On Einstein's Theory of Gravitation, and its Astronomical Consequences. By W. de Sitter, Assoc. R.A.S. Second Paper.*

Contents of Second Paper.

20. We will now derive the gravitational field of n bodies, which have arbitrarily given motions relatively to the system of co-ordinate axes. We will suppose the bodies to be rigid spheres. By this we mean that in a system of co-ordinates relatively to which any one body is at rest, its outer surface would be described as a sphere. In a system relatively to which it moves it then undergoes the well-known Lorentz-contraction. By the attribute "rigid" I mean that the form of the bodies will not be influenced by their mutual gravitation. We further suppose that their dimensions are small compared with their mutual distances, and also their velocities are supposed to be small compared with the velocity of light. The different bodies will be distinguished by indices μ or ν; the indices i and j refer to the co-ordinate axes as before.

We will take rectangular Cartesian co-ordinates, $x_1 = x$, $x_2 = y$, $x_3 = z$, and $x_4 = ct$, t being the time. The g_{ij} will be

$$\begin{aligned} g_{ii} &= -1 + \gamma_{ii}, & g_{ij} &= \gamma_{ij}, \\ g_{i4} &= \chi_i, & g_{44} &= 1 + \gamma, \end{aligned} \qquad (i, j = 1, 2, 3).$$

The γ_{ij} and γ will be of the first order, χ_i will be of the order $\frac{3}{2}$. Only in γ we require the terms of the second order.

We must now form the equations (first paper, p. 706)

$$\text{(II.)} \qquad G_{ij} = -\kappa T_{ij} + \tfrac{1}{2}\kappa g_{ij} T.$$

* See first paper, *M.N.*, vol. lxxvi. p. 699. For this second paper I have, beyond the help from my colleagues and friends already acknowledged in the first paper, had the great privilege of several conversations with Professor Einstein himself, during a visit of his to Leiden.

To the list of papers given in the footnote to page 700 must be added:

VII. Einstein, "Hamiltonsches Prinzip und allgemeine Relativitätstheorie," *Sitzungsber. Berlin*, 1916 Oct., p. 1111.

To the required order of accuracy we have

$$(63) \qquad \begin{cases} T_{ij} = \rho \dot{x}_i \dot{x}_j, & T_{i4} = -\rho \dot{x}_i, \\ T_{44} = \rho(1 + \gamma + \phi^2), & T = \rho, \end{cases}$$

where we have again put

$$\dot{x}_i = \frac{dx_i}{cdt}, \qquad \phi^2 = \sum_\alpha{}' \dot{x}_\alpha{}^2,$$

The velocities $\dot{x}_i$ are of the order $\frac{1}{2}$. We have neglected all terms derived from the inner constitution of the gravitating bodies. By the same reasoning as in art. 10 (first paper) we can show that they can be taken into account by slightly altering the meaning of the symbol ρ.

We will suppose that γ_{ij}, χ_i, and γ do not depend explicitly on $x_4 = ct$, but only through the co-ordinates x_1, x_2, x_3 of the n bodies, which are functions of t. Consequently, a differentiation with respect to x_4 increases the order of smallness by $\frac{1}{2}$, and such a term as, *e.g.*, $\frac{\partial^2\gamma}{c^2\partial t^2}$, is of the second order. To the required order of accuracy the equations are found to be

$$(64) \quad \begin{cases} G_{ij} = \frac{1}{2}\frac{\partial^2\gamma}{\partial x_i \partial x_j} - \frac{1}{2}\sum_\alpha{}' \frac{\partial^2\gamma_{\alpha\alpha}}{\partial x_i \partial x_j} + \frac{1}{2}\sum_\alpha{}' \left[\frac{\partial^2\gamma_{i\alpha}}{\partial x_\alpha \partial x_j} + \frac{\partial^2\gamma_{j\alpha}}{\partial x_\alpha \partial x_i} - \frac{\partial^2\gamma_{ij}}{\partial x_\alpha{}^2}\right] = -\frac{1}{2}\delta_{ij}\kappa\rho, \\ G_{i4} = -\frac{1}{2}\sum_\alpha{}' \frac{\partial^2\gamma_{\alpha\alpha}}{\partial x_i c\partial t} + \frac{1}{2}\sum_\alpha{}' \left[\frac{\partial^2\gamma_{i\alpha}}{\partial x_\alpha c\partial t} + \frac{\partial^2\chi_\alpha}{\partial x_i \partial x_\alpha} - \frac{\partial^2\chi_i}{\partial x_\alpha{}^2}\right] = \kappa\rho\dot{x}_i, \\ G_{44} = -\frac{1}{2}\sum_\alpha{}' \frac{\partial^2\gamma}{\partial x_\alpha{}^2} - \frac{1}{2}\sum_\alpha{}' \frac{\partial^2\gamma_{\alpha\alpha}}{c^2\partial t^2} + \sum_\alpha{}' \frac{\partial^2\chi_\alpha}{\partial x_\alpha c\partial t} + \frac{1}{4}\sum_\alpha{}' \left(\frac{\partial\gamma}{\partial x_\alpha}\right)^2 \\ \qquad + \frac{1}{4}\sum_\alpha{}' \sum_\beta{}' \frac{\partial\gamma}{\partial x_\alpha}\frac{\partial\gamma_{\beta\beta}}{\partial x_\beta} - \frac{1}{2}\sum_\alpha{}' \sum_\beta{}' \frac{\partial}{\partial x_\alpha}\left[\gamma_{\alpha\beta}\frac{\partial\gamma}{\partial x_\beta}\right] = -\frac{1}{2}\kappa\rho(1 + \gamma + 2\phi^2). \end{cases}$$

It will be remembered that $\delta_{ii} = 1$ and $\delta_{ij} = 0$ for $i \neq j$, and that Σ' denotes a sum over the values 1, 2, 3 only.

If we neglected all but the terms of the lowest order, the equations would be linear. Then the field of the n bodies would be got by superposing the fields of the separate bodies, each computed as if the others did not exist. The field of a single body has been found in art. 11. Thus, if we take the system B of art. 11 we find that the γ_{ij} must be of the form

$$\gamma_{ii} = \gamma, \quad \gamma_{ij} = 0 \text{ for } i \neq j.$$

21. If this is introduced into the equations for G_{ij} and the terms of the first order of G_{44}, these equations all reduce to

$$(65) \qquad \sum_\alpha{}' \frac{\partial^2\gamma}{\partial x_\alpha{}^2} = \kappa\rho.$$

The integral is

$$\gamma = -\frac{\kappa}{4\pi}\int\frac{\rho dV}{r}.$$

Here r is the three-dimensional distance from the point (x_1, x_2, x_3) at which the value of γ is required to the point (x_1', x_2', x_3') at which the density is ρ. Thus $r^2 = \sum_a'(x_a - x_a')^2$. Further, dV is the three-dimensional element of volume. The integral must be taken over the whole of three-dimensional space. Outside the n bodies, however, $\rho = 0$, and as the bodies are supposed never to coincide, the integral consists of n separate integrals, each over one of the bodies. Since the bodies are small, we can take for r the distance to any point within the body, say the centre. In performing the integration $\int \rho dV$ we must remember that the body moves with the velocity ϕ_ν. In a system of co-ordinates in which the body would be at rest $\int \rho dV$ would be the mass. Now, however, we have in consequence of the Lorentz-contraction *

$$\int_\nu \rho dV = m_\nu \sqrt{1 - \phi_\nu^2}.$$

Therefore, for points outside the n bodies,

$$(66) \qquad \gamma = -2\lambda^2 \sum_\nu \frac{m_\nu}{r_\nu}(1 - \tfrac{1}{2}\phi_\nu^2).$$

Here we have put, as before, $\kappa = 8\pi\lambda^2$ and $r_\nu^2 = \sum_a'(x_a - x_{a\nu})^2$, $x_{1\nu}, x_{2\nu}, x_{3\nu}$ being the co-ordinates of the body with the index ν, m_ν its mass, and ϕ_ν its velocity. For points inside one of the bodies we shall have the same value in which the sum is to be taken over the $n - 1$ other bodies, the term relating to the body itself being replaced by a term of the form (29) [first paper, art. 11].

The last term in (67) is of the second order. It will be combined below with other terms of the second order, and we will put

$$(67') \qquad \begin{cases} \gamma = \gamma_0 + \gamma', \\ \gamma_0 = -2\lambda^2 \sum_\nu \dfrac{m_\nu}{r_\nu}. \end{cases}$$

The second order term γ' is a sum of several terms, one of which is thus

$$\gamma_1' = \lambda^2 \sum_\nu \frac{m_\nu \phi_\nu^2}{r_\nu}.$$

* For any transformation $\sqrt{g}\, d\tau$ is invariant, if $d\tau$ is the four-dimensional element of volume. Thus for the three-dimensional element of volume we have

$$\left(\frac{dV'}{dV}\right)^2 = \left(\frac{dt}{dt'}\right)^2 \frac{g}{g'}.$$

If we use a Lorentz-transformation, then $g = g'$, and to the required order $\frac{dt}{dt'} = \sqrt{(1 - \phi\nu^2)}$, therefore $dV' = dV\sqrt{(1 - \phi\nu^2)}$, where the letters without accents refer to the system in which the body is at rest.

