents Emden's solution,

$y = y(t + C)$ is another solution for any C . Hence, if $y = f(\xi)$ is Emden's solution,

$$\left. \begin{aligned} y &= f(C\xi) \\ \theta &= \frac{1}{\xi} f(C\xi) \end{aligned} \right\} \quad (C \text{ any constant}) \quad (19)$$

is another solution. Any such solution is an E-solution of Theorem II. There exists therefore a family of E-solutions of the form (19), where f is the function constructed by Emden. In the (y, ξ) -plane they are of the simple half-sine-curve form shown in fig. 2. Every curve of the family is identical in type, only the ξ -scale changing from member to member. They have an envelope $y = m$, where m is the root of $f'(\xi) = 0$, a root easily shown to be the unique stationary value of $f(\xi)$ for the range $0 \leq \xi \leq \xi_0$, in which alone we are interested. Their various intersections must lie in the relative positions shown, and there

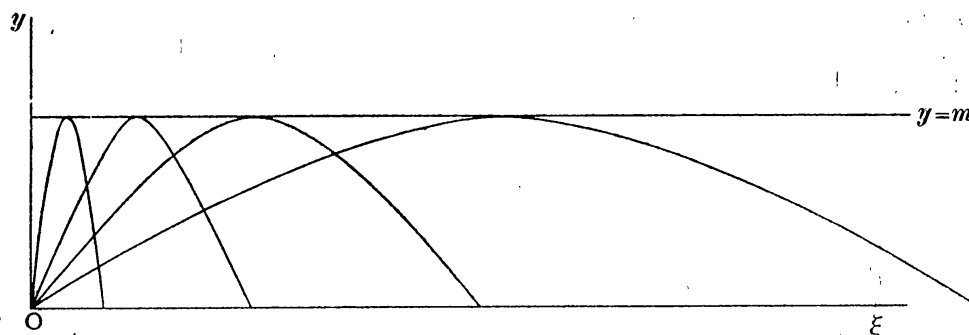


FIG. 2.

can be no others. Any one curve cuts in order all curves for greater C before it touches the envelope itself and after they have touched it. It cuts in order all curves of lesser C after it touches the envelope itself and before they touch it.*

Multiplication by $1/\xi$ to form the (θ, ξ) -curves distorts the picture, but cannot affect contacts or relative arrangements of intersections. It therefore yields the curves shown in fig. 3, no other arrangements or intersections being possible. The envelope is the rectangular hyperbola

$$\theta = m/\xi \quad (20)$$

The remaining part of the argument is most easily carried through in the (θ, ξ) diagram.

There are in fact no other E-solutions except this family. This is guaranteed by the uniqueness theorem for the solution of a differential equation (see § 6 for details), for our family includes all initial conditions of type $\theta(0) = C$, $\theta'(0) = 0$, which are characteristic of any E-solution. We need, however, to appeal to this theorem with care as it has here to be applied to a singular point $\xi = 0$. It is then sufficient to group the

* All the facts are intuitively obvious from the geometry in this simple case. In the more complicated case of § 9 a more detailed discussion is given which covers this case too.

solutions into this family of E-solutions and all other solutions, and proceed to discuss *what all the others can do*, remembering that they cannot be E-solutions.

With this object in view let us start from any initial point P ($\xi = \xi_1$, $\theta \geq 0$) and follow any solution through P inwards towards $\xi = 0$. It is convenient in this section to reckon slopes as positive when θ increases going towards the origin. The principle of the argumentation

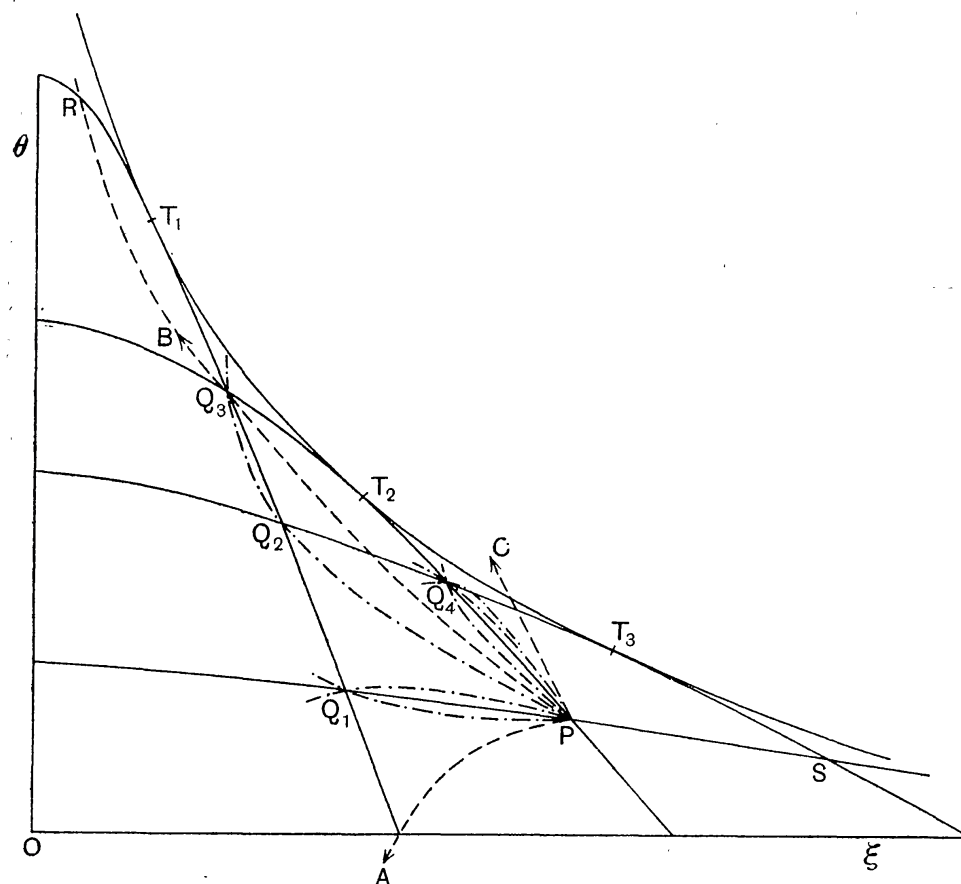


FIG. 3.

is that by the uniqueness theorem no other solution may ever touch any member of the family of E-solutions, for if it does it must be an E-solution itself.

Consider first any position of the point P below the envelope. Through such a P there pass two, and only two, E-solutions of slopes α and β , $\alpha > \beta$, both *positive* in the present sense. The corresponding E-solutions themselves may be referred to as E_α and E_β respectively—if necessary as $E_\alpha(P)$, $E_\beta(P)$. In the limiting case, when P is on the envelope, E_α and E_β coincide and $\alpha = \beta$. In the limiting case when P is on $\theta = 0$, $\beta = 0$, and E_β is $\theta = 0$. In fact, $\theta = 0$ is the limiting form of the E-solutions as $C \rightarrow 0$ and $\xi = 0$ their limit as $C \rightarrow +\infty$. The detailed behaviour of *any* solution through P can be deduced quite simply from the results of Part I. and the uniqueness theorem.

