

Then

$$\frac{1}{a} = \frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{e}.$$

$$I = \frac{\pi \rho \delta^{k+2}}{(k+1)\sqrt{(abca)}} \left[\frac{(1-L_0^2)^{k+2}}{2(k+2)} - L_0 \int_0^{\theta_0} \sin^{2k+3} \theta d\theta \right]$$

where $\cos \theta_0 = L_0$ and θ_0 is between 0 and π .

The Secular Perturbations of the Four Inner Planets.

By Harold Jeffreys, B.A., M.Sc.

It was shown by Newcomb* that the observed secular variations in the orbits of the inner planets differ slightly from those calculated from the law of gravitation, the best available values of the masses being adopted. The principal discrepancies are in the movements of the perihelion of Mercury and the node of Venus, but certain others also have notable values. Two important attempts have been made to account for the differences. The first was by Seeliger,† who assumed that about the sun are two superposed oblate homogeneous ellipsoids of gravitating matter. In addition he assumed that the axes of reference, which are fixed relative to the mean of the stars observed, are actually rotating in the plane of the ecliptic. By means of a method of least squares he found the best values of the densities of the ellipsoids and the rotation; all the residuals, with the exception of those in the eccentricity of Mercury and Mars, the explanation of which does not seem to have been attempted by anyone, were less than their mean errors. It thus appeared very satisfactory. Nevertheless, it seems to require serious modification, on account of the doubtful character of some of the physical hypotheses involved.

(a) The rotation found appears too great to be possible. De Sitter‡ found, however, that it is not necessary to the theory, for on rejecting it he obtained a satisfactory representation by means of the ellipsoids alone.

(b) The density of the outer ellipsoid, which extends outside of the orbit of the earth, was found by De Sitter to be 3.7×10^{-15} , and by Seeliger to be 3.1×10^{-15} of that of the sun. It will be shown later that this density, small as it seems, is greater than can actually be allowed.

On the other hand, Einstein§ has recently, by means of a form of the Principle of Relativity, given an excellent representation of the perihelion of Mercury and a fair one of that of Mars; but

* *Astronomical Constants*, p. 109.

† *Sitzungsber. K. Bayer. Akad., Math.-Phys. Klasse*, 1906, p. 595.

‡ *Observatory*, vol. xxxvi., 1913, p. 296. This also gives an account of the earlier work on the subject.

§ *Annalen der Physik*, Bd. xlix., 1916, p. 769.

naturally no hypothesis of this type can affect the position of the plane of an orbit, so that the empirical part of the motion of the node of Venus remains unexplained. For this reason it appears desirable to try to find the distribution of gravitating matter that will best account for the changes in the inclinations and nodes, for it seems impossible to account for them otherwise; this can then be used to determine the motions of the perihelia, to ascertain how much of these can be accounted for by such means. The disturbing matter must be assumed to be close to the sun, as the density near the orbit of Mercury and outside it will be shown to be so small that it can have no perceptible effect. When this is done, the distribution that gives the best approximation to the movements of the planes of the orbits is capable of causing an advance of the perihelion of Mercury $1\frac{1}{2}$ times as great as the observed excess.

The Density of the Matter causing the Zodiacal Light.—Seeliger adopted the special dimensions that he chose for his outer ellipsoid because the observed phenomenon of the zodiacal light indicates the presence of matter as far out as he indicated, and in particular between the orbits of Mercury and the earth. Within the orbit of Mercury the observations are not so decisive, as they must be made so soon after sunset or before sunrise that matter there is practically invisible. Presumably, however, the density increases rapidly towards the sun, as Seeliger supposed. The nature of the substance causing the light is uncertain, but it may be supposed either to be composed of particles of meteoric type, large compared with the wave-length of light, or of very fine dust or gaseous matter, the particles being small compared with the wave-length. The treatment of the problem is different in the two cases.