We now use this value of γ in the equation for G_{i4}, which then becomes

$$\sum_{\alpha}{}' \frac{\partial}{\partial x_{\alpha}}\left[\frac{\partial \chi_{\alpha}}{\partial x_i} - \frac{\partial \chi_i}{\partial x_{\alpha}}\right] = 2\kappa\rho\dot{x}_i + 2\frac{\partial^2\gamma_0}{\partial x_i c\partial t}.$$

In the partial differentiation of γ_0 with respect to t, we must consider $x_{i\nu}$ as a function of t, thus $\frac{\partial x_{i\nu}}{c\partial t} = \dot{x}_{i\nu}$, but x_i does not depend explicitly on t. The integral is found to be

$$\chi_i = \frac{\kappa}{2\pi}\int\frac{\rho\dot{x}_i}{\mathrm{r}}d\mathrm{V} = 4\lambda^2\sum_{\nu}\frac{m_{\nu}\dot{x}_{i\nu}}{\mathrm{r}_{\nu}} \tag{68}$$

The correction for the Lorentz-contraction can here be neglected, as it would give a term of the order $\frac{5}{2}$. From (68) we find

$$\sum_{\alpha}{}' \frac{\partial^2\chi_{\alpha}}{\partial x_{\alpha}c\partial t} = 2\frac{\partial^2\gamma_0}{c^2\partial t^2}.$$

This we introduce into the equation G_{44}. We then divide by $(1+\gamma)$, and subtract (65). Then we find

$$\sum_{\alpha}{}' \frac{\partial^2(\gamma' - \gamma_1')}{\partial x_{\alpha}^2} = 2\kappa\rho\phi^2 + \frac{\partial^2\gamma_0}{c^2\partial t^2} + \sum_{\alpha}{}'\left(\frac{\partial\gamma_0}{\partial x_{\alpha}}\right)^2. \tag{69}$$

The first term on the right gives

$$\gamma_2' = -4\lambda^2\sum_{\nu}\frac{m_{\nu}\phi_{\nu}^2}{\mathrm{r}_{\nu}},$$

which can be combined with γ_1'. The second term gives

$$\gamma_3' = -\lambda^2\sum_{\nu} m_{\nu}\frac{\partial^2\mathrm{r}_{\nu}}{c^2\partial t^2},$$

The last term can, by means of (65), be transformed to

$$\sum_{\alpha}{}' \frac{\partial^2}{\partial x_{\alpha}^2}\left(\tfrac{1}{2}\gamma_0^2\right) - \kappa\rho\gamma_0,$$

We have thus

$$\gamma_4' = \tfrac{1}{2}\gamma_0^2,$$

and

$$\sum_{\alpha}{}' \frac{\partial^2\gamma_5'}{\partial x_{\alpha}^2} = -\kappa\rho\gamma_0.$$

Now whenever ρ differs from zero, say for a point within the body ν, γ_0 is of the form

$$\gamma_0 = \gamma_{0\nu} - 2\lambda^2\sum_{\mu\neq\nu}\frac{m_{\mu}}{\mathrm{r}_{\mu}},$$

The first part does not depend on the other bodies; for any one point it is a constant, and it can thus be taken into account at once in (65) by writing $\rho' = \rho(1 - \gamma_{0\nu})$ for ρ. We will suppose that this has been done, and then we will call ρ' again ρ. There remains

$$\sum_a{}' \frac{\partial^2 \gamma_5'}{\partial x_a^2} = 2\lambda^2 \kappa\rho \sum_{\mu \neq \nu} \frac{m_\mu}{\Delta_{\mu\nu}},$$

where $\Delta_{\mu\nu}$ has been written for r_μ, since the latter is required only for points within the body ν. We then find

$$\gamma_5' = -4\lambda^2 \sum_\nu \frac{m_\nu}{r_\nu} \sum_{\mu \neq \nu} \frac{m_\mu}{\Delta_{\mu\nu}}.$$

Collecting results we have

$$(70) \quad \begin{cases} \gamma_{ii} = \gamma_0, \quad \gamma_{ij} = 0, \\ \gamma_0 = -2\lambda^2 \sum_\nu \frac{m_\nu}{r_\nu}, \\ \chi_i = 4\lambda^2 \sum_\nu \frac{m_\nu \dot{x}_{i\nu}}{r_\nu}, \\ \gamma = \gamma_0 + \gamma', \\ \gamma' = \frac{1}{2}\gamma_0^2 - 3\lambda^2 \sum_\nu \frac{m_\nu \phi_\nu^2}{r_\nu} - 4\lambda^2 \sum_\nu \frac{m_\nu}{r_\nu} \sum_{\mu \neq \nu} \frac{m_\mu}{\Delta_{\mu\nu}} - \lambda^2 \sum_\nu m_\nu \frac{\partial^2 r_\nu}{c^2 \partial t^2}. \end{cases}$$

If we put

$$(71) \qquad \sum_a{}' (x_{i\nu} - x_i)\dot{x}_{i\nu} = r_\nu \epsilon_\nu, \quad \sum_a{}' (x_{i\nu} - x_i)\ddot{x}_{i\nu} = r_\nu \eta_\nu,$$

and if by P_ν we denote the angle at the body ν between the velocity of that body and the radius r_ν, then we have

$$\epsilon_\nu = -\phi_\nu \cos P_\nu$$

and

$$(72) \qquad \frac{\partial^2 r_\nu}{c^2 \partial t^2} = \frac{\phi_\nu^2 - \epsilon_\nu^2}{r_\nu} + \eta_\nu.$$

22. The analysis of the preceding article is due to Dr. J. Droste.* Practically simultaneously Einstein † has given another solution of the same equations. Einstein does not introduce the condition that $\dot{x}_i$ shall be small; on the other hand, he gives his solution only to the first order in κ. It is, however, easy to complete it by adding the terms of the second order in g_{44}. Einstein takes as co-ordinates $x_1 = x$, $x_2 = y$, $x_3 = z$, $x_4 = ict$ ($i = \sqrt{-1}$). Then he puts

$$g_{pq} = -\delta_{pq} + \gamma_{pq}.$$

* J. Droste, *Proceedings Academy of Sciences at Amsterdam*, June 1916, (vol. xix. p. 447).

† Einstein VI., *Sitzungsber. Berlin*, June 1916.

Thus for i and j different from 4 these γ_{ij} are the same as those in the preceding article. Further,

$$\gamma_{p4} = i\chi_p , \quad \gamma_{44} = -\gamma .$$

Einstein then puts

$$\gamma_{pq} = \gamma_{pq}' + \delta_{pq}\psi, \tag{73}$$

and he introduces the conditions

$$\sum_\alpha \frac{\partial \gamma_{p\alpha}'}{\partial x_\alpha} = 0 \quad \text{or} \quad \sum_\alpha \frac{\partial \gamma_{p\alpha}}{\partial x_\alpha} = \frac{\partial \psi}{\partial x_p}.$$

The function ψ which has been arbitrarily introduced requires one more condition, which is

$$\psi = -\tfrac{1}{2}\sum_\alpha \gamma_{\alpha\alpha}'. \tag{74}$$

The equations then become, neglecting terms of the second order,

$$\sum_\alpha \frac{\partial^2}{\partial x_\alpha{}^2}\left(\gamma_{pq}' - \tfrac{1}{2}\delta_{pq}\sum_\beta \gamma_{\beta\beta}'\right) = 2\kappa\left(\mathrm{T}_{pq} - \tfrac{1}{2}\delta_{pq}\sum_\beta \mathrm{T}_{\beta\beta}\right), \tag{75}$$

or

$$\sum_\alpha \frac{\partial^2}{\partial x_\alpha{}^2}\gamma_{pq}' = 2\kappa \mathrm{T}_{pq} ;$$

or, if we return to the variable $ct = -ix_4$,

$$\sum_\alpha \frac{\partial^2 \gamma_{pq}'}{\partial x_\alpha{}^2} - \frac{\partial^2 \gamma_{pq}'}{c^2 dt^2} = 2\kappa \mathrm{T}_{pq} . \tag{76}$$

The integral is

$$\gamma_{pq}' = -\frac{\kappa}{2\pi}\int \frac{\mathrm{T}_{pq}'}{\mathrm{r}'} d\mathrm{V}. \tag{77}$$

Here $\mathrm{r}'^2 = \sum_\alpha' (x_\alpha - x_\alpha')^2$, T_{pq}' is the value at the point (x_1', x_2', x_3') *and at the time* $t - \frac{\mathrm{r}'}{c}$. For p and q different from 4, T_{pq}' are the same as T_{pq} in the preceding article. Further,

$$\mathrm{T}_{p4}' = -i\mathrm{T}_{p4} , \quad \mathrm{T}_{44}' = -\mathrm{T}_{44} .$$

The integration in (77) must be performed with care, the time $t - \frac{\mathrm{r}'}{c}$ being different for each point in any of the bodies. The space over which the integral must be extended—*i.e.* in which ρ differs from zero—therefore does not coincide with the space occupied by the body at any particular time. For r′ we can take its value for the centre of the body. Further, we find,* neglecting quantities of the order of the square of the velocity,

$$\int_\nu \rho d\mathrm{V} = \frac{m_\nu}{1+\epsilon_\nu}.$$

Therefore we have, correct to the order $\frac{3}{2}$,

$$\gamma_{pq}{}' = 0, \quad \gamma_{p4}{}' = -4i\lambda^2 \sum_\nu \frac{m_\nu \dot{x}_{i\nu}}{\mathrm{r}_\nu{}'},$$

$$\gamma_{44}{}' = 4\lambda^2 \sum_\nu \frac{m_\nu}{\mathrm{r}_\nu{}'}(1-\epsilon_\nu).$$

Consequently,

$$(78) \qquad \begin{cases} \psi = -\frac{1}{2}\gamma_{44}{}' = -2\lambda^2 \sum_\nu \frac{m_\nu}{\mathrm{r}_\nu{}'}(1-\epsilon^\nu), \\ \gamma_{ii} = \psi, \ \gamma_{ij} = 0, \ \gamma = -\gamma_{44} = \psi, \\ \chi_\iota = 4\lambda^2 \sum_\nu \frac{m_\nu \dot{x}_{i\nu}}{\mathrm{r}_\nu{}'}(1-\epsilon_\nu). \end{cases}$$

It will be seen that this solution differs from (70)—if there also we neglect the terms of the second order—by the substitution of $\mathrm{r}_\nu{}'$ for r_ν and by the factor $(1-\epsilon_\nu)$. Now, to the first order in the velocities we have

$$\mathrm{r}' = \mathrm{r} - \tfrac{1}{2}\mathrm{r}' \frac{\partial \mathrm{r}}{c\partial t} = \mathrm{r}(1-\epsilon).$$

Consequently, the factor $(1-\epsilon)$ is exactly cancelled, and (78) agrees with (70), neglecting only the terms of the second order (and in χ_i of the order $\frac{5}{2}$). We could now add the terms of the second order in the same way as was done in the preceding article, and we should find the same result, with the exception of the term $\gamma_3{}'$, which is here included in the first approximation. The two systems may differ in the terms of the second order owing to the difference in the additional conditions introduced to fix the system of reference.

In both systems the velocity of light is

$$v = c\left[1 + \gamma + 4\lambda^2 \sum_\nu \frac{m_\nu \phi_\nu}{\mathrm{r}_\nu} \cos \mathrm{V}_\nu\right],$$

where V_ν is the angle between the element $d\sigma$ of the ray of light (taken positive in the direction of the propagation of light) and the velocity of the body ν.

In the system (78) the g_{ij} are functions of the distance r' between the point at which gravitation acts at the time t and the body from which it emanates at the time $t-\frac{\mathrm{r}'}{c}$. This shows that

* See Lorentz, *Theory of Electrons*, note 19 (p. 254), where $\frac{\mathrm{v}_r}{c} = -\epsilon$.

gravitation is propagated with the velocity of light. Nevertheless, in the system (70) the g_{ij} depend only on the distance r between the *simultaneous* positions of the point acted upon and the source of the action, and we should apparently be forced to make the conclusion that the velocity of propagation is instantaneous. The paradox is, however, only apparent. If we speak of propagation of gravitation at all, we must conceive the g_{ij} as variable with the time, and in the given problem this variability can only arise from the motions of the bodies producing the gravitational field. Consequently, by the method of art. 21, where these velocities are assumed to be small, and third differentials with respect to the time are neglected entirely, it is *a priori* impossible to derive anything at all about the velocity of propagation.

