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On Einstein's Theory of Gravitation, and its Astronomical Consequences. By W. de Sitter, Assoc. R.A.S. First Paper.

Contents of First Paper.

1. Since Minkowski the conception of space and time as a four-dimensional continuity has been widely accepted. The ideal

put forward by him in his celebrated lecture[*] of 1908, "that space and time each separately should vanish to shadows, and only a union of the two should preserve reality," has, however, only been completely realised by the latest theory of Einstein,[†] the "Allgemeine Relativitätstheorie" of 1915, by which, moreover, gravitation is also incorporated in the union.

The points of space occupied by a given material point at successive times form in the four-dimensional time-space a continuity of one dimension, which is called the *world-line* of the point. Also a light-vibration has its world-line, the projection of which on three-dimensional space is a ray of light. Now what we *observe* are always *intersections* of world-lines. Take, *e.g.*, an observation of an occultation of a star by the moon, and let us imagine, to simplify the argument, that the face of the clock is illuminated by the light of the star.[‡] Then the world-line of a certain light-vibration starts from a point on the world-line of the star, it then intersects successively the world-line of a point on the moon's edge, that of the clock's hand, and that of a point on the clock's face. The last two intersections may be said to coincide, so that three world-lines have one point in common. About the course of world-lines between the points of intersection we know nothing, and no observation can ever tell us anything.

Now we must necessarily describe the world-lines and their intersections by means of a system of co-ordinates. The laws of nature are also necessarily expressed by means of these co-ordinates. We can imagine two physicists each making a model of

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[*] Raum und Zeit, *Vortrag gehalten auf der* 80^{en} *Versammlung Deutscher Naturforscher und Aerzte*, 1908 Sept. 21. Reprinted in Lorentz-Einstein-Minkowski, *Das Relativitätsprincip*, Teubner, 1913.

[†] The theory has been gradually developed in several papers, of which the most important are:—

Einstein und Grossmann, "Entwurf einer verallgemeinerten Relativitätstheorie und einer Theorie der Gravitation," *Zeitschr. für Math. u. Physik*, 1914 Jan.

I. Einstein, "Die formale Grundlage der allgemeinen Relativitätstheorie," *Sitzungsber. Berlin*, 1914 Nov., p. 1030.

II. Einstein, "Zur allgemeinen Relativitätstheorie," *ibid.*, 1915 Nov. 4, p. 778.

III. Einstein, "Erklärung der Perihelbewegung der Merkur aus der allgemeinen Relativitätstheorie," *ibid.*, 1915 Nov. 11, p. 831.

IV. Einstein, "Die Feldgleichungen der Gravitation," *ibid.*, 1915 Nov. 25, p. 844.

Only in the last paper the theory is given its definitive form. The whole theory is summarised in a paper:—

V. Einstein, "Die Grundlage der allgemeinen Relativitätstheorie," *Annalen der Physik*, xlix., 1916, p. 769 (also published separately by Barth).

Afterwards Einstein has published:—

VI. Einstein, "Näherungsweise Integration der Feldgleichungen der Gravitation," *Sitzungsber. Berlin*, 1916 June, p. 688.

These papers will be quoted as "Einstein I.," etc.

[‡] This example is given by Lorentz in a paper, "On Einstein's Theory of Gravitation, I.," *Academy of Sciences, Amsterdam*, 1916 Feb. This, and other papers by Lorentz, Droste, and de Sitter, have, up to the time of writing, only been published in the Dutch *Verslagen* of the Academy, and have not yet been translated into English.

all world-lines with their coincidences, and the two models must be both correct, and therefore essentially identical, whenever they both represent all intersections in the right order. The course of the world-lines themselves may be entirely different in the two models. These considerations have led Einstein to his postulate of general relativity, which requires the laws of nature to be invariant for *all* transformations of co-ordinates.

2. Let the four co-ordinates be x_1, x_2, x_3, x_4. For the fourth we may choose the time measured in such a unit that the velocity of light in a space, where there is no matter and no gravitation, is unity: or $x_4 = ct$. The other co-ordinates are then pure space-co-ordinates, for which we can, *e.g.*, take ordinary rectangular Cartesian co-ordinates. The four-dimensional distance between two neighbouring points will be called ds. We have generally *

$$ds^2 = \sum_\alpha \sum_\beta g_{\alpha\beta} dx_\alpha dx_\beta, \tag{1}$$

where necessarily $g_{\alpha\beta} = g_{\beta\alpha}$. There are thus ten coefficients $g_{\alpha\beta}$, which are functions of the co-ordinates $x_1 \ldots x_4$. The line-element ds must be invariant for all transformations, and it entirely characterises the metric properties of the four-dimensional time-space. If we introduce other co-ordinates $x_1' \ldots x_4'$ by an arbitrary transformation

$$x_i' = f_i(x_1, x_2, x_3, x_4),$$

so that

$$\begin{cases} dx_i = \sum_j p_{ij} dx_j', \\ dx_i' = \sum_j \pi_{ji} dx_j, \end{cases} \tag{2}$$

we shall find in the new co-ordinates

$$ds^2 = \sum_\alpha \sum_\beta g'_{\alpha\beta} dx_\alpha' dx_\beta',$$

and

$$g'_{ij} = \sum_\alpha \sum_\beta p_{\alpha i} p_{\beta j} g_{\alpha\beta}. \tag{3}$$

In the four-dimensional time-space we consider tensors of different orders. The tensor of order zero is a pure number (scalar), the tensor of the first order is a vector, which has 4 components, the tensor of the second order has 16 components, and so on. The ten coefficients g_{ij} form a tensor of the second order. Since $g_{ij} = g_{ji}$, this tensor is symmetrical. We need not go into details regarding the calculus of these tensors,† which has been developed by Riemann, Christoffel, Levi-Civita, Ricci, and others. The central fact is that the transformation-formulas for tensors are easily derived from those

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* Throughout this paper $\sum_\alpha$ will denote a sum over the values 1, 2, 3, 4 of the index α, and $\sum_\alpha'$ a sum over the values 1, 2, 3 only.

† See, *e.g.*, Einstein und Grossmann (quoted above), Einstein I. and V.

for the co-ordinates [thus, *e.g.*, any set of 16 quantities which are transformed by the equations (3) form a covariant tensor of the second order], and that these transformation-formulas express the components of the transformed tensor as homogeneous linear functions of the components of the original tensor. Therefore, if for one system of co-ordinates a certain tensor is zero, it is zero for any system of co-ordinates. Consequently, if once we have expressed the laws of nature in the form of linear relations between tensors, they will be invariant for all transformations. Thus with the aid of the calculus of tensors Einstein has succeeded in satisfying the postulate of general relativity. The fundamental tensor g_{ij}, which defines the line-element, and therefore the metric properties of the reference-system of space-time co-ordinates, naturally occupies a prominent place in all formulas.

3. The characteristic feature of Einstein's theory is the intimate connection which he has traced between this fundamental tensor and the gravitational field. In all other theories, also in the "old" theory of relativity, gravitation is a "force," like, *e.g.*, electromagnetic forces, which requires its own laws, and these laws have no greater inherent necessity than those of any other natural phenomenon. In Einstein's new theory, gravitation is of a much more fundamental nature: it becomes almost a property of space. Gravitation certainly differs from all other forces of nature by its generality and its independence of anything else. At a given point in a gravitational field every material point receives the same acceleration whatever its chemical or physical properties may be. Now, if we introduce a new system of co-ordinates which at this point has exactly this acceleration, then the material point subjected to gravitation would be at rest relatively to this new system of co-ordinates, and would thus in this new system be apparently not subjected to gravitation. By the principle of general relativity there is no essential difference between the two systems of co-ordinates: we have no right to say that either of them is at rest and the other moves. Whether there is at a given point a gravitational field or not thus depends on the choice of the reference-system.

In the old mechanics space is Euclidean, and a material point subjected to no forces describes a straight line with uniform velocity, *i.e.* its world-line in a Euclidean four-dimensional time-space [the system of reference of the old theory of relativity] is a straight line. In Einstein's theory, if there is gravitation, the four-dimensional time-space is not Euclidean, and the world-line of a point subjected to no other forces than gravitation is a geodetic line. If there is no gravitation, the time-space is Euclidean, and the geodetic line is a straight line as in the old theory. Gravitation is thus, properly speaking, *not* a "force" in the new theory.

4. The equations of the geodetic line are, of course, derived in terms of the coefficients g_{ij} by writing down the condition that $\int ds$ shall be a minimum. We will not enter into the details of this computation, but we will only explain so much of the opera-

tions involved as is necessary for the good understanding of the subsequent reasoning.*

The determinant of the g_{ij} will be called g.

If $d\tau$ be the four-dimensional element of volume (thus $d\tau = dx_1 dx_2 dx_3 dx_4$), then

$$\sqrt{g} \,.\, d\tau = \sqrt{g'} \,.\, d\tau'$$

is also invariant for all transformations.

It is always possible to perform a transformation by which at any *one* given point the g_{ij} are reduced to their values in the old theory of relativity, *i.e.* to

$$ds^2 = -dx_1^2 - dx_2^2 - dx_3^2 + dx_4^2.$$

It follows that g is always negative, if we restrict ourselves to transformations in which $d\tau$ remains real and positive.