Case (1), $\theta' < \beta$. There are three *a priori* possibilities :

- (i) $\theta = \theta(\xi)$ cuts E_β again at ξ_2 , (Q_1), and $\theta(\xi) > 0$, $\xi_2 \leq \xi \leq \xi_1$.
- (ii) $\theta = \theta(\xi)$ remains between $\theta = E_\beta(\xi)$ and $\theta = 0$ as $\xi \rightarrow 0$.
- (iii) $\theta(\xi_2) = 0$, $0 < \xi_2 < \xi_1$.

(i) This is impossible by the uniqueness theorem. For consider the continuum of E-solutions for which $0 \leq C \leq C_\beta(P)$ (with an obvious notation). As C diminishes, their intersections with the two fixed curves $E_\alpha(P)$ and $E_\alpha(Q_1)$ move steadily downwards towards the line $\theta = 0$, which is the limit of the E-solutions as $C \rightarrow 0$. It is geometrically obvious that one of these E-solutions must touch the proposed solution, and this is impossible. The argument can be given rigorously, as follows. If for $C = C'$ the E-solution has points in common with $\theta = \theta_1(\xi)$, then it also has points in common with it for $C > C'$. If for $C = C'$ the E-solution has no points in common with $\theta = \theta(\xi)$ then it still has no points in common for $C < C'$. For a sufficiently small C the second type of E-solution exists. We can therefore make a Dedekind section in the C 's by the criterion of having or not having common points, and the section determines a C_0 for which the corresponding E-solution can only touch $\theta = \theta(\xi)$.

(ii) This is impossible, because by Theorem I. the solution must then be either an M-solution or an E-solution. It cannot be an M-solution because it remains below a given E-solution. It cannot be an E-solution since it is not (near P) a member of the family of E-solutions.

We conclude that if $\theta'(\xi_1) < \beta$ at P , $\theta = \theta(\xi)$ certainly cuts $\theta = 0$ at ξ_2 , $0 < \xi_2 < \xi_1$.

Case (2), $\alpha > \theta' > \beta$. There are five *a priori* possibilities, as follows. The lettering of points refers to fig. 3.

- (i) $\theta = \theta(\xi)$ remains between E_α and E_β as $\xi \rightarrow 0$.
- (ii) $\theta = \theta(\xi)$ first cuts E_β at ξ_2 , Q_1 , remaining below E_α for $\xi_2 \leq \xi < \xi_1$.
- (iii) $\theta = \theta(\xi)$ first cuts E_α at ξ_2 , Q_4 , between P and T_2 where E_α touches the envelope, or at T_2 , remaining above E_β for $\xi_2 \leq \xi < \xi_1$.
- (iv) $\theta = \theta(\xi)$ first cuts E_α at ξ_2 , Q_3 , beyond T_2 at such a slope that $\theta'(\xi_2) > \alpha(Q_3)$.
- (v) $\theta = \theta(\xi)$ first cuts E_α at ξ_2 , Q_3 , beyond T_2 at such a slope that $\alpha(Q_3) > \theta'(\xi_2) > \beta(Q_3)$.

(i) is obviously impossible by the argument already used in Case (1) (ii) above.

(ii) is impossible by the uniqueness theorem. For if we take the E-solution PQ_1 and increase its C , its intersections with the fixed E-solutions Q_1Q_2 and PQ_4 move up through Q_2Q_4 to a limiting position Q_3T_2 as C approaches $C_\alpha(P)$. Thus we must reach an E-solution which has no points in common with $\theta = \theta(\xi)$ between $Q_1Q_2Q_3$, and PQ_4T_2 , and therefore, by a Dedekind section, one that touches $\theta = \theta(\xi)$.

(iii) is impossible by the same argument. We keep the E-solutions

PQ_1 , Q_4Q_2 fixed and move their intersections with PQ_4 to the left by increasing the C of this E -solution. Since PQ_4 then has $\xi = 0$ as a limit, the E -solution must clear $\theta = \theta(\xi)$ and contact must occur.

(iv) is also impossible by the same argument. For if $\theta = \theta(\xi)$ is to cut $E_a(P) (\equiv E_\beta(Q_3))$ again at Q_3 at a slope greater than $\alpha(Q_3)$ it must have previously cut $E_a(Q_3)$ in order to get below it, at Q_2 say. Applying the same argument to the arc Q_2Q_3 we reach a contact and the case fails.

We conclude that if $\beta < \theta'(\xi_1) < \alpha$ at P then $\theta = \theta(\xi)$ remains above E_β and below E_α until it cuts the latter again at a point Q_3 beyond the contact of E_α with the envelope, and in such a way that $\beta(Q_3) < \theta'(\xi_2) < \alpha(Q_3)$.

We can, however, say much more than this, for the argument can be repeated for Q_3 as for P , leading to a new similar intersection at R , say, and so on indefinitely. The solution $\theta = \theta(\xi)$, starting in this way at P , as $\xi \rightarrow 0$, must ultimately pass above every E -solution and lie below the envelope; we conclude that $\theta = \theta(\xi) \rightarrow +\infty$ and is an M -solution.

Case (3), $\theta' > \alpha$. This case is rather more complicated and it is well to dispose of the easier alternatives first.

(i) $\theta = \theta(\xi)$ passes above the envelope. The following argument applies also to any solution started from a point P above the envelope. Such a solution cannot remain above the envelope, for then $y (= \xi\theta)$ does not tend to zero. It must therefore recross the envelope from above downwards (without touching it and therefore an E -solution). Since all slopes are continuous and on the envelope $\alpha = \beta = \text{slope of envelope}$, one can find a point P on the solution near and below the envelope where we have the inequalities of Case (1). There is, therefore, for any such solution a point $\xi_2 < \xi_1$ at which $\theta(\xi_2) = 0$.

(ii) $\theta = \theta(\xi)$ must not touch the envelope.

(iii) $\theta = \theta(\xi)$ remains below the envelope. In this case it must cut $E_a(P)$ again before T_2 , at Q_4 , $\xi = \xi_2$, say. It may cut it so that $\theta'(\xi_2) < \beta(Q_4)$. In that case we are back again in Case (1) and $\theta(\xi_2') = 0$, $\xi_2' < \xi_1$. It may finally cut $E_a(P)$ again at Q_4 , so that

$$\alpha(Q_4) > \theta'(\xi_2) > \beta(Q_4).$$

If it does this, as shown in fig. 3, we have a contradiction with the uniqueness theorem. For keeping Q_4T_3 , PS fixed let the E -solution Q_4P move to the right by diminishing C towards its limiting position T_3S . Unless $\theta = \theta(\xi)$ also intersects T_3S we form a Dedekind section and have a contradiction with the uniqueness theorem. We are left, therefore, with the curious alternative shown more clearly in fig. 4. If there were more than one such arc $Q_4'P'$ we could make a definite choice of that one nearest to T_3 .