23. In the planetary theories we wish to determine the motion of one of the n bodies in the field produced by the others and itself. If the body is of finite dimensions, the accelerations impressed on its different points will not be the same. The body being rigid, its points must, however, all have the same motion. There are thus produced strains in it which affect the T_{ij}, and through these again the gravitational field. These effects are, of course, of the second order and, if the dimensions of the bodies are small compared with their mutual distances, they will be negligible. Then the equations of motion of a body are the same as those of any point in it, say the centre. The equations can be written in any of the forms given in the first paper. We will use the formulas (47), and for χ_i and γ we use the values (70).

If the body whose motion is considered has the index μ, then in γ_0 we must leave out the term with that index. The same remark applies to the single sums in χ_i and in the second and fourth terms of γ'. In the double sum, which forms the third term of γ', we must in the first sum omit the body μ, but in the second (inner) sum it must be included. We will bring the term so arising into evidence by taking it out of the sum. Then the index μ must be excluded from *all* sums. The complete equation then becomes:

$$
\begin{aligned}
(82)\quad \frac{d^2 x_{i\mu}}{c^2 dt^2} + \lambda^2 \sum_\nu m_\nu \frac{x_{i\mu} - x_{i\nu}}{\mathrm{r}_\nu^{\,2}} &= 4\lambda^2 \sum_\nu \sum_\kappa m_\nu m_\kappa \frac{x_{i\mu} - x_{i\nu}}{\mathrm{r}_\nu^{\,3} \mathrm{r}_\kappa} \\
&+ 4\lambda^2 \sum_\nu m_\nu {\sum_p}' \frac{x_{p\mu} - x_{p\nu}}{\mathrm{r}_\nu^{\,3}} \dot{x}_{i\mu} \dot{x}_{p\mu} \\
&- \lambda^2 \sum_\nu m_\nu \frac{x_{i\mu} - x_{i\nu}}{\mathrm{r}_\nu^{\,3}} \left\{ \phi_\mu^{\,2} + 2\phi_\nu^{\,2} - \tfrac{3}{2}\epsilon_\nu^{\,2} + \tfrac{1}{2}\mathrm{r}_\nu \eta_\nu \right\} \\
&+ \lambda^2 \sum_\nu m_\nu \left\{ \frac{7}{2} \frac{\ddot{x}_{i\nu}}{\mathrm{r}_\nu} + 3 \frac{\dot{x}_{i\mu} - \dot{x}_{i\nu}}{\mathrm{r}_\nu^{\,2}} \epsilon_\nu + 4 {\sum_p}' \frac{(x_{i\mu} - x_{i\nu}) \dot{x}_{p\nu} - (x_{p\mu} - x_{p\nu}) \dot{x}_{i\nu}}{\mathrm{r}_\nu^{\,3}} \dot{x}_{p\mu} \right\} \\
&- 2\lambda^4 \sum_\nu m_\nu \frac{x_{i\mu} - x_{i\nu}}{\mathrm{r}_\nu^{\,3}} \sum_{\kappa \neq \nu} \frac{m_\kappa}{\Delta_{\kappa\nu}} - 4\lambda^4 m_\mu \sum_\nu m_\nu \frac{x_{i\mu} - x_{i\nu}}{\mathrm{r}_\nu^{\,4}}.
\end{aligned}
$$

It should be noted that r_ν is identical with $\Delta_{\mu\nu}$.

The left-hand members put equal to zero give the motion according to Newton's law of gravitation. The first three terms on the right are the generalisation of the terms already found in the first paper for the motion of an infinitesimal planet.

In applying this equation to the solar system we must take account of the smallness of most of the masses. The sun will be given the index $\mu = 0$.

We will introduce co-ordinates relatively to the sun by putting

$$x_i' = x_i - x_{i0},$$

from which

$$\frac{d^2 x_i'}{c^2 dt^2} = \frac{d^2 x_i}{c^2 dt^2} - \frac{d^2 x_{i0}}{c^2 dt^2}.$$

In the right-hand member we must then put

$$\dot{x}_i = \dot{x}_i' + \dot{x}_{i0}, \quad \ddot{x}_i = \ddot{x}_i' + \ddot{x}_{i0}.$$

In these right-hand members we can use the values given by the Newtonian theory. The system of co-ordinates to which (82) is referred was fixed in art. 20 by the additional conditions introduced during the integration. One of these was that at a large distance from all masses the geodetic lines are straight lines. In Newtonian mechanics this condition is satisfied relatively to a system of co-ordinates of which the origin is in the centre of gravity of all masses. Thus for the Newtonian values we will have to take

$$\dot{x}_{i0} + m_\mu \dot{x}_{i\mu} + \sum_\nu m_\nu \dot{x}_{i\nu} = 0,$$

or, if we put

$$M = 1 + m_\mu + \sum_\nu m_\nu,$$

$$\dot{x}_{i\mu} = \left(1 - \frac{m_\mu}{M}\right)\dot{x}_{i\mu}' - \sum_\nu \frac{m_\nu}{M}\dot{x}_{i\nu}',$$

and the same for second differentials. The sums exclude the values 0 and μ. After the substitution we drop the accents.

We must now write the equation (82) first for the index μ and then for the index 0, and subtract the latter from the former. I will slightly alter the notation of the mutual distances, to bring it into harmony with the usual notation of planetary theories. In equation (82) we first replace r_ν by $\Delta_{\mu\nu}$. Then we replace

$$\Delta_{0\mu},\ \Delta_{0\nu},\ \Delta_{\mu\nu},\ \Delta_{\kappa\nu};\ m_0,\ m_\mu,\ m_\nu$$

by

$$r,\quad r_\nu,\ \Delta_\nu,\ \Delta_{\kappa\nu};\ 1,\quad m,\ m_\nu$$

respectively, and we also drop the index μ of the co-ordinates, and we replace λ^2 by λ_0^2. The masses m and m_ν are very small. We will suppose them to be of the order $\frac{1}{2}$. Then in the right-hand members all terms beyond the first three will be of the order $2\frac{1}{2}$, and the resulting equation becomes, correct to the second order,

$$(83) \qquad \frac{d^2x_i}{c^2dt^2} + \lambda_0^2(1+m)\frac{x_i}{r^3} + \lambda_0^2\sum_\nu m_\nu \frac{x_i - x_{i\nu}}{\Delta_\nu^3} - \lambda_0^2\sum_\nu m_\nu \frac{x_{i\nu}}{r_\nu^3}$$
$$= 4\lambda_0^4\frac{x_i}{r^4} + 4\lambda_0^2\sum_p{}' \frac{x_p}{r^3}\dot{x}_p\dot{x}_i - \lambda_0^2\frac{x_i}{r^3}\phi^2.$$

The left-hand member gives the unperturbed motion and the perturbing forces according to Newton's law. The right-hand member contains nothing beyond the terms which have already been found in the first paper (art. 15) for an infinitesimal planet.

If we wish to take account of the terms of the order $\lambda^4 m$, then we must: 1st, in the ordinary perturbing function on the left hand use the co-ordinates increased by the "perturbations" produced by the right-hand member, *i.e.* we must use the formula (52, B) instead of Kepler's third law, and we must take account of the motion of the perihelion found in art. 16; 2nd, we must in the right-hand member of (83) use the co-ordinates increased by their Newtonian perturbations of the first order; and 3rd, we must add to this right-hand member the following terms:—

$$(83') \quad \left\{\begin{aligned}
&\lambda_0^2 m\left\{2\frac{\dot{x}_i\dot{r}}{r^2} - \frac{x_i}{r^3}\left[4\phi^2 - \tfrac{3}{2}\dot{r}^2\right]\right\} + 8\lambda^4 m\frac{x_i}{r^4} \\
&+ \lambda_0^2\sum_\nu m_\nu\left\{\frac{x_i - x_{i\nu}}{\Delta_\nu^3}\left[4\chi_\nu + \tfrac{3}{2}\epsilon_\nu^2 - \phi^2 - 2\phi_\nu^2\right] + \epsilon_\nu\left[\frac{3\dot{x}_i + \dot{x}_{i\nu}}{\Delta_\nu^2} + \frac{7\Delta_\nu\dot{x}_i}{r^3}\right]\right. \\
&\qquad\qquad \left. + \dot{x}_i\left[\frac{4\epsilon_\nu'}{\Delta_\nu^3} - \frac{7r_\nu\dot{r}_\nu}{r^3}\right] + 6\frac{x_i}{r^3}\chi_\nu + \frac{x_{i\nu}}{r_\nu^3}\left[\tfrac{3}{2}\dot{r}_\nu^2 - 2\phi_\nu^2\right] + 3\frac{\dot{r}_\nu\dot{x}_{i\nu}}{r_\nu^2}\right\} \\
&+ \lambda_0^4\sum_\nu m_\nu\left\{\frac{x_i - x_{i\nu}}{4\Delta_\nu^3}\left[\frac{3}{r_\nu} - \frac{2r\cos\psi_\nu}{r_\nu^2}\right] - \frac{2x_i}{r_\nu^3\Delta_\nu} - \frac{x_i\cos\psi_\nu}{2r^2r_\nu^2}\right. \\
&\qquad\qquad \left. + \tfrac{7}{2}\frac{x_{i\nu}}{r_\nu^3}\left[\frac{2}{r_\nu} + \frac{1}{r} - \frac{1}{\Delta_\nu}\right]\right\},
\end{aligned}\right.$$

where for abbreviation we have put

$$\Delta_\nu\epsilon_\nu = \sum_p{}'(x_{p\nu} - x_p)\dot{x}_{p\nu}, \qquad \Delta_\nu\epsilon_\nu' = \sum_p{}'(x_p - x_{p\nu})\dot{x}_p,$$
$$rr_\nu\cos\psi_\nu = \sum_p{}' x_p x_{p\nu}, \qquad \chi_\nu = \sum_p{}'\dot{x}_p\dot{x}_{p\nu}.$$

All these terms are, of course, extremely small, and none of them will give rise to an effect which is at all within the reach of observations.