In the first case, let the number of particles per unit volume be ν , and their mean diameter σ . Then it was shown* from the observed luminosity of the zodiacal light that the value of $\nu\sigma^2$ is about $3.5 \times 10^{-22} \div 1$ cm. for the neighbourhood of the orbit of Venus. The determination was not an accurate one, but the true value is not likely to be more than 10 times this or less than $\frac{1}{10}$ of it. The density is $\frac{1}{8}\pi\rho\nu\sigma^3$, where ρ is the density of a single particle. Now, putting this equal to De Sitter's density of 3.7×10^{-15} of that of the sun, we see that the above value of $\nu\sigma^2$ would require the diameters of the particles to be about 10^7 cm. This value is so great that the particles would appear as small separate planets instead of as a continuous sheet. Any size that would be acceptable, taken in conjunction with the above value of $\nu\sigma^2$, would make the density so small that it is quite evident that the large particles that may help to cause the zodiacal light must be so few that their effect on the orbits of the planets must be hundreds of times less than was required of the outer ellipsoid.

If the particles are small compared with the wave-length, the

* P. 94.

problem is considerably altered. The light is no longer diffusely reflected from the surface, but is scattered instead.* The intensity is now proportional to $\nu\sigma^6$ instead of $\nu\sigma^2$, and a comparison of the theory with observation may be derived from the scattering of light by the earth's atmosphere. If the diameters of the particles were the same in the two cases, the amounts of light received from equal small angles, first, of the blue sky in daytime, and second, of the zodiacal light, would be proportional to the masses of columns of equal cross-section through the two structures considered. This ratio, on Seeliger's hypothesis, would be 1.2×10^4 . The observed ratio of the luminosity of the sky to that of an equal area of the sun is given by L. V. King† to be 80×10^{-8} at the Mount Wilson level, which was considered sufficiently high above sea-level for dust to be of no importance. The intensity of the light from the sky is therefore 8×10^6 of that of the zodiacal light, so that the latter is of the order of 600 times fainter than would be expected on the hypothesis that both it and the light from the sky are due to molecular scattering. Lord Kelvin's estimate‡ gives values ranging from 4×10^{-7} to 2.6×10^{-7} for the ratio of sky to sun, which would reduce the above ratio to about 200. If the zodiacal light were caused by particles much larger than the molecules in the air, the discrepancy would be still greater, as $\nu\sigma^6$ would be greater for the same value of $\nu\sigma^3$. It seems very improbable then that the matter causing the zodiacal light can have a density in any way comparable with that of the outer ellipsoid, and its effect on the motions of the planets may accordingly be neglected.

The Secular Motions of a Planet consequent on an Oblate Distribution close to the Sun.—The hypothesis here adopted to give the distribution of the disturbing matter is a modification of that of Seeliger, so as to make it consistent with photometric observations. Any important perturbing mass must then be far inside the orbit of Mercury. The potential can then be expressed in terms of a series of spherical solid harmonics of the form

$$V = f \frac{M}{r} + \frac{S_2}{r^3} + \frac{S_3}{r^4} + \frac{S_4}{r^5} + \dots,$$

where S_2, S_3, S_4 , etc., are surface harmonics of the second, third, fourth, and higher orders respectively. Further, it can be seen that as it is highly improbable that a mass of this type is not rotating about an axis through the centre of the sun, the harmonics involved must be zonal. It can also be shown without much difficulty that odd harmonics can give rise to no secular terms in the motion of a planet that moves near the equatorial plane of the distribution. S_3 may therefore be omitted. In this investigation S_4 will not be considered, as its effect will be small on every planet except Mercury; if later observations show this to be

* R. W. Wood, *Physical Optics*, pp. 478–480.

† *Phil. Trans.*, 212A, 1912–13, p. 411.

‡ *Baltimore Lectures*, p. 314.

desirable, however, this term will have to be included and determined with the rest.

Let now the sum of the masses of the sun and the perturbing matter be M ,
the moments of inertia of the perturbing mass about its polar and equatorial axes be C and A ,
the colatitude of any point, measured from the polar axis, be θ ,

and put $\alpha = \frac{1}{2}(C - A)$.

α is positive when the distribution is oblate.