We introduce further the symmetrical contravariant tensor g^{ij}, of which the components are the minors of the g_{ij} divided by g. We may also state the relation between g_{ij} and g^{ij} thus:

$$\text{if} \quad y_i = \sum_j g_{ij} z_j,$$

$$\text{then} \quad z_i = \sum_j g^{ij} y_j.$$

The transformation-formulas for the g^{ij} are

$$g'^{ij} = \sum_\alpha \sum_\beta \pi_{\alpha i} \pi_{\beta j} g^{\alpha\beta}. \tag{4}$$

We will now proceed to give the definition of certain symbols which are of frequent occurrence in the formulas. We put

$$\left\{ \begin{array}{l} \begin{bmatrix} ij \\ p \end{bmatrix} = \frac{1}{2}\frac{\partial g_{ip}}{\partial x_j} + \frac{1}{2}\frac{\partial g_{jp}}{\partial x_i} - \frac{1}{2}\frac{\partial g_{ij}}{\partial x_p}, \\ \begin{Bmatrix} ij \\ p \end{Bmatrix} = \sum_\alpha g^{\alpha p} \begin{bmatrix} ij \\ \alpha \end{bmatrix}. \end{array} \right. \tag{5}$$

The symbols $\begin{Bmatrix} ij \\ p \end{Bmatrix}$ thus contain the g_{ij} and their first derivatives with respect to the co-ordinates. The differential equations of the geodetic line are then found to be

$$\frac{d^2 x_i}{ds^2} + \sum_\alpha \sum_\beta \begin{Bmatrix} \alpha\beta \\ i \end{Bmatrix} \frac{dx_\alpha}{ds} \frac{dx_\beta}{ds} = 0. \qquad (i = 1 \;.\;.\;.\; 4.) \tag{I.}$$

If we wish to separate the time-co-ordinate from the space-co-ordinates, we take $x_4 = ct$. Further, we put

$$\frac{dx_i}{cdt} = \dot{x}_i, \text{ so that } \dot{x}_4 = 1.$$

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* See, *e.g.*, Einstein V., §§ 8 and 9.

Then we find easily from (I.)

$$(6) \qquad \frac{d^2x_i}{c^2dt^2} = -\sum_p\sum_q\left[\begin{Bmatrix}pq\\i\end{Bmatrix} - \begin{Bmatrix}pq\\4\end{Bmatrix}\dot{x}_i\right]\dot{x}_p\dot{x}_q. \qquad (i = 1, 2, 3.$$

These are the equations of motion of a material point in the gravitational field.

For moderate velocities the differential coefficients $\dot{x}_i$ for $i = 1, 2, 3$ are small on account of the denominator c. We will consider as a small quantity of the first order the square of the Gaussian constant of attraction k, divided by the square of the velocity of light:

$$\lambda^2 = \frac{k^2}{c^2} = 9{\cdot}853 \times 10^{-9}. \qquad \text{(Astronomical units.)}$$

Then $\dot{x}_i$ is of the order $\frac{1}{2}$. The brackets $\begin{Bmatrix}pq\\i\end{Bmatrix}$ are, for those systems of co-ordinates which we will mostly use, at least of the first order. In the equation (6) the left-hand member is of the first order, and in the right-hand member we will retain quantities of the second order. The g_{ij} for rectangular Cartesian co-ordinates are

$$(7) \qquad \begin{cases} g_{ii} = -1 + \gamma_{ii}, & g_{ij} = \gamma_{ij}, \\ g_{i4} = \gamma_{i4}, & g_{44} = 1 + \gamma_{44}. \end{cases}$$

We shall often use the notation

$$\delta_{ii} = 1, \quad \delta_{ij} = 0 \text{ for } i \neq j.$$

Then we can write for (7)

$$(7') \qquad g_{ij} = \mp\delta_{ij} + \gamma_{ij},$$

where the lower sign must be taken for $i = j = 4$. This double sign might have been avoided by taking $x_4 = ct \, . \sqrt{-1}$; we will, however, generally use $x_4 = ct$.

The γ_{ij} are at least of the first order.

5. The element ds integrated along the geodetic line gives $s = \int ds$. This is what Minkowski calls the *proper-time* of the material particle. The relation between t and s is given by the fourth of the equations (I.), viz.:

$$\frac{d^2x_4}{ds^2} = -\sum_p\sum_q\begin{Bmatrix}pq\\4\end{Bmatrix}\frac{dx_p}{ds}\frac{dx_q}{ds}.$$

The brackets $\begin{Bmatrix}pq\\4\end{Bmatrix}$ for p and q different from 4 are generally zero, and in any case the terms in which they occur are of a higher order than the others. We have thus

$$\frac{d^2x_4}{ds^2} = -2\sum_v\begin{Bmatrix}p4\\4\end{Bmatrix}\frac{dx_p}{ds}\frac{dx_4}{ds}.$$

Further, since also $g^{\alpha 4}$ and the square brackets $\begin{bmatrix} p4 \\ \alpha \end{bmatrix}$ for α different from 4 are of a higher order, we have

$$\begin{Bmatrix} p4 \\ 4 \end{Bmatrix} = g^{44}\begin{bmatrix} p4 \\ 4 \end{bmatrix} = \tfrac{1}{2}g^{44}\frac{\partial g_{44}}{\partial x_p}.$$

Further, with sufficient accuracy

$$g^{44} = \frac{1}{g_{44}}.$$

Therefore our equation becomes

$$\frac{d^2x_4}{ds^2} = -\frac{1}{g_{44}}\frac{dg_{44}}{ds}\frac{dx_4}{ds},$$

from which

$$\log\frac{dx_4}{ds} = -\log g_{44}$$

or

$$\frac{dt}{ds} = \frac{1}{c\,.\,g_{44}} = \frac{1}{c(1+\gamma)}. \tag{8}$$

6. As a first approximation we can in the right-hand members of (6) take only the terms of the first order. The equations then become

$$\frac{d^2x_i}{c^2dt^2} = -\begin{Bmatrix} 44 \\ i \end{Bmatrix}.$$

Now we have, by (5)

$$\begin{Bmatrix} 44 \\ i \end{Bmatrix} = g^{ii}\begin{bmatrix} 44 \\ i \end{bmatrix} = -\begin{bmatrix} 44 \\ i \end{bmatrix} = -\tfrac{1}{2}\frac{\partial g_{44}}{\partial x_i}.$$

Thus if we take

$$\gamma_{44} = \gamma = \frac{2\Omega}{c^2},$$

we find

$$\frac{d^2x_i}{dt^2} = -\frac{\partial\Omega}{\partial x_i}. \tag{9}$$

The first approximation leads thus to the ordinary Newtonian theory of gravitation, and $\Omega = \frac{1}{2}c^2\gamma$ is the potential function. It thus appears that the g_{ij} determine the gravitational field in much the same way as the potential does in classical mechanics. The essential feature of the theory is that these same g_{ij} also determine the metric properties of the reference system of space-time coordinates.

7. In classical mechanics the potential is determined by Poisson's equation

$$\nabla^2\Omega = -4\pi k^2\rho.$$

The equations analogous to this, which determine the g_{ij} in Einstein's theory, are the following. We put

$$(10) \quad \left\{ \begin{aligned} R_{pq} &= -\sum_{\alpha} \frac{\partial \left\{ {pq \atop \alpha} \right\}}{\partial x_{\alpha}} + \sum_{\alpha} \sum_{\beta} \left\{ {p\alpha \atop \beta} \right\} \left\{ {q\beta \atop \alpha} \right\}, \\ S_{pq} &= \sum_{\alpha} \frac{\partial \left\{ {p\alpha \atop \alpha} \right\}}{\partial x_{q}} - \sum_{\alpha} \sum_{\beta} \left\{ {pq \atop \alpha} \right\} \left\{ {\alpha\beta \atop \beta} \right\}. \end{aligned} \right.$$

$$(11) \qquad G_{pq} = R_{pq} + S_{pq}, \qquad G = \sum_{\alpha}\sum_{\beta} g^{\alpha\beta} G_{\alpha\beta}.$$

The G_{pq} form a symmetrical tensor of the second order $[G_{pq} = G_{qp}]$. The R_{pq} and the S_{pq} separately are, however, not tensors for every system of co-ordinates. Einstein has, as we will see below, often specialised the system of co-ordinates in such a way that

$$(12) \qquad \sqrt{-g} = 1.$$

We find by a simple computation that

$$S_{pq} = \frac{\partial^2}{\partial x_p \partial x_q} \log \sqrt{-g} - \sum_{\alpha} \left\{ {pq \atop \alpha} \right\} \frac{\partial \log \sqrt{-g}}{dx_{\alpha}},$$

Therefore, if we adopt (12), we have $S_{pq} = 0$, $G_{pq} = R_{pq}$.

The equations determining the g_{ij} are now

$$(\text{II.}) \qquad G_{pq} = -\kappa[T_{pq} - \tfrac{1}{2} g_{pq} T],$$

where T_{pq} is a tensor depending on the matter (and energy) producing the gravitational field, and

$$T = \sum_{\alpha}\sum_{\beta} g^{\alpha\beta} T_{\alpha\beta}.$$

If we multiply (II.) by g^{pq} and sum over p and q, we find

$$(13) \qquad G = \kappa T.$$

Therefore (II.) can also be written

$$(\text{II}'.) \qquad G_{pq} - \tfrac{1}{2} g_{pq} G = -\kappa T_{pq}.$$

The tensor G_{pq} is the only tensor of the second order which contains no higher derivatives of the g_{ij} than the second. The introduction of the tensor T_{ij} by Einstein is based on certain considerations connected with the laws of conservation of energy, etc., on which we need not enter here. Its expression, if we neglect pressures, molecular forces, etc., is

$$(14) \qquad T_{ij} = \sum_{\alpha}\sum_{\beta} g_{\alpha i} g_{\beta j} \frac{dx_{\alpha}}{ds} \frac{dx_{\beta}}{ds} \cdot \rho,$$

from which we easily find

$$\mathrm{T} = \rho,$$

where ρ is the density. In the first approximation we find that of the equations (II.) we need only one, and that one reduces to the equation of Poisson,

$$\sum_{\alpha}{}' \frac{\partial^2 \gamma}{\partial x_\alpha{}^2} = -\kappa\rho.$$

Consequently we have

$$\kappa = 8\pi \frac{k^2}{c^2} = 8\pi\lambda^2.$$

Exactly the same equations (II.) have also been derived by Lorentz * from a very different point of view. Lorentz considers a certain function H, which must be a minimum, or at least stationary, within any limited portion of the four-dimensional time-space. This function consists of three parts, $\mathrm{H} = \mathrm{H}_1 + \mathrm{H}_2 + \mathrm{H}_3$. The first part refers to matter, the second to the electromagnetic field, and the third to the gravitational field. We will here only consider the functions H_1 and H_3. Lorentz puts

$$\mathrm{H}_1 = \Sigma\mu\int ds,$$

where the sum is to be taken over all material points which are present within the four-dimensional volume considered, μ is a factor depending on the mass of the material point, and $\int ds$ is the length of that part of the world-line of the point which falls within the considered volume. Further,

$$\mathrm{H}_3 = \int\sqrt{-g} \,.\, \mathrm{G}d\tau,$$

where $d\tau$ is the four-dimensional element of volume, and the integral is to be taken over the whole volume. It appears that the mathematical interpretation of G is the *curvature* of the four-dimensional system of reference. From (II.) it appears that at points where there is no matter $[\mathrm{T}_{ij} = 0]$ this curvature is zero.

