

AN APPLICATION OF THE QUANTUM THEORY TO THE LUMINOSITY OF DIFFUSE NEBULAE

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ABSTRACT

H. N. Russell, on theoretical grounds, made the suggestion that the *emission spectrum* observed in nebulae is, in some way, *induced by radiations from neighboring stars of a high temperature*. This conclusion was reached independently by E. P. Hubble, who, by an extended study of observational data, settled this question beyond any reasonable doubt. He also found a number of *qualitative and quantitative relations* to exist between the *characteristics of the nebula and of the star involved in its luminosity*. In line with this idea, already widely accepted, the writer has aimed to work out a *quantitative treatment of the emission spectrum produced in diffuse nebulae*, using Hubble's quantitative results as a guide. The problem is simplified by assuming that the star emits radiation like a black body and that the nebula consists of hydrogen of which an appreciable amount is in the atomic form. It is assumed that the ultra-violet starlight beyond the head of the Lyman series is completely absorbed by the nebula, causing ionization of the nebular hydrogen atoms into protons and free electrons. The subsequent recombination of electrons and protons causes the re-emission by the nebula of a number of line spectra and continuous spectra, among others the Balmer series and the continuous spectrum at its head. Finally, an *approximate expression for the ratio of the photographic intensity of the nebula to the photographic intensity of the intercepted starlight* is obtained, and this ratio is calculated for different temperatures (T) of the star; it is unity for $T=33,000^\circ$. Hubble found this ratio to be approximately unity for stars of type B0 and O associated with diffuse nebulae, and hence the *theory requires for those stars an average temperature of about $33,000^\circ$ C*. A more detailed discussion of the observational data leads to the following rough values for the *temperature of stars associated with diffuse nebulae*: B1, $21,000^\circ$; B0, $28,000^\circ$; O, $34,000^\circ$. Stars in planetary nebulae would, in general, have a still higher temperature. These values are in fair agreement with the temperature determinations from the theory of thermal ionization of stellar atmospheres, and this agreement serves as a check for the mechanism of luminosity proposed. The fact that the continuous spectrum at the head of the Balmer series is more conspicuous in planetaries than in diffuse nebulae led to the assumption that free electrons of higher velocity are captured to a large extent by the lower levels of the hydrogen atom. This offers an explanation for the result of H. H. Plaskett in work not yet published, who finds that the first lines of the Balmer series in nebulae become relatively stronger, as the temperature of the associated star becomes higher. The assumption is also in harmony with the theoretical considerations of H. A. Kramers.

I. BRIEF REVIEW OF OBSERVATIONAL FACTS AND THEORETICAL IDEAS REGARDING GALACTIC NEBULAE

The galactic nebulae are usually subdivided into two classes, diffuse and planetary nebulae.

The planetary nebulae have a definite structure, appearing as

¹ National Research Fellow. The present discussion of nebular luminosity was, in its main features, given in a paper read before a meeting of the American Physical Society, under the same title (see *Physical Review*, 27, 644, 1926).

ovals, rings, or disks. Their spectrum consists of a number of single lines, and frequently there is also a continuous spectrum, starting approximately at the head of the Balmer series of hydrogen and extending into the ultra-violet. Following Hubble, a spectrum of this type will be classified as an "emission" spectrum. Each planetary nebula has a star approximately at its center.

The diffuse nebulae, on the other hand, do not show a definite sharp structure. Hubble divides them into dark nebulae, obscuring the background, and luminous nebulae. The luminous nebulae have either emission spectra, similar in a general way to those of the planetaries, or continuous spectra of approximately the same type as, although less intense than, those of ordinary stars. Frequently the three types of diffuse nebulosity, "dark," "continuous," and "emission," are found in the same regions of the sky and even merge more or less into one another, as for example in the large Orion nebula.

It is generally recognized at present that both the continuous and the emission spectrum observed in nebulae are, in some way, caused by neighboring stars.

In a number of cases V. M. Slipher¹ has found that the continuous spectra of certain nebulosities are approximately the same as those of stars in their neighborhood and suggests, for this reason, that those nebulae owe their luminosity to reflected starlight. E. Hertzsprung² estimated that the luminosity of the Pleiades nebulae is from 3.5 to 5.6 mag. fainter than would be the case for a perfect diffuse reflection, although a more recent statistical study of a large amount of observational material by Hubble leads to a much stronger luminosity for diffuse nebulae in general.

If the nebular spectrum is of the emission type, an explanation of the luminosity by simple reflection of starlight is out of the question, since the nebular spectrum is then entirely different from that of the star. H. N. Russell³ has suggested that such emission nebulae are excited in some way by radiation, either short wave-length or

¹ *Publications of the Astronomical Society of the Pacific*, **30**, 63, 1918.

² *Astronomische Nachrichten*, **195**, 448, 1913.

³ *Observatory*, **44**, 72, 1921; *Proceedings of the National Academy of Sciences*, **8**, 115, 1922.

corpuscular, the most reasonable source of such radiation being a very hot body, such as a star of early type.