Then

$$V = f \frac{M}{r} + f \frac{\alpha(1 - 3 \cos^2 \theta)}{r^3}.$$

If now x, y, z be the Cartesian co-ordinates of a point, and (l, m, n) the direction cosines of the polar axis, the disturbing function is

$$R = far^{-5} \{r^2 - 3(lx + my + nz)^2\}.$$

Now, in the usual notation of the planetary theory,

$$\begin{aligned} r &= a(1 - e^2)/(1 + e \cos w), \\ x &= r \{ \cos \Omega \cos(\nu - \Omega) - \sin \Omega \sin(\nu - \Omega) \cos i \}, \\ y &= r \{ \sin \Omega \cos(\nu - \Omega) + \cos \Omega \sin(\nu - \Omega) \cos i \}, \\ z &= r \sin(\nu - \Omega) \sin i. \end{aligned}$$

If ω be the mean motion of the planet, so that $\omega = (\mu/\alpha^3)^{\frac{1}{2}}$, the disturbing function is periodic in a time $2\pi/\omega$. The secular perturbations depend only on the secular part of R , which is the average of R taken with regard to the time. If this be S , we have

$$\begin{aligned} S &= \frac{\omega}{2\pi} \int_0^{2\pi/\omega} R dt, \\ &= \frac{fa}{a^3(1 - e^2)^{\frac{3}{2}}} \left[1 - \frac{3}{2}(l \cos \Omega + m \sin \Omega)^2 - \frac{3}{2}(-l \sin \Omega \cos i + m \cos \Omega \cos i + n \sin i)^2 \right]. \end{aligned}$$

The perturbations are now given by the equations *

$$\begin{aligned} \frac{da}{dt} &= \frac{2}{\omega a} \frac{\partial S}{\partial e}, \\ \sin i \frac{d\Omega}{dt} &= \frac{1}{\omega a^2 \sqrt{1 - e^2}} \frac{\partial S}{\partial i}, \\ \frac{d\varpi}{dt} &= \frac{\sqrt{1 - e^2}}{\omega a^2} \frac{\partial S}{\partial e} + \frac{\tan \frac{1}{2}i}{\omega a^2 \sqrt{1 - e^2}} \frac{\partial S}{\partial i}, \\ \frac{de}{dt} &= -\frac{\sqrt{1 - e^2}}{\omega a^2 e} \frac{\partial S}{\partial \varpi} - \sqrt{1 - e^2} \frac{1 - \sqrt{1 - e^2}}{\omega a^2 e} \frac{\partial S}{\partial e}, \\ \frac{di}{dt} &= -\frac{1}{\omega a^2 \sqrt{1 - e^2} \sin i} \frac{\partial S}{\partial \Omega} - \frac{\tan \frac{1}{2}i}{\omega a^2 \sqrt{1 - e^2}} \left(\frac{\partial S}{\partial \varpi} + \frac{\partial S}{\partial e} \right), \\ \frac{d\epsilon}{dt} &= -\frac{2}{\omega a} \frac{\partial S}{\partial a} + \frac{\tan \frac{1}{2}i}{\omega a^2 \sqrt{1 - e^2}} \frac{\partial S}{\partial i} + \sqrt{1 - e^2} \frac{1 - \sqrt{1 - e^2}}{\omega a^2 e} \frac{\partial S}{\partial e}. \end{aligned}$$

* Tisserand, *Mécanique Céleste*, t. 1, p. 369.

We see at once that $\frac{da}{dt}$ and $\frac{de}{dt}$ are zero. In the other variations, the parts depending on the squares of the small quantities i , l , m can be omitted, except in the case of the perihelion of Mercury. In this case squares will be retained, but no higher powers.

$$\begin{aligned}\sin i \frac{d\Omega}{dt} &= -\frac{3fa}{\omega a^5(1-e^2)^2}(\sin i - l \sin \Omega + m \cos \Omega), \\ \frac{di}{dt} &= -\frac{3fa}{\omega a^5(1-e^2)^2}(l \cos \Omega + m \sin \Omega), \\ \frac{d\varpi}{dt} &= \frac{3fa}{\omega a^5(1-e^2)^2} \left[1 - \frac{3}{2}(l - \sin i \sin \Omega)^2 - \frac{3}{2}(m + \sin i \cos \Omega)^2 \right. \\ &\quad \left. - \frac{1}{2} \sin i (\sin i - l \sin \Omega + m \cos \Omega) \right].\end{aligned}$$

Now choose the units so that the unit of length is the earth's mean distance from the sun, that of time is such that the earth's mean motion is unity, and the sum of the masses of the sun and earth is the unit of mass. Then $f=1$, and for any planet $\omega = a^{-\frac{3}{2}}$.