Consider more closely the arc $P'Q_4'$. The relationship of $\theta'(\xi)$ to α and β at P' is the same as at P . It must also be the same at Q_4' as at Q_4 or we should meet obvious contradictions. We therefore can repeat the whole argument and find a contradiction with the uniqueness theorem, unless an internal part of the arc $P'Q_4'$ of $\theta = \theta(\xi)$ lies to the right of $T_3'S'$. The argument can be repeated indefinitely, unless we

meet with a contradiction with the uniqueness theorem after a finite number of steps. At each step the points $P^{(s)}Q_4^{(s)}$ move monotonically towards each other along the given solution $\theta = \theta(\xi)$. They cannot have non-coincident limits $P^{(l)}Q_4^{(l)}$, for we could always repeat the argument for the limiting arc itself. We should therefore necessarily have $P^{(l)} \equiv Q_4^{(l)}$ at which point $\theta = \theta(\xi)$ would touch an E-solution and contradict the uniqueness theorem. If, therefore, $\theta'(\xi_1) > \alpha(P)$

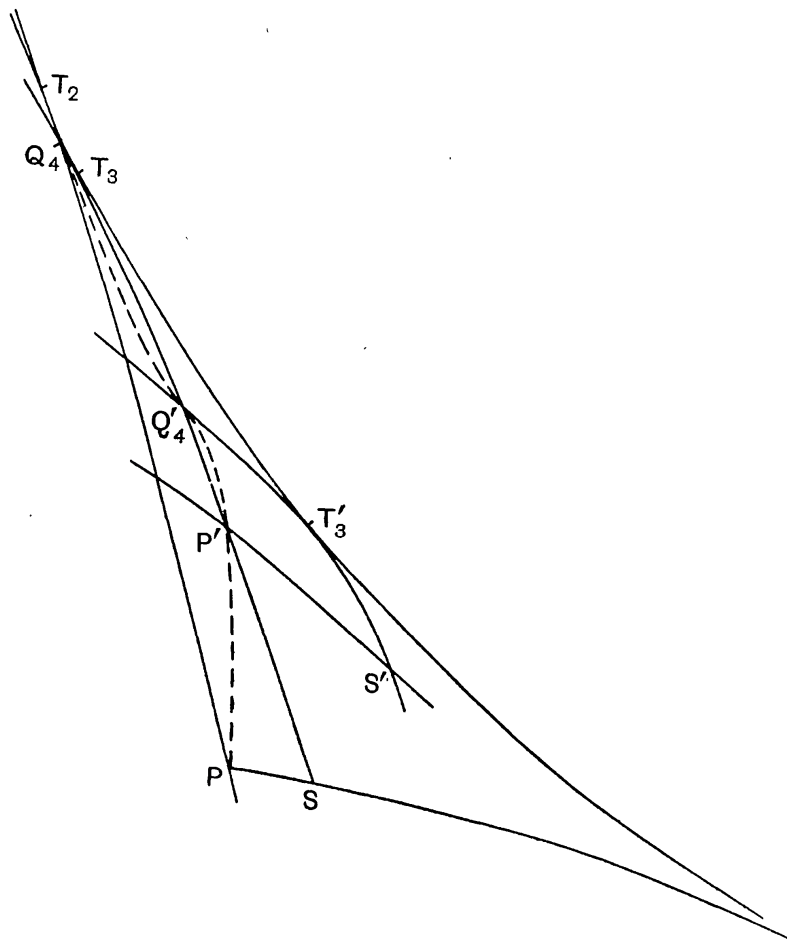


FIG. 4.

and $\theta = \theta(\xi)$ remains below the envelope, it is impossible that $\alpha(Q_4) > \theta'(\xi_2) > \beta(Q_4)$. We conclude that if $\theta'(\xi_1) > \alpha(P)$ there exists a $\xi_2' < \xi_1$ such that $\theta(\xi_2') = 0$.*

We collect the results of this section in Theorem III.

Theorem III.—Through any point P , ξ_1 , below the envelope, $\theta = m/\xi$, of the E-solutions of equation (A) pass two E-solutions with slopes α, β , $\alpha > \beta$. (1) If $\alpha > \theta'(\xi_1) > \beta$, $\theta = \theta(\xi)$ is an M-solution and $\theta(\xi) \rightarrow +\infty$ as $\xi \rightarrow 0$. (2) If $\theta'(\xi_1) = \alpha$ or $\theta'(\xi_1) = \beta$ the solution is an E-solution.

* It seems to be possible to prove by a similar argument the more exact result that, if $\theta'(\xi_1) > \alpha(P)$, $\theta = \theta(\xi)$ rises above the envelope. This result, however, would not be of any great value to us here.

(3) If $\theta'(\xi_1) > \alpha$ or $\theta'(\xi_1) < \beta$ the solution cuts $\theta = 0$ again at a point ξ_2 such that $0 < \xi_2 < \xi_1$.

Any solution through a point P, ξ_1 , above or on the envelope cuts $\theta = 0$ again at ξ_2 , where $0 < \xi_2 < \xi_1$, except that through a point P on the envelope there passes the single E-solution which touches the envelope there.

This theorem is the part of our work which is of chief importance to astrophysicists.

§ 4. *Study of Equation (A). Part III. Non-existence of Oscillating Solutions as $\xi \rightarrow 0$, $t \rightarrow \infty$.*—We can now use the results of the last section to complete our study. We proved in Part I. that if $y = y(t)$ continually oscillates about $y = 0$ as $t \rightarrow \infty$, then $|y'(t)| \rightarrow 0$. Therefore as $\xi \rightarrow 0$

$$|y'(\xi_s)| = o\left(\frac{1}{\xi}\right), \quad |\theta'(\xi_s)| = o\left(\frac{1}{\xi^2}\right).$$

But, at the zero point of an E-solution $\theta = \theta_E(\xi)$,

$$f(C\xi) = 0, \quad C\xi = q;$$

$$\theta_E'(\xi) = \frac{d}{d\xi} \left\{ \frac{1}{\xi} f(C\xi) \right\} = \frac{C}{\xi} f'(q) = \frac{q}{\xi^2} f'(q).$$

Hence if the solution continually recrosses the axis $\theta = 0$ as $\xi \rightarrow 0$ we must arrive at a zero, while still $\xi \geq \epsilon > 0$, at which

$$|\theta'(\xi)| \leq |\theta_E'(\xi)|.$$

But this must be the last zero; for the conditions of Theorem III. (1) or (2) are now satisfied and the solution now in general tends to $\pm \infty$ as an M-solution or, in the particular case of equality to $\pm C$, as an E-solution. Though we have argued all the time about positive values of y , all arguments transfer at once to negative values so long as $(-y)^n = -(y)^n$.