We now write down the condition

$$\delta \mathrm{H} = 0, \tag{15}$$

where the variation is to be effected with regard to the co-ordinates x_i of all material points, and to the ten g_{ij}'s, and at the boundary the values of x_i, g_{ij}, and their derivatives must not be varied. Then, from this generalised principle of Hamilton, we derive Lagrangian

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* In a course of lectures given at the University of Leiden from March to June 1916. See also *Proceedings of the Academy of Sciences at Amsterdam*, 1916, Feb. Many of the results contained in the present paper are wholly or partially derived from these lectures. Others were developed during, or suggested by, conversations with my colleagues Lorentz and Ehrenfest. Much is also due to Mr. J. Droste. As much as possible I have added footnotes quoting the authority: but as there has been so much free interchange of ideas, it is not always possible to assign to each his exact share.

equations, precisely in the same manner as is done in classical mechanics.

The variation with regard to the co-ordinates, which occur explicitly in H_1 only, gives the equations (I.), and that with regard to the g_{ij}, which occur both in H_1 and H_3, leads to the equations (II.).

The function H_2, which is here neglected, contains also the g_{ij}, and in addition to these four parameters q_i and their first derivatives, which describe the magnetic field. If H_2 is included, H must be a minimum also with respect to these parameters q_i. The variation with regard to them gives four more Lagrangian equations, which are equivalent to Maxwell's equations for the electromagnetic field. This latter thus depends necessarily on the g_{ij}, *i.e.* on the gravitational field, and conversely the g_{ij} depend on the q_i.

Thus the whole of the theories of electromagnetism (and light) and gravitation (mechanics) is included in the one equation (15).

Hilbert* has given very much the same reasoning, only he does not consider the presence of matter, and accordingly omits the function H_1.

8. An essential feature of the new theory is that of the ten equations (II.) only six are mutually independent. The g_{ij} are therefore not determined by these equations: there remains a large amount of freedom. This is essential. Once the g_{ij} are fixed upon, all the properties of the four-dimensional representative time-space are determined. The principle of general relativity, however, requires that the system of reference shall be arbitrary. The equations (I.) and (II.) completely describe all phenomena. They must, therefore, not imply a choice of the system of co-ordinates, *i.e.* they must leave the g_{ij} to a certain extent undetermined. To determine them completely, and thereby fix the system of co-ordinates, four additional conditions are required.

Hilbert, in the paper already quoted, mentions a mathematical theorem which asserts that, if J be an invariant for all transformations of the four variables x_i, and Lagrangian equations are formed from the condition $\delta \int J\sqrt{-g}d\tau = 0$, where the variation is to be effected with regard to n parameters, then always four of these equations are dependent on the other $n-4$. The $\sqrt{-g}$ is needed to make the whole function under the integral sign an invariant, $d\tau$ by itself being not invariant. Now G is an invariant. The function H_3 thus falls under the theorem, and consequently, if we consider the gravitational field outside the material points (where $H_1 = 0$), only six of the ten equations (II.) are mutually independent.†

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* *Göttinger Nachrichten*, 1915 Nov. 20, p. 395.

† If H_2 also is included, we have fourteen equations, of which only ten are mutually independent. For the four which are dependent on the ten others we can choose the equations of the electromagnetic field, and we can thus say, as Hilbert does, that by the new theory the electromagnetic phenomena are effects of gravitation. This must, however, not be so understood that the q_i are determined by the g_{ij}. From the first ten equations we can solve the

If the function H_1 is included, Hilbert's theorem is still applicable, since ds is an invariant. The function H_1 produces the right-hand members of the equations (II.) or (II'.), but not the complete right-hand members. Hilbert's theorem thus applies to the left-hand members of these equations only. But it is evident that it must also apply to the complete equations. For if from six of the left-hand members we can, by simple operations, derive the four others, then these same operations applied to the six corresponding right-hand members must also reproduce the right-hand members of the four other equations, else the equations would be contradictory. In the practical applications most of the right-hand members are of a higher order than need be taken into account, and can be put equal to zero. But if we wish to be correct to all orders, evidently the applicability of Hilbert's theorem, *i.e.* the principle of general relativity, imposes certain conditions on the tensor T_{ij}, which is then no longer determined by the equation (14) alone, but also contains pressures, molecular forces, etc. These considerations lead to the expectation that the theory of general relativity may eventually lead also to a definite theory of molecular forces and the constitution of the atom, or at least it restricts such theories within certain limits. I may add that Lorentz, in order to get the equations (II.) exact to all orders in agreement with Einstein, is compelled to add to his function H a further term which corresponds precisely to these pressures and molecular forces. I will, however, not insist on these points, which are not of immediate importance for the astronomical applications.

9. The equations (I.) and (II.) are invariant for all transformations of the four co-ordinates $x_1 \ldots x_4$. (I.) are the equations of motion of a material point. To construct these for any particular system of reference, we must first solve the g_{ij} from (II.), together with the four additional conditions defining the chosen system. If then we wish to transform to another system, we can alter our additional conditions, solve g_{ij} anew from (II.) and the new conditions, and substitute in (I.). Or we can transform the g_{ij} by (3). Or, finally, we can express the g_{ij} in (I.) in terms of the co-ordinates, and then transform (I.) directly by means of (2) and their first derivatives. The result is the same in all cases. The equations (I.) expressed in g'_{ij} and x_i' are of the same form as in g_{ij} and x_i, but if the g_{ij} are replaced by their expressions in terms

g_{ij} in terms of the q_i: if these values are substituted in the four other equations, these become identities, *i.e.* they are satisfied by *any* values of the q_i. We might thus choose the q_i, *i.e.* the electromagnetic field, entirely arbitrarily, and the choice would be equivalent to a determination of the system of co-ordinates. In other words, we can always introduce such a system of reference that the quantities q_i determining the electromagnetic field have any prescribed values. Then in this system of reference the gravitational field is entirely determinate. If, on the other hand, we fix the system of reference by four other additional conditions, then both the g_{ij} and the q_i are entirely determinate. It should be pointed out that it is an accidental coincidence that the electromagnetic field is described by *four* parameters, and there are also *four* degrees of indeterminateness. I will not dwell longer on these questions, which are not yet entirely cleared up and are outside my subject.

of the x_i, and similarly the g'_{ij} in the x_i', then of course the two sets of equations are entirely different.

From the transformation-formulas of the g_{ij} and g^{ij} we derive those for the brackets, which are found to be

$$(16) \qquad \left\{ {pq \atop i} \right\}' = \sum_{\alpha} \sum_{\beta} \sum_{\kappa} p_{\alpha p} p_{\beta q} \pi_{\kappa i} \left\{ {\alpha\beta \atop \kappa} \right\} + \delta \left\{ {pq \atop i} \right\}.$$

The second term would be zero if the p_{ij}, and therefore also the π_{ij}, were constants. The brackets would then form a tensor of the third order, and the transformation of the equations (I.) would be

$$\frac{d^2 x_i'}{c^2 dt'^2} = \sum_{\lambda} \pi_{\lambda i} \frac{d^2 x_\lambda}{c^2 dt^2}.$$

The second term in (16) must be computed for each transformation separately from (3) and (4), by taking only those parts of, the differential coefficients of g'_{ij} which arise through the differentiation of the p_{ij}.

Take, *e.g.*,

$$x_i' = x_i - \psi_i(ct), \qquad x_4' = x_4 = ct,$$

where the ψ_i are given functions of the time. Then we find

$$\delta \left\{ {44 \atop i} \right\} = \ddot{\psi}_i, \text{ all other } \delta \left\{ {pq \atop i} \right\} = 0.$$

Further, we have $\pi_{ii} = 1$, $\pi_{\lambda i} = 0$ for $\lambda \neq i$, therefore

$$\frac{d^2 x_i'}{c^2 dt^2} = \frac{d^2 x_i}{c^2 dt^2} - \ddot{\psi}_i,$$

as it evidently must be.

As another example we will derive the general transformation-formula for a transformation of the space co-ordinates alone, thus: $p_{4i} = p_{i4} = 0$, $p_{44} = 1$. We will suppose that for rectangular co-ordinates the g_{ij} have the following values, which are somewhat less general than (7):

$$g_{ij} = -\delta_{ij} + \gamma_{ij}, \qquad g_{i4} = 0, \qquad g_{44} = 1 + \gamma.$$

For other systems of co-ordinates the g_{ij} will also consist of a part of the order zero and a part of the first or higher orders; thus, for all values of i and j:

$$g_{ij} = g_{ij}^0 + \gamma_{ij}, \qquad g^{ij} = g_0^{ij} + \gamma^{ij}.$$

Consequently, also the brackets $\left\{ {pq \atop i} \right\}$ will similarly consist of a part of the order zero and a part of the first or higher orders, thus:

$$\left\{ {pq \atop i} \right\} = \left\{ {pq \atop i} \right\}_0 + \left\{ {pq \atop i} \right\}_1.$$

The equations (6) then become, neglecting terms higher than the second order:

$$(17)\quad \frac{d^2x_i}{c^2dt^2}+\sum_p{}'\sum_q{}'\left\{\begin{matrix}pq\\i\end{matrix}\right\}_0\dot{x}_p\dot{x}_q-\tfrac{1}{2}\sum_j g_0{}^{ij}\frac{\partial\gamma}{\partial x_j}$$

$$=\sum_p{}'\frac{\partial\gamma}{\partial x_p}\dot{x}_i\dot{x}_p+\tfrac{1}{2}\sum_j{}'\gamma^{ij}\frac{\partial\gamma}{\partial x_j}-\sum_p{}'\sum_q{}'\left\{\begin{matrix}pq\\i\end{matrix}\right\}_1\dot{x}_p\dot{x}_q.$$

The left-hand members put equal to zero are the complete equations of motion for Newtonian mechanics, if for $c^2\gamma$ we write 2Ω, as in (9). It is easily verified that they agree with the equations of classical mechanics for any system of co-ordinates. The right-hand members, which are of the second order, are the terms added by the new theory. The addition of these terms, and those of the third and higher orders, which have not been written, is necessary in order to make the equations invariant for all transformations.