An extended study by E. P. Hubble,¹ based on a large amount of observational material, establishes beyond any reasonable doubt that the luminosity of the different types of galactic nebulae is due to neighboring stars. In the first of his two papers a classification of nebulae is given, and he finds that with practically every luminous nebula a star is associated which can be held responsible for the nebular luminosity. There is a definite relation, which holds with very few exceptions, between the spectral characteristics of this star and of the nebulosity; stars of type B1 and later are usually associated with diffuse nebulosity of the continuous type, stars of type B0 and Oe5 with diffuse nebulosity of the emission type, and still earlier stars with planetary nebulae.

In the second paper a quantitative relation between the luminosity of a nebula and the luminosity of the associated star is derived by a statistical study of observational data.

This relation may be formulated as follows: (a) The light emerging from a patch of diffuse nebulosity having a continuous spectrum is the equivalent of the starlight intercepted (star type B1 and later).² (b) The light emerging from a patch of diffuse nebulosity having an emission spectrum is the equivalent of the starlight intercepted (stellar type notably B0 and Oe5). (c) The light emerging from a part of the nebulosity in a planetary nebula is, on the average, 4 to 5 mag. brighter than the equivalent of the starlight intercepted (type of central star earlier than Oe5).

Quantities of light are called "equivalent" if they produce the same blackening on the photographic plate under the conditions of observation.

This empirical relation will be our guide in discussing the theoretical mechanism by which the luminosity of the nebula may be caused.

¹ *Astrophysical Journal*, 56, 162, 400, 1922.

² Hubble's interpretation implies that the light emerging from the nebula is re-emitted evenly in all directions. A different result will therefore be obtained, if the continuous spectrum is assumed to be due to reflection of the starlight by the nebula. This question will be discussed in sec. 4.

2. LUMINOSITY PRODUCED IN A NEBULA CONSISTING OF
HYDROGEN BY A STAR EMITTING BLACK-
BODY RADIATION

Attention will be confined mainly to the production of an emission spectrum in the nebula. Apparently there are no cases known where a normal star of type B1 or later produces a nebular emission spectrum of appreciable intensity. The hotter stars of type B0 and Oe5 are able to produce a strong nebular emission spectrum. The stars in planetary nebulae, which have a higher temperature still, produce also a more intense nebular spectrum. This strongly suggests that the temperature of the associated star is the main factor in determining the intensity of the nebular emission spectrum.

In the discussion which follows the star will be replaced by a black body of surface temperature T . If the star be a sphere of radius R , the total radiation per second within the frequency interval $d\nu$ is

$$\pi R^2 c \rho_\nu d\nu, \quad (1a) \quad \rho_\nu = \frac{8\pi h}{c^3} \cdot \frac{\nu^3}{e^{\frac{h\nu}{kT}} - 1}, \quad (1b)$$

$$h = 6.54 \times 10^{-27}, \quad k = 1.372 \times 10^{-16},$$

the formula (1b) being Planck's formula for the energy density $\rho_\nu d\nu$ of black-body radiation.¹

The total number of quanta emitted per second within $d\nu$ is obtained by dividing this by $h\nu$ and therefore becomes

$$\frac{8\pi^2 R^2 k^3}{c^2 h^3} T^3 \frac{x^2}{e^x - 1} dx, \quad (2) \quad x = \frac{h\nu}{kT}. \quad (2a)$$

If Ω be the solid angle subtended by a patch of nebulosity from

¹ Well-known considerations of radiation equilibrium lead to the following result: The energy within $d\nu$ emitted by an area $d\sigma_1$ of a black body of temperature T and intercepted by a surface element $d\sigma_2$ at distance r is $I_\nu d\nu d\sigma_{1n} d\sigma_{2n}/r^2$, where $I_\nu = \rho_\nu c/4\pi$ and $d\sigma_{1n} d\sigma_{2n}$ are the projections of $d\sigma_1$ and $d\sigma_2$ on a plane perpendicular to r . The formula (1a) for the radiation emitted by a spherical body is easily obtained from this by considering as σ_1 the surface of the black body and as σ_2 a large concentric sphere, through which the radiation passes outward.

the center of the star, the number of quanta within $d\nu$ intercepted per second by this patch is

$$\Omega \frac{2\pi R^2 k^3}{c^2 h^3} T^3 \frac{x^2}{e^x - 1} dx, \quad (3)$$

as is found by multiplying (2) by $\Omega/4\pi$.

The number of intercepted quanta which, under observational conditions, can effect the photographic plate will be taken as a rough measure of the photographic intensity of the intercepted starlight. The long wave-length limit of an ordinary photographic plate is about 5050 Å, that for the atmospheric absorption is about 3300 Å corresponding to the frequencies $\nu_1 = 5.95 \times 10^{14}$ per second and $\nu_2 = 9.10 \times 10^{14}$ per second. If a reflector is used, only quanta having frequencies between those two limits can affect the photographic plate. The number of such "photographic quanta" in the starlight intercepted by the patch of nebulosity per second is, therefore,

$$N_{ph} = \Omega \frac{2\pi R^2 k^3}{c^2 h^3} T^3 \int_{x_1}^{x_2} \frac{x^2}{e^x - 1} dx, \quad (4) \quad x_1 = \frac{h\nu_1}{kT}, \quad (4a) \quad x_2 = \frac{h\nu_2}{kT}, \quad (4b)$$

which is an approximate measure of the photographic intensity of the starlight intercepted.