The variations found from the last equations will then be expressed in radians per unit of time, whereas they are recorded by Newcomb in seconds of arc per century, the former unit being 1.296×10^8 times the latter. It will therefore be found convenient to introduce a new quantity $\beta = 1.296 \times 10^8 \alpha$.

The following table gives the elements of the orbits of the four inner planets, with their unexplained secular variations in a century and their mean errors:—

Planet.	a .	e .	$\frac{3}{\omega a^5(1-e^2)^2}$.	Ω .	i .	$\sin i \delta \Omega$.	δi .	δe .	$e \delta \varpi$.
Mercury	3871	206	72.01	47° 8' 45"	7° 0' 10"	0.61 ± 0.51	0.38 ± 0.80	-0.88 ± 0.50	8.48 ± 0.43
Venus	7233	007	10.46	75° 46' 47"	3° 23' 27"	0.60 ± 0.17	0.38 ± 0.33	0.21 ± 0.31	-0.05 ± 0.25
Earth	10000	017	3.002	...	0° 0' 0"	...	-0.22* ± 0.27	0.02 ± 0.10	0.10 ± 0.13
Mars	15237	093	55.57	48° 47' 9"	1° 15' 10"	0.03 ± 0.22	-0.01 ± 0.20	0.29 ± 0.27	0.75 ± 0.34

Let us first consider the motions of the planes of the orbits separately.

The secular changes in i and Ω give then the following conditional equations to determine the unknowns β , l , m :—

$$\begin{aligned}(-49.9l - 52.7m)\beta &= 0.38 \pm 0.80 \\ (+52.7l - 49.9m - 8.79)\beta &= 0.61 \pm 0.51 \quad \left. \vphantom{\begin{aligned} (-49.9l - 52.7m)\beta &= 0.38 \pm 0.80 \\ (+52.7l - 49.9m - 8.79)\beta &= 0.61 \pm 0.51 \end{aligned}} \right\} \text{from Mercury.} \\ (-2.57l - 10.14m)\beta &= 0.38 \pm 0.33 \\ (10.14l - 2.57m - 0.62)\beta &= 0.60 \pm 0.17 \quad \left. \vphantom{\begin{aligned} (-2.57l - 10.14m)\beta &= 0.38 \pm 0.33 \\ (10.14l - 2.57m - 0.62)\beta &= 0.60 \pm 0.17 \end{aligned}} \right\} \text{from Venus.} \\ (-.363l - .417m)\beta &= -0.01 \pm 0.20 \\ (.417l - .363m - .0121)\beta &= 0.03 \pm 0.22 \quad \left. \vphantom{\begin{aligned} (-.363l - .417m)\beta &= -0.01 \pm 0.20 \\ (.417l - .363m - .0121)\beta &= 0.03 \pm 0.22 \end{aligned}} \right\} \text{from Mars.}\end{aligned}$$

* This entry is not δi , but δe .

The normal equations derived from these, weights being assigned according to the mean errors, are—

$$\begin{aligned}(36.22l - 13.17m - 4.095)\beta - 0.594 &= 0. \\ (-13.17l + 29.30m + 3.473)\beta + 0.466 &= 0. \\ (-4.095l + 3.473m + 0.640)\beta + 0.069 &= 0.\end{aligned}$$

From these we find $l = +0.09503$; $m = -0.09005$; $\beta = .915$. Thus the equatorial plane of the disturbing matter is inclined at $7^\circ 31'$ to the ecliptic, and the longitude of its ascending node is $46^\circ 33'$. It therefore nearly coincides with the orbit of Mercury.