10. We will now apply the equations (II.) to derive the gravitational field of a sphere (say the sun), which we will suppose to be at rest at the origin of the system of space co-ordinates. For the fourth co-ordinate we will take $x_4=ct$. For the space co-ordinates we will use either rectangular Cartesian co-ordinates,

$$x_1=x,\quad x_2=y,\quad x_3=z,$$

or polar co-ordinates,

$$x_1=r,\quad x_2=\theta,\quad x_3=\phi.$$

The formulas of transformation are

$$\begin{aligned}x&=r\cos\phi\cos\theta,\\ y&=r\cos\phi\sin\theta,\\ z&=r\sin\phi.\end{aligned}$$

If there is no gravitation, *i.e.* if the mass of the sun were zero, the expression of the line-element is, in these two systems,

$$(18)\quad \begin{cases}ds^2=-dx^2-dy^2-dz^2+c^2dt^2,\\ ds^2=-dr^2-r^2\cos^2\phi d\theta^2-r^2d\phi^2+c^2dt^2.\end{cases}$$

The actual values of the g_{ij} will differ from those implied by (18) by quantities of at least the first order. We will take $g_{i4}=0$. We will for the moment not enter into details regarding this assumption, and the consequences implied by it. At the present stage we may accept it as constituting three of our four additional conditions. From reasons of symmetry it is apparent at once that, if the gravitational field of the sun is taken into account, the line-element must be of the form *

$$(19)\quad ds^2=-(1+\alpha)dr^2-(1+\beta)[r^2\cos^2\phi d\theta^2+r^2d\phi^2]+(1+\gamma)c^2dt^2.$$

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* See also J. Droste, *Proceedings of the Academy of Sciences at Amsterdam*, 1914 Dec., vol. xlii. p. 998.

The corresponding values of the g_{ij} for rectangular co-ordinates are given by

$$(19') \quad ds^2 = -(1+\beta)(dx^2+dy^2+dz^2) + \frac{\beta-\alpha}{r^2}\sum_i{}'\sum_j{}' x_i x_j dx_i dx_j + (1+\gamma)c^2dt^2.$$

Evidently α, β, and γ must be functions of r alone. Differential quotients with respect to r will be denoted by accents: $\alpha' = \frac{d\alpha}{dr}$, etc. Then, in the system of polar co-ordinates, we find the following values for the brackets:

$$(20) \quad \left\{\begin{aligned}
&\left\{\begin{matrix}11\\1\end{matrix}\right\} = \frac{1}{2}\frac{\alpha'}{1+\alpha},\\
&\left\{\begin{matrix}22\\1\end{matrix}\right\} = -r\cos^2\phi\frac{1+\beta}{1+\alpha} - \frac{1}{2}r^2\cos^2\phi\frac{\beta'}{1+\alpha},\\
&\left\{\begin{matrix}33\\1\end{matrix}\right\} = -r\frac{1+\beta}{1+\alpha} - \frac{1}{2}r^2\frac{\beta'}{1+\alpha},\\
&\left\{\begin{matrix}12\\2\end{matrix}\right\} = \left\{\begin{matrix}13\\3\end{matrix}\right\} = \frac{1}{r} + \frac{1}{2}\frac{\beta'}{1+\beta},\\
&\left\{\begin{matrix}22\\3\end{matrix}\right\} = \sin\phi\cos\phi, \qquad \left\{\begin{matrix}23\\2\end{matrix}\right\} = -\tan\phi,\\
&\left\{\begin{matrix}44\\1\end{matrix}\right\} = \frac{1}{2}\frac{\gamma'}{1+\alpha}, \qquad \left\{\begin{matrix}14\\4\end{matrix}\right\} = \frac{1}{2}\frac{\gamma'}{1+\gamma}.
\end{aligned}\right.$$

Those which have not been written are zero.

The computations are somewhat simplified if we introduce

$$\lambda = \log(1+\alpha), \quad \mu = \log(1+\beta), \quad \nu = \log(1+\gamma).$$

Then we have

$$g_{11} = -e^\lambda, \quad g_{22} = -r^2\cos^2\phi e^\mu, \quad g_{33} = -r^2e^\mu, \quad g_{44} = e^\nu.$$

Further, since all g_{ij} for $i \neq j$ are zero,

$$g^{ii} = \frac{1}{g_{ii}}.$$

We thus find

$$G_{11} = \mu'' + \tfrac{1}{2}\nu'' + \frac{2}{r}\mu' - \frac{1}{r}\lambda' + \tfrac{1}{2}\mu'^2 - \tfrac{1}{2}\lambda'\mu' - \tfrac{1}{4}\lambda'\nu' + \tfrac{1}{4}\nu'^2,$$

$$\begin{aligned} G_{22} = -\cos^2\phi + \cos^2\phi\,.\,e^{\mu-\lambda}\big[1 &+ 2r\mu' + \tfrac{1}{2}r(\nu'-\lambda') \\ &+ \tfrac{1}{2}r^2\mu'' + \tfrac{1}{2}r^2\mu'(\mu' + \tfrac{1}{2}\nu' - \tfrac{1}{2}\lambda')\big],\end{aligned}$$

$$G_{33} = -1 + e^{\mu-\lambda}\big[1 + 2r\mu' + \tfrac{1}{2}r^2\mu'' + \tfrac{1}{2}r(\nu'-\lambda') + \tfrac{1}{2}r^2\mu'(\mu' + \tfrac{1}{2}\nu' - \tfrac{1}{2}\lambda')\big],$$

$$G_{44} = -e^{\nu-\lambda}\left[\frac{\nu'}{r} + \tfrac{1}{2}\nu'' + \tfrac{1}{2}\nu'(\mu' + \tfrac{1}{2}\nu' - \tfrac{1}{2}\lambda')\right].$$

The others are zero.

Now the equations (II′.) are

$$G_{ii} - \tfrac{1}{2} g_{ii} G = -\kappa T_{ii}, \qquad G = \sum_i \frac{G_{ii}}{g_{ii}}.$$

Thus, if we put

$$y_i = \frac{G_{ii}}{g_{ii}}, \quad Y_i = -\frac{\kappa T_{ii}}{g_{ii}},$$

our equations become

$$(21) \quad \left\{ \begin{aligned} y_1 - y_2 - y_3 - y_4 &= 2Y_1 \\ -y_1 + y_2 - y_3 - y_4 &= 2Y_2 \\ -y_1 - y_2 + y_3 - y_4 &= 2Y_3 \\ -y_1 - y_2 - y_3 + y_4 &= 2Y_4. \end{aligned} \right.$$

We have

$$y_2 = y_3, \quad Y_2 = Y_3.$$

Of the equations (21) there are thus three different from each other, and we have three unknowns, λ, μ, ν. After some simple transformations we find

$$(22) \quad \left\{ \begin{aligned} &\mu'' + \tfrac{1}{2}\nu'' + \frac{2}{r}(\mu' - \tfrac{1}{2}\lambda') + \tfrac{1}{2}\mu'^2 + \tfrac{1}{4}(\lambda' + \mu')^2 = X_1 \\ &1 - e^{\lambda - \mu} + \tfrac{1}{2}r^2\mu'' + 2r\mu' + \tfrac{1}{2}r(\nu' - \lambda') + \tfrac{1}{2}r^2\mu'(\mu' + \tfrac{1}{2}\nu' - \tfrac{1}{2}\lambda') = X_3 \\ &\tfrac{1}{2}\nu'' + \frac{\nu'}{r} + \tfrac{1}{2}\nu'(\mu' + \tfrac{1}{2}\nu' - \tfrac{1}{2}\lambda') = X_4 \end{aligned} \right.$$

where the right-hand members are

$$(22') \quad \left\{ \begin{aligned} X_1 &= \tfrac{1}{2}\kappa\left[T_{11} - \frac{e^{\lambda - \mu}}{r^2}(\sec^2\phi T_{22} + T_{33}) + e^{\lambda - \nu}T_{44}\right], \\ X_3 &= \tfrac{1}{2}\kappa r^2[T_{11} - e^{\lambda - \nu}T_{44}], \\ X_4 &= \tfrac{1}{2}\kappa\left[T_{11} + \frac{e^{\lambda - \mu}}{r^2}(\sec^2\phi T_{22} + T_{33}) + e^{\lambda - \nu}T_{14}\right]. \end{aligned} \right.$$

We must now introduce the tensor T_{ij}. This is given by (14), together with terms depending on the internal forces (pressures, etc.) within the matter composing the sun. We must thus make a hypothesis regarding these internal forces. If we assume the sun to consist of a fluid whose changes of volume occur adiabatically,* we must add the terms

$$T_{11} = p, \quad T_{22} = r^2 \cos^2\phi \,.\, p, \quad T_{33} = r^2 p, \quad T_{44} = \rho P,$$

where p is the pressure and

$$P = -\int_{\tau_0}^{\tau} p\, d\tau,$$

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* See Einstein I., pp. 1061–1062. See also J. Droste, *Proceedings . . . Amsterdam*, 1916 May, and K. Schwarzschild, "Ueber das Gravitationsfeld einer Kugel," *Sitzungsber. Berlin*, 1916 Feb. 24, p. 424.