24. For the lunar theory * I will take a system of reference which has its origin in the sun. The masses of the sun, the earth, and the moon will be m, 1, and μ respectively. We will further

* See also De Sitter, "Planetary Motion and the Motion of the Moon according to Einstein's Theory," *Proceedings Acad. Amsterdam*, June 1916 (vol. xix. p. 367).

put $\lambda_0^2 = \lambda^2 m$, $\lambda_1^2 = \lambda^2$, if λ^2 is expressed with the mass of the earth as unit. Consequently, λ_0^2 will have the same numerical value as before, and $\lambda_1^2 = \lambda_0^2/m$. We will put

x_i, Δ = heliocentric co-ordinates and distance of the moon,
ξ_i, ρ = " " " " earth,
x_i, r = geocentric " " " moon.

Further, we put

$$\sum_\alpha{}' \dot{x}_\alpha^2 = \phi^2, \quad \sum_\alpha{}' \dot{\mathrm{x}}_\alpha^2 = \phi_1^2, \quad \sum_\alpha{}' \dot{\xi}_\alpha^2 = \phi_0^2,$$

whence

$$\phi_1^2 = \phi_0^2 + \phi^2 + 2\sum_\alpha{}' \dot{x}_\alpha \dot{\xi}_\alpha .$$

We now must compute the difference

$$\frac{d^2 \mathrm{x}_i}{c^2 dt^2} - \frac{d^2 \xi_i}{c^2 dt^2},$$

both differential quotients being computed by (82). The terms of (82) will be divided into three groups, viz.:—

a. Those which only contain the masses 1 and μ.

b. Those in $\dfrac{d^2 \mathrm{x}_i}{c^2 dt^2}$ containing m or $m \cdot \mu$, and in $\dfrac{d^2 \xi_i}{c^2 dt^2}$ containing m and $m \cdot 1$,

c. The others, *i.e.* in $\dfrac{d^2 \mathrm{x}_i}{c^2 dt^2}$, the terms with $m \cdot 1$, and in $\dfrac{d^2 \xi_i}{c^2 dt^2}$ those with $m \cdot \mu$.

The first group corresponds to the problem of the *two* bodies, earth and moon. The second group would give the perturbations by the sun if the complete field of the earth and sun were equal to the sum of the superposed fields of each of these bodies computed as if the other did not exist. The third group contains the terms which arise from the "interference" of the two fields.

a. In the first group I take, in the terms of the second order,

$$\ddot{x}_i = -\lambda_1^2 (1 + \mu) \frac{x_i}{r^3} .$$

Further, I put

$$\epsilon = \sum_p{}' x_p \dot{\xi}_p .$$

We then find, after some reductions,

$$(84) \quad \frac{d^2 x_i}{c^2 dt^2} + \lambda_1^2 (1 + \mu) \frac{x_i}{r^3} = 4\lambda_1^4 (1 - \mu + 2\mu^2) \frac{x_i}{r^4} + 4\lambda_1^2 (1 + \tfrac{3}{4}\mu) \frac{\dot{x}_i \dot{r}}{r^2} - \lambda_1^2 (1 + 2\mu) \frac{x_i}{r^3} \phi^2$$
$$+ \lambda_1^2 \frac{x_i}{r^3} \Big[(1 + \mu)\phi_0^2 + 2\sum_p{}' \dot{x}_p \dot{\xi}_p + 3\lambda_1^2 \mu \frac{\dot{r}\epsilon}{r} + \tfrac{3}{2}(1 + \mu)\frac{\epsilon^2}{r^2} + \tfrac{1}{2}(1 - \mu)\sum_p{}' x_p \ddot{\xi}_p \Big]$$
$$+ \lambda_1^2 \Big[(1 + 2\mu) \frac{\dot{x}_i \epsilon}{r^3} + \tfrac{7}{2}(1 - \mu) \frac{\ddot{\xi}_i}{r} \Big].$$

If we took $\mu = 0$, $\dot{\xi}_i = \ddot{\xi}_i = 0$, this would become the formula for the motion of an infinitesimal planet. We can take

$$\phi_0{}^2 = \dot{\rho}^2 + \rho^2 \dot{\theta}_0{}^2, \quad \ddot{\xi} = -\lambda_0{}^2 \frac{\xi_i}{\rho^3}.$$

If we neglect the excentricity of the earth's orbit, we have

$$\dot{\rho} = 0, \quad \dot{\theta}_0 = \nu_0,$$

therefore

$$\phi_0{}^2 = \frac{\lambda_0{}^2}{\rho}.$$

The formula (84) then becomes, if we neglect in the second member the terms having μ as a factor,

$$(85) \quad \frac{d^2 x_i}{c^2 dt^2} + \lambda_1{}^2 (1 + \mu) \frac{x_i}{r^3} = 4\lambda_1{}^4 \frac{x_i}{r^4} + 4\lambda_1{}^2 \frac{\dot{x}_i \dot{r}}{r^2} - \lambda_1{}^2 \frac{x_i}{r^3} \phi^2$$
$$+ \lambda_1{}^2 \frac{x_i}{r^3} \sum_p{}' \left(2\dot{x}_p \dot{\xi}_p - \frac{\lambda_0{}^2 x_p \xi_p}{2\rho^3} \right) + \lambda_1{}^2 \frac{x_i \epsilon}{r^3} + \tfrac{7}{2} \lambda_1{}^2 \lambda_0{}^2 \frac{\xi_i}{r\rho^3}$$
$$+ \lambda_1{}^2 \frac{x_i}{r^3} \left[\frac{{}_0{}^2}{\rho} + \frac{3}{2} \frac{\epsilon^2}{r^2} \right].$$

We can, in the terms of the second order, safely neglect the excentricity and the inclination of the moon's orbit. Then, if

$$\mathrm{D} = l_{☾} - l_{\odot}$$

be the difference of the mean longitudes of the moon and sun, we have

$$\sum_p{}' x_p \xi_p = r\rho \cos \mathrm{D}, \quad \epsilon = r\rho\nu_0 \sin \mathrm{D}, \quad \sum_p{}' \dot{x}_p \dot{\xi}_p = r\rho\nu\nu_0 \cos \mathrm{D},$$

where we can take

$$\nu = \lambda_1 a^{\frac{3}{2}}, \quad \nu_0 = \lambda_0 \rho^{\frac{3}{2}}.$$

The terms in the second line of (85) are periodic in D, and can therefore certainly be neglected. The terms in the last line contain secular terms. The secular part of ϵ^2 is

$$\epsilon^2 = \tfrac{1}{2} r^2 \rho^2 \nu_0{}^2 = \tfrac{1}{2} \lambda_0{}^2 \frac{r^2}{\rho}.$$

The two terms can therefore be combined, and can be transported to the left-hand member. We then have

$$(86) \quad \frac{d^2 x_i}{c^2 dt^2} + \lambda_1{}^2 \left(1 + \mu - \frac{7}{4} \frac{\lambda_0{}^2}{\rho} \right) \frac{x_i}{r^3} = 4\lambda_1{}^4 \frac{x_i}{r^4} + 4\lambda_1{}^2 \frac{\dot{x}_i \dot{r}}{r^2} - \lambda_1{}^2 \frac{x_i}{r^3} \phi^2.$$

The right-hand member is the same as for an infinitesimal planet. The only secular effect is thus a motion of the perigee amounting to

$$\delta\varpi = 3 \frac{\lambda_1{}^2}{p} v.$$

The motion in one century is

$$0''\cdot 06.$$

b. Many of the terms of the second group are zero because the origin of the system of reference has been taken in the sun. We find

$$(87)\quad \frac{d^2x_i}{c^2dt^2}+\lambda_1{}^2m\left(\frac{\mathrm{x}_i}{\Delta^3}-\frac{\xi_i}{\rho^3}\right)=4\lambda_1{}^4m\frac{\xi_i}{\rho^4}+4\lambda_1{}^4m^2\left(\frac{\mathrm{x}_i}{\Delta^4}-\frac{\xi_i}{\rho^4}\right)$$
$$+4\lambda_1{}^2m\left(\frac{\dot\Delta\dot{\mathrm{x}}_i}{\Delta^2}-\frac{\dot\rho\dot\xi_i}{\rho^2}\right)-\lambda_1{}^2m\left(\frac{\mathrm{x}_i}{\Delta^3}\phi_1{}^2-\frac{\xi_i}{\rho^3}\phi_0{}^2\right)-4\lambda_1{}^4m\mu\frac{\mathrm{x}_i}{\Delta^4}.$$

The left-hand member contains the ordinary perturbing function. In the usual developments of this function the factor $\lambda_0{}^2/\rho^3$ is replaced by $\nu_0{}^2$. This, however, is not correct in the new theory, but we must take, according to (52, B),

$$\lambda_1{}^2m=\lambda_0{}^2=\rho^3\nu_0{}^2\left(1+\frac{3\lambda_0{}^2}{\rho}\right).$$

The first term on the right contains no co-ordinates of the moon, and can thus only give rise to periodic terms, which we neglect. The last term could be added to the second. It is, however, about 27 million times smaller than this, and can be neglected.

The other terms in the right-hand member produce perturbations in the motion of the moon, which will be investigated in the next article.

c. The terms of the third group are

$$(88)\quad \frac{d^2x_i}{c^2dt^2}=4\lambda_1{}^4m\left(\frac{\mathrm{x}_i}{r\,\Delta^3}-\mu\frac{\xi_i}{r\rho^3}\right)-2\lambda_1{}^4m\left(\frac{\mathrm{x}_i}{\rho\,\Delta^3}-\mu\frac{\xi_i}{\Delta\,\rho^3}\right)$$
$$+4\lambda_1{}^4m\frac{x_i}{r^3}\left(\frac{1}{\Delta}+\frac{\mu}{\rho}\right)-2\lambda_1{}^4m\frac{x_i}{r^3}\left(\frac{1}{\rho}+\frac{\mu}{\Delta}\right).$$

The last four terms have x_i/r^3 as a factor. If we develop the coefficient in powers of the parallax a/ρ, we can neglect all but the lowest power, and we can thus in these small terms take $\Delta=\rho$. The four terms together then give

$$2\lambda_1{}^2\frac{\lambda_0{}^2}{\rho}\,\frac{x_i}{r^3}(1+\mu).$$

This can be transported to the left-hand member. The coefficient of x_i/r^3 then becomes

$$\lambda_1{}^2\left(1+\mu-\frac{15}{4}\,\frac{\lambda_0{}^2}{\rho}-2\mu\frac{\lambda_0{}^2}{\rho}\right),$$

for which we can write

$$\lambda_1{}^2(1+\mu'),$$

and the difference between μ and μ' is very much smaller than the uncertainty of the numerical value of μ. Of the other terms of (88) the third is of the same nature as the second of (87), but 328000 times as small, and therefore entirely negligible. The fourth term is still smaller, and if we replace Δ by ρ, it is of the same nature as the first of (87), and can be neglected for the same reason. The second term, which contains the factor μ and is therefore very small, can, moreover, produce only periodic perturbations. Of all the terms in (88) the first is the largest. It is still about 800 times smaller than the second of (88), and can be neglected. It can, however, also very easily be taken into account. Developing in powers of $1/\rho$, we find for the lowest power

$$-2\lambda_1^2\lambda_0^2\frac{x_i}{r\rho^3} = -2\lambda_1^2\nu_0^2\frac{x_i}{r}.$$

The resulting radial and transversal perturbing forces are (if we neglect the inclination of the moon's orbit),

$$S = -2\nu_0^2, \qquad T = 0.$$

The effect on the motion of the perigee is

$$e d\varpi = 2\nu_0^2 r^2 \cos v dv,$$

which gives only extremely small periodic terms, that can be neglected.