As a study in the form near infinity of the proper solutions of our differential equations (A) we can combine our results into the following theorem:—

Theorem IV.—As $x, t \rightarrow \infty$ no proper solution of the equations (A)

$$\theta'' + x^{-n-1}\theta^n = 0, \quad y'' + y' + y^n = 0,$$

where $n = p/q > 1$, and p, q are odd, can oscillate about zero. In terms of (y, t) , every solution $y = y(t) \rightarrow 0$ as $t \rightarrow \infty$ in one or other of the asymptotic forms

$$y = Ce^{-t} - \frac{aC^ne^{-nt}}{n(n-1)} \quad (a \sim 1),$$

$$\pm y \sim \left[\frac{1}{(n-1)(t+C)} \right]^{\frac{1}{n-1}}.$$

The first form is exceptional.

The proof of this theorem is so far completed for $n = 3$ and for any other values of n to which Emden's construction applies. We shall complete the justification of the theorem for the full range $n > 1$

by proving *ad hoc* an existence and uniqueness theorem for the family of E-solutions, in § 6.

§ 5. *Study of Equations (A) when $n = p/q > 1$ but p is even and q is odd, or q is even.* The study of the first case, characterized by

$$(-y)^n = y^n$$

is somewhat simpler than the case whose discussion has just been completed. For positive y there is no essential difference. Every argument which we have used in Theorems I.–IV. remains unaffected so far as it applies to the region $y > 0$. It is only when $y = y(t)$ crosses the axis of y that any difference can enter. When, however, this happens we see on putting $y = -\eta$ that we have then the form

$$\eta'' + \eta' - \eta^n = 0 \quad \eta > 0 \quad . \quad . \quad . \quad (21)$$

Let us consider what happens to any solution $\eta = \eta(t)$ of equation (21), which starts at $t = t_0$, say, with the initial conditions

$$\eta(t_0) = 0, \quad \eta'(t_0) > 0.$$

It is easy to see that $\eta'(t) > 0$ for $t > t_0$. For

$$\begin{aligned} \eta'' + \eta' &> 0, \\ \frac{d}{dt}(e^t \eta') &> 0, \\ \eta' &> e^{t_0-t} \eta'(t_0) > 0 \quad (t > t_0). \end{aligned}$$

Thus either $\eta(t) \rightarrow l$ or $\eta(t) \rightarrow +\infty$. But if $\eta(t) \rightarrow l$, it follows that

$$\begin{aligned} \frac{d}{dt}(e^t \eta') &> e^t(l^n - \epsilon) \quad (t > t_0), \\ \eta' &> \kappa > 0 \quad (t > t_0'), \end{aligned}$$

which contradicts $\eta(t) \rightarrow l$. Therefore $\eta(t) \rightarrow \infty$. The solution must therefore cease to be a proper solution unless also $t \rightarrow \infty$. But we can now consider

$$v \frac{dv}{d\eta} + v - \eta^n = 0 \quad (\eta \rightarrow +\infty),$$

using Hardy's theorem. There are three cases :

(i) $v \sim \eta^n \succ v \frac{dv}{d\eta}$. This implies

$$\begin{aligned} \frac{d\eta}{\eta^n} &\sim dt, \\ -\frac{1}{n-1} \left(\frac{1}{\eta^{n-1}} - \frac{1}{\eta_0^{n-1}} \right) &\sim t - t_0, \\ t &\rightarrow t_1. \end{aligned}$$

(ii) $\frac{dv}{d\eta} \sim -1$, $v \succ \eta^n$. But the first relation implies $v \sim -\eta$ which

Then

$$-v_s' < 1, \quad -v_s < y,$$

and

$$-v_s' > 1 - \frac{1}{A}y^{n-1}, \quad -v_s > y - \frac{1}{nA}y^n.$$

We can therefore choose A and y_0 , for example, by taking

$$y_0^{n-1} < nA(1 - A),$$

so that

$$Ay \leq -v_s \leq y \quad (0 < A < 1).$$

Therefore, since relation (23) is satisfied by v_1 , it is satisfied by v_s for all s .

It can next be shown that $\text{Lt}_{s \rightarrow \infty} v_s$ exists and is the function v we require.

For

$$\text{Lt}_{s \rightarrow \infty} v_s = v_1 + \sum_{s=1}^{\infty} (v_{s+1} - v_s) \quad . \quad . \quad . \quad (24)$$

and

$$v_{s+1}' - v_s' = \frac{y^n}{v_s v_{s-1}} (v_s - v_{s-1}),$$

$$|v_{s+1}' - v_s'| < \frac{y^{n-2}}{A^2} |v_s - v_{s-1}| \quad (0 \leq y \leq y_0).$$

If therefore

$$|v_s - v_{s-1}| < k_{s-1} \quad (0 \leq y \leq y_0),$$

$$|v_{s+1} - v_s| < k_{s-1} \cdot \frac{1}{A^2} \int_0^{y_0} y^{n-2} dy = k_{s-1} \cdot \frac{y_0^{n-1}}{(n-1)A^2},$$

$$< \epsilon \cdot k_{s-1} \quad (\epsilon < 1),$$

provided y_0 be chosen sufficiently small, since $n > 1$. Thus

$$|v_{s+1} - v_s| < \epsilon^s k_0,$$

which guarantees that the series (24) converges absolutely and uniformly, and that a function $v(y)$ exists such that

$$\text{Lt}_{s \rightarrow \infty} v_s = v(y)$$

uniformly in the range $0 \leq y \leq y_0$. Further, since in (23) A can be chosen as near 1 as we please by suitable choice of y_0 , we have also

$$v(y) \sim -y.$$

Further, $v = v(y)$ satisfies the differential equation, for

$$v(y) = -y - \text{Lt}_{s \rightarrow \infty} \int_0^y y^n dy / v_s(y),$$

$$= -y - \int_0^y \frac{y^n dy}{v(y)} - \text{Lt}_{s \rightarrow \infty} \int_0^y \frac{y^n dy}{v_s v(y)} \{v(y) - v_s\}.$$

But for $0 \leq y \leq y_0$, v_s , and therefore $v(y)$, satisfies (23), and since the convergence of (24) is uniform $|v(y) - v_s| < \epsilon$ for sufficiently large s over the whole of the same interval. The limit of the last term is therefore zero, and

$$v(y) = -y - \int_0^y \frac{y^n dy}{v(y)},$$

which is what we required to prove.