τ being the volume and τ_0 the volume for $p = 0$. If the fluid is incompressible, we have $P = 0$. In the formulas (14) we have, the sun being at rest, $dx_i = 0$ for $i = 1, 2, 3$, and $ds^2 = (1 + \gamma)dx_4^2$. The complete values of T_{ii} to be used in (22′) are thus:

$$(23) \qquad T_{11} = p, \quad \sec^2 \phi T_{22} + T_{33} = 2r^2 p, \quad T_{44} = \rho(1 + \gamma + P).$$

The pressure p arises through gravitation, and is thus of the order of the constant of gravitation κ, *i.e.* of the first order. We require α and β only to the first order, and γ to the second order. Introducing the values (23), returning to the original unknowns α, β, γ, and at the same time omitting terms of higher orders than are needed,* we find, after some reduction, the following three equations:

$$(24) \qquad \alpha' - 2\beta' - \gamma' - r(\beta'' + \gamma'') = 0,$$

$$(25) \qquad \alpha - \beta - r(\beta' + \gamma') = 0,$$

$$(26) \qquad r\gamma' + \tfrac{1}{2}r^2\gamma'' + \tfrac{1}{2}r^2\gamma'(\beta' - \tfrac{1}{2}\alpha' - \tfrac{1}{2}\gamma') = \tfrac{1}{2}\kappa r^2 \rho'(1 + \alpha + \gamma),$$

where we have put

$$\rho' = \rho(1 + P) - 3p.$$

The difference between ρ' and ρ is of the first order. If we had made another assumption regarding the inner constitution of the sun, we would have found another value for ρ'. No sharp definition has as yet been given of the "density" ρ. There is no objection to calling ρ' "density."

We will accordingly drop the accent, and write ρ for ρ'.

11. It is apparent at once that (24) is nothing but the first derivative of (25). The three equations are thus not sufficient to determine the three unknowns: we must add one condition. As γ is needed to the second order, we will put

$$\gamma = \gamma_1 + \gamma_2,$$

where γ_1 is of the first and γ_2 of the second order. We will also replace κ by its value $8\pi\lambda^2$. Instead of (26) we then get the two equations

$$(27) \qquad r\gamma_1' + \tfrac{1}{2}r^2\gamma_1'' = 4\pi\lambda^2 r^2 \rho.$$

$$(28) \qquad r\gamma_2'' + \tfrac{1}{2}r^2\gamma_2'' + \tfrac{1}{2}r^2\gamma_1'(\beta' - \tfrac{1}{2}\alpha' - \tfrac{1}{2}\gamma_1') = 4\pi\lambda^2 r^2 \rho(\alpha + \gamma_1).$$

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* The *rigorous* equations for the two systems considered in the next article are:

$$\text{A.} \quad \begin{cases} \lambda + 2\mu + \nu = 0 \\ \nu'' + \dfrac{2}{r}\nu' - \lambda'\nu' = \kappa\rho e^{\lambda} \\ 1 - e^{\lambda - \mu} + r(\mu' + \nu') + \tfrac{1}{2}r^2\mu'(\nu' + \tfrac{1}{2}\mu') = 0. \end{cases}$$

$$\text{B.} \quad \begin{cases} \lambda = \mu \\ \quad + \dfrac{2}{r}\nu' + \tfrac{1}{2}\nu'(\nu' + \mu') = \kappa\rho e^{\mu} \\ \mu' + \nu' + \tfrac{1}{2}r\mu'(\nu' + \tfrac{1}{2}\mu') = 0. \end{cases}$$

From (27) we find

$$(29)\qquad \begin{cases} \gamma_1{}' = \dfrac{2\lambda^2}{r^2} m(r) \\[2ex] \gamma_1 = -\dfrac{2\lambda^2}{r} m(r) - 2\lambda_0{}^2[\mathrm{N} - n(r)], \end{cases}$$

where we have put

$$m(r) = 4\pi \int_0^r r^2 \rho dr,$$

$$n(r) = 4\pi \int_0^r r\rho dr, \qquad \mathrm{N} = n(\mathrm{R}),$$

R being the radius of the sun. For $r \geqq \mathrm{R}$ we have $n(r) = \mathrm{N}$.

Then (25) and (28) become

$$(30)\qquad r\beta' + \beta - \alpha = -\frac{2\lambda^2}{r} m(r),$$

$$(31)\qquad r^2\gamma_2{}' = -2\lambda^2 m(r)[\beta - \tfrac{1}{2}(\alpha + \gamma_1)] + 4\pi\lambda^2 \int_0^r r^2 \rho[\beta + \tfrac{1}{2}(\alpha + \gamma_1)]dr.$$

These equations are not sufficient to determine α, β, and γ_2, we must add one condition, which will define the manner in which the radius r should be measured. Lorentz in his lectures took the condition that the radial velocity of light should be constant, *i.e.* r is measured by the time taken by a light vibration to reach a point on the radius from the sun. This leads to the condition $\alpha = \gamma$. Einstein takes the condition (12): $\sqrt{-g} = 1$, as is also done by Schwarzschild.* We will here consider two different conditions, viz.:

$$(32)\qquad \begin{cases} \text{A.} \quad \beta = 0, \\ \text{B.} \quad \alpha + \gamma_1 = 0. \end{cases}$$

In both cases we get α and β at once from (30) and (32). We find:

$$(33)\qquad \begin{cases} \text{A.} \quad \alpha = +\dfrac{2\lambda^2}{r} m(r), \\[2ex] \text{B.} \quad \alpha = \beta = \dfrac{2\lambda^2}{r} m(r) - 2\lambda^2[\mathrm{N} - n(r)]. \end{cases}$$

Then we can solve γ_2 from (31). We find

$$\text{A.} \quad r^2\gamma_2{}' = -4\lambda^4 m(r)[\mathrm{N} - n(r)] - 2\lambda^4 q(r),$$

$$\text{B.} \quad r^2\gamma_2{}' = -\frac{4\lambda^4}{r} m(r)^2 + 8\lambda^4 q(r),$$

where we have put

$$q(r) = 4\pi \int_0^r r\rho m(r) dr.$$

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* Einstein III., p. 832. Schwarzschild, "Das Gravitationsfeld eines Massenpunktes nach der Einstein'schen Theorie," *Sitzungsber. Berlin*, 1916 Jan. 13, p. 189.

For the equations of motion (6) we need only γ_2' and not γ_2 itself. Also we shall use the equations only outside the sun. We will put

$$m(\mathrm{R}) = m, \quad q(\mathrm{R}) = q.$$

Outside the sun we have thus $m(r) = m$, $n(r) = \mathrm{N}$, $q(r) = q$. We put further

$$m' = m - \lambda^2 q, \quad m'' = m + 4\lambda^2 q.$$

Then we have, outside the sun,

$$\text{A.} \quad \gamma_2 = \frac{2\lambda^4 q}{r^2}, \quad \gamma = -\frac{2\lambda^2 m'}{r}.$$

$$\text{B.} \quad \gamma_2 = \frac{2\lambda^4 m^2}{r^2} - \frac{8\lambda^4 q}{r}, \quad \gamma = -\frac{2\lambda^2 m''}{r} + \frac{2\lambda^4 m^2}{r^2}.$$

Now m' and m'' differ from m only in quantities of the first order. Since α and β need only be known to the first order, we may in their expressions, and also in γ_2, replace m by m' or m''. We may then call m' (or m'') the "mass," and we may drop the accents. Further, I put

$$\lambda_0^2 = \lambda^2 m.$$

Then we have finally,

$$(34) \quad \left\{ \begin{array}{l} \text{A.} \quad \alpha = -\gamma, \quad \beta = 0, \quad \gamma = -\dfrac{2\lambda_0^2}{r}. \\ \text{B.} \quad \alpha = \beta = -\gamma, \quad \gamma = -\dfrac{2\lambda_0^2}{r} + \dfrac{2\lambda_0^4}{r^2}. \end{array} \right.$$

The condition (12) is, to the first order, $\beta + \frac{1}{2}(\alpha + \gamma) = 0$. This is satisfied by A,* but not by B. We will further give the expressions of the line-element ds. The three-dimensional line-element will be called $d\sigma$, thus

$$d\sigma^2 = dx^2 + dy^2 + dz^2 = dr^2 + r^2(\cos^2\phi\, d\theta^2 + d\phi^2).$$

Then we have

$$(35) \quad \left\{ \begin{array}{l} \text{A.} \quad ds^2 = -d\sigma^2 + c^2dt^2 + \gamma(dr^2 + c^2dt^2). \\ \text{B.} \quad ds^2 = -d\sigma^2 + c^2dt^2 + \gamma(d\sigma^2 + c^2dt^2). \end{array} \right.$$

The two systems differ only in the way in which r is measured. We can transform the one to the other by a transformation of r alone. If in the formulas A we substitute

$$(36) \quad r = r'\left(1 + \frac{\lambda_0^2}{r}\right) = r' + \lambda_0^2,$$

and then write r for r', we find the formulas B, remembering that we must reject terms with λ_0^4, unless multiplied by c^2.

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* Inside the sun, it is also not satisfied by A.

All our formulas are approximations: they can thus only be used if α, β, γ are actually small, *i.e.* if r is considerably larger than $2\lambda_0^2$. For the sun λ_0^2 is about 1·5 km.

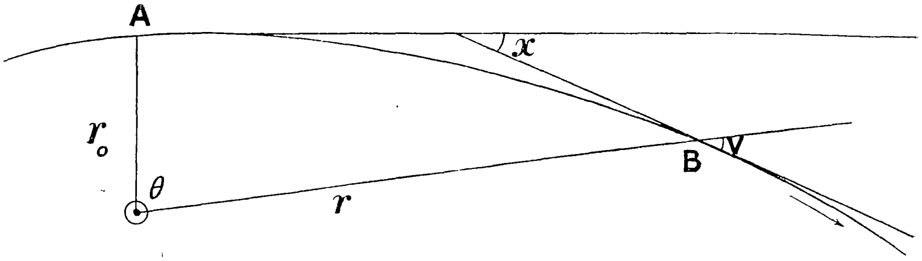
12. The velocity of light measured in the representative system of co-ordinates is

$$v = \frac{d\sigma}{dt}.$$

When there is no gravitation, this velocity is constant, $v = c$, and therefore, by (18), $ds = 0$. In the gravitational field we have also, for a ray of light,

$$ds = 0,$$

and we can then determine v from (35). Consider a ray of light in the plane $\phi = 0$, and let the angle θ be reckoned from the radius r_0, which is perpendicular to the ray. Let, further, V be the angle at a point on the ray between the tangent at that point and the radius-vector (see figure).