There thus remains only the equation (87), *i.e.* the effect of the pure superposition of the fields of the earth and the sun.

25. We can write this equation in the following form:—

$$\frac{d^2x_i}{c^2dt^2} + \lambda_1^2 m\left(\frac{\mathrm{x}_i}{\Delta^3} - \frac{\xi_i}{\rho^3}\right) = \lambda_1^2[4A_i + 4B_i - C_i], \tag{89}$$

where

$$A_i = \lambda_1^2 m^2\left(\frac{\mathrm{x}_i}{\Delta^4} - \frac{\xi_i}{\rho^4}\right),$$

$$B_i = \lambda_1^2 m\left(\frac{\Delta\dot{\Delta}\dot{\mathrm{x}}_i}{\Delta^3} - \frac{\rho\dot{\rho}\dot{\xi}_i}{\rho^3}\right),$$

$$C_i = \lambda_1^2 m\left(\frac{\mathrm{x}_i}{\Delta^3}\phi_1^2 - \frac{\xi_i}{\rho^3}\phi_0^2\right).$$

We must develop A, B, and C in powers of r/ρ. We will only retain such terms as can give secular perturbations, and of these only those of the lowest degree in r/ρ. We will write x, y, z for x_1, x_2, x_3, and similarly x, y, z, etc.; and for A_1, A_2, A_3 we will write A_x, A_y, A_z, and similarly B_x, . . ., etc. As plane of xy we take the orbital plane of the moon, thus $z = 0$. Then the radial, transversal, and orthogonal perturbing forces are

$$S_A = \frac{x}{r}A_x + \frac{y}{r}A_y, \quad T_A = \frac{x}{r}A_y - \frac{y}{r}A_x, \quad W_A = A_z,$$

and similarly for B and C. Let i and Ω be the inclination and ascending node of the moon's orbit on the ecliptic. We will neglect i^2. Also we neglect the excentricity of the earth. Then we have:

$$\begin{aligned} \xi &= \rho \cos \mathrm{L}, & \dot{\xi} &= -\rho\nu_0 \sin \mathrm{L}. \\ \eta &= \rho \sin \mathrm{L}, & \dot{\eta} &= \rho\nu_0 \cos \mathrm{L}. \\ \zeta &= -\rho \sin i \sin (\mathrm{L} - \Omega), & \dot{\zeta} &= -\rho\nu_0 \sin i \cos (\mathrm{L} - \Omega). \end{aligned}$$

$$\rho^3 \nu_0^2 = \lambda_0^2 = \lambda_1^2 m.$$

Further, we have

$$\Delta^2 = \rho^2 \left[1 + 2\frac{x \cos \mathrm{L} + y \sin \mathrm{L}}{\rho} + \frac{r^2}{\rho^2} \right],$$

and

$$\mathrm{x} = \xi + x, \quad \mathrm{y} = \eta + y, \quad \mathrm{z} = \zeta.$$

Taking first A_x, we find for the terms of the lowest degree in r/ρ,

$$\frac{\mathrm{x}}{\Delta^4} - \frac{\xi}{\rho^4} = \frac{x}{\rho^4} - 4\frac{\xi(x \cos \mathrm{L} + y \sin \mathrm{L})}{\rho^5}.$$

The secular part (*i.e.* the part independent of L) of the numerator of the second term is $\frac{1}{2}\rho x$, therefore

$$\mathrm{A}_x = -\lambda_0^2 m \frac{x}{\rho^4} = -\frac{m\nu_0^2}{\rho} x.$$

Similarly we find

$$\mathrm{A}_y = -\frac{m\nu_0^2}{\rho} y.$$

Further,

$$\frac{\mathrm{z}}{\Delta^4} - \frac{\zeta}{\rho^4} = \zeta\left(\frac{1}{\Delta^4} - \frac{1}{\rho^4}\right) = 4\rho \sin i \sin (\mathrm{L} - \Omega) \frac{x \cos \mathrm{L} + y \sin \mathrm{L}}{\rho^5}.$$

The numerator of the last fraction is

$$r \cos (w - \mathrm{L}),$$

where w is the moon's longitude. The secular part thus becomes

$$\frac{2r \sin i \sin (w - \Omega)}{\rho^4}.$$

Now we put

$$z' = r \sin i \sin (w - \Omega),$$

so that z' is the co-ordinate of the moon perpendicular to the ecliptic. Then we have

$$\mathrm{A}_z = 2\frac{m\nu_0^2}{\rho} z'.$$

Consequently

$$
(90) \quad \begin{cases} S_A = -\dfrac{mv_0^2}{\rho} r, \\ T_A = 0, \\ W_A = 2\dfrac{mv_0^2}{\rho} z'. \end{cases}
$$

It is not necessary to relate the computations of the terms B and C at the same length. In B we have $\dot{\rho} = 0$, so there remains only one term to be developed. We have $\Delta\dot{\Delta} = \mathrm{x}\dot{\mathrm{x}} + \mathrm{y}\dot{\mathrm{y}} + \mathrm{z}\dot{\mathrm{z}}$. After development we find:

$$
\begin{aligned}
B_x &= \tfrac{1}{2}\frac{mv_0^2}{\rho} x - \tfrac{1}{2}\frac{mv_0}{\rho}\dot{y}, \\
B_y &= \tfrac{1}{2}\frac{mv_0^2}{\rho} y + \tfrac{1}{2}\frac{mv_0}{\rho}\dot{x}, \\
B_z &= -\tfrac{1}{2}\frac{mv_0^2}{\rho} z' + \tfrac{1}{2}\frac{mv_0}{\rho} z'\nu.
\end{aligned}
$$

The factor z' in the last term of B_z is found by neglecting the moon's excentricity in the expressions for $\dot{x}$ and $\dot{y}$, which is allowed on account of the factor $\sin i$. With the same accuracy we can take $\nu = \dot{\theta}$. Then, since

$$
x\dot{y} - y\dot{x} = r^2\dot{\theta}, \quad x\dot{x} + y\dot{y} = r\dot{r},
$$

we find

$$
(91) \quad \begin{cases} S_B = \tfrac{1}{2}\dfrac{mv_0^2}{\rho} r - \tfrac{1}{2}\dfrac{mv_0}{\rho} r\dot{\theta}, \\ T_B = \tfrac{1}{2}\dfrac{mv_0}{\rho}\dot{r}, \\ W_B = -\tfrac{1}{2}\dfrac{mv_0^2}{\rho} z' + \tfrac{1}{2}\dfrac{mv_0}{\rho} z'\dot{\theta}. \end{cases}
$$

Finally, for C we find similarly

$$
\begin{aligned}
C_x &= -\tfrac{1}{2}\frac{mv_0^2}{\rho} x + \frac{mv_0}{\rho}\dot{y}, \\
C_y &= -\tfrac{1}{2}\frac{mv_0^2}{\rho} y - \frac{mv_0}{\rho}\dot{x}, \\
C_z &= \tfrac{3}{2}\frac{mv_0^2}{\rho} z' - \frac{mv_0}{\rho} z'\nu,
\end{aligned}
$$

and

$$
(92) \quad \begin{cases} S_C = -\tfrac{1}{2}\dfrac{mv_0^2}{\rho} r + \dfrac{mv_0}{\rho} r\dot{\theta}, \\ T_C = -\dfrac{mv_0}{\rho}\dot{r}, \\ W_C = \tfrac{3}{2}\dfrac{mv_0^2}{\rho} z' - \dfrac{mv_0}{\rho} z'\dot{\theta}. \end{cases}
$$

Collecting results, we find for the total perturbing forces $(4A+4B-C)$,

$$(93) \quad \begin{cases} S = -\frac{3}{2}\frac{m\nu_0^2}{\rho}r - 3\frac{m\nu_0}{\rho}r\dot{\theta}, \\ T = \qquad\qquad\quad + 3\frac{m\nu_0}{\rho}\dot{r}, \\ W = +\frac{9}{2}\frac{m\nu_0^2}{\rho}z' + 3\frac{m\nu_0}{\rho}z'\dot{\theta}. \end{cases}$$

It has already been remarked that the ordinary Newtonian perturbing function, which stands in the left-hand member of (89), also requires a correction. We call the last two terms in this left-hand member

$$-\lambda_1^2 R_i,$$

and we develop R_x, R_y, R_z in powers of r/ρ as we have done for A, B, C. Then transporting to the right-hand member, we find for the secular terms of the lowest degree,

$$R_x = \frac{1}{2}\frac{mx}{\rho^3}, \quad R_y = \frac{1}{2}\frac{my}{\rho^3}, \quad R_z = -\frac{3}{2}\frac{mz'}{\rho^3}.$$

Therefore

$$S_R = \frac{1}{2}\frac{m}{\rho^3}r, \quad T_R = 0, \quad W_R = -\frac{3}{2}\frac{m}{\rho^3}z'.$$

These are the perturbing functions as used in the ordinary lunar theories. Usually m/ρ^3 is replaced by n_0^2, by Kepler's third law. Now, however, we must take by (52) of the first paper

$$\frac{\lambda_1^2 m}{\rho^3} = \nu_0^2\left(1 + \frac{3\lambda_0^2}{\rho}\right).$$

The correction therefore is

$$\delta\frac{m}{\rho^3} = 3\frac{\lambda_0^2}{\lambda_1^2}\frac{\nu_0^2}{\rho} = 3\frac{m\nu_0^2}{\rho}.$$

Therefore

$$(94) \qquad \delta S_R = \frac{3}{2}\frac{m\nu_0^2}{\rho}r, \quad \delta T_R = 0, \quad \delta W_R = -\frac{9}{2}\frac{m\nu_0^2}{\rho^3}z'.$$

It will be seen that these exactly cancel the first terms of (93). There then remain only the second terms, viz.:

$$(95) \qquad S = -3\frac{m\nu_0}{\rho}r\dot{\theta}, \quad T = 3\frac{m\nu_0}{\rho}\dot{r}, \quad W = 3\frac{m\nu_0}{\rho}z'\dot{\theta}.$$

26. We found in art. 24 that the new terms in the motion of the moon are completely given by the superposition of the fields of the sun and the earth, both computed as if the other did not exist. The expression for these fields which has been used above is that given by the formulas B of art. 11 (first paper). If we had used the system A of the same article, we would have found, instead of (89),

$$\text{(96)} \qquad \frac{d^2x_i}{c^2dt^2} + \lambda_1{}^2m\left(\frac{\mathrm{x}_i}{\Delta^3} - \frac{\xi_i}{\rho^3}\right) = \lambda_1{}^2[2\mathrm{A}_i + 2\mathrm{B}_i - 2\mathrm{C}_i + 3\mathrm{D}_i],$$

where

$$\mathrm{D}_i = m\left[\frac{\mathrm{x}_i\dot{\Delta}}{\Delta^3} - \frac{\xi_i\dot{\rho}}{\rho^3}\right].$$

It is easily found that the term D is of a higher order in r/ρ than A, B, and C, and can consequently be neglected. The other terms give for S, T, and W exactly the values (95). On the other hand, in this system Kepler's third law is exact (if we neglect the moon's excentricity), as is shown by (52, A), and therefore the terms (94) do not exist in this case. The two systems thus give the same result, as they ought to do.