The mechanism by which the radiation intercepted by a patch of nebulosity can be transformed into an emission spectrum will now be considered. The observed emission spectra, in which the Balmer series of hydrogen is very conspicuous, indicate that the diffuse nebulae are composed of certain gases, of which atomic hydrogen is the most active in producing the luminosity. Therefore the problem will be simplified by assuming the nebula to consist wholly of hydrogen, an appreciable amount of which is atomic. Such a nebula may absorb a part of the radiation which it intercepts, and for this only atoms in the normal state need to be considered.¹ The quantum theory provides two mechanisms by which this absorption may occur. In the first place, a hydrogen atom in its normal state may

¹ Because the incident radiation has a very low density, it may reasonably be expected that the fraction of neutral atoms in the excited states is negligible, although it must be admitted that the present knowledge of the statistics of the atom is insufficient to decide this question.

absorb the different frequencies of the Lyman series, and the atom thus excited will subsequently return to its normal state, either directly or in steps, the electron falling back to the different energy-levels before it reaches the lowest level. In doing this it re-emits a number of line spectra, among others the Lyman series and the Balmer series. The mechanism of ordinary excitation accounts in this way for some luminosity in the nebula. This effect, however, is much too weak to account for the total luminosity observed, and it will not be further considered.

The second, and much the more powerful, mechanism which the quantum theory provides is that of ionization and subsequent recombination. An atom in its normal state is capable of absorbing all the energy of a wave-length shorter than the head of the Lyman series, frequency $\nu_0 = 32.84 \times 10^{14}$ per second. The number of quanta per second thus absorbed from the ultra-violet starlight by a patch of nebulosity is, in view of (3),

$$N_{ul} = \Omega \frac{2\pi R^2 k^3}{c^2 h^3} T^3 \int_{x_0}^{\infty} \frac{x^2}{e^x - 1} dx, \quad (5) \quad x_0 = \frac{h\nu_0}{kT}, \quad (5a)$$

provided that the absorbing layer contains a sufficient number of normal atoms for the absorption to be practically complete.

For each quantum $h\nu$ thus absorbed, an electron, mass m , is knocked out of the atom and becomes a free electron with the velocity v given by the photo-electric equation

$$h\nu = W + \frac{1}{2} m v^2, \quad (6)$$

where $-W = -W_1 = h\nu_0$ is the energy of the first atomic level. These free electrons will recombine with the nuclei by falling back to one of the different levels, energies $-W_1, -W_2, -W_3$, etc., emitting the continuous spectrum at the head of the Lyman series, at the head of the Balmer series, at the head of the Paschen series, etc., according to the same equation (6). Subsequently the electrons on the higher levels 2, 3, 4, etc., will return to the normal level 1, either directly, under emission of the Lyman series, or in steps, emitting the other hydrogen lines, among others the Balmer series. In this way a number of continuous spectra and line spectra will be re-emitted,

and this would account for the Balmer series in the nebular spectrum as well as the continuous spectrum sometimes observed at its head.

The next question is the intensity of the re-emitted spectra. To solve this completely would require a complete knowledge of the probabilities of recombination with the levels 1, 2, 3, etc., as well as of the probabilities of the transitions in the hydrogen atom, or at any rate of their ratios. Without going into details, however, a few general statements can be made. Any ionized atom will have to return, sooner or later, to its normal state, where the electron is in its first energy-level. The very last electron jump is therefore either a jump of a free electron to this first level or a jump from one of the higher levels to this first level, the corresponding frequencies emitted being those of the continuous spectrum at the head of the Lyman series and the lines of the Lyman series. Assuming a stationary state of the nebula, viz., that just as many atoms become ionized per unit time as return from the ionized to the normal state, then, since the number of electrons emitted photo-electrically from the first level is equal to the number of ultra-violet quanta absorbed, N_{ul} , the number of electrons falling back to the first level is also N_{ul} . Hence the number of quanta re-emitted by the nebula, either as the continuous spectrum at the head of the Lyman series or as lines of the Lyman series, is equal to N_{ul} , or symbolically

$$Ly_c + Ly = N_{ul}, \quad (7)$$

where Ly_c and Ly represent the numbers of quanta of the two different kinds.

There is reason to believe, however, that practically all the Lyman quanta will ultimately leave the nebula as the first line of the Lyman series $Ly\alpha$. To fix the ideas, take the line $Ly\beta$ as an example. This line, after being produced for the first time, can, on its way out of the nebula, excite another hydrogen atom to the level 3; this second atom may then either re-emit the first line of the Balmer series $H\alpha$ and subsequently $Ly\alpha$, or it may re-emit $Ly\beta$. In the second case the quantum $Ly\beta$, on its way out of the nebula, may again be absorbed by another hydrogen atom and be re-emitted, either as a quantum $H\alpha$ and a quantum $Ly\alpha$, or as the quantum $Ly\beta$. If this secondary absorption and re-emission goes on a sufficient

number of times, all the original radiation $Ly\beta$ will have been split up into $H\alpha$ and $Ly\alpha$. In other words, the splitting up of all quanta $Ly\beta$ into $H\alpha$ and $Ly\alpha$ will be complete, if the absorption coefficient of $Ly\beta$ is sufficiently large compared with the original absorption of the continuous spectrum. This condition is presumably fulfilled, because the absorption coefficient for a single spectral line may in general be expected to be much larger and perhaps even of an entirely different order than the absorption coefficient for the continuous spectrum at the head of the series. Carrying out a similar reasoning for the lines $Ly\gamma$, $Ly\delta$, etc., we arrive at the conclusion

$$Ly = Ly\alpha . \quad (8)$$

The next point of interest is the Balmer series and the continuous spectrum at its head. These are produced, if an electron falls back to the level 2, a process which always precedes the production of one quantum $Ly\alpha$. Hence the number of quanta of these two kinds Ba_c and Ba is equal to the number of quanta $Ly\alpha$.

