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RADIATION THROUGH A FOGGY ATMOSPHERE

By ARTHUR SCHUSTER

1. In discussing the transmission of light through a mass of gas, it is usual to consider only the effects of emission and absorption, and to neglect all effects of scattering. But when the absorbing mass holds fine particles of matter in suspension, the scattered light materially affects the character of the transmitted radiation. I propose to discuss the conditions under which "bright line" spectra or "dark line" spectra may be obtained from a radiating mass of gas, taking account of scattering. I call an atmosphere "foggy" when scattering takes place to an appreciable extent. The applications of the results of this investigation are, however, much wider than the title chosen would seem to imply, for there is some scattering even from the molecules of a homogeneous substance, and to that extent all bodies fall within the definition and may be called "foggy."

According to the investigations of Lord Rayleigh, the greater part of the light we receive from the sky is due to light scattered by the molecules of the air. This involves a diminution in the intensity of the direct rays amounting in our atmosphere to roughly 5 per cent. The effective thickness of stellar atmospheres may be great compared with that of the shell of air which surrounds our globe, and hence the effects of scattering may be of primary importance in interpreting the nature of stellar atmospheres.

2. The following notation will be used:

E = the total energy of radiation within a certain small range of wave-lengths sent out by unit surface of a completely black surface. E is a function of the temperature and wave-length.

S = the total energy of radiation incident within the same limit of wave-lengths on unit surface of a plane layer of the foggy gas.

R = the energy which leaves the plane layer per unit surface.

R_o = the particular value of R for the case that there is no absorption.

R_c = the particular value of R for rays which are completely absorbed by an infinitely thin layer.

κ = the coefficient of absorption.

s = the coefficient of scattering.

The object of the investigation is to determine R in terms of S and E, if E refers to the temperature of the foggy gas. It will save needless repetition if it is understood once for all that our statements always refer to unit surface of the radiating or absorbing layer.

κ is a function of the wave-length which also depends on the density of the medium. If the medium is uniform, all molecules absorbing alike, κ would be proportional to the density. But in a mixture of different gases, κ must be considered proportional only to the quantity (measured per unit volume) of the particular substance which absorbs the wave-length in question. Similarly s depends on the number of the scattering particles, the scattering and absorbing particles not necessarily being of the same nature. If the scattering is of the nature of that which causes the blue color of the sky, the value of s varies inversely as the fourth power of λ, but in case of an ordinary cloud or mist, the dependence on wave-length is much less marked.

If S be the total intensity of radiation incident on a layer of small thickness dx, the radiation absorbed by the layer is $\kappa S dx$. The light emitted by the same layer in each direction is thus $\kappa E dx$. This follows from the law connecting absorption and radiation.

The light scattered by the layer is $sSdx$, of which one-half is sent forward and one-half returned backward.

The following variables will be introduced for convenience of expression:

$$\beta = \kappa/s \ ,$$

$$\alpha = \sqrt{\kappa/(\kappa+s)} = \sqrt{\beta/(1+\beta)} \ ,$$

$$\therefore \beta = \alpha^2/(1-\alpha^2) \ .$$

β varies from zero to infinity, but a must lie between zero and unity.

$$\gamma=(1+a)/(1-a)\ ,$$
$$\therefore\gamma=1+2\beta+\sqrt{\beta+\beta^2}\ .$$

3. Let (Fig. 1) S_1S_2 be a surface sending out the radiation S, and let this radiation after passing through part of the foggy atmosphere be reduced to a value A and fall on a thin layer of thickness dx. The effect of the layer is to absorb energy amounting to $\kappa A dx$, and additionally to reduce the incident light by a quantity $sAdx$, which is not absorbed, but sent in equal amounts backward and forward as scattered light. If the stream of radiant energy in the opposite direction is B, we have similarly a diminution of energy equal to $(\kappa+s)\ B$, of which, however, $\frac{1}{2}sB$ is sent both forward and backward as scattered light. The layer also radiates energy in both directions equal to $\kappa E dx$. Collecting these effects, we obtain the equations:

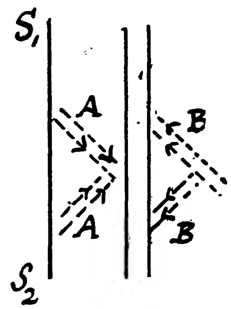

Fig. 1.

$$\frac{dA}{dx}=\kappa(E-A)+\tfrac{1}{2}s(B-A) \tag{1}$$

$$\frac{dB}{dx}=\kappa(B-E)+\tfrac{1}{2}s(B-A)\ . \tag{2}$$

Combining (1) and (2) we find:

$$\frac{d(A+B)}{dx}=(\kappa+s)(B-A) \tag{3}$$

$$\frac{d(A-B)}{dx}=2\kappa E-\kappa(A+B)\ . \tag{4}$$

Differentiating (3) and with the help of (4)

$$\frac{d^2(A+B)}{dx^2}=\kappa(\kappa+s)(A+B-2E)\ . \tag{5}$$

If E is constant or varies uniformly with x, the last equation may be integrated, and we derive:

$$(A+B-2E)=Ke^{(\kappa+s)ax}+K_1e^{-(\kappa+s)ax} \tag{6}$$

where K and K_1 are two constants and a has the value assigned to it in § 2.

If the temperature of the medium is constant, so that E has the same value throughout the scattering medium, differentiation gives

$$\frac{d(A+B)}{dx}=a(\kappa+s)(K_{\mathrm{I}}e^{(\kappa+s)ax}-K_{\mathrm{I}}e^{-(\kappa+s)ax}) \tag{7}$$

and hence by introducing (3)

$$B-A=a(Ke^{(\kappa+s)ax}-K_{\mathrm{I}}e^{-(\kappa+s)ax}) \ . \tag{8}$$

Equations (6) and (8) now allow us to obtain A and B separately, and we thus find:

$$\left.\begin{array}{l} 2A=2E+(\mathrm{I}-a)Ke^{(\kappa+s)ax}+(\mathrm{I}+a)K_{\mathrm{I}}e^{-(\kappa+s)ax} \\ 2B=2E+(\mathrm{I}+a)Ke^{(\kappa+s)ax}+(\mathrm{I}-a)K_{\mathrm{I}}e^{-(\kappa+s)ax} \end{array}\right\} \ . \tag{9}$$

We consider x to be measured from the front surface of the foggy medium in the direction in which the radiation A proceeds. If no radiation enters the medium from the opposite direction, and if the radiation incident in the first absorbing layer be S, we have the conditions:

$$\left.\begin{array}{ll} \text{for } x=0; & B=0 \\ \text{for } x=-t; & A=S \end{array}\right\} \tag{10}$$

the thickness of the medium being denoted by t.

We require to determine the emergent radiation which is equal to the value which A acquires when $x=0$. Denoting this by R, we have from the first of equations (9)

$$2R=2E+(\mathrm{I}-a)K+(\mathrm{I}+a)K_{\mathrm{I}} \ . \tag{11}$$

Introducing (10) into (9) allows us to determine K and K_{I}. We obtain in the first place the equations

$$0=2E+(\mathrm{I}+a)K+(\mathrm{I}-a)K_{\mathrm{I}}$$

$$2S=2E+(\mathrm{I}-a)Ke^{-a(\kappa+s)t}+(\mathrm{I}+a)K_{\mathrm{I}}e^{a(\kappa+s)t} \ ,$$

and these give

$$K=\frac{2\left[(\mathrm{I}-a)-(\mathrm{I}+a)e^{a(\kappa+s)t}\right]E-2(\mathrm{I}-a)S}{(\mathrm{I}+a)^2e^{a(\kappa+s)t}-(\mathrm{I}-a)^2e^{-a(\kappa+s)t}} \ ,$$

$$K_{\mathrm{I}}=\frac{2\left[(\mathrm{I}-a)e^{-a(\kappa+s)t}-(\mathrm{I}+a)\right]E+2(\mathrm{I}+a)S}{(\mathrm{I}+a)^2e^{a(\kappa+s)t}-(\mathrm{I}-a)^2e^{-a(\kappa+s)t}} \ .$$

