

On the Advance of Periastron in Eclipsing Binaries.

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It is well known that if a small satellite moves about an oblate planet in an orbit of small eccentricity with a period P , the periastron will advance with a period P' such that

$$P/P' = \frac{r^2}{a^2}(e - \frac{1}{2}\phi),$$

where a is the radius of the orbit, r that of the planet, e the superficial ellipticity of the latter, and ϕ the ratio of centrifugal force to gravity at its equator. The value of $e - \frac{1}{2}\phi$ depends on the internal constitution of the planet, and ranges from $\frac{3}{4}\phi$ for a homogeneous body to zero for one in which all the mass is concentrated at the centre.

For two oblate spheroids in orbital motion about one another, the motion of periastron will be given, to the first order of small quantities, by

$$P/P' = \frac{1}{a^2}\{r_1^2(e_1 - \frac{1}{2}\phi_1) + r_2^2(e_2 - \frac{1}{2}\phi_2)\},$$

or setting $e - \frac{1}{2}\phi = k\phi$,

$$P/P' = \frac{1}{a^2}(k_1 r_1^2 \phi_1 + k_2 r_2^2 \phi_2).$$

In eclipsing binaries there is usually direct evidence from the radiation effect that the periods of revolution and rotation of the stars coincide (at least on the surface). If m_1 and m_2 are the masses,

$$\phi_1 = \frac{m_1 + m_2}{m_1} \frac{r_1^3}{a^3},$$

and we have

$$P/P' = \frac{m_1 + m_2}{a^5} \left\{ k_1 \frac{r_1^5}{m_1} + k_2 \frac{r_2^5}{m_2} \right\}.$$

If the ellipticity of the stars was due to their rotation alone, this equation would give the rate of the advance of periastron; but an additional ellipticity arises from the tidal distortion of each star by the other. The upward tidal force at the point on the equator of star 1 nearest to star 2 is $\frac{2m_2}{m_1 + m_2}$ times the centrifugal force at this point to the first order, while the downward force at all points 90° from this is half as great. To this order, the effect of this distortion will be independent of the other, and in the same proportion to the disturbing force, so that the difference between the longest diameter and the mean of the other two, due to tidal distortion, will be $\frac{3m_2}{m_1 + m_2}$ times that of the equatorial and polar diameters arising from the rotation. The perturbative effect is proportional to the difference between the diameter lying in the line of

centres and the mean of the other two, so that the tidal effect in star 1 is $\frac{6m_2}{m_1 + m_2}$ times the rotational. We find, then, for the whole rate of advance of periastron,

$$P/P' = k_1 \frac{r_1^5}{a^5} \left(1 + 7 \frac{m_2}{m_1}\right) + k_2 \frac{r_2^5}{a^5} \left(1 + 7 \frac{m_1}{m_2}\right).$$

The only eclipsing binary for which an advance of periastron appears to be definitely indicated is Y Cygni, for which Dugan and Waterfield (*Harvard Bulletin*, 856) find $P' = 5400P$. The two stars in this system are substantially equal in mass, and, according to Shapley (*Princeton Contributions*, No. 3, 89, 1915), they are equal in size, $\frac{r}{a} = 0.167$. Assuming that the k 's are equal, we find $k = 0.076$.

For a homogeneous body $k = 0.75$; for the Earth it is 0.47, for Jupiter 0.26, and for Saturn 0.12. It appears, therefore, that these stars must be even more condensed toward the centre than Saturn.

It is of much interest to compare this result with the theoretical values for stars of different constitution. The ellipticity of the surfaces of equal density within a mass of fluid rotating with uniform angular velocity, and with any assigned law of variation of density from the centre to the surface, may be compiled by equations due to Radau (Tisserand, *Traité de Mécanique Céleste*, tome II., p. 221, Paris, 1891).

For any such body

$$k = \frac{e - \frac{1}{2}\phi}{\phi} = \frac{3 - \eta_1}{2\eta_1 + 4},$$

where η_1 is the value at the surface of a quantity which is defined by the differential equation

$$a \frac{d\eta}{da} + \eta^2 - \eta - 6 + \frac{6\rho}{D}(\eta + 1) = 0.$$

Here a is the distance from the centre, on such a scale that $a = 1$ at the surface, D is the mean density and ρ the density at any point, and the integration starts with $a = 0, \eta = 0$.

Quadratures for the determination of η were made by the writer several years ago in connection with a study of stars built on Eddington's early model, in which the ratio β of gas pressure to total pressure was constant, and the equation of state of the gas was taken to be

$$P = \frac{R}{m} \rho T \frac{\rho_0}{\rho_0 - \rho},$$

where ρ_0 is the limiting density attainable under very great pressure.

When the inapplicability of this model became evident these computations were laid aside; now at last they may serve some useful purpose. For different values of the ratio of the central density ρ_c to ρ_0 they give spheres of different degrees of central condensation, which may be roughly indicated by the value of ρ_c/ρ_m , the ratio of the central to the mean density.

The quadratures gave the following results:

TABLE I.

$\frac{\rho_c}{\rho_0}$	$\frac{\rho_c}{\rho_m}$	k	$\frac{\rho_c}{\rho_0}$	$\frac{\rho_c}{\rho_m}$	k
0.00	54.2	0.0144	0.60	5.61	0.091
0.10	39.8	0.0175	0.70	3.65	0.147
0.20	27.5	0.0220	0.80	2.37	0.248
0.30	19.0	0.0290	0.90	1.53	0.430
0.40	12.8	0.0401	1.00	1.00	0.750
0.50	8.55	0.0586			

The first line of the table corresponds to a perfectly gaseous star on Eddington's model, the last to a homogeneous sphere.

The value derived from observation for Y Cygni corresponds approximately to $\rho_c = 6\rho_m$. This is a surprisingly small degree of central condensation. Other models (for example, Emden's polytropes) would give a different value of ρ_c/ρ_m for this value of k , but, in all probability, one of the same order of magnitude.

For stars built on Eddington's model, the rate of advance of the apse should be only one-fifth of the observed value.

For Y Cygni the spectral class is O9, and the mean density 0.17 that of the Sun. It appears improbable that the central condensation is as small as these calculations indicate. An explanation may be found in Jeans's proof (*M.N.*, **86**, 456, 1926) that, owing to "radiative viscosity" the inner portion of a star should rotate more rapidly than the outer layers. Except near the centre, the angular velocity is inversely proportional to the square of the radius, and in the central regions it may be ten times as great as at the surface.

An exact allowance for this effect would be difficult. To assume that the whole star rotates as fast as the inner layers will evidently exaggerate it; it may be concluded roughly that on Jeans's hypothesis the effect of rotational ellipticity may be ten times or even fifty times as great as has previously been calculated.

The tidal effect, which on the previous hypothesis amounts to three-quarters of the whole, will be unaltered, so that the rate of advance of periastron comes out from three to thirteen times that previously calculated.

The actual central condensations of Y Cygni should on this hypothesis be comparable with that of a uniformly rotating star for which k lies between 0.025 and 0.006, but is not likely to be much smaller. The values of ρ_c/ρ_m in Table I. are roughly proportional to those of $1/k$. On Jeans's theory, then, the central density in Y Cygni may be from 25 to 120 times the mean density, or from 6 to 30 times that of water. These are high values, but much lower than those suggested in a recent paper by Jeans (*M.N.*, **87**, 43, 1926), which are as high as 1000 for Sirius and 5000 for V Puppis (a star very similar to Y Cygni). With such high densities as these, most of the mass would be concentrated in a small volume near the centre, and even though the internal rotation were rapid the motion of the periastron would be slow.