$$Ba_c + Ba = Ly\alpha .$$

This, together with (7) and (8), leads to

$$Ly_c + Ba_c + Ba = N_{ul} . \quad (9)$$

Therefore a number of quanta N_{ul} emerge from the patch of nebulosity, partly as Ly_c , partly as Ba_c , and partly as Ba .

Only those quanta with frequencies in the interval $\lambda_1 = 5050 \text{ \AA}$ to $\lambda_2 = 3300 \text{ \AA}$ can affect the ordinary photographic plate under observational conditions. Hence the quanta Ly_c and the quanta of the first line of the Balmer series $H\alpha$ do not affect the photographic plate, but the remaining quanta Ba_c and the quanta of the other lines of the Balmer series $H\beta$, $H\gamma$, $H\delta$, etc., are all "photographic quanta." The number of these emerging from the nebula is therefore

$$N'_{ph} = N_{ul} - (Ly_c + H\alpha) , \quad (10)$$

which is smaller than the number of absorbed ultra-violet quanta by the amount $Ly_c + H\alpha$.

The line $H\alpha$ in the nebular spectrum, taken with panchromatic plates, though it is strong, shows an intensity comparable with the

other lines $H\beta$ and $H\gamma$. The continuous spectrum at the head of the Lyman series, arising from the capture of free electrons by level 1, is not accessible to direct observation, but it is reasonable to expect that its intensity in quanta is of the same order as that of the continuous spectrum Ba_c at the head of the Balmer series, which, in a similar way, is produced by the capture of free electrons by level 2. The spectrum Ba_c in diffuse nebulae, as far as is known, has only been observed in one case.¹ For planetary nebulae, where the temperature of the associated star is much higher, this spectrum Ba_c becomes more conspicuous. It seems safe, therefore, to assume that, for diffuse nebulae, the quantity Ba_c , and, by indirect argument, the quantity Ly_c is either negligible or of the same order as the number of quanta in the lines $H\beta$ and $H\gamma$. The physical nature of this assumption is probably that electrons of rather small velocity, as produced by a star of lower temperature, are mostly recaptured on the higher levels of the hydrogen atom, whereas, for electrons of higher velocity, freed in the planetary nebulae, the chance of being captured by the lower levels becomes much larger. On this picture, the nebular spectrum approaches more to a pure line spectrum as the temperature of the associated star gets lower, which is in line with the observational evidence. The considerations just given indicate that, for diffuse nebulae at any rate, the quantity $Ly_c + Ha$ occurring in equation (10) is of the same order as the quantity N'_{ph} , and, therefore, as regards order of magnitude

$$N'_{ph} \sim N_{ul} . \quad (11)$$

Combining (11) with (4) and (5), the ratio of the number N'_{ph} of photographic quanta emerging from a patch of nebulosity to the number N_{ph} of photographic quanta in the intercepted starlight is found to be approximately

$$\frac{N'_{ph}}{N_{ph}} \sim \frac{N_{ul}}{N_{ph}} = \frac{\int_{x_0}^{\infty} \frac{x^2}{e^x - 1} dx}{\int_{x_1}^{x_2} \frac{x^2}{e^x - 1} dx} , \quad (12)$$

$$x = \frac{h\nu}{kT} , \quad \nu_0 = 32.84 \times 10^{14} , \quad \nu_1 = 5.95 \times 10^{14} , \quad \nu_2 = 9.10 \times 10^{14} .$$

¹ E. P. Hubble, *Publications of the Astronomical Society of the Pacific*, **32**, 155, 1920.

This is also a rough measure of the ratio L of the photographic intensity of the emission spectrum of a patch of nebulosity to the photographic intensity of the intercepted starlight.

If quanta of an entirely different frequency were compared, the assumption that the number of quanta between the two assigned limits is a measure of photographic intensity would not be allowed, even as an approximation. But in the final equation (12) the inaccuracy introduced by this assumption of "photographic quanta" is only due to the difference in distribution of intensity of the continuous spectrum of the star as compared with the nebular emission spectrum, both covering a large range of frequency, and here the procedure followed, although very inaccurate, should give the right order of magnitude for the luminosity ratio L .

The values of the integral occurring in (10), taken between the limits x and ∞ , are given in Table I for different values of x . The integrals between subsequent values of x were obtained graphically with an accuracy of at least 1 per cent, except for the first two intervals 0–0.2 and 0.2–0.3, where the accuracy is 2 per cent. For values of x larger than 6 the numerator in the integral was replaced by e^x and the integral solved by the ordinary process; the accuracy is then at least $\frac{1}{4}$ per cent, since $e^6 = 403.4$. In the table, only four decimals were given.