We must now find the effect of the perturbing forces (95) on the elements of the moon's orbit. We can in the perturbing forces use the Keplerian values for the co-ordinates and velocities. We thus have

$$r\dot{\theta} = \frac{\lambda_1\sqrt{p}}{r}, \qquad \dot{r} = \frac{\lambda_1 e \sin v}{\sqrt{p}}.$$

Substituting these values, we find that secular terms arise in the elements ϖ and Ω. We find

from S: $\qquad e\,d\varpi = 3\dfrac{mv_0}{\rho}\lambda_1\sqrt{p}\,.\,r\cos v\,dv,$

from T: $\qquad e\,d\varpi = 3\dfrac{mv_0}{\rho}\dfrac{\lambda_1}{\sqrt{p}}r^2\left(1+\dfrac{r}{p}\right)e\sin^2 v\,dv,$

from W: $\qquad \sin i\,d\Omega = 3\dfrac{mv_0}{\rho}\dfrac{\lambda_1}{\sqrt{p}}r^2\sin i\sin^2(w-\Omega)dv,$

where $w = v + \varpi$. It would not be difficult to integrate these equations rigorously. We get a sufficient approximation, however, by neglecting the excentricity and the inclination. We then find the following secular terms—

$$\text{(97)} \qquad \delta\varpi = \delta\Omega = \tfrac{3}{2}\frac{mv_0\nu}{\rho}\,.\,nt,$$

where we have put $a^{\frac{1}{2}}\lambda_1 = \nu$. The numerical value is

$$+1''\cdot 91 \text{ per century},$$

to which for the perigee we must add the $0''\cdot 06$ found above.

The observed values are *

Brown, Cowell	. .	$\delta\varpi = +14643536'' \pm 2''$
Newcomb, de Vos	. .	$+14643530'' \pm 2''$
Newcomb, Brown	. .	$\delta\Omega = -6967944'' \pm 2''$

* See De Sitter, "The Motions of the Lunar Perigee and Node, and the Figure of the Moon," *Proc. Acad. Amsterdam*, vol. xvii. p. 1309 (April 1915).

The theoretical values, including the new terms found here, are

$$\delta\varpi = +14643534'' \pm 2''$$
$$\delta\Omega = -6967939'' \pm 2''$$

It is to be regretted that it is not possible from Brown's lunar theory to find the theoretical motion with one more decimal place. The residuals

$$\Delta\varpi = \begin{cases} +2'' \pm 3'' \\ -4'' \pm 3'' \end{cases}$$
$$\Delta\Omega = \quad -5'' \pm 3''$$

are undoubtedly partly due to the inaccuracy of the theoretical value, owing to accumulation of errors by the neglect of the decimal of the second in the many terms of which the complete value is the sum. The residuals as they stand are therefore not decisive either for or against Einstein's theory.

27. In the equations of art. 21 we will assume that there is only one body, the sun, to which we will assign the index 0, Further, we put

$$\mathrm{x}_i = x_i - x_{i0}, \quad x_{i0} = a_i t, \quad \mathrm{r}^2 = \sum_p{}' \mathrm{x}_p{}^2,$$
$$\sum_p{}' a_p{}^2 = q^2, \quad \sum_p{}' a_p \mathrm{x}_p = -\mathrm{r}\epsilon.$$

Then

$$\frac{\partial^2 \mathrm{r}}{c^2 \partial t^2} = \frac{q^2 - \epsilon^2}{\mathrm{r}}$$

and the g_{ij} become

$$(98) \qquad (\mathrm{D}) \begin{cases} -1+\gamma & 0 & 0 & -2a_1\gamma \\ 0 & -1+\gamma & 0 & -2a_2\gamma \\ 0 & 0 & -1+\gamma & -2a_3\gamma \\ -2a_1\gamma & -2a_2\gamma & -2a_3\gamma & 1+\gamma+\frac{1}{2}\gamma^2+\gamma(2q^2-\frac{1}{2}\epsilon^2) \end{cases}$$

where

$$\gamma = -\frac{2\lambda_0{}^2}{\mathrm{r}}.$$

This will be called the system D.

In the first paper we found the field of the sun at rest at the origin of the system of co-ordinates, *i.e.* for $a_i = 0$. The line-element in this gravitational field was

$$(99) \qquad ds^2 = -(1-\gamma_0)d\sigma^2 + (1+\gamma_0+\tfrac{1}{2}\gamma_0{}^2)c^2dt^2,$$

where

$$\gamma_0 = -\frac{2\lambda_0{}^2}{r}, \quad r^2 = \sum_p{}' x_p{}^2, \quad x_{i0} = 0.$$

If to (99) we apply a Newton-transformation

$$x_i' = x_i - a_i t,$$

and then after the transformation we drop the accents, we find $\gamma_0 = \gamma$, and the g_{ij} become

$$(100)\quad (\mathrm{N}) \begin{cases} -1+\gamma & 0 & 0 & \alpha_1(1-\gamma) \\ 0 & -1+\gamma & 0 & \alpha_2(1-\gamma) \\ 0 & 0 & -1+\gamma & \alpha_3(1-\gamma) \\ \alpha_1(1-\gamma) & \alpha_2(1-\gamma) & \alpha_3(1-\gamma) & 1+\gamma+\frac{1}{2}\gamma^2-q^2(1-\gamma) \end{cases}$$

This will be called the system N. It will be seen that it does not agree with the system D (98). In (N) the g_{i4} are of the order $\frac{1}{2}$, and the difference of g_{44} with its value in (99) is of the first order; in (D) these terms are of the orders $\frac{3}{2}$ and 2 respectively. The velocity of light in the two systems is

$$(101)\qquad \begin{cases} (\mathrm{D}) & v = c(1+\gamma), \\ (\mathrm{N}) & v = c(1+\gamma+q\cos \mathrm{V}+\frac{1}{2}q^2\cos^2 \mathrm{V}), \end{cases}$$

where V is the angle between the ray of light and the velocity q.

If, on the other hand, we transform (99) by a Lorentz-transformation,* we find exactly the system D. This is due to the fact that the Lorentz-transformation is an *orthogonal* transformation of the four co-ordinates x_1, x_2, x_3, x_4.

The equations of motion in the systems (D) and (N) are different, also the course of rays of light is different. The *observable* phenomena must be exactly the same in the two systems. It will be interesting to demonstrate this by a simple example.

Let a planet move in a circle, and let it be observed by an observer on the sun. If the radius of the circle be a, then the differential equation for θ is in the two systems

$$(102)\qquad \begin{cases} (\mathrm{D}) & \dfrac{d^2\theta}{c^2dt^2} = -\dfrac{\lambda^2\epsilon}{a^2}\dot{\theta}, \\ (\mathrm{N}) & \dfrac{d^2\theta}{c^2dt^2} = 0. \end{cases}$$

* The formulas for a Lorentz-transformation are given in my paper of 1911 on the (old) principle of relativity, *M.N.*, vol. lxxi. p. 391. It must be remarked that what is here called $\alpha_1, \alpha_2, \alpha_3$ is the same as $\alpha q, \beta q, \gamma q$ in that paper. By the application of these formulas, and the transformation formula (3) of the first paper (vol. lxxvi. p. 701) the g_{ij} become

$$\mathrm{A} \begin{cases} -1+\gamma_0 & 0 & 0 & -2\alpha_1\gamma_0 \\ 0 & -1+\gamma_0 & 0 & -2\alpha_2\gamma_0 \\ 0 & 0 & -1+\gamma_0 & -2\alpha_3\gamma_0 \\ -2\alpha_1\gamma_0 & -2\alpha_2\gamma_0 & -2\alpha_3\gamma_0 & 1+\gamma_0+\frac{1}{2}\gamma_0{}^2+2q^2\gamma_0. \end{cases}$$

Now, however, γ_0 is not the same as γ, since in γ_0 the co-ordinates x_i and x_{i0} are simultaneous relatively to the time t, and in γ relatively to the time t'. We must then apply the formula (6) of vol. lxxi. p. 393, where the index 1 must be taken to refer to the planet and 2 to the sun. Then $-\alpha q_1 \Delta_q = \frac{1}{2}\alpha_1 r\epsilon$ in our present notation, and, since $\sum_p' \alpha_p x_p = -r\epsilon$, we have, accurate to the second degree in q, $\mathrm{r}^2 = r^2(1-\epsilon^2)$, and consequently $\gamma_0 = \gamma(1-\frac{1}{2}\epsilon^2)$. This reduces (A) to (98).

The light which leaves the planet at the time t reaches the observer at a time t_0, which is different in the two systems, since the velocity of light is different. In both systems, however, we can apply the rule of planetary aberration, which asserts that the true direction at the time t is the apparent direction at the time t_0. In both systems, therefore, the observed value θ_0 is the same as the true value θ belonging to the time t, as derived from the equation (102). We can thus omit the index 0 of θ_0.