Finally, we can show that if $v^*(y)$ satisfies the conditions $v^*(y) \rightarrow 0$ ($y \rightarrow 0$), $|v^*| > Ay$ ($0 \leq y \leq y_0$), and is also a solution of (22), then $v^*(y) \equiv v(y)$, so that $v(y)$ is unique. For let Δ be the numerical value of the difference $v^*(y) - v(y)$. Then

$$\begin{aligned} \frac{d\Delta}{dy} &= \frac{y^n}{v^*(y)v(y)} \Delta \quad (0 \leq y \leq y_0), \\ &< Ky^{n-2} \Delta \quad (0 \leq y \leq y_0), \end{aligned}$$

where K is a definite constant. Thus

$$\log \frac{\Delta(y)}{\Delta(y_1)} < \frac{K}{n-1} (y^{n-1} - y_1^{n-1}) \quad (0 \leq y, y_1 \leq y_0).$$

Now let $y_1 \rightarrow 0$. Then $\Delta(y_1) \rightarrow 0$, but the right-hand side remains bounded since $n > 1$. Therefore $\Delta(y) = 0$, and the solutions are identical. We combine these results into the following lemma. Once it is away from the critical point the solution can be continued uniquely by the usual methods.

Lemma I.—There exists a solution $v = v(y)$ of the equation

$$vv' + v + y^n = 0 \quad (n > 1),$$

which satisfies $v \sim -y$ as $y \rightarrow 0$, and this solution is unique.

The unique $v = v(y)$ leads of course to the corpus of E-solutions. It is well to trace through this last step. We have

$$v = \frac{dy}{dt} = -\xi \frac{dy}{d\xi} = v(y) = -y + \frac{\alpha y^n}{n} \quad (25)$$

where $\alpha \sim 1$ as $y \rightarrow 0$. Therefore

$$\int_{\xi_0}^{\xi} \frac{d\xi}{\xi} = \int_{y_0}^y \frac{dy}{-v(y)},$$

or, say,

$$\log \frac{\xi}{\xi_0} = g(y) - g(y_0),$$

where, as $y_0 \rightarrow 0$,

$$g(y_0) \sim \log y_0 + o(1).$$

There must exist a certain range, $0 \leq y \leq y_1$, in which $g(y)$ is a definite one-valued monotonic function of y . Now let $y_0 \rightarrow 0$. Then must also $\xi_0 \rightarrow 0$ and, in order that there may be any solution of equation (25), in such a way that

$$y_0/\xi_0 \rightarrow C. \quad (26)$$

In that case $\log C\xi = g(y)$.

Since $g(y)$ is a continuous monotonic function of y in the range $0 \leq y \leq y_1$ we can take exponentials and invert this functional relation, obtaining a unique relation

$$y = f(C\xi) \quad . \quad . \quad . \quad (27)$$

such that

$$y/\xi \rightarrow C, \quad (\xi \rightarrow 0), \quad (n > 1) \quad . \quad . \quad (28)$$

which is the family of E-solutions previously assumed.

It does not follow, of course, from what we have already proved that for all values of $n > 1$ the family (27) consists of curves which have a zero for $C\xi = X > 0$, as shown in fig. 2, and therefore with all the essential properties of Emden's curves for $n = 3$. This, however, follows at once for p, q odd from the fact that as $t \rightarrow \infty$ all the solutions of $y'' - y' + y^n = 0$ must oscillate about $y = 0$,* and therefore all the solutions of $y'' + y' + y^n = 0$ as $t \rightarrow -\infty$, and therefore $y = y(\xi)$ as $\xi \rightarrow +\infty$. For other values of n we have to consider the equation $y'' - y' + y^n = 0$, known not to have oscillating solutions, and ask further whether it can have a monotonic solution for which $y > 0$ as $t \rightarrow +\infty$. The various possibilities can all be handled with the help of Hardy's theorem applied to the equation

$$\frac{dv}{dy} = +1 - \frac{y^n}{v},$$

and it is easily found that there is no possibility of such solutions. Monotonic proper solutions exist only for $y < 0$ as $t \rightarrow +\infty$. The E-solutions, therefore, which start positive near $\xi = 0$ must either become negative or cease to be proper solutions—in either case by reaching a point ξ_0 at which $y(\xi_0) = 0$. From the (y, ξ) form of equation (A) y' must continually decrease from $\xi = 0$ to $\xi = \xi_0$, so that the simple geometrical form of fig. 2 of the family (27) is assured.

From the family (27) and condition (28) we can at once deduce the existence and nature of the θ, ξ curves. It is to be observed, however, that though they all possess the property

$$\theta(\xi) \rightarrow C \quad (\xi \rightarrow 0)$$

they do not all possess the property

$$\theta'(\xi) \rightarrow 0 \quad (\xi \rightarrow 0),$$

which, in fact, they possess only when $n > 2$, as may be seen from their asymptotic form (17). We find that

$$\begin{aligned} \theta'(\xi) &\rightarrow -\frac{1}{2}C^2 & (n = 2), \\ \theta'(\xi) &\rightarrow -\infty & (1 < n < 2). \end{aligned}$$

In this case there is a *cusp* at $\xi = 0$ with a vertical tangent. The figures 3 and 4 therefore no longer properly represent the θ, ξ curves.

* Fowler, *loc. cit.*, p. 344, Table VI.

Hence the terms $|\theta'(x_s)|$ form a decreasing monotonic sequence as $s \rightarrow \infty$.

Let us now trace such a solution more closely as it leaves $x = x_s$. Since $\theta'' < 0$ until $x = x_{s+1}$ we have

$$\theta'(x) \leq \theta'(x_s), \quad \theta(x) \leq \theta'(x_s)(x - x_s) \quad (x_s < x \leq x_{s+1}).$$

Therefore in this range

$$-\theta'' = x^\sigma \theta^n \leq x^\sigma (x - x_s)^n [\theta'(x_s)]^n.$$

Integrating this inequality,

$$\theta'(x_s) - \theta'(x) \leq [\theta'(x_s)]^n \int_{x_s}^x x^\sigma (x - x_s)^n dx.$$

It follows that $\theta'(x)$ cannot become negative until

$$\theta'(x_s) < [\theta'(x_s)]^n \int_{x_s}^x x^\sigma (x - x_s)^n dx;$$

that is never, so that the solution is after all monotonic, if

$$\begin{aligned} 1 &> [\theta'(x_s)]^{n-1} \int_{x_s}^{\infty} x^\sigma (x - x_s)^n dx, \\ &= [\theta'(x_s)]^{n-1} x_s^{\sigma+n+1} M, \end{aligned}$$

where M is a number depending only on σ and n . Since on a would-be oscillating solution $\theta'(x_s)$ diminishes as $s \rightarrow \infty$ and $x_s^{\sigma+n+1} \rightarrow 0$, this condition must eventually be satisfied and the solution must be ultimately monotonic, of one or other of the forms specified.

We can next show that monotonic solutions of type (iii), $\pm \theta = x^{-\frac{\sigma+2}{n-1}+\epsilon}$ ($\epsilon \rightarrow 0$), are impossible. The argument is the same for either sign when p, q are odd. Suppose, then, that

$$\theta = x^{-\frac{\sigma+2}{n-1}+\epsilon};$$

then

$$-\theta'' = x^{-2-\frac{\sigma+2}{n-1}+\epsilon'} \quad (\epsilon' \rightarrow 0),$$

and, since $\sigma + n + 1 < 0$,

$$-2 - \frac{\sigma+2}{n-1} > -1.$$

The last relation can therefore be integrated and takes the form

$$-\theta' = x^{-1-\frac{\sigma+2}{n-1}+\epsilon''} \quad (\epsilon'' \rightarrow 0),$$

which can be integrated again, giving

$$-\theta = x^{-\frac{\sigma+2}{n-1}+\epsilon'''} \quad (\epsilon''' \rightarrow 0).$$

On account of the sign this is necessarily inconsistent with the proposed form, which is therefore impossible.