With the aid of Table I the values of $\int_{x_0}^{\infty}$ and $\int_{x_1}^{x_2}$ were determined by graphical interpolation for a series of temperatures T from $15,000^\circ$ up to $200,000^\circ$. The values of those integrals and their ratio $\int_{x_0}^{\infty} / \int_{x_1}^{x_2}$ are given in Table II. The last column of Table II contains the value of this ratio, expressed in magnitudes.

Table II shows that the ratio L is a function of the temperature T of the associated star which increases very rapidly with T , hence if L is known from observations, the temperature of the associated star required by the mechanism of ionization and recombination may be calculated. For diffuse nebulae with emission spectra, where the stars are predominantly of the types Bo and Oe5, Hubble found empirically that the nebular luminosity is the equivalent of the inter-

cepted starlight, and hence $L = 1$ (sec. 1, b). The corresponding temperature, as found from Table II, is approximately $33,000^{\circ}$. To account for the nebular luminosity observed, the theoretical mechanism thus requires an average temperature of $33,000^{\circ}$ for Bo and O

TABLE I
VALUES OF $\int_x^{\infty} \frac{x^2}{e^x - 1} dx$

x	$\int_x^{\infty}$	x	$\int_x^{\infty}$	x	$\int_x^{\infty}$
0.0.....	2.4002	2.....	1.4176	7.....	0.0593
0.2.....	2.3824	3.....	0.8634	8.....	.0275
0.3.....	2.3606	4.....	.4792	9.....	.0125
0.6.....	2.2562	5.....	.2496	10.....	0.0055
1.....	2.0516	6.....	0.1240		

TABLE II

T	$\int_{x_0}^{\infty}$	$\int_{x_1}^{x_2}$	Ratio $\int_{x_0}^{\infty} / \int_{x_1}^{x_2} \propto L$	Ratio in Magnitudes
15,000°.....	0.0043	0.57	0.0075	5.31
20,000.....	.032	.48	.066	2.95
25,000.....	.103	.38	.271	1.42
30,000.....	.216	.30	0.72	0.36
35,000.....	.35	.25	1.41	— .37
40,000.....	.50	.20	2.50	—0.99
50,000.....	0.80	.150	5.4	—1.83
70,000.....	1.27	.083	15.3	—2.96
100,000.....	1.68	.044	38	—3.95
150,000.....	2.02	.020	101	—5.01
200,000.....	2.16	0.0115	185	—5.68

T is the temperature of the star producing the luminosity. The ratio in the fourth and fifth columns represents the approximate theoretical value for the ratio L of the photographic intensity of the nebula to the photographic intensity of the starlight intercepted.

stars associated with diffuse nebulae. This value of the temperature seems quite reasonable, and serves as a first check of the proposed mechanism of luminosity. In the next section, a more detailed discussion of the observational data will be given.

3. COMPARISON WITH OBSERVATION; TEMPERATURE
OF BO AND O STARS

Hubble's method for obtaining the empirical value of L , the ratio of nebular luminosity to starlight intercepted, involves the

photographic magnitude m of the star and the maximum angular extent a_1 of the nebular luminosity on the photographic plate for an exposure of one hour. Assuming that the light re-emitted by the nebula is the photographic equivalent of the starlight intercepted, Hubble¹ derived the following expression:

$$m + 5 \log a_1 = 11.61 ,$$

by spreading the luminosity of the star over a spherical shell of a radius a_1 , on the implied assumption that the light is re-emitted by the nebula evenly in all directions. He corrects for the fact that the line connecting the star with the extreme point of the nebula is not perpendicular to the line of sight, by subtracting 0.98 mag. from the right-hand side of this equation. We find, however, that this correction amounts only to 0.52 mag.² Hence we shall take

$$m + 5 \log a_1 = 11.09 , \quad (13)$$

if the light re-emitted by the nebula is the photographic equivalent of the intercepted starlight.

In the observational material contained in Table I of Hubble's second paper, there are ten O stars (seven stars classed as Oe5, one star Ob, and two stars Odp) and five Bo stars connected with diffuse-emission nebulae.

¹ *Astrophysical Journal*, **56**, 410, 1922.

² Assuming different directions of the line r connecting star and extreme nebulous patch to be equally probable, the average projection of r perpendicular to the line of sight is $\frac{\pi}{4}r$. This may be taken into consideration by adding $5 \log \frac{\pi}{4} = -0.52$ to the right-hand side of the equation. The value of $\frac{\pi}{4}r$ is obtained in the following way. The star may be considered as the center of a sphere of radius r , and the patch of nebula is then at the surface of the sphere. For the position of the nebular patch, equal areas $d\sigma$ of the sphere are equally probable. The line of sight being fixed, then if the angle between r and the line of sight be ϕ , the projection of r perpendicular to the line of sight is $r \sin \phi$. The average value of this projection is accordingly $\int r \sin \phi d\sigma / \int d\sigma$, the integration being extended over the whole sphere. Hubble, in his original paper, gave the value $\frac{2}{\pi}r$ for the average projection, but by the method just outlined we obtain the larger value $\frac{\pi}{4}r$, and accordingly a smaller correction.

Table III shows the comparison of the observational data with the theory. The second column contains the average observed value of $m+5 \log a_1$ taken from Hubble's table. An excess of this observed value over the value given by (13) means that the nebula is brighter than the intercepted starlight by this excess. In this way the empirical value of the ratio L has been obtained. Once this is known, the temperature T of the star can be calculated from Table II by interpolation. These values of T are given in the last column.