The observer does not know the time t, but only the time t_0. We must therefore write the differential equation with respect to this time. We thus find

$$\frac{d^2\theta}{c^2 dt_0{}^2}=\frac{d^2\theta}{c^2dt^2}\left(\frac{dt}{dt_0}\right)^2+\frac{d\theta}{c^2dt}\frac{d^2t}{dt_0{}^2}.$$

In the system (D) the velocity of light is the same in all directions, consequently $t_0 - t$ is a constant, and

$$\text{(103)} \qquad \text{(D)} \quad \frac{d^2\theta}{c^2dt_0{}^2}=\frac{d^2\theta}{c^2dt^2}=-\frac{\lambda^2\epsilon}{a^2}\dot{\theta}.$$

In the system (N) we have, to the first order in q,

$$v=c(1+q\cos \mathrm{V})=c(1+\epsilon),$$

and consequently

$$t_0=t+\int_0^a\frac{dr}{v}=t+\frac{a}{c}(1-\epsilon).$$

Therefore

$$\frac{d^2t}{dt_0{}^2}=\frac{a}{c}\frac{d^2\epsilon}{dt_0{}^2}=ac\frac{d^2\epsilon}{c^2dt^2}=-\frac{c\lambda^2\epsilon}{a^2}.$$

Consequently in the system (N) we find

$$\text{(103}'\text{)} \qquad \text{(N)} \quad \frac{d^2\theta}{c^2dt_0{}^2}=-\frac{\lambda^2\epsilon}{a^2}\dot{\theta},$$

or exactly the same as in the system (D).

28. In this article we will consider the gravitational field produced by the fixed stars. We will suppose that the distribution of the stars in space is symmetrical round the sun. Of course this is very far from being true in the actual galactic system, but, if we took a distribution in ellipsoidal, instead of spherical, shells we would find practically the same result, and also for more irregular distributions the results would undoubtedly be of the same order of magnitude. Since exact numerical results are, of course, entirely out of the question, and the order of magnitude is all we can hope to derive, it is not necessary at present to consider any but the simplest possible case.

We will thus compute the field produced by a hollow spherical shell, whose total mass may be M, and whose inner and outer radii are R_1 and R_2 respectively. The analysis of art. 11 of the first paper can be used unaltered, and—as it will appear that we can neglect terms having λ^4 as a factor—we will find

$$ds^2 = -d\sigma^2 + c^2dt^2 + \gamma(d\sigma^2 + c^2dt^2),$$

where

$$\gamma = -\frac{2\lambda^2}{r}m(r) + 2\lambda^2[n(r) - \mathrm{N}].$$

Integrating by parts we find

$$m(r) = rn(r) - s(r),$$

where

$$s(r) = \int_0^r n(r)dr.$$

Therefore

$$\gamma = -2\lambda^2\mathrm{N} + \frac{2\lambda^2}{r}s(r). \tag{104}$$

Inside the shell we have $s(r) = 0$, and outside $s(r) = r\mathrm{N} - \mathrm{M}$. If the density ρ were constant we would have $s(r) = \frac{1}{2}m(r)$. If ρ decreases with increasing r, the ratio $s(r)/m(r)$ will increase. We thus have always

$$m(r) \leqq 2s(r), \tag{105}$$

and $m(r)$ and $s(r)$ will remain of the same order of magnitude so long as ρ differs appreciably from zero. It will be remembered that $m(r)$ is the total mass within the sphere of radius r.

Inside the shell we have

$$\gamma_0 = -2\lambda^2\mathrm{N}.$$

Therefore γ_0 is a constant, and there will be *no* gravitational effects. For, if we perform the transformation

$$x_i' = x_i\sqrt{(1 - \gamma_0)}, \quad t' = t\sqrt{(1 + \gamma_0)},$$

the g_{ij} will be reduced to the values given by

$$ds^2 = -d\sigma^2 + c^2dt^2.$$

The units of space and time will be changed, but this cannot be verified by observations of the motions of bodies inside the shell relatively to each other.

The difference in the unit of time can, however, be verified by observing light which emanates from a point within the mass, or outside the shell. The frequency of light-vibrations is proportional to $\sqrt{(1 + \gamma)}$ [see first paper, art. 13]. Now $s(r)$ is always positive, therefore all light coming from distances exceeding R_1 must appear displaced *towards the violet* as compared with light from a source inside the shell. If γ_1 be the value of γ at the source of the light, then the ratio of the frequencies is

$$\frac{\mathrm{N}_1}{\mathrm{N}_0} = \sqrt{\frac{1 + \gamma_1}{1 + \gamma_0}}. \tag{106}$$

Now there is certainly no systematic displacement of the stellar spectral lines towards the violet. Of the very faint stars, of which no measures of spectral lines can be made, we know, of course, nothing; but of stars within a certain distance, say r_1, we can say with certainty that a systematic displacement towards the violet is smaller than, say, $\frac{1}{3}$ km./sec. On the displacement towards the violet produced by the general field is, of course, superposed the displacement towards the red, produced by each star separately (*cf.* art. 13). This may easily cancel the displacement towards the violet. It is well known that stars of all spectra actually show a small systematic displacement towards the red. Still, to be quite safe, we will increase the limit, and only assert that the general field of the stars certainly does not cause a displacement exceeding 1 km./sec. towards the violet. We can then say

$$\sqrt{\frac{1+\gamma_1}{1+\gamma_0}} < 1 + 3 \cdot 10^{-6}$$

or

$$\gamma_1 - \gamma_0 < 6 \cdot 10^{-6}(1+\gamma_0). \tag{107}$$

In (106) no assumption is made as to the smallness of γ_1 or γ_0; (107) shows, however, that they must be small, and consequently the neglect of the terms of the second order is justified. Now we have

$$\gamma_1 - \gamma_0 = \frac{2\lambda^2}{r} s(r).$$

Therefore, dividing by $\lambda^2 = 10^{-8}$, and using (105), we have

$$m(r_1) < 600\, r_1.$$

Here r_1 is expressed in astronomical units. If we express it in parsecs we have, roughly,

$$m(r_1) < 10^8 . r_1. \tag{108}$$

If we take, *e.g.*, for r_1 the distance corresponding to a parallax of $0''\cdot01$ we would find

$$m(0''\cdot01) < 10^{10}.$$

It appears probable that the actual mass within this distance is much smaller. It should be pointed out that, although the limit has been derived from the observations of stellar spectra, $m(r)$ includes *all* mass within a sphere of radius r. It thus also includes all faint or dark stars, and all nebulæ and meteoric or other matter.

29. Having now discussed the principal astronomical consequences of the new theory, we will return briefly to the fundamental concepts. The principle of general relativity asserts that the space co-ordinates and the time are entirely irrelevant, and have no physical meaning whatever. In fact, we never observe anything but a combination of time and space. Nobody has ever measured a pure distance, nor a pure interval of time. Always the distance, or the interval of time, is measured from a material point (event) m_1 at time t_1 and at the place (x_1, y_1, z_1) to another point m_2 at another time t_2 and another place (x_2, y_2, z_2).

Let us, by way of illustration, consider the concept "velocity." The differential quotient $\frac{d\sigma}{dt}$ does not correspond to any physical reality. It is a consequence of the theory, or of the old theory of relativity, that no material velocity can exceed the velocity of light. This does not mean that $\frac{d\sigma}{dt}$ can never exceed the value c, but that its value for a material body at any point of the four-dimensional time-space and in any direction is always smaller than the value of the same differential quotient for a ray of light at the same point and in the same direction. In other words, no material body can ever overtake a ray of light, and every moving body can be overtaken by light.

Take, *e.g.*, two systems of co-ordinates A and B. The system A we will suppose to have been so chosen relatively to existing material bodies that at large distances from these bodies the geodetic lines are practically straight lines. Such a system is by Einstein called a *Galilean* system. Let the system B be non-Galilean. Then B must be derived from A, and A from B, by a non-linear transformation. In both systems we take cylindrical co-ordinates r, θ, z, and let the transformation which reduces the one system to the other be a rotation. Thus, if θ' refers to B and θ to A, we have

$$\theta' = \theta - \omega ct.$$

The values of the line-element in the two systems, if we neglect the gravitational effect of material bodies, are

$$(109) \qquad \begin{cases} \text{A. } ds^2 = -d\sigma^2 + c^2dt^2, \\ \text{B. } ds^2 = -d\sigma'^2 - 2r^2\omega d\theta' cdt + (1 - r^2\omega^2)c^2dt^2, \end{cases}$$

where $d\sigma^2 = dr^2 + r^2d\theta^2 + dz^2$, and similarly $d\sigma'$.

Now consider a "velocity" perpendicular to the radius-vector in the plane $z = 0$, thus $d\sigma = \pm r d\theta$. The velocity of light in the two systems is

$$\begin{aligned} &\text{A. } v = c, \\ &\text{B. } v' = c(1 \mp r\omega), \end{aligned}$$

where the upper sign must be taken for light travelling in the direction of increasing θ'. It follows that for values of r exceeding $1/\omega$ light is propagated tangentially in only one direction. Similarly we can show that for $r\omega > 1$ it is a physical impossibility for a material body to be at rest in the system B. It must necessarily move towards the negative θ' with a velocity exceeding $(r\omega - 1)c$, and smaller than $(r\omega + 1)c$. This shows the irreality of the co-ordinates and of the "velocity." Still, it may very well be that, for smaller values of r, the system B is more appropriate for the description of the actual phenomena than A.

The difference between the two systems is characterised by the value of g_{24}.* In A we have $g_{24}=0$ and in B $g_{24}=-r^2\omega$. Both values are solutions of the equation

$$G_{24}=0,$$

which, of course, is the same in all systems of co-ordinates. The right-hand member is zero, since we neglect the presence of matter. It can be shown that the equation is satisfied by

$$g_{24}=kr^2,$$

k being an arbitrary constant. The theory only gives the differential equation, and does not prescribe the constant of integration. This must, of course, be so determined that the solution represents the observed relative motions of material bodies and light-rays as described in the adopted system of co-ordinates. The particular solution which does so in one system must, if this system is transformed into another, by the same transformation be reduced to the particular solution which fits the observed phenomena in the new system. This implies that the constants of integration are also subjected to the transformation, and are therefore as a rule different in different systems of co-ordinates.†

In classical mechanics the constants of integration are prescribed, since only one system of co-ordinates is admitted. This prescription is usually given in the form of the values of the variable quantities at infinity. Thus the solution of Poisson's equation is subjected to the condition that the potential shall be zero at infinity.‡ In the theory of general relativity the values at infinity are, of course, different in different systems of co-ordinates.

If we consider physical phenomena at the surface of the earth, and describe them in a system of co-ordinates in which this surface

* We need only consider g_{24}. We could, *e.g.*, introduce the condition (12) $g=-1$, or any other, then g_{44} would follow immediately from g_{24}.