Finally by substitution into (11)

$$R=2a\frac{\left[(\mathrm{I}+a)e^{a(\kappa+s)t}+(\mathrm{I}-a)e^{-a(\kappa+s)t}\right]E+2(S-E)}{(\mathrm{I}+a)^2e^{a(\kappa+s)t}-(\mathrm{I}-a)^2e^{-a(\kappa+s)t}} \ . \tag{12}$$

Equation (12) contains the solution of our problem.

4. The equations of the last paragraph have been deduced under the assumption that the radiation throughout the absorbing mass is uniformly distributed in such a way that it does not depend on the angle between any direction considered and the normal drawn toward the same side. This supposition is obviously incorrect, for it appears that, even if it were to hold at any surface, e. g., the first surface of the layer dx (Fig. 1), absorption in that layer would destroy the uniformity owing to the greater absorption which the oblique rays suffer. To some extent the effect of scattering would act in the sense of partly restoring the equality of distribution; nevertheless serious errors might be introduced, if we attempted to obtain accurate values of κ and s by means of the application of equation (12). The complete investigation leads to equations of such complexity that a discussion becomes impossible, and I shall only use the solution obtained under the simplified conditions to deduce certain consequences which cannot be affected by the assumption made. The error committed might be allowed for by taking s and κ to be functions of the distance. When considered in this light, it is seen how useless the more complete calculation would be, because in the more important cases to which we have to apply our results, the coefficients of scattering and absorption vary in an unknown manner, and the error committed by the simplification of this problem becomes merged in other unavo dable uncertainties.

5. Before discussing the general results contained in equation (12) we may treat separately of some simple special cases. When the coefficient of absorption, and consequently a, is zero, we require to express the exponentials of (12) in a series, the first two terms being retained. But it is easier in this case to proceed directly. Equations (3) and (4) in this case become

$$\frac{d(A+B)}{dx}=s(B-A) \ ,$$

$$\frac{d(A-B)}{dx}=0 \ .$$

The second equation shows that $A-B$ is a constant which must be equal to R_0, the value of A at the front surface. The first equation may now be integrated, and gives

$$A+B=a-sR_0x \ .$$

As for $x=0$, $B=0$, and $A=R_0$, it follows that $a=R_0$; or replacing B by $A-R$,

$$2(A-R_0)=-sR_0x \ .$$

When $x=-t$, the value of A is equal to S, the incident radiation, hence

$$2(S-R_0)=sA_0t \ ;$$

or finally:

$$R_0=\frac{2}{2+st}S \ . \tag{13}$$

The equation shows that the emergent radiation diminishes with increasing thickness, but not so quickly as it would do if scattering acted in the same manner as absorption. If, for instance, we give to st the numerical value of ninety-eight, so that the emergent light is 2 per cent. of the incident light, doubling the layer would still give us 1 per cent. for the transmitted light, and with greater thicknesses the light would, roughly speaking, be inversely proportional to the thickness. But in the case of absorption, the double layer would only transmit 2 per cent. of 2 per cent., and the transmitted light would diminish in a geometric ratio, while the thickness increases in an arithmetic ratio.

6. When either st or κt is so large that practically no part of the original light is transmitted, we may neglect in (12) all terms except those containing an exponential with a positive argument, and this gives at once

$$R_c=\frac{2\alpha}{1+\alpha}E \tag{14}$$

When κ is large compared with s, α approaches unity and ultimately $R_c=E$. The radiation in that case becomes equal to that of a completely black surface, which agrees with the well-known law that absorption irrespective of scattering tends to make the radiation of all bodies equal to that of a black body when the thickness is increased.