The temperatures for Bo and O stars associated with diffuse nebulosity calculated in this manner agree fairly well with those obtained from the theory of thermal ionization. R. H. Fowler and

TABLE III

Diffuse-Emission Nebulae	Observed Average $m+5 \log a_1$	Empirical Ratio L from (13)	Stellar Temp. T Calculated from Table II
O stars (10)	11.35	—0.26	34,000°
Bo stars (5)	10.37	0.72	28,000

E. A. Milne¹ give the approximate values Bo—O9, 26,500°; O5, more than 35,000°; Miss C. H. Payne² gives for O stars 35,000° (from observed He^+ lines) and, according to another method (observed Si^{+++}), 25,000°.

The values derived in the present paper are also approximate. Too much importance should not be placed on that calculated for the Bo stars, because of their small number. Also, two of them are of a peculiar type. Another, ζ Orionis, associated with the nebula N.G.C. 2024, should probably be discarded because of intervening nebulosity, as Dr. Hubble has informed me, and this would make the average temperature somewhat higher. On the other hand, if the color-excess of the remaining four stars is taken into account, and their observed magnitude corrected accordingly, the average temperature would be lowered by about the same amount. The empirical material for the O stars is much better, and, moreover, a correction on account of their color-excess would be insignificant.

¹ *Monthly Notices, R.A.S.*, **84**, 515, 1924.

² *Stellar Atmospheres*, p. 139.

Although not too much value should be attached to the observational material of the Bo stars, the fact that a lower temperature is found for them than is found for the O stars, as of course should be the case, is definitely in favor of the proposed mechanism.

No detailed discussion of observational data for planetary nebulae will be undertaken. However, the empirical material definitely indicates that the photographic intensity of the nebula may be greatly in excess of that of the intercepted starlight. In the planetary nebulae the "nebulium" spectrum of unknown origin becomes predominant, and hence they cannot be treated satisfactorily with the approximations that have been introduced. The significant fact remains, however, that the mechanism of ionization and recombination can account for a photographic nebular luminosity greatly in excess of that of the intercepted starlight, as is shown by Table II. Although no quantitative statement can be made at present, there is no doubt that the mechanism requires, on the average, a temperature of the central star of planetaries very much higher than $33,000^{\circ}$. This is in line with present evidence, e.g., from the distribution of intensity in the continuous spectrum of the central star.¹

Mr. H. H. Plaskett has recently obtained the following interesting observational result which, through his courtesy, I am permitted to mention, though it has not yet been published. Investigating the intensity ratios of the lines of the Balmer series in one typical planetary and one typical diffuse nebula, he found that the decrease in intensity toward the higher lines of the series is much more rapid in the planetary than in the diffuse nebula. At the same time the temperature of the associated star in the planetary must be much the higher, since the line 4686 of ionized helium in this nebula is very strong. So it appears that in the typical case investigated by him, a higher temperature of the associated star is accompanied by a strengthening of the first lines of the Balmer series as compared with the higher lines. This fact seems to fit in quite well with the mechanism of luminosity proposed in the present discussion. According to it, the electrons freed photo-electrically by the ultra-

¹ See, for example, W. H. Wright, *Publications of the Lick Observatory*, **13**, 252, 253, 1918.

violet starlight are recaptured by the different levels of the hydrogen atom and fall back to the lowest level, either directly or in steps, thus producing line spectra. The more detailed physical picture made of this recapture implies that the probability of recapture by the lower levels becomes relatively larger, as the velocity of the free electron becomes higher. The higher the temperature of the star, therefore, the more electrons will be present at the lower levels after this recapture. But an electron on a lower level cannot possibly produce a line of the Balmer series originating on a higher level. Hence, as the temperature of the star increases, the capture of the electrons by the lower levels will result in a strengthening of the first lines of the Balmer series as compared with the higher lines and thus will account, in a qualitative way, for the observed phenomenon. Plaskett has pointed out that the distribution of intensity as observed by him could not be understood if the mechanism of luminosity were one of ordinary excitation, because in that case one would expect the lines requiring higher excitation to become relatively stronger as the temperature of the star increases. In the present discussion, the ordinary excitation by ultra-violet starlight was discarded as being totally inadequate to account for the observed intensity of luminosity. Mr. Plaskett's observation is additional and apparently conclusive evidence that the main mechanism causing the luminosity is one of ionization and recombination, and not that of ordinary excitation.

Certain theoretical considerations of H. A. Kramers¹ bear upon this interpretation. By an application of Bohr's correspondence principle Kramers derives an expression for the probability of capture of a free electron by the different energy-levels of a positive nucleus. The probability $Q_{i \rightarrow n}$ that a free electron of velocity v is captured in one second by the n th energy-level of the nucleus is given by the following formulae:

$$Q_{i \rightarrow n} = \frac{\beta v}{V}, \quad \beta = \frac{K}{v^2 n^3 \nu} g', \quad h\nu = W_n + \frac{1}{2} m v^2, \quad W_n = hN \frac{1}{n^2},$$

which are Kramers' formulae (13), (56) or (60), and (11), and the formula for the ionization energy W_n of the n th level of the hydrogen

¹ *Philosophical Magazine*, 46, 836, 1923.

atom. In those expressions, V is the volume considered; K , a constant expressible in terms of universal constants and the atomic number; h , Planck's constant; N , the Rydberg constant; and m , the mass of the electron. The quantity g' is equal to 1 in Kramers' formula (56) and approaches unity in (60), if $\frac{h}{e^2}v$ is sufficiently small (e =electronic charge).