The values of the perturbations caused in a century by the distribution thus found are as follows, expressed in seconds of arc:—

Planet.	$\sin i \delta \Omega$.	δi .	$ed\omega$.
Mercury	0.64	0.00	13.6
Venus	0.52	0.61	0.06
Earth	...	...	0.05
Mars	0.05	0.00	0.05

The distribution found thus accounts for all the unexplained secular motions with the exceptions of those of the perihelia of Mercury and Mars and the variations in the eccentricities of the same planets. The residuals in all other cases are less than the mean errors. On the other hand, it is seen that the motions of the orbital planes of the planets actually imply a secular movement of the perihelion of Mercury half as large again as the unexplained part already known to exist.

Next, consider the motions of the orbital planes and of the perihelia together in order to determine the best values of l , m , and β . In this case the equations for $ed\omega/dt$ will give four conditional equations in addition to those already used.

$$\begin{aligned}(14.72 + .660l - .613)\beta &= 8.48 \pm 0.43 \\ (\text{squares of } l - \sin i \sin \Omega \text{ and } m + \sin i \cos \Omega \text{ being omitted}) \\ .07\beta &= -0.05 \pm 0.25 \\ .05\beta &= 0.10 \pm 0.13 \\ .05\beta &= 0.75 \pm 0.34\end{aligned}$$

The normal equations now found are—

$$\begin{aligned}(36.22l - 13.17m - 3.99)\beta - 0.654 &= 0, \\ (-13.17l + 29.30m + 3.375)\beta + 0.522 &= 0, \\ (-3.990l + 3.375m + 2.984)\beta - 1.282 &= 0,\end{aligned}$$

whence

$$l = +0.1057; m = -0.0986; \beta = 0.575.$$

The perturbations caused by this distribution, with the same units as before, are given in the following table:—

Planet.	$\sin i \delta \Omega.$	$\delta i.$	$\delta \delta.$
Mercury	0.94	0.00	8.53
Venus	0.41	0.42	0.04
Earth	...	...	0.03
Mars	0.04	0.00	0.03

The secular variations are thus all accounted for within their mean errors except the motions of the perihelion of Mars and the node of Venus. In the latter case, however, the residual is only just greater than the mean error. The equatorial plane of the perturbing matter is inclined at $8^{\circ} 19'$ to the ecliptic, and the longitude of its ascending node is $47^{\circ} 0'$. Thus it is sufficiently near to the plane of the orbit of Mercury for the quantity $(l - \sin i \sin \Omega)^2 + (m + \sin i \cos \Omega)^2$ to be neglected. The value found for β shows that $\alpha = 4.43 \times 10^{-9}$. If the disturbing potential was to be attributed to an ellipticity of the sun, the requisite value of this would be 9.93×10^{-3} (the sun being assumed homogeneous), which is much too great to be possible. The matter must therefore be outside of the sun, though its diameter must be much less than that of the orbit of Mercury. A somewhat better representation could be obtained by including a fourth harmonic, but in view of the size of the other unexplained discrepancies this scarcely seems worth doing at present.

If then Einstein's hypothesis be accepted, any perturbing matter that may exist must be such as to produce considerable effects on the positions of the planes of the orbits, without affecting the much more sensitive perihelion of Mercury. This can only occur if a fourth harmonic is included in the potential, and so accurately adjusted that its effect on the perihelion of Mercury just balances that of the second harmonic. In the figure of the disturbing mass the fourth harmonic must then be very much more prominent than the second harmonic, which appears unlikely. It appears therefore that on the hypothesis of Einstein it will be found very difficult to find a plausible explanation of the motions of the planes of the orbits, and in particular of the node of Venus.

The Irregular Variable RS Camelopardalis.

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1. The valuable paper by C. Martin and H. C. Plummer in the May number of the *Mon. Not.* (p. 612) gives not only a fine series of observations of this star extending over more than four years, but a careful analysis of them resulting in an apparent solution of the problem of irregular variability, possibly extending to other