† We must be careful to distinguish between the indeterminateness of the constants of integration (or rather the arbitrary functions, or boundary conditions) corresponding to the fact that the g_{ij} are given by *differential* equations, and the fundamental indeterminateness involved by the principle of general relativity, which has been explained in art. 8 (first paper), and which allows a free choice of the system of co-ordinates, and of the measure of space and time. Thus, *e.g.*, in art. 10 the assumption $g_{i4}=0$ is of mixed character. The primary assumption was that we should have a system possessing three-dimensional spherical symmetry, and in which the field should be stationary. This imposes certain conditions on the g_{i4}, as well as on the other g_{ij}, which might, if desired, be expressed in the form of differential equations. That among all possible solutions of these equations we chose $g_{i4}=0$ is of the nature of a boundary condition.

‡ Of course we know nothing about infinity, and never will. The true condition determining the arbitrary constants or functions is that the actually observed phenomena shall be correctly represented, and only for mathematical convenience this condition is brought in the form of a boundary condition for infinity.

is fixed, we find that the formula (109, B) is the one we need to represent the observations [apart from gravitational effects], and not (109, A), if only we take for ω the appropriate value. This value we call the rotational velocity of the earth.

Suppose we use a system of co-ordinates relatively to which the earth has a velocity of rotation ω_1. This means that the co-ordinate θ of any point on the earth would in that system be represented by the formula $\theta = \theta_0 + \omega_1 t$. This ω_1 is, of course, entirely arbitrary. It is also not ascertainable by observations, since the co-ordinate axes cannot be observed. It must thus in a true theory of relativity disappear from the final equations describing the observed or observable phenomena. The differential equation gives $g_{24} = kr^2$, and the constant k is also undetermined, and must also disappear from the final equations. Now, if we come to compare the theory with the observations, we will find that, in order to make it fit, we have to take

$$k = \omega_1 - \omega,$$

and if this is introduced, both k and ω_1 will disappear and only the observable quantity ω will remain.*

Rotation in its purely kinematical aspect is thus relative in the new theory, quite as relative as linear translation. We can transform it away as we can transform away linear translation. But there is a difference. If a linear translation is transformed away (by a Lorentz-transformation), it is utterly gone; no trace of it remains. Not so in the case of rotation. The transformation which does away with a rotation at the same time alters the equation of relative motion in a definite manner. This shows that rotation is not a purely kinematical fact, but an essential physical reality. Its amount ω is a physical constant, proper to the earth, like its mass.

I have purposely in the above only referred to such methods of ascertaining and measuring ω which depend exclusively on observations made on the surface of the earth (Foucault's pendulum, gyroscopes, the ellipsoidal figure of the earth), without any reference to celestial bodies. That by all these methods, and also by the apparent daily motion of the stars, we find the same value of ω, is a confirmation of the correctness of the theories by means of which the various phenomena, which are used in the different methods, are explained, especially of the theory of inertia. But all this is entirely independent of the system of co-ordinates.

The case is somewhat analogous with the gravitational effect on planetary motion of a rotation of the sun. Whether the solar system has a translation or not can only be ascertained by observations of bodies outside that system. Therefore such a translation can have no effect on the relative motions within the system.

* See also De Sitter, "The Relativity of Rotation in Einstein's Theory," *Acad. of Sci., Amsterdam*, Sept. 1916 (vol. xix. p. 527).

But whether the sun rotates or not can be ascertained by observations within the solar system, and accordingly the principle of relativity does admit an effect of the rotation of the sun on the relative motions of the planets.

30. Newton in his *Principia* gave only formal laws for gravitation and inertia. He made no attempt at explanation: "Hypotheses non fingo." When Newton's integral law of gravitation was replaced by Poisson's equation, the concept of a *field* of gravitation, of which the material bodies were the sources, was introduced. At the same time the absolute space was substantialised and called "ether." At very large distances from any material body the potential is zero: the whole of the numerical value of the potential at any point can thus be derived from the material sources.

Expressed in the language of Einstein's theory, we would say, at infinity the g_{ij} are:

$$\left\{\begin{array}{cccc} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 \end{array}\right. \tag{110}$$

These are the natural values, or the g_{ij} of absolute space, or of the ether. In classical mechanics the g_{ij} in the neighbourhood of matter are the same as (110), with the exception of g_{44}, which becomes

$$g_{44} = 1 + \gamma,$$

and γ is produced by the material bodies through Poisson's equation. In Einstein's theory all g_{ij} differ from the values (110), and they are all determined by differential equations, of which the right-hand members (κT_{ij}) depend on matter. Thus matter here also appears as the source of the g_{ij}, *i.e.* of inertia. But can we say that the *whole* of the g_{ij} is derived from these sources? The differential equations determine the g_{ij} apart from constants of integration, or rather arbitrary functions, or boundary conditions, which can be mathematically defined by stating the values of g_{ij} at infinity. Evidently we could only say that the whole of the g_{ij} is of material origin if these values at infinity were *the same for all systems of co-ordinates.* It is not necessary to insist on the values (110). We can prescribe any other set of degenerated values which the g_{ij} must have at infinity, only they must be the same for all systems of reference, *i.e.* they must be invariant for all transformations, or at least for a so comprehensive group of transformations that restriction to this group does not mean giving up the principle of relativity.

The values (110) are certainly not invariant. Einstein has, however,* pointed out a set of degenerated g_{ij} which are actually

* In a conversation with the writer.

invariant for all transformations in which, at infinity, x_4 is a pure function of x_4'. They are:

$$(111) \qquad \begin{cases} 0 & 0 & 0 & \infty \\ 0 & 0 & 0 & \infty \\ 0 & 0 & 0 & \infty \\ \infty & \infty & \infty & \infty^2 \end{cases}$$

Einstein makes the hypothesis that at infinity the g_{ij} actually have the values (111). These are then the "natural" values, and any deviation from them must be due to material sources. The meaning of this degenerated set of g_{ij} is that at infinity the four-dimensional time-space is dissolved into a three-dimensional space and a one-dimensional time. In the three-dimensional space all transformations are still allowed, also those involving arbitrary functions of t: rotations, accelerated motions, etc. At infinity we would thus have an absolute time,* but no absolute space. Of course, so far as our observations reach (in space and time), there would still be complete four-dimensional relativity, but at very large distances from all matter the g_{ij} would gradually converge towards the degenerated values (111).

The hypothesis can thus be said to be consistent with the formal principle of relativity. It is, however, not a consequence of it, and also denying it is not contrary to the principle. If we adopt the hypothesis, then in any system of co-ordinates the g_{ij} would have exactly determined values, there being no constants of integration left by which they can be made to fit the observed phenomena. Now it is certain that, in many systems of reference (*e.g.* in all Galilean systems), the g_{ij} at large distances from all material bodies known to us actually have the values (110). On Einstein's hypothesis these are special values which, since they differ from (111), must be produced by some material bodies. Consequently there must exist, at still larger distances, certain unknown masses which are the source of the values (110), *i.e.* of all inertia. These hypothetical masses would form the boundary of the universe, which would thus be necessarily finite and limited. Outside them there would be *nothing* but the field of g_{ij} gradually degenerating to the values (111).

We must insist on the impossibility that any of the known fixed stars or nebulæ can form part of these hypothetical masses. The light even from the farthest stars and nebulæ has approximately the same wave-length as light produced by terrestrial sources.

* This must be so understood that we are free to introduce any other time t', which is an arbitrary function of t, but not of x_1, x_2, x_3. For finite values of x_1, x_2, x_3, x_4 *all* transformations are still allowed, but they must all at infinity degenerate in such a way that t' becomes independent of the space-co-ordinates. Thus, *e.g.*, an ordinary Lorentz-transformation would be no longer allowed, since it does not degenerate at infinity; but a transformation which for all finite values of x_1, x_2, x_3, x_4 coincides with a Lorentz-transformation, while degenerating at infinity in the required manner, would be allowable.

Consequently the ratio of the units of time and space is in these nebulæ approximately the same as here; the deviation of the g_{ij} from the Galilean values (110) is of the same order as here, and they must therefore be still inside the limiting envelope which separates our universe from the outer parts of space, where the g_{ij} have the values (111).

If the hypothesis is accepted, then rotation is relative not only in its kinematical aspect, but also physically. The constant ω in the value of g_{24} is then induced by, or derived from, these distant masses, and it is at the same time the velocity of rotation relatively to these masses. These thus take up the rôle of the absolute space in the old theory of inertia. And, to my mind at least, they are quite as objectionable as this absolute space, if not more so. If we believe in the existence of these supernatural masses, which control the whole physical universe without having ever been observed, then the temptation must be very great indeed to give preference to a system of co-ordinates relatively to which they are at rest, and to distinguish it by a special name, such as "inertial system" or "ether." * Formally the principle of relativity would remain true, but as a matter of fact we would have returned to the absolute space under another name.

Theoretically it is, of course, important that the *possibility* † has been shown to ascribe the whole of the deviation of the actual g_{ij} from certain standard values to material sources, but evidently this "explanation" of the origin of inertia by hypothetical masses which must be imagined for that purpose and have not been otherwise observed, is practically equivalent to no explanation at all, or to a confession of our ignorance. The hypothesis is not an essential element of Einstein's theory. We can accept that theory without the hypothesis. We need not, therefore, give up the hope that some day a satisfactory explanation of inertia will be found which, of course, must, as all physical theories, be consistent with the principle of relativity. In the meantime, we can be content to put ourselves again on the point of view of Newton, accepting the formal law and making no hypotheses.

Even if Einstein has not explained the origin of inertia and gravitation, his theory represents an enormous progress over the physics of yesterday. Perceiving the irrelevance of the representation by co-ordinates in which our science was clothed, he has penetrated to the deeper realities which lay hidden behind it, and not only has he entirely explained the exceptional and universal nature of gravitation by the principle of the identity of gravitation and inertia, but he has laid bare intimate connections between

* Of course, once the source of inertia had been pointed out, the need would be felt to explain *how* the inertia is transferred from these distant masses to the known material bodies, and a hypothetical mechanism would probably be built up quite analogous to the ether in the old theories.

† At least the *logical* possibility. Whether it will be possible to imagine an actual distribution of ordinary physical matter having the desired effect, remains to be investigated. If at all finite, they must certainly enormously exceed the whole known physical universe.

branches of science which up to now were considered as entirely independent from each other, and has thus made an important step towards the unity of nature. Finally, his theory not only explains all that the old theory of relativity could explain (experiment of Michelson, etc.), but, *without introducing any new hypothesis or empirical constant*, it explains the anomalous motion of the perihelion of Mercury, and it predicts a number of phenomena which have not yet been observed. It has thus at once proved to be a very powerful instrument of discovery.

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