But, as has been mentioned, scattering always exists, and has to be taken into account. It appears from the definition of α that it is always a fraction, and hence the factor of E in (14) is always smaller than one. It follows that the emergent radiation increases with the value of κ, and hence a luminous gas always gives a spectrum of bright lines, and does not approach with increasing thickness to the radiation from a black body, as it would do in the absence of scattering.

7. It is not possible to discuss equation (12) in its general form. In order to draw the appropriate conclusions in certain typical cases, we introduce other variables.

Put

$$e^{st}=r\ ;\qquad \beta=\kappa/s\ ,$$

and introduce a quantity γ defined by

$$\gamma=\frac{1+a}{1-a}\ .$$

We have then

$$a(\kappa+s)t=st(1+\beta)a\ ;$$

and also

$$a=\sqrt{\beta/(1+\beta)}\ .$$

Hence

$$\gamma=1+2\beta+2\sqrt{\beta+\beta^2} \tag{15}$$

$$\frac{2a}{1-a}=\gamma-1\ ,$$

$$\frac{2a(1+a)}{(1-a)^2}=\frac{1+a}{1-a}\cdot\frac{2a}{1-a}=\gamma(\gamma-1)\ ,$$

$$\frac{4a}{(1-a)^2}=\gamma^2-1\ .$$

Equation (12) now becomes

$$R=(\gamma-1)\frac{(\gamma r^{\sqrt{\beta+\beta^2}}+r^{-\sqrt{\beta+\beta^2}})E-(\gamma+1)(E-S)}{\gamma^2 r^{\sqrt{\beta+\beta^2}}-r^{-\sqrt{\beta+\beta^2}}} \tag{16}$$

As (15) gives γ in terms of β, all factors of E and S are now expressed in terms of β and st. I have carried out the calculations for the three cases that st is equal to $\frac{1}{2}$, 1, and 2, respectively, and for a number of different values of β. If we calculate the coefficient in (16) and write it (16) in the form

$$R=aE+bS\ ,$$

Table I gives the coefficients of a and b.

TABLE I

		0.1	0.8	1	1.2	2	10
st = 0.5	a	0.0486	0.3258	0.3872	0.4428	0.6165	0.9762
	b	0.7604	0.5333	0.4820	0.4355	0.2911	0
st = 1	a	0.0944	0.5276	0.6019	0.6626	0.8142	0.9762
	b	0.6000	0.2902	0.2364	0.1926	0.0855	0
st = 2	a	0.1760	0.7147	0.7716	0.8111	0.8895	0.9762
	b	0.3971	0.0872	0.0574	0.0379	0.0074	0

The relative intensities of the dark and bright lines as they would appear in the special cases considered will be most easily under-

