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Note on the Numerical Integration of $\frac{d^2x}{dt^2}=f(x, t)$.

By J. Jackson, M.A., D.Sc.

After the discovery of the eighth satellite of Jupiter, Cowell introduced numerical integration into dynamical astronomy as a method for taking account of all the forces, including the action of the central body. This method he applied later to the investigation of the motion of Halley's Comet between 1759 and 1910. This work of Cowell's has led other workers to deal with the subject of numerical integration, as it affords a ready solution to problems otherwise intractable. Cowell, instead of using the formulæ for numerical integration which had been left in a practically perfect state by Gauss,* reinvestigated the problem and introduced formulæ for changing the second differential coefficients into second differences, so that the integral was obtained at once by a simple double summation. After the work had been done, Cowell found out the disadvantages of the method he introduced, and incorporated the long-established formula in the first of two suggestions for future computations, which are given at the end of the essay by himself and Crommelin on the motion of Halley's Comet, published as an appendix to the Greenwich Observations for 1909. As these suggestions do not appear to be well known they are repeated here.

(1) In place of the fundamental formula

$$x_2=\mathrm{X}+\tfrac{1}{12}\mathrm{X}_2-\tfrac{1}{240}\mathrm{X}_4+\tfrac{1}{19\,51}\mathrm{X}_6\ \ldots$$

it would have been far better to use

$$x=\mathrm{X}_{-2}+\tfrac{1}{12}\mathrm{X}-\tfrac{1}{240}\mathrm{X}_2+\tfrac{1}{19\,51}\mathrm{X}_4\ \ldots$$

(2) Short time intervals ought to be employed, as errors in that case are far easier to detect. The intervals in the *Ast. Ges.* essay were far too long, and those in the revised calculations of this appendix not too short. The intervals should be 1 day at perihelion, and may be doubled at 40, 80, 160 days, and so on from perihelion. The interval should not be changed during a close approach to a planet.

Of these suggestions (1) is a return to a long-established formula and (2) is a point of practical importance, the truth of which is borne home with greater and greater force the more one computes. A numerical mistake made at any point may vitiate all subsequent work. Professor Noumerov has recently introduced "special co-ordinates" in

* The formulæ for numerical integration go right back to the beginnings of the calculus. Formulæ were developed by Newton, Gregory, and others in the seventeenth and eighteenth centuries. The formulæ advocated here were known to Gauss early in the nineteenth century. The subject was carefully discussed by Encke in the Berlin *Jahrbuch* for 1837, and is given in great detail by Oppolzer at the beginning of the second volume of his *Lehrbuch zur Bahnbestimmung der Kometen und Planeten.*

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an attempt to get over the difficulties of extrapolation required in estimating the values of X_2, X_4, . . . in the formulæ given above. It will be seen that these special co-ordinates make the first and largest (though not necessarily the most important) step backwards indicated by Cowell in (1) above. Noumerov has then attempted to introduce the special co-ordinates into the differential equation, with the result that his formulæ appear rather complicated. It does not appear that the long-established formulæ can be improved.

Notation.—The fundamental formulæ, although long known, are so rarely given in English textbooks that they may be given here, together with a convenient notation. The values actually computed from the differential equation, at equal intervals of the time, are denoted by $f(1), f(2), f(3)$, etc. The values are given below in a vertical column, and the differences formed according to the general formula $f_{n+1}(t)=f_n(t+\frac{1}{2})-f_n(t-\frac{1}{2})$ are given to the right. The sums are given to the left, and are computed from ${}_{n+1}f(t)={}_{n+1}f(t-1)+{}_nf(t-\frac{1}{2})$. The quantities in brackets $[{}_2f(1)]$ and $[{}_2f(2)]$ have to be chosen to suit the initial values of x and its velocity. In the actual mechanical integration the numerical values of the functions are tabulated according to the schemes:

$[{}_2f(1)]$		$f(1)$					
	${}_1f(\frac{3}{2})$		$f_1(\frac{3}{2})$				
$[{}_2f(2)]$		$f(2)$		$f_2(2)$			
	${}_1f(\frac{5}{2})$		$f_1(\frac{5}{2})$		$f_3(\frac{5}{2})$		
${}_2f(3)$		$f(3)$		$f_2(3)$		$f_4(3)$.	
	${}_1f(\frac{7}{2})$		$f_1(\frac{7}{2})$		$f_3(\frac{7}{2})$		
${}_2f(4)$		$f(4)$		$f_2(4)$			
	${}_1f(\frac{9}{2})$		$f_1(\frac{9}{2})$				
${}_2f(5)$		$f(5)$					

If we choose $[{}_2f(1)]$ and $[{}_2f(2)]$, so that

$$x(1)=[{}_2f(1)]+\tfrac{1}{12}f(1)-\tfrac{1}{240}f_2(1)+\tfrac{31}{60480}f_4(1)\ \ldots$$
$$x(2)=[{}_2f(2)]+\tfrac{1}{12}f(2)-\tfrac{1}{240}f_2(2)+\tfrac{31}{60480}f_4(2)\ \ldots,$$

then we have for all integral values of n

$$x(n)={}_2f(n)+\tfrac{1}{12}f(n)-\tfrac{1}{240}f_2(n)+\tfrac{31}{60480}f_4(n) \quad . \quad . \quad (1)$$

Cowell and Crommelin in their work, instead of applying the correcting terms to ${}_2f(n)$, applied them to $f(n)$, so that the double summation gave the co-ordinate directly by double summation, *i.e.* they computed

$$X(n)=f(n)+\tfrac{1}{12}f_2(n)-\tfrac{1}{240}f_4(n)+\tfrac{31}{60480}f_4(n) \quad . \quad . \quad (2)$$

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and then obtained x from a double summation, thus

$$\begin{array}{ccc} x(1) & & \mathrm{X}(1) \\ & {}_1\mathrm{X}(\tfrac{3}{2}) & \\ x(2) & & \mathrm{X}(2) \\ & {}_1\mathrm{X}(\tfrac{5}{2}) & \\ x(3) & & \mathrm{X}(3). \end{array}$$

It is clear that this procedure is exactly similar to that advocated in formula (1), except that the "correcting terms" in (2) are obtained by differencing twice more than for (1) and have to be finally summed twice. The serious disadvantages in the use of (2) as compared with (1) lies in the influence of neglected higher differences, for any error made in deducing $\mathrm{X}(n)$ from $f(n)$ enters into ${}_1\mathrm{X}(n+\frac{1}{2})$ and all subsequent values of this function and produce steadily increasing errors in x. On the other hand, neglected terms in the deduction of $x(n)$ from ${}_2f(n)$ by (1) are not passed on to subsequent values of x. In fact, $x(n)$ as required for computing $f(x, t)$ is not required very accurately, as the time interval should be chosen so that $f(x, t)$, which is roughly proportional to x, does not usually exceed in numerical value $x/100$.

Professor Noumerov has used a "special co-ordinate," $\bar{x} = x - \frac{1}{12}f(x, t)$, so that $\bar{x}$ differs from ${}_2f(n)$ by a term beginning with $\frac{1}{240}f_2(n)$. In the paper just presented to the society (p. 592) he writes about the possible extension of the use of "special co-ordinates," so as to make the differential coefficient approximate more closely still to the second difference. When he has completed the process he will find that $\bar{x}$ is the ${}_2f(n)$ of formula (1).