Taking those formulae as they stand and putting $g'=1$, which is strictly justified only for sufficiently small velocities v , we obtain for the fraction p_n of the total number of electrons which is captured by the n th level

$$p_n = \frac{\frac{1}{n+Cn^3}}{\sum_{n=1}^{\infty} \frac{1}{n+Cn^3}}, \quad C = \frac{\frac{1}{2}mv^2}{W_1}.$$

The constant C is the ratio between the kinetic energy of the free electron and the ionizing energy, $W_1 = h\nu_0$, of the first level. If the velocity v of the electron and hence C approaches zero, the numerator approaches infinity, and p_n approaches zero for every level. This means that, considering any finite level n , the capture of electrons takes place in the higher levels up to infinity, and the contribution of a finite number of lower levels is negligible. On the other hand, if C is not zero, the numerator converges and, if the kinetic energy $\frac{1}{2}mv^2$ of the electron is comparable with W_1 , an appreciable fraction will be captured by the first levels and this fraction drops off very rapidly, for the higher levels even as fast as $\frac{1}{n^3}$.

The foregoing indicates that the assumption that electrons with large velocities are to a great extent captured by the lower levels is in harmony with present theoretical views based on the correspondence principle.

E. A. Milne[†] has also discussed the capture of electrons, but not on the basis of the correspondence principle. For Kramers' quantity β (in Milne's notation $2\pi F/v$), which may be considered as an

[†] *Philosophical Magazine*, 47, 209, 1924.

effective area of capture for a given energy-level, he derives an expression

$$\beta = \frac{\text{const.}}{v^2},$$

which, according to him, should hold for small velocities. Thus far Milne's and Kramers' results agree, as is seen from Kramers' expressions for β and $h\nu$, when v approaches zero. Milne, however, introduces the additional assumption that this expression holds for all velocities, from which would follow that the relative distribution of captured electrons on the different levels is independent of the velocity of the electron. It appears, therefore, that Milne's view is hard to reconcile with the observations of nebulae.

4. NEBULOSITY OF THE CONTINUOUS TYPE; TEMPERATURE OF A STAR B1

According to Hubble, stars of class B1 and later are usually connected with nebulosity emitting a continuous spectrum. There is hardly any doubt that the spectrum of such a nebula is produced by reflection of the starlight, and the formula (13) must be modified accordingly. If the nebular patch acts like a perfect reflector, the light emerging from the nebula is the equivalent of the intercepted starlight, but it is re-emitted only in a forward direction and not backward. Not aiming at an accurate statistical treatment, this may be taken into account by multiplying the surface luminosity of the nebula by 2, hence the appropriate formula is obtained by adding $2.5 \log 2 = 0.75$ to the right-hand side of equation (13). We have, therefore,

$$m + 5 \log a_1 = 11.84, \quad (14)$$

for the nebular patch acting as a perfect reflector. This formula, on the present interpretation, should hold for all nebulae of the continuous type.

The data for continuous nebulae are contained in Table I of Hubble's second paper, in which all stars of class B1 or later are assumed to produce a continuous spectrum, even if this has not been directly observed. The ten such stars of color-excess zero give an average value of $A = 10.73$ for $m + 5 \log a_1$. The other forty stars for

which the color-excess is known show a systematic discrepancy from this value. This evidently is due to detached patches of nebulosity which happen to be between the star and the observer. Such a patch of nebulosity absorbs and scatters the starlight, causing the observed m to be larger than the true m , and at the same time the scattering produces a positive color-excess. On this interpretation, one may assume that the true magnitude of the star is given by $m_t = m - k\epsilon$, where m is the observed magnitude and ϵ the color-excess, and therefore $(m - k\epsilon) + 5 \log a_1 = A$. A least-squares solution based on the data for all fifty stars gives the values $k = 1.4$ and $A = 10.61$. This means that a correction of -1.4 times the color-excess must be applied to the observed magnitude m in order to obtain the true magnitude m_t , and further that for stars producing a continuous nebular spectrum

$$m_t + 5 \log a_1 = 10.61 ,$$

on the average. This value of 10.61 for fifty stars is sufficiently close to that of 10.73 obtained for the ten stars with color-excess zero, and it should be the more accurate.¹

If the light of the nebular patch is assumed to be re-emitted evenly in all directions, and is the equivalent of the starlight intercepted, the theoretical formula (13) holds. On this interpretation the empirical value 10.61 means that the continuous spectrum re-emitted by the nebular patch is less intense than the intercepted starlight by 0.48 mag., or that it is 64 per cent of the intercepted starlight. This is much the same inference as that of Hubble, who concluded that the light re-emitted was the equivalent of the light intercepted (sec. 1, a).

For continuous nebulae, however, the formula (14), based on the interpretation of reflection, appears to be preferable. The empirical value 10.61 means, then, that the reflection is not perfect, but that there is an albedo of $11.84 - 10.61 = 1.23$ mag., or about 30 per cent. The reflection theory leads, therefore, to a reasonable value of the albedo.