We have

$$\sin \mathrm{V} = r\frac{d\theta}{d\sigma}, \quad \cos \mathrm{V} = \frac{dr}{d\sigma}, \quad \tan \mathrm{V} = r\frac{d\theta}{dr}.$$

Putting $ds = 0$ in (35) we have, in the two cases,

$$\text{A.} \quad v^2(1 - \gamma \cos^2 \mathrm{V}) - c^2(1 + \gamma) = 0,$$
$$\text{B.} \quad v^2(1 - \gamma) - c^2(1 + \gamma) = 0,$$

or

$$(37) \qquad \begin{cases} \text{A.} \quad v = c[1 + \tfrac{1}{2}\gamma(1 + \cos^2 \mathrm{V})], \\ \text{B.} \quad v = c[1 + \gamma]. \end{cases}$$

It is easily verified that by (36) the first formula is transformed into the second. In the system of co-ordinates B the velocity of light is the same in all directions; in the system A it has a maximum for directions perpendicular to the radius-vector.

The time required for the light to get from A to B is

$$t - t_0 = \int_{r_0}^{r} \frac{dt}{dr} dr = \int_{r_0}^{r} \frac{d\sigma}{dr} \cdot \frac{1}{v} dr = \int_{0}^{r} \frac{\sec \mathrm{V}}{v} dr.$$

This time must be a minimum for the actual ray. We use the system B, thus

$$\frac{\sec \mathrm{V}}{v} = (1 - \gamma) \sqrt{1 + r^2\theta'^2}.$$

We consider another ray between the same points, and we take as corresponding points on the two rays, those points which have the same r, then $\delta r = 0$, and we find easily

$$\delta \frac{\sec \mathrm{V}}{v} = (1 - \gamma)\, r \sin \mathrm{V} \delta\theta'.$$

We must have $\delta(t - t_0) = 0$. We integrate by parts; the integrated term vanishes, because at the terminal points the variation is zero. We have thus

$$\int_{r_0}^{r} \frac{d}{dr}\left[(1 - \gamma) r \sin \mathrm{V}\right] \delta\theta dr = 0.$$

The variations $\delta\theta$ are entirely arbitrary. Consequently the equation of the ray of light is

$$(1 - \gamma) r \sin \mathrm{V} = a, \tag{38}$$

a being a constant. At the point A we have $\mathrm{V} = 90°$, $r = r_0 = a(1 + \gamma)$. The constant a is therefore practically the minimum distance of the ray from the sun. The ray is curved. We will compute the deviation x, *i.e.* the angle between the tangent to the ray and the tangent at A. We have

$$x = \mathrm{V} + \theta - 90°,$$

or

$$\frac{dx}{d\mathrm{V}} = 1 + \frac{d\theta}{d\mathrm{V}} = 1 + \frac{\tan \mathrm{V}}{r} \frac{dr}{d\mathrm{V}}.$$

If we put

$$r_1 = r(1 - \gamma) = r + 2\lambda_0^2, \quad dr_1 = dr,$$

we have from (38)

$$\sin \mathrm{V} = \frac{a}{r_1},$$

from which

$$\frac{dr}{d\mathrm{V}} = -\frac{r_1^2}{a} \cos \mathrm{V} = -r_1 \cot \mathrm{V}.$$

Therefore

$$\frac{dx}{d\mathrm{V}} = 1 - \frac{r_1}{r} = \gamma = -\frac{2\lambda_0^2}{a} \sin \mathrm{V}$$

to the required order of accuracy. Therefore

$$x = \frac{2\lambda_0^2}{a} \cos \mathrm{V}. \tag{39}$$

For $r = \infty$ we have $\mathrm{V} = 0°$, and $x = \frac{2\lambda_0^2}{a}$. This is thus the total deviation. For a ray of light passing the sun, *i.e.* going from $-\infty$ to $+\infty$, the total deviation is twice this amount. If the ray just grazes the surface of the sun, the minimum distance is $a = \mathrm{R}$, and the total double deviation is

$$\frac{4\lambda_0^2}{\mathrm{R}} = 1''\cdot 75.$$

Suppose that during a solar eclipse the sun is placed between two stars at the distances a_1 and a_2 from its centre. The mutual distance of the stars, if the sun were not between them, would be $a_1 + a_2$. During the eclipse the apparent distance would be

$$a_1 + a_2 - 1''\cdot 75\left(\frac{\mathrm{R}}{a_1} + \frac{\mathrm{R}}{a_2}\right).$$

The difference could probably very well be measured.

13. Consider a fixed point in (three-dimensional) space. Thus $dx = dy = dz = 0$, or $d\sigma = 0$. Then from (35) we have, for both systems,

$$\frac{dt}{ds} = \frac{1}{c\sqrt{1+\gamma}} = \frac{1}{c}\left(1 + \frac{\lambda_0^2}{r}\right).$$

At the surface of the sun we have thus

$$\frac{dt}{ds} = \frac{1}{c}\left(1 + \frac{\lambda_0^2}{\mathrm{R}}\right) = \frac{1\cdot 00000212}{c},$$

and at an infinite distance from the sun

$$\frac{dt}{ds} = \frac{1}{c}.$$

The measure of time is thus different at different places in the gravitational field, and consequently also the frequency of periodic phenomena for which ds is constant. Consequently, spectral lines originating on the solar surface will be displaced towards the red, and the displacement will be the same as would be produced by a radial velocity of $0\cdot 00000212\, c$ or 0·634 km./sec.

As is well known, several solar lines are actually displaced towards the red. The amount is, however, different for different lines, and the displacement has generally been ascribed to pressure-effects. For a star whose mass and density measured in those of the sun as units are M and ρ, the displacement would be

$$\mathrm{K} = 0\cdot 634\,\mathrm{M}^{\frac{2}{3}}\rho^{\frac{1}{3}}. \tag{40}$$

For helium stars there is a systematic displacement towards the red of about 4·5 km. From spectroscopic doubles of the B class we can take, as a very rough approximation,

$$\mathrm{M} = 10, \quad \rho = \tfrac{1}{10}.$$

This would give

$$\mathrm{K} = 1\cdot 4 \text{ km.},$$

or about one-third of the observed value. Of course it is not impossible that on the average single stars have larger masses and higher densities than spectroscopic doubles. Still, at the present stage of the question it is hardly a correct representation of the facts to say that Einstein's theory explains the observed systematic motion of the B stars.*

14. We will now compute the effect on the gravitational field of an axial rotation of the sun. The velocity of rotation may be $d\theta = c\omega dt$, or $\dot{\theta} = \omega$, and we will neglect quantities of the order of ω^2. To the values of T_{ij} previously computed we must now add

$$T_{24} = -r^2\rho\cos^2\phi \, . \, \omega.$$

For reasons of symmetry all g_{ij} must be either zero or of the order of ω^2 at least, only excepting g_{24}. The G_{ij} also remain unaltered, with the exception of

$$G_{24} = -\tfrac{1}{2}\frac{\partial^2 g_{24}}{\partial r^2} - \frac{1}{2r^2}\left[\frac{\partial^2 g_{24}}{\partial\phi^2} + \tan\phi\frac{\partial g_{24}}{\partial\phi}\right].$$

We have thus the new equation

$$G_{24} = -\kappa T_{24}.$$

We put

$$g_{24} = \frac{\kappa}{2\pi}\omega\cos^2\phi \, . \, \chi,$$

where χ is a function of r alone.

The equation then becomes

$$\chi'' - \frac{2\chi}{r^2} = -4\pi r^2\rho, \tag{41}$$

from which we find easily

$$\chi = \tfrac{1}{3}r^2[N - n(r)] + \tfrac{1}{3}\frac{p(r)}{r},$$

where we have put

$$p(r) = 4\pi\int_0^r r^4\rho dr.$$

Outside the sun, for $r \geqq R$, $p(r)$ becomes a constant $p(r) = P$, and we have

$$g_{24} = \tfrac{4}{3}\lambda^2 P\frac{\omega\cos^2\phi}{r} = \frac{b\cos^2\phi}{r}. \tag{42}$$

To compute the effect on the curvature of the rays of light, we can neglect the ordinary gravitational field, since we require only the terms of the lowest order. We have thus

$$0 = ds^2 = -d\sigma^2 + 2\frac{b\cos^2\phi}{r}d\theta \, . \, cdt + c^2dt^2.$$

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* Compare a paper by Freundlich, "Ueber die Gravitationsverschiebung der Spektrallinien bei Fixsternen," *A.N.*, 4826, and its criticism by Seeliger, *A.N.*, 4829. It should be pointed out that *all* stars appear to show a slight systematic displacement towards the red.

Consider a ray of light in the equatorial plane, so that $\cos^2\phi = 1$. Then we find, if we neglect b^2,

$$v=\frac{d\sigma}{dt}=c\pm cb\frac{\sin \mathrm{V}}{r^2}. \tag{43}$$

The upper sign must be taken for light travelling in the direction of increasing θ. By the same reasoning as before, we find for the equation of the ray of light

$$\left(1\mp\frac{b}{ar}\right)r\sin \mathrm{V}=a. \tag{44}$$

Comparing with (38), we find that we have only to replace γ by $\pm\frac{b}{ar}$, *i.e.* $2\lambda_0^2$ by $\mp\frac{b}{a}$. We have therefore

$$x=\mp\frac{b}{a^2}\cos \mathrm{V}. \tag{45}$$

This deviation is very much smaller than that produced by the ordinary gravitational field, which is given by (39). The ratio is

$$l=\mp\frac{b}{2a\lambda_0^2}=\mp\frac{2\mathrm{P}\omega}{3am}.$$

The ratio P/m depends on the distribution of mass within the sun; it is greater for a homogeneous sun than for one with a central condensation. For the case of homogeneity we have

$$\frac{\mathrm{P}}{m}=\tfrac{3}{5}\mathrm{R}^2.$$

Therefore, if we take $a=\mathrm{R}$, we have

$$l=\mp\tfrac{2}{5}\mathrm{R}\omega.$$

This is a very small fraction. For the sun it is about 1/400,000.

15. We will now consider the motion of an infinitesimal planet in the gravitational field of the sun. From the last article it has appeared that the rotation of the sun produces a g_{24} which is of the order $\frac{3}{2}$, ω being of the order $\frac{1}{2}$. It will appear later that this is a particular case of a general result.