A similar argument obviously shows that the relation

$$\theta \geq \lambda x^{-\frac{\sigma+2}{n-1}} \quad (x > x_0),$$

where λ is any positive constant, is impossible for any solution.

We can now assert therefore that all proper solutions are ultimately monotonic and of type (i) or (ii). We can further refine these asymptotic forms.

Type (i), $\theta = x^\epsilon$ ($\epsilon \rightarrow 0$). It follows that

$$\begin{aligned} -\theta'' &= x^{\sigma+\epsilon'} & (\sigma < -2), \\ -\theta' &= D - x^{\sigma+1+\epsilon''}, \\ -\theta &= Dx - C + x^{\sigma+2+\epsilon''}. \end{aligned}$$

This is inconsistent with our hypothesis unless $D = 0$, when it takes a form which we may write as

$$\theta = C - x^{\sigma+2+\epsilon}.$$

This can be made still more exact by repetition of the argument. For instance, we have by one more step

$$\theta = C - \frac{\alpha C^n x^{\sigma+2}}{(\sigma+1)(\sigma+2)} \quad (\alpha \sim 1) \quad . \quad . \quad (30)$$

This type is of course the family of E-solutions whose existence we shall establish rigorously in § 8. In terms of (y, ξ) this asymptotic form is

$$y = C\xi \left[1 - \alpha \frac{C^{n-1} \xi^{-(\sigma+2)}}{(\sigma+1)(\sigma+2)} \right] \quad (\alpha \sim 1) \quad . \quad . \quad (31)$$

Type (ii), $\theta = x^{1+\epsilon}$ ($\epsilon \rightarrow 0$). It follows that

$$\begin{aligned} \left. \begin{aligned} -\theta'' &= x^{\sigma+n+\epsilon'} & (\sigma+n < -1), \\ -\theta' &= -D - x^{\sigma+n+1+\epsilon''}, \\ \theta &= Dx + C - x^{\sigma+n+2+\epsilon''}, & (\sigma+n+2 < 0) \\ &= Dx + x^{\sigma+n+2+\epsilon''} & (\sigma+n+2 > 0) \end{aligned} \right\} \quad . \quad (30a) \end{aligned}$$

$$\begin{aligned} \left. \begin{aligned} \theta &= \frac{D}{\xi} + C - \xi^{-\sigma-n-2+\epsilon''} & (\sigma+n+2 < 0) \\ \theta &= \frac{D}{\xi} + \xi^{-\sigma-n-2+\epsilon''} & (\sigma+n+2 > 0) \end{aligned} \right\} \quad . \quad (31a) \end{aligned}$$

Both cases are consistent with the type form. These are the M-solutions. In either case they depend on two constants of integration, but when $\sigma+n+2 > 0$ the second must logically be absorbed in the ϵ'' at this stage of the approximation. More exact forms can be obtained by repetition of the argument.

§ 8. *Study of Equation (B). Part II. The Existence and Uniqueness of the E-solutions.*—It is already clear that, if the corpus of E-solutions have properties in any way similar to the corpus of E-solutions of

equation (A) when $\sigma + n + 1 = 0$, we shall be able to prove that the M-solutions and the E-solutions are related in the same way as before. This is the astrophysically important theorem. It will be well therefore to take the existence theorem next, as it cannot be handled quite so simply in two stages as for equation (A). Again there is no novelty in the arguments or the result [Lemma II], but I am unaware of a convenient treatment for reference.

Consider therefore the equation

$$y'' + \xi^{-\sigma-n-3}y^n = 0,$$

and attempt to construct solutions $y = y(\xi)$, satisfying $y \sim C\xi$, $y' \sim C$, by Picard's iterative process. To do this we may split the equation into a pair of simultaneous equations by putting $y' = z$ ($z \sim C$), thus

$$\left. \begin{aligned} y' &= z \\ z' &= -\xi^{-\sigma-n-3}y^n \end{aligned} \right\}, \quad (31b)$$

and solve by iteration with

$$\left. \begin{aligned} y_1 &= C\xi, & z_1 &= C, \\ y'_s &= z_{s-1} & (y_s(0) &= 0) \\ z'_s &= -\xi^{-\sigma-n-3}y_{s-1}^n & (z_s(0) &= C) \end{aligned} \right\}$$

We can then prove the following series of propositions. First, we can choose ξ_0 independent of s so that if

$$0 \leq z_{s-1} \leq C, \quad 0 \leq y_{s-1} \leq C\xi \quad (0 \leq \xi \leq \xi_0),$$

then, for the same ξ_0 ,

$$0 \leq z_s \leq C, \quad 0 \leq y_s \leq C\xi \quad (0 \leq \xi \leq \xi_0) \quad (32)$$

The proof of this is sufficiently obvious. Secondly, we can prove that the sequences y_s, z_s tend uniformly to limit functions $y(\xi), z(\xi)$ in a range $0 \leq \xi \leq \xi'_0$. For suppose that

$$|z_{s-1} - z_{s-2}| < k, \quad |y_{s-1} - y_{s-2}| < k\xi.$$

Then, since

$$(y_s - y_{s-1})' = z_{s-1} - z_{s-2},$$

$$|y_s - y_{s-1}| < k\xi.$$

Also

$$\begin{aligned} |(z_s - z_{s-1})'| &< \xi^{-\sigma-n-3}n(C\xi)^{n-1} |y_{s-1} - y_{s-2}|, \\ &< nkC^{n-1}\xi^{-(\sigma+3)}, \end{aligned}$$

$$\begin{aligned} |z_s - z_{s-1}| &< \frac{nkC^{n-1}}{-(\sigma+2)}\xi^{-(\sigma+2)}, \\ &< \delta k \quad (\delta < 1, \xi < \xi_0), \end{aligned}$$

where ξ_0 can be fixed when k and C are chosen. Repeating the argument we find that

$$|y_{s+1} - y_s| < \delta k\xi, \quad |z_{s+1} - z_s| < \delta k \quad (33)$$

The argument can be repeated indefinitely and the absolute and uniform convergence of the sequences follows. We can prove further, from the first of the relations (33), that

$$|y_s - y(\xi)| < \epsilon \xi \quad (0 \leq \xi \leq \xi_0) \quad (34)$$

uniformly in this interval, provided that s be great enough.