The trouble in the use of all these formulæ is the extrapolation necessary for the determination of the higher differences required for converting the differential coefficients into differences or the sums into integrals. The great advantage of formula (1) over the others is that this extrapolation only affects the deduction of $x(n)$ from ${}_2f(n)$, so that in all cases ${}_2f(n)$ is accurate, while $x(n)$ does not need to be very accurately known for the computing of $f(n)$, which leads accurately to ${}_2f(n+1)$ at once. It may seem that the need to apply so large a correction as $\frac{1}{12}f(n)$ to ${}_2f(n)$ in order to obtain $x(n)$ by formula (1) would lead to considerable uncertainty, for $f(n)$ may contain six or even seven significant figures. But when the arithmetic is carried out with a suitable time interval this can be done with all the required accuracy. Suppose the differences fall off in such a way that f_6 can be extrapolated without serious error. Then, when $f(n-1)$ is known so also are the values of f_6 to $f_6(n-4)$. When once we have extrapolated $f_6(n-3)$ we can compute with equal accuracy $f_5(n-\frac{5}{2})$, $f_4(n-2)$, $f_3(n-\frac{3}{2})$, $f_2(n-1)$, $f_1(n-\frac{1}{2})$, $f(n)$. An error of 12 units in the last decimal place of the extrapolated value of $f_6(n-3)$ will only produce 1 unit in the last decimal place of $x(n)$. Another extrapolation of f_1 to $f_6(n-2)$ will enable $f_2(n)$ to be computed, and only $\frac{1}{240}$th part of this has to be applied in formula (1). A little consideration will show that the unknown terms of (1) can be extrapolated with all the accuracy required.

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As we have seen, Professor Noumerov has made the first step in the transition form (2) to (1) by introducing $x(n)-\frac{1}{12}f(n)$ as a "special co-ordinate" $\bar{x}$. By making use of the fact that except for perturbations $f(x, t)=-k^2x/r^3$, and similar expressions for the other three co-ordinates, he deduces $\frac{d^2x}{dt^2}=\sigma\bar{x}$ where

$$\sigma(1-\tfrac{1}{12}\sigma)^2=k^2/\bar{r}^3,$$

and

$$\bar{r}^2=\bar{x}^2+\bar{y}^2+\bar{z}^2.$$

$\frac{d^2x}{dt^2}$ is thus computed by means of $\bar{x}$, $\bar{y}$, $\bar{z}$ without transforming to x, y, z.

Professor Noumerov has taken considerable trouble to apply the formulæ which use $\bar{x}$, $\bar{y}$, $\bar{z}$, and the success he has met with has been due to the use of the above cubic equation in σ. His process could not be easily extended so as to use ${}_2f(n)$ as a "special co-ordinate" in place of x. As it is, his formulæ are rather involved compared with the original one, and the process for deriving $\bar{x}_n$ from earlier values much more complicated. Experience has shown that (1) can be applied with ease, and it appears that attempts to improve it will complicate its application so much as to increase the labour of integration.

As an example, take the values of $\frac{d^2x}{dt^2}$ given by Cowell and Crommelin at the beginning of their discussion of Halley's Comet. The time interval is 32 days, and the first date is 230 days after the perihelion of 1759. The interval is rather long, as is shown by the fact that f_6 is changing rather rapidly, and in fact Cowell advocated that the interval should be reduced. From nine values of f, three values of f_6 can be obtained. These three values indicate that f_6 is being approximately halved at each step, and in the table we have extrapolated other three steps, assuming that f_6 is actually halved each time. We have then computed three extra values for all the other differences, although they have not been bracketed.

f.	f_1.	f_2.	f_3.	f_4.	f_5.	f_6.
+2290 736						
	−409 073					
1881 663		+108 610				
	300 463		−36 226			
1581 200		72 384		+14 089		
	228 079		22 137		−6 123	
1353 121		50 247		7 966		+2 880
	177 832		14 171		3 243	
1175 289		36 076		4 723		1 478
	141 756		9 448		1 765	
1033 533		26 628		2 958		694
	115 128		6 490		1 071	
918 405		20 138		1 887		(347)
	94 990		4 603		724	
823 415		15 535		1 163		(174)
	79 455		3 440		550	
743 960		12 095		613		(87)
	67 360		2 827		463	
(676 600)		9 268		150		
	58 092		2 677			
(618 508)		6 591				
	51 501					
(567 007)						