¹ Hubble, treating all the stars producing continuous or emission spectrum in the same way, obtained $k = 0.9$. We thought it better to leave out the stars producing emission spectrum, where the main cause of the difference in $m + 5 \log a_1$ appears to be the temperature of the star, and not the color-excess.

The continuous spectrum of a nebula under the influence of stars B1 or later is most likely due to reflection of the starlight by solid or liquid particles. If gases also are present, a faint nebular emission spectrum may be expected which, if sufficiently strong, will show up against the continuous background. The occurrence of both spectra together permits a rough value for the intensity of the emission spectrum to be obtained by matching it with the continuous spectrum, and thus leads to a theoretical value for the temperature of the associated star. The star B.D. $+46^{\circ}3474$, associated with the nebula I.C. 5146, is a case in point. Dr. Hubble has furnished the following unpublished description of the nebular spectrum. The photograph shows $H\beta$ and $H\gamma$ and a trace of $H\delta$ against a continuous background. The width of the first two lines on the photographic plate is about one-fortieth of the total spectrum. Their intensity appears to be of about the same order as that of the continuous background on which they are superimposed. According to this description, the intensity of the lines $H\beta$ and $H\gamma$ together may be roughly estimated as one-twentieth of that of the continuous spectrum. If twice this amount is allowed for the other lines of the Balmer series, the total intensity of the Balmer series would be roughly 15 per cent of that of the continuous spectrum. It has just been concluded that the continuous spectrum, which is re-emitted by the nebula in the forward direction only, contains about 30 per cent of the intercepted starlight. The part of the Balmer series which also is re-emitted forward is 15 per cent of this, but an equal amount is re-emitted backward, hence the total intensity of the Balmer series may be estimated to be roughly 2×15 per cent $\times 30$ per cent, or about one-tenth of the intercepted starlight, and hence $L = \frac{1}{6}$. According to Table II, this corresponds to a temperature of roughly $21,000^{\circ}$ for the associated star, B.D. $+46^{\circ}3474$, of type B1. Another case of a B1 star producing a nebular spectrum crossed by emission lines is c' Orionis, associated with the nebula N.G.C. 1977.¹ In three other cases of B1 stars where the nebular spectra have been observed (cometary nebula, I.C. 446 and B.D. $-12^{\circ}1771$) they have been listed as continuous. If observational conditions were about the same, we would expect for c' Orionis also a temperature of about

¹ V. M. Slipher, *Publications of the Astronomical Society of the Pacific*, 31, 22, 1919.

21,000°. For the other three B1 stars it may be that the temperature is lower than 21,000°, or possibly that no gases are present.

Fowler and Milne, in their provisional temperature scale, give a temperature of roughly 19,000° for the stellar type B1. The agreement is as good as might be expected, the more so since B1 stars producing a nebular spectrum with emission lines are probably of a slightly earlier type than the normal B1.

For stars of a normal spectral type later than B1 and therefore of lower temperature, a continuous spectrum with emission lines would not be expected under the same observational conditions. This is in agreement with the observations as far as listed in Hubble's table. However, if a narrower slit could be used with longer exposure, the possibility is not excluded that for those stars also emission lines might be observed on the continuous background and a temperature estimate be carried out. The absence of emission lines, as previously remarked, does not necessarily mean a low temperature of the star, but if the emission lines are present, it appears probable that the absorption of the ultra-violet starlight is complete and that the line intensities can be used for temperature determinations.

Collecting the results, it is found that, as far as can be judged, the proposed mechanism of luminosity requires the following approximate values for the temperatures of stars associated with diffuse nebulosity: B1, 21,000° C. Abs.; B0, 28,000°; O, 34,000°; among which the value for B0 is probably not very reliable. Those values agree fairly well with the approximate temperatures obtained by Fowler and Milne: B1, 19,000°; B0, O9, 25,000°; O5, more than 35,000°.

The agreement of temperatures obtained from the luminosity of nebulae with those obtained from the theory of thermal ionization serves as a check for the mechanism of luminosity proposed. To this is added the further evidence of H. H. Plaskett's observations regarding the distribution of intensities in the Balmer series of the nebular spectrum.

Once convinced that the consequences of the proposed mechanism fit in fairly well with the present observations and theories, the temperatures which were obtained may be considered on their own merit. The luminosity of nebulae provides a method for obtaining

the temperatures of stars. An advantage of this method is that physical conditions in a nebula are very simple an account of the extremely low concentration of the nebular atoms. Certain points neglected in the present treatment may be taken into account when the physical theories of recombination and probability of transition become more reliable and more definite. The "nebulium" spectrum offers additional difficulties. The assumption that the ultra-violet starlight is practically complete may need further check. Hubble's statistical treatment of diffuse nebulae followed in this paper is based on a theoretical average value of $m+5 \log a_1$, which is only approximate. The greatest source of error, however, appears to arise from the difference in distribution of intensity of nebular spectrum and star spectrum, where the number of quanta between certain limits of frequency is introduced as an approximate measure of photographic intensity. This error may be removed by the introduction of better photographic methods. It is hard to give an estimate of the accuracy of the results, but it is believed that the total error may amount to 1 or even $1\frac{1}{2}$ mag., which corresponds to a rather wide range in temperature, according to the first and last column of Table II.

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