For the field of n moving bodies (with moderate velocities) it is possible to choose a system of reference so that the g_{ij} have the values, to the required order of accuracy (for rectangular coordinates),

$$\left\{\begin{array}{cccc} -1+\gamma & 0 & 0 & \chi_1 \\ 0 & -1+\gamma & 0 & \chi_2 \\ 0 & 0 & -1+\gamma & \chi_3 \\ \chi_1 & \lambda_2 & \chi_3 & 1+\gamma, \end{array}\right. \tag{46}$$

where γ is of the first order and χ_i of the order $\frac{3}{2}$. The equations (6) then become

$$(47)\quad \begin{cases} \dfrac{d^2x_i}{c^2dt^2}+\frac{1}{2}\dfrac{\partial\gamma}{\partial x_i}=-\frac{1}{2}\gamma\dfrac{\partial\gamma}{\partial x_i}+2\dot{x}_i{\sum_p}'\dfrac{\partial\gamma}{\partial x_p}\dot{x}_p-\frac{1}{2}\dfrac{\partial\gamma}{\partial x_i}\phi^2+\frac{3}{2}\dfrac{\partial\gamma}{c\partial t}\dot{x}_i \\ \qquad\qquad +{\sum_p}'\left(\dfrac{\partial\chi_i}{\partial x_p}-\dfrac{\partial\chi_p}{\partial x_i}\right)\dot{x}_p+\dfrac{\partial\chi_i}{c\partial t}, \end{cases}$$

where

$$\phi^2={\sum_p}'\dot{x}_p{}^2.$$

If we consider the field of the sun only, which is at rest at the origin of co-ordinates, and for the moment neglect the axial rotation, then we can use the equations (17) of article 9. Taking polar co-ordinates, these become

$$(48)\quad \begin{cases} \dfrac{d^2r}{c^2dt^2}-r\cos^2\phi\left(\dfrac{d\theta}{cdt}\right)^2-r\left(\dfrac{d\phi}{cdt}\right)^2+\frac{1}{2}\dfrac{\partial\gamma}{\partial r} \\ \qquad =\frac{1}{2}\alpha\gamma'+(\gamma'-\frac{1}{2}\alpha')\dot{r}^2+[(\beta-\alpha)r+\frac{1}{2}r^2\beta'](\cos^2\phi\dot{\theta}^2+\dot{\phi}^2), \\ \dfrac{d^2\theta}{c^2dt^2}+\dfrac{2}{r}\dfrac{dr}{cdt}\dfrac{d\theta}{cdt}-2\tan\phi\dfrac{d\theta}{cdt}\dfrac{d\phi}{cdt}+\dfrac{\sec^2\phi}{2r^2}\dfrac{\partial\gamma}{\partial\theta}=(\gamma'-\beta')\dot{r}\dot{\theta}, \\ \dfrac{d^2\phi}{c^2dt^2}+\dfrac{2}{r}\dfrac{dr}{cdt}\dfrac{d\phi}{cdt}+\sin\phi\cos\phi\left(\dfrac{d\theta}{cdt}\right)^2+\dfrac{1}{2r^2}\dfrac{\partial\gamma}{\partial\phi}=(\gamma'-\beta')\dot{r}\dot{\phi}. \end{cases}$$

In the right-hand members we have supposed that α, β, γ are functions of r alone. The left-hand members put equal to zero are the well-known equations of Newtonian mechanics.

From the last equation it appears that, in case γ does not depend on ϕ, ϕ remains zero when at one time $\phi=\dot{\phi}=0$. The orbit is plane. We can thus always take $\phi=0$. The first two equations then are, if we also take $\dfrac{\partial\gamma}{\partial\theta}=0$:—

$$(49)\quad \begin{cases} \ddot{r}-r\dot{\theta}^2+\frac{1}{2}\gamma'=\frac{1}{2}\alpha\gamma'+(\gamma'-\frac{1}{2}\alpha')\dot{r}^2+[(\beta-\alpha)r+\frac{1}{2}r^2\beta']\dot{\theta}^2, \\ \ddot{\theta}+\dfrac{2}{r}\dot{r}\dot{\theta}=(\gamma'-\beta')\dot{r}\dot{\theta}. \end{cases}$$

Now put

$$r^2\dot{\theta}=\mathrm{G}.$$

The second of the equations (49) then is

$$\frac{1}{\mathrm{G}}\frac{d\mathrm{G}}{cdt}=(\gamma'-\beta')\dot{r}=\frac{d}{cdt}(\gamma-\beta)$$

from which

$$(50)\qquad \mathrm{G}=\mathrm{G}_0e^{\gamma-\beta}.$$

This is the integral of areas (Kepler's second law) for the new theory. If we wished to make the area described in the unit of time a constant, we would have to introduce the condition $\gamma=\beta$. We will, however, not investigate the corresponding system of reference, but restrict ourselves to the systems A and B of article 11. For these two systems we will derive the analogue of Kepler's third law for the special case that the orbit is circular. Then $\dot{r}=0$, and the second equation of (49) gives $\dot{\theta}=$ a constant. Then from the first of (49) we have

$$r\dot{\theta}^2[1+\beta-\alpha+\tfrac{1}{2}r\beta'] = \tfrac{1}{2}\gamma'(1-\alpha),$$

or

$$r^3\dot{\theta}^2 = \tfrac{1}{2}r^2\gamma'[1-\beta-\tfrac{1}{2}r\beta'];$$

or, in the more usual astronomical notation,

$$a^3n^2 = \tfrac{1}{2}r^2c^2\gamma'[1-\beta-\tfrac{1}{2}r\beta']. \tag{51}$$

For the two systems A and B we find

$$\left\{\begin{array}{ll} \text{A.} & a^3n^2 = k^2m. \\ \text{B.} & a^3n^2 = k^2m\left(1-3\dfrac{\lambda_0^2}{a}\right). \end{array}\right. \tag{52}$$

Thus in the system A Kepler's third law is exact (for a circular orbit). The transformation-formula (36) transforms A to B.

16. We will now use the system B.* The equations then become

$$\left\{\begin{aligned} &\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 + \frac{1}{r^2} = k^2m\left[\frac{4\lambda_0^2}{r^3} + 3\frac{\dot{r}^2}{r^2} - \dot{\theta}^2\right], \\ &\frac{d^2\theta}{dt^2} + \frac{2}{r}\frac{dr}{dt}\frac{d\theta}{dt} = k^2m\,.\,\frac{4\dot{r}\dot{\theta}}{r^2}. \end{aligned}\right. \tag{53}$$

These equations we treat in the same way as the equations without the right-hand members are usually treated. We first derive the integrals of living force and of areas. These are

$$\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 - k^2m\left(\frac{2}{r} - \frac{1}{a_0}\right) = k^2m\left[\frac{6\lambda_0^2}{a_0r} - \frac{10\lambda_0^2}{r^2}\right], \tag{54}$$

$$r^2\frac{d\theta}{dt} = k\sqrt{m}\sqrt{p_0}\left(1-\frac{4\lambda_0^2}{r}\right), \tag{55}$$

where a_0 and p_0 are constants of integration. We then eliminate the time to get the equation of the orbit. We find

$$\left(\frac{dr}{d\theta}\right)^2 + r^2 - \frac{2r^3}{p_1} + \frac{r^4}{a_0p_0} = 6\lambda_0^2\frac{r^2}{p_1}, \tag{56}$$

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* See also de Sitter, "Planetary Motion and the Motion of the Moon according to Einstein's Theory," *Proceedings of the Academy of Sciences, Amsterdam*, 1916 June.

where

$$p_1 = p_0\left(1 + \frac{\lambda_0^2}{a_0}\right).$$

The integration of (56) gives

$$\frac{1}{r} = \frac{1 + e\cos(g\theta - \varpi)}{p} \tag{57}$$

where

$$g = 1 - \frac{3\lambda_0^2}{p}, \tag{58}$$

and

$$p = g^2 p_1, \quad a = a_0 - \lambda_0^2.$$

The equation (57) shows that the orbit is an ellipse with a moving perihelion. The motion of the perihelion during one period of revolution is

$$2\pi(1 - g) = \frac{3\lambda_0^2}{p} \cdot 2\pi.$$

In the system A the orbit is not an ellipse; the equations corresponding to (56) must be integrated by means of elliptic functions.*

If in (55) we introduce instead of p_0 the final elements a and p, we find

$$r^2\frac{d\theta}{dt} = k\sqrt{m}\sqrt{p}\left(1 - \tfrac{1}{2}\frac{\lambda_0^2}{a} + 3\frac{\lambda_0^2}{p} - 4\frac{\lambda_0^2}{r}\right).$$

For a circle this becomes ($r = p = a$),

$$a^2\frac{d\theta}{dt} = k\sqrt{m}\sqrt{a}\left(1 - \tfrac{3}{2}\frac{\lambda_0^2}{a}\right),$$

which is the same as (52) above.

17. The left-hand members of (48) or (53) put equal to zero are the equations of elliptic motion, if in (48) we take $\gamma = -\dfrac{2k^2m}{c^2r}$. The right-hand members can be considered as perturbing forces. If S and T denote the radial and transversal perturbing forces, the equations are

$$\left\{\begin{aligned} &\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 + \frac{\partial\Omega}{\partial r} = k^2m\mathrm{S}, \\ &\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) + \frac{1}{r}\frac{\partial\Omega}{\partial\theta} = k^2m\mathrm{T}. \end{aligned}\right. \tag{59}$$

We take

$$\gamma = \frac{2\Omega}{c^2} + \gamma_2.$$

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* See Einstein III., and Droste, *Proceedings . . . Amsterdam*, 1916 May.