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When we form the sums we get the following values, the first column being a repetition from above:

f.	$_1f$.	$_2f$.
+2290 736		
	−29 522 122	
1881 663		3·66 917 633
	27 640 459	
1581 200		3·94 558 092
	26 059 259	
1353 121		4·20 617 351
	24 706 138	
1175 289		4·45 323 489
	23 530 849	
1033 533		4·68 854 338
	22 497 316	
918 405		4·91 351 654
	21 578 911	
823 415		5·12 930 565
	20 755 496	
743 960		5·33 686 061
	20 011 536	
(676 600)		5·53 697 597
	19 334 936	
(618 508)		5·73 032 533
	18 716 428	
(567 007)		5·91 748 961
	18 149 421	
		6·09 898 382

The last four values of $_2f$ are extrapolations beyond the dates for which x has been used. The value of x computed from $_2f$ for three dates beyond those used are:

$_2f$	−5·53 697 597	−5·73 032 533	−5·917 48 961
$+\frac{1}{12}f$	+ 56 383·3	+ 51 542·3	+ 47 250·6
$-\frac{1}{240}f_2$	− 38·6	− 27·5	− (16·4)
$+\frac{1}{1051}f_4$	+ ·1	...	...
	−5·53 641 252	−5·72 981 018	−5·917 01 727

All these values are extrapolated except the first value of $_2f$.

The final values obtained by Cowell and Crommelin were

−5·53 641 244	−5·72 980 858	−5·917 00 633

It will be seen that the first two extrapolated values are sufficiently accurate for the computation of the next two values of $\frac{d^2x}{dt^2}$, although in reality we only need to extrapolate one step at a time. The values of f_6 obtained by actual differentiation of the tabulated values are 455, 203, 169, instead of 347, 174, 87 assumed above. Of course slight errors in the computed values of $\frac{d^2x}{dt^2}$ affect f_6 considerably.

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Some Eye-pieces made by Sir William Herschel.

By W. H. Steavenson. (Plate 17.)

The claims made by Sir William Herschel to have used powers up to and over 6000 on one of his 7-foot telescopes have always aroused the interest and curiosity of practical astronomers. From the title * and substance of a paper written in 1782, and published in *Phil. Trans.*, **72**, pp. 173–178, it is evident that from the very first there had been considerable hesitation on the part of scientific men in accepting his statements on this point. In this paper Herschel gives the results of a re-measurement of his powers by a method different from that formerly used. The powers, as thus re-measured, come out substantially the same, and in fact the values are all actually higher except for the power previously referred to as "6450," which was now found to be 5787. He goes on to point out that the use of such powers was not his regular habit, but was rather in the nature of an experiment, and that "farther inferences ought not as yet to be drawn." But, in spite of the very clear and circumstantial statements contained in this paper, some incredulity with regard to the matter has always persisted, and in modern text-books it is common to find doubts expressed as to the possibility of the construction and use of such lenses, or of the measurement of their powers.

In the month of May 1924, with the kind permission of the Rev. Sir John Herschel, I began a systematic examination of such of the instruments of Sir William Herschel as are still preserved at Slough. Among those already recognised as the handiwork of Sir William were many eye-pieces in brass and wooden mounts, but none of the extremely high powers already mentioned. However, while examining the contents of a cabinet, supposed to contain only some chemical specimens and apparatus used by Sir J. F. W. Herschel, I came across nine small wooden cylinders, about 0·8 by 0·9 inch. A closer examination showed these to be eye-pieces fitted with lenses of exceedingly short focal length, and I think there can be no doubt but that these are some of the famous high powers made and used by Sir William.

All are constructed on the same plan. The main body of each is of cocus wood, turned no doubt by Herschel on the lathe which he used for all such work, and which is still preserved at Slough in excellent condition. This same wood was used for many of the larger eye-pieces, and it seems probable that its use as a suitable material was first suggested to Herschel by his familiarity with the oboe, for the making of which instrument it was, and still is, largely employed. At the lower end of each mount is cut a thread, of the same standard pitch and diameter as that used for many other of his eye-pieces, both brass and wood. On the upper end of each cylinder there screws down a wooden rim, which serves to hold in position two circular plates of very thin brass. These plates, which are exactly alike, have all

* "A Paper to obviate some Doubts concerning the great Magnifying Powers used." By Mr. Herschel, F.R.S.

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