Thirdly, $y = y(\xi)$, $z = z(\xi)$, so constructed, satisfy the differential equations. For we have necessarily

$$y(\xi) = \text{Lt}_{s \rightarrow \infty} \int_0^\xi z_s d\xi, \quad z(\xi) = C - \text{Lt}_{s \rightarrow \infty} \int_0^\xi \xi^{-\sigma-n-3} y_s^n d\xi.$$

Therefore

$$\begin{aligned} y(\xi) &= \int_0^\xi z(\xi) d\xi + \text{Lt}_{s \rightarrow \infty} \int_0^\xi (z_s - z_s(\xi)) d\xi, \\ z(\xi) &= C - \int_0^\infty \xi^{-\sigma-n-3} [y(\xi)]^n d\xi + \text{Lt}_{s \rightarrow \infty} \int_0^\xi \xi^{-\sigma-n-3} \{[y(\xi)]^n - y_s^n\} d\xi, \end{aligned}$$

and with the help of the inequalities (32) and (34) it is easily shown that the limits are zero.

Finally, we can prove that any solution $y = y^*(\xi)$, $z = z^*(\xi)$ of the differential equations (31b) with the same initial conditions $y^*(0) = 0$, $z^*(0) = C$, which also satisfies the relation

$$y^*(\xi) = O(\xi),$$

must be identical with the solution we have just constructed. For if Δy , Δz are the differences of the two solutions, $\Delta y(0) = \Delta z(0) = 0$ and

$$\begin{aligned} (\Delta y)' &= \Delta z, \quad |(\Delta z)'| \leq k \xi^{-\sigma-n-3} |\Delta y| y^{n-1}, \\ &\leq k' \xi^{-(\sigma+4)} |\Delta y|. \end{aligned}$$

In the first place, from the O-relation we know that

$$|(\Delta z)'| \leq k'' \xi^{-(\sigma+3)}, \quad |\Delta z| \leq \frac{k''}{-(\sigma+2)} \xi^{-(\sigma+2)} < \mu$$

say, provided that $0 \leq \xi \leq \xi_0$. Therefore by the other equation for the same range

$$|\Delta y| < \mu \xi \quad (0 \leq \xi \leq \xi_0).$$

Therefore from the second equation again

$$|\Delta z| < \mu \delta \quad (\delta < 1),$$

provided only ξ_0 be chosen so that

$$\frac{k'}{-(\sigma+2)} \xi_0^{-(\sigma+2)} \leq \delta.$$

Without further tampering with ξ_0 we can repeat the process indefinitely, proving in succession that

$$|\Delta y| < \mu \delta \xi, \quad |\Delta z| < \mu \delta^2, \quad |\Delta y| < \mu \delta^2 \xi, \dots$$

and therefore that $\Delta y = \Delta z = 0$. Once we are away from the singularity, the unique continuation of this solution is guaranteed by the usual existence and uniqueness theorems. We summarize the results of this section in

Lemma II.—*The equation (B) has a set of proper solutions of the asymptotic form (30) or (31), and these solutions are the only solutions which satisfy the specified initial conditions and the relationship $\theta(\xi) = O(1)$.*

§ 9. *Study of Equation (B). Part III. The General Form of the E-solutions.*—We have already shown, equation (31), that the second approximation to the general E-solution of equation (B) is

$$y = C\xi \left[1 - \frac{C^{n-1}\xi^{-(\sigma+2)}}{(\sigma+1)(\sigma+2)} \right] \quad (35)$$

This is also the result of the second stage of the iterative process. The next stage of the iterative process is equivalent to writing

$$y'' = -\xi^{-(\sigma+3)} C^n \left[1 - \frac{C^{n-1}\xi^{-(\sigma+2)}}{(\sigma+1)(\sigma+2)} \right]^n,$$

integrating twice and adding the proper integration constants which give $C\xi$, the leading term. The form of the result of these operations must be

$$y = C\xi - C^n \xi^{-(\sigma+1)} P_0(C^{n-1}\xi^{-(\sigma+2)}),$$

where $P_0(x)$ is a power series in x , beginning with a non-zero constant term. It follows that

$$y = C\xi P_0(C^{n-1}\xi^{-(\sigma+2)}) \quad (36)$$

so that (35) and (36) are of the same essential form, which has been unaltered by a step of the iteration process. It follows that the final solution, constructed as the limit of the iteration process, must have the same form, which we may write

$$y = C\xi f(C\xi^{-\frac{\sigma+2}{n-1}}), \quad \theta = Cf(C\xi^{-\frac{\sigma+2}{n-1}}) \quad (37)$$

The E-solutions therefore are all essentially the same curve, being derivable one from another by mere changes of scale of both co-ordinates, instead of by change of ξ -scale only, as for equation (A).

From the form of the differential equation, we know that the (y, ξ) -solution has $y'' < 0$ so long as $y > 0$, so that its slope is continually diminishing from its value C at $\xi = 0$. It remains only to show that it has a zero. This follows from the form of the solutions of $y'' + \xi^{-\sigma-n-3}y^n = 0$ as $\xi \rightarrow \infty$. If $-\sigma - n - 3 = \sigma'$ then $\sigma' + 2 > 0$, $\sigma' + n + 1 > 0$. It therefore follows that as $\xi \rightarrow \infty$ the solutions must oscillate,* and therefore there exists a least q such that

$f(q) = 0$. From $\xi = 0$ to $C\xi^{-\frac{\sigma+2}{n-1}} = q$, where they first vanish, the Emden solutions all have a steadily decreasing $y'(\xi)$, and consist of a single loop of half-sine-curve type.

* See Fowler, *loc. cit.*, p. 325, Table III.

The E-solutions have an envelope, which should be given by

$$f(C\xi^{-\frac{\sigma+2}{n+1}}) + C\xi^{-\frac{\sigma+2}{n-1}}f'(C\xi^{-\frac{\sigma+2}{n-1}}) = 0.$$

This equation is of the form

$$\frac{d}{d\mu}\{\mu f(\mu)\} = 0 \quad . \quad . \quad . \quad . \quad (38)$$

and must have a unique root. For $\mu f(\mu)$ is the same as $\xi^{\tilde{\omega}-1}y$, where we have written $\tilde{\omega}$ for $-(\sigma+2)/(n-1)$ for shortness. A root of (38) is equivalent to a root of

$$\frac{d}{d\xi}(z) = \frac{d}{d\xi}(\xi^{\tilde{\omega}-1}y) = 0,$$

and z satisfies the differential equation

$$z'' + \frac{2(1-\tilde{\omega})}{\xi}z' + \frac{1}{\xi^2}[(-\tilde{\omega})(1-\tilde{\omega})z + z^n] = 0.$$