Then we have, if we take the general case, in which γ also depends on θ and ϕ,

$$(60)\quad \begin{cases} \lambda_0^2 S = -\tfrac{1}{2}\gamma_2' + \tfrac{1}{2}a\dfrac{\partial\gamma}{\partial r} + a\dot{\gamma} - \tfrac{1}{2}a'\dot{r}^2 + [(\beta - a)r + \tfrac{1}{2}r^2\beta']\dot{\theta}^2, \\ \lambda_0^2 T = -\dfrac{1}{2r}\dfrac{\delta\gamma_2}{\partial\theta} + \dfrac{\beta}{2r}\dfrac{\partial\gamma}{\partial\theta} + (\dot{\gamma} - \beta'\dot{r})r\dot{\theta}, \end{cases}$$

where we have put

$$\dot{\gamma} = \frac{\partial\gamma}{\partial r}\dot{r} + \frac{\partial\gamma}{\partial\theta}\dot{\theta} + \frac{\partial\gamma}{\partial\phi}\dot{\phi}.$$

In our case, of course, we have $\dot{\gamma} = \gamma'\dot{r}$, and also $\dfrac{\partial\gamma}{\partial\theta} = 0$. S must be an even function of $\dot{r}$ and $r\dot{\theta}$, and T must be an uneven function of $r\dot{\theta}$. Both are of the first order. The only possible forms are thus

$$S_1 = \frac{r^2\dot{\theta}^2}{r^2}, \quad S_2 = \frac{\dot{r}^2}{r^2}, \quad S_3 = \frac{2\lambda_0^2}{r^3}, \quad T_1 = \frac{\dot{r}r\dot{\theta}}{r^2}.$$

We have for the two systems

$$(61)\quad \begin{cases} \text{A.} \quad S = -2S_1 + 3S_2 + S_3, \quad T = 2T_1. \\ \text{B.} \quad S = -S_1 + 3S_2 + 2S_3, \quad T = 4T_1. \end{cases}$$

The "perturbations" produced by these forces are : *

δa: from S_1: $-\frac{2}{3}\lambda_0^2(1 - e^2)\dfrac{a^3}{r^3}$

S_2: $\frac{2}{3}\lambda_0^2(1 - e^2)\dfrac{a^3}{r^3} - 2\lambda^2\dfrac{a^2}{r^2}e\cos u$

S_3: $-\dfrac{4\lambda_0^2}{(1 - e^2)^2}(e\cos v + \frac{1}{4}e^2\cos 2v)$

T_1: $-\frac{2}{3}\lambda_0^2(1 - e^2)\dfrac{a^3}{r^3}$.

δe : from S_1, S_2, S_3 : $\frac{1}{2}\dfrac{1 - e^2}{ae} \times$ corresponding term in δa

T_1 : $\frac{1}{2}\dfrac{1 - e^2}{ae} \cdot \delta a + \dfrac{\lambda_0^2}{a}\cos v$.

δp : from S_1, S_2, S_3 : *zero*

T_1 : $-2\lambda_0^2 e\cos v$.

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* See also de Sitter, "The Bearing of the Principle of Relativity on Gravitational Astronomy," *M.N.*, 1911, vol. lxxi. pp. 403-404. As there are some misprints on these pages, the formulas are here repeated. What is called S_1 in that paper is $S_1 + S_2$ in the present notation.

$$e\delta\varpi : \text{from } \mathrm{S}_1 : \; -\frac{\lambda_0^2}{p}\left[\sin v + \tfrac{1}{2}e \sin 2v + e^2(\sin v - \tfrac{1}{3}\sin^3 v) + ev\right]$$

$$\mathrm{S}_2 : \; -\frac{\lambda_0^2}{p} \cdot \tfrac{1}{3}e^2 \sin^3 v$$

$$\mathrm{S}_3 : \; -\frac{\lambda_0^2}{p}\left[2\sin v + \tfrac{1}{2}e \sin 2v + ev\right]$$

$$\mathrm{T}_1 : \; \frac{\lambda_0^2}{p}\left[-\tfrac{1}{2}e \sin 2v + \tfrac{1}{3}e^2 \sin^3 v + ev\right]$$

$$\delta\epsilon_1 : \text{from } \mathrm{S}_1 : \; -\frac{2\lambda_0^2}{a\sqrt{1-e^2}}(v + e \sin v)$$

$$\mathrm{S}_2 : \; \frac{2\lambda_0^2}{a}\left(u - \frac{v}{\sqrt{1-e^2}}\right) + \frac{2\lambda_0^2}{a\sqrt{1-e^2}}\, e \sin v$$

$$\mathrm{S}_3 : \; -\frac{4\lambda_0^2}{a\sqrt{1-e^2}}v$$

T_1 : *zero.*

In these formulas u is the excentric and v the true anomaly. The formulas are not developments in powers of e, but are rigorous for all values of e. We have put

$$\delta\epsilon = \delta\epsilon_1 + \frac{e^2}{1+\sqrt{1-e^2}}\delta\varpi.$$

The only secular perturbation is a motion of the perihelion. This is for the two systems

$$\text{A.} \quad \delta\varpi = (+2 - 1 + 2)\frac{\lambda_0^2}{p} \cdot v = 3\frac{\lambda_0^2}{p} \cdot v,$$

$$\text{B.} \quad \delta\varpi = (+1 - 2 + 4)\frac{\lambda_0^2}{p} \cdot v = 3\frac{\lambda_0^2}{p} \cdot v,$$

the same for both systems and the same as has been found in the preceding article by a different reasoning.

The "periodic perturbations" all have periods which are submultiples of the period of revolution. They will for the greater part be absorbed in elliptic motion by a slight alteration of the meaning of the elements a, e, etc. The remainder will, of course, be different for the two systems, but by the formula of transformation (36) the two perturbed orbits must become identical. To verify this would require rather long developments, which we will take for granted. All these periodic terms are, of course, extremely small, and far beyond the reach of verification by the observations.

18. Finally, we will consider the effect of the axial rotation of the sun. We have then to add to the previously used values of g_{ij}

$$g_{24} = \frac{b\cos^2\phi}{r}.$$

By the equations (47) we find easily that we must add to the right-hand members of (48) the terms

$$\frac{b\cos^2\phi}{r^2}\dot{\theta}, \qquad -\frac{b}{r^4}\dot{r} - \frac{2b\tan\phi}{r^3}\dot{\phi}, \qquad \frac{b\sin 2\phi}{r^3}\dot{\theta}.$$

Take a planet moving in the equatorial plane of the sun. Then $\phi = \dot{\phi} = 0$, and we have to add to the perturbing forces

$$\lambda_0^2\delta S = \frac{b\dot{\theta}}{r^2}, \quad \lambda_0^2\delta T = -\frac{b\dot{r}}{r^3}.$$

For $\dot{r}$ and $\dot{\theta}$ we can use the unperturbed values $\dot{r} = \frac{\lambda_0 e\sin v}{\sqrt{p}}$, $r^2\dot{\theta} = \lambda_0\sqrt{p}$. Thus we find, substituting the value of b, and remembering that $\lambda_0^2 = \lambda^2 m$

$$\delta S = \frac{b\sqrt{p}}{\lambda_0 r^4} = \tfrac{4}{3}\frac{\lambda P\omega}{m}\frac{\sqrt{p}}{r^4} = \frac{h}{r^4}$$

$$\delta T = -\frac{be\sin v}{\lambda_0 r^3\sqrt{p}} = -\frac{he\sin v}{pr^3}.$$

The effect on the elements includes a secular motion of the perihelion, which is given by

$$ed\varpi = -\frac{h}{p^2}[(1+e)\cos v + 2e]dv,$$

from which

$$\delta_1\varpi = -\frac{2h}{p^2}v. \tag{62}$$

We can take $\lambda_0 = a^{\frac{3}{2}}\nu$, where $\nu = n/c$. Thus we have

$$\delta_1\varpi = -\tfrac{8}{3}\lambda_0^2\frac{\omega}{\nu}\cdot\frac{P}{m}\cdot\frac{1}{a^3(1-e^2)^{\frac{3}{2}}}\cdot v.$$

The ratio of this to the motion of the perihelion found above is, if we take $\frac{P}{m} = \frac{3}{5}R^2$ (*i.e.* for a homogeneous sun),

$$\frac{\delta_1\varpi}{\delta\varpi} = -\frac{8}{15}\frac{\omega}{\nu}\frac{R^2}{a^2\sqrt{1-e^2}},$$

which is a very small fraction. For Mercury it is 0·000262. We can thus entirely neglect the effect of the axial rotation of the sun.

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19. The only effect on planetary motion, which comes at all within the reach of verification by the observations, is the secular motion of the perihelia. These are, per century :—

	$d\tilde{\varpi}$.	$e d\tilde{\varpi}$.
Mercury	+42″·9	+8″·82
Venus	+ 8·6	+0·05
Earth	+ 3·8	+0·07
Mars	$+ 1{\cdot}3^{5}$	+0·13

I adopt the theoretical values corresponding to Newton's theory from Newcomb.* The observed values I also take from Newcomb. As given there, they imply the value of the constant of precession used by him in their derivation, *i.e.* 5023″·71 (for 1850·0). I have adopted 5024″·90. The comparison, then, stands thus :

		Observed.	Theory.	Difference.	Differences as given by Newcomb.
Mercury	$e d\tilde{\varpi}$	+118″·00	+118″·58	−0″·58±0″·43	+8″·48
	$i d\Omega$	− 92·04	− 92·50	+0·46± ·52	+0·61
Venus	$e d\tilde{\varpi}$	+ 0·28	+ 0·39	−0·11± ·25	−0·05
	$i d\Omega$	−105·47	−106·00	+0·53± ·17	+0·60
Earth	$e d\tilde{\varpi}$	+ 19·46	+ 19·46	0·00± ·13	+0·10
Mars	$e d\tilde{\varpi}$	+149·44	+148·93	+0·51± ·35	+0·75
	$i d\Omega$	− 72·64	− 72·63	−0·01± ·22	+0·03

The mean errors have been adopted from Newcomb. The differences, as found by Newcomb, are added for comparison. Though some of the differences between the observed values and those given by the new theory still exceed their mean errors, the agreement is satisfactory on the whole. Only the node of Venus still shows a considerable discrepancy. The differences have no tendency to show the same sign; there is thus not the slightest reason to adopt a rotation of the system of the fixed stars. Also Seeliger's explanation of the anomalous motion of the perihelion of Mercury by the attraction of nebulous matter in the neighbourhood of the sun now becomes superfluous. The node of Venus, of course, remains outstanding, but none of the hypotheses put forward in explanation of the anomalies in the motions of the inner planets can put it right without at the same time introducing greater discrepancies in other elements.

Domburg :
1916 *August.*

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* *Astronomical Constants*, p. 109. See also de Sitter, "The Secular Variations of the Elements of the four Inner Planets," *Observatory*, 1913 July, p. 290.