Since $\tilde{\omega} > 1$, $z'' + 2(1-\tilde{\omega})z'/\xi = 0$ only when $z = 0$, and therefore

$$\frac{d}{d\xi}(\xi^{2(1-\tilde{\omega})}z') = 0 \quad . \quad . \quad . \quad . \quad (38a)$$

only when $z = 0$. It follows that on the (y, ξ) loop on which z vanishes only at the ends z' vanishes once, and once only, so that there is a unique root of (38); otherwise there would be at least two internal zeros of z' and therefore a root of (38a), where $z \neq 0$. Any E-solution therefore touches this envelope where

$$C\xi^{-\frac{\sigma+2}{n-1}} = \nu,$$

$$\theta = \frac{y}{\xi} = Cf(\nu) = \frac{\nu f(\nu)}{\xi^{-\frac{\sigma+2}{n-1}}} \quad . \quad . \quad . \quad . \quad (39)$$

The various E-solutions can cut each other once only (other than at $y = 0$, $\xi = 0$), and no three can go through the same point. This is most easily seen by writing their equations in the form

$$\xi^{\tilde{\omega}-1}y = z(C\xi^{\tilde{\omega}}).$$

An intersection of two E-solutions means that

$$z(C_1\xi^{\tilde{\omega}}) = z(C_2\xi^{\tilde{\omega}}), \quad . \quad . \quad . \quad . \quad (40)$$

or that the function z assumes the same value for two different values of its argument. Since the (z, ξ) curve is a simple half-sine curve on which z' vanishes just once, equation (40) can hold for given ξ and C_2 for only one value of C_1 ($\neq C_2$), so that no third E-solution can pass through the intersection of any two. Again the family of (z, ξ) curves are simple half-sine curves identical in form with mere changes of ξ -scale.

made on equation (A); at a zero of θ the solution must cease to be a proper solution. When p is even this does not happen, but a crossing of the axis $\theta = 0$ from the positive side as x increases (ξ decreases) must lead to $\theta \rightarrow -\infty$. For on putting $\theta = -\phi$ we have the equation

$$\phi'' - x^\sigma \phi^n = 0 \quad (41)$$

and have to discuss a solution which starts at x_0 with $\phi(x_0) = 0$, $\phi'(x_0) = \delta > 0$. But equation (41) implies that so long as ϕ does not vanish, $\phi'' > 0$, $\phi' \geq \delta(x - x_0)$. Therefore $\phi \geq \delta(x - x_0)$ and cannot vanish and in fact $\phi \rightarrow +\infty$ as stated. So long as $\theta > 0$ all the arguments of the previous case remain unaltered. We summarize in

Theorem VIII.—The equation (B)

$$\theta'' + x^\sigma \theta^n = 0 \quad (\sigma + n + 1 < 0),$$

where $n = p/q > 1$ and one of p, q is even, cannot have a proper solution oscillating about $\theta = 0$ as $x \rightarrow \infty$. For positive values of θ proper solutions exist as $x \rightarrow \infty$. They are monotonic, are arranged in the same way, and have the same asymptotic forms as the corresponding solutions described in Theorems VI., VII.

§ 11. *Concluding Remarks.*—It obviously remains to discuss the equation $\theta'' + x^\sigma \theta^n = 0$ and its equivalent forms when $\sigma + n + 1 > 0$, though these have not yet proved of astrophysical interest. There are now three cases to consider: $\sigma + 2 > 0$, $\sigma + 2 = 0$, $\sigma + 2 < 0$. The last (if any) is probably the case of astrophysical importance, since usually $\sigma = -4$. When $\sigma + 2 > 0$ and n is odd, all the solutions oscillate;* when n is even all the solutions are of the form

$\theta = -x^{-\frac{\sigma+2}{n-1}+\epsilon}$. Fractional indices have not yet been considered, but clearly there can never in such a case be any astrophysically possible solutions with permanently positive θ as $x \rightarrow \infty$. When $\sigma + 2 = 0$ the equation can be reduced by the substitution $x = e^t$ to $\theta'' - \theta' + \theta^n = 0$, which has only oscillating solutions when n is odd,† and, when n is even, only monotonic solutions of the type

$$\theta = -(\log x)^{-\frac{1}{n-1}+\epsilon};$$

again nothing is astrophysically possible. The discussion of these cases is already fairly complete. When $\sigma + 2 < 0$ the discussion is still incomplete and one may attempt to push it further by means of the methods of this paper. But it is clear that considerable additional complication is possible, and since this paper has already run to inordinate length, any such attempts must be reserved for a separate communication.

In conclusion, I should record my thanks to Professor Milne, for it was only on hearing from him of the success of his renewed attack on Emden's equation that I recalled the possible relevance of my own previous studies of these equations and was led to attack them afresh, with the results given in this paper.

* Fowler, *loc. cit.*, p. 325, Tables II. and III.

† Fowler, *loc. cit.*, p. 343, Table V.

On a Point-Source Model of a Star. By T. G. Cowling, B.A.*(Communicated by Professor E. A. Milne.)*

1. At present very little is known about stellar models. In spite of controversies on the subject of stellar structure, very few attempts have been made to investigate and classify the various stellar types corresponding to a given model of a star. The model proposed by Eddington, for which $\kappa\eta = \text{const.}$, is the only model which has been investigated with any attempt at completeness: and in the discussion of this model there have been important gaps which are only now being filled. An attempt is made here to give a complete discussion of a particular model.

The model discussed below is a point-source model, *i.e.* a model all of whose energy is supposed to be generated at its centre. It is assumed that κ , the coefficient of opacity, and μ , the mean molecular weight of the gas, are constant through the model. It is not suggested that the model represents physical conditions accurately: but a full discussion of this model should give insight into the nature of stellar structure. The point-source model has been of some importance in considerations of stellar structure.* As Eddington pointed out, the model is a limiting case of stellar structure: it represents the maximum possible concentration of sources of energy towards the centre of the star. In consequence, any result derived from this model should supply an outer limit for results derived from other models. Again, there is reason to believe that the outer layers of a star do not produce any appreciable part of its energy. In consequence, throughout these layers the same laws will govern the variation of density and temperature as prevail in the corresponding layers of a point-source model. A knowledge of the behaviour of density and temperature in the outer layers of a point-source model will therefore be equivalent to a knowledge of their behaviour in the outer layers of a star.

2. It will be assumed that the model is composed of perfect gas. It will also be assumed that at any point there is a hydrostatic radiation-pressure, and local thermodynamic equilibrium. Then we have the equations

$$p = \frac{R}{\mu} \rho T \quad . \quad . \quad . \quad . \quad (1)$$

$$p' = \frac{1}{3} a T^4 \quad . \quad . \quad . \quad . \quad (2)$$

$$\frac{dp'}{dr} = - \frac{\kappa L \rho}{4\pi c r^2} \quad . \quad . \quad . \quad . \quad (3)$$

$$\frac{dp}{dr} + \frac{dp'}{dr} = - \frac{GM(r)\rho}{r^2} \quad . \quad . \quad . \quad . \quad (4)$$

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho \quad . \quad . \quad . \quad . \quad (5)$$

* Eddington, *M.N.*, **85**, 408, 1925; *Observatory*, **53**, 208, 1930. Larmor, *Observatory*, **53**, 167, 249, 1930.