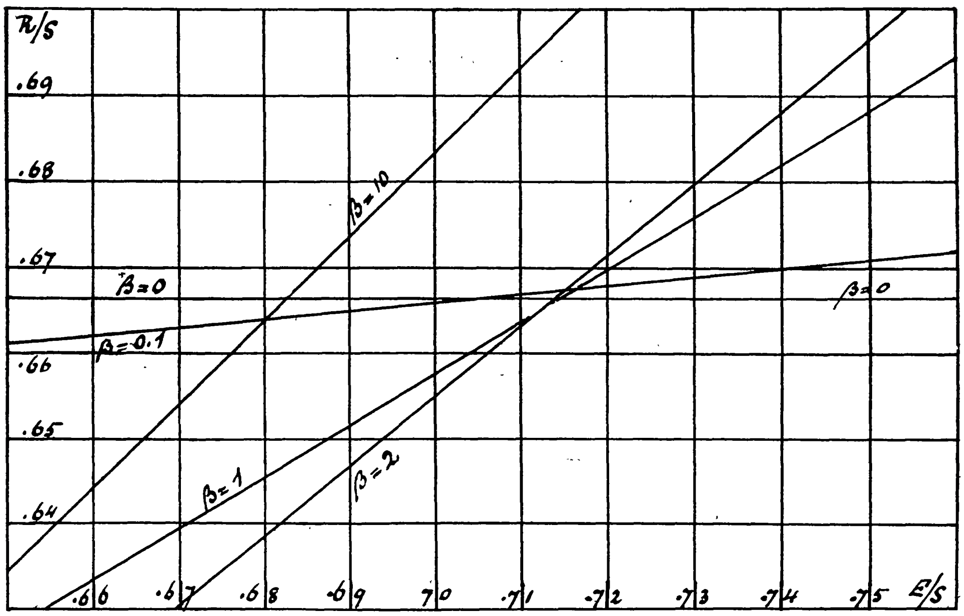

FIG. 2. $st=1$.

stood if straight lines are drawn (Fig. 2) with E/S as abscissæ and R/S as ordinates. That figure refers to the case $st=1$. The horizontal line marked $\beta=0$ gives R_0/S, which defines the intensity of the transmitted light when there is no absorption. The value of R_0/S is obtained from (13), which shows that for $st=1$, only two-thirds of the incident light traverses the scattering medium. If this medium is capable of sending out any vibration defined as regards radiative power by the fraction $\beta=\kappa/s$, the corresponding intensity may be obtained from the curve by taking on the horizontal axis the magnitude E/S, which is the intensity of the black radiation at the temperature of the absorbing and scattering medium in terms of that of the incident light. The corresponding ordinate gives the transmitted light in terms of the same unit. If the point of the straight line corresponding to any particular value of β lies above the horizontal line marked $\beta=0$, the appearance will be that of a bright line; if it lies below, a dark line would be observed. Starting with a comparatively low temperature of the foggy medium and gradually increasing it (i. e., gradually increasing the ordinates), it is seen that at first all homogeneous vibrations appear as dark lines, and if the temperature is sufficiently low (not shown in figure) the highest values of β give the greatest deficiency of light. This is in accordance with what takes place in the absence of scattering.

When the temperature is gradually raised, the most intense line represented in the figure ($\beta=10$) ceases to be the darkest line, and ultimately when E/S is about 0.682, this line becomes brighter than the background. The next line to change from darkness to brightness is the line of lowest intensity $\beta=0.1$, and when E/S is more than 0.715, all the lines are bright. The change from dark to bright lines takes place within a comparatively small range of temperature; nevertheless the possibility of the simultaneous appearance of dark and bright lines according to the intensity of absorption is shown by the figure. If, instead of a homogeneous line, we contemplate the case of narrow bands such as frequently occur, we must consider β to have a maximum value at the center of the band (e. g., $\beta=10$) and to fall off on either side more or less rapidly to zero. At very low temperatures of the medium, the center of the band in this case would be darkest and at high temperatures brightest. But intermediate temperatures would give the appearance of a bright central line on a dark absorption band. Thus at a temperature of 0.69, the brightness of the center ($\beta=10$) in terms of the intensity of the transmitted light is 0.674, which means that it is 1.2 per cent. brighter than the background, while towards both sides, where β has fallen to 2, the intensity is 0.637 or 4.5 per cent. darker than the background. The appearance is therefore that of an absorption band with a reversed line in the center.

Fig. 3 gives the diagram of radiation for $st = 0.5$. It is drawn to a somewhat larger scale than Fig. 2.

Fig. 4 represents the connection between transmitted light and coefficient of absorption when $st = 2$. In this case the unabsorbed radiation is scattered to the extent that only half the incident light is transmitted. The possibility of the simultaneous appearance of dark and bright lines, which carries with it the possibility of an absorption band with a reversed line at the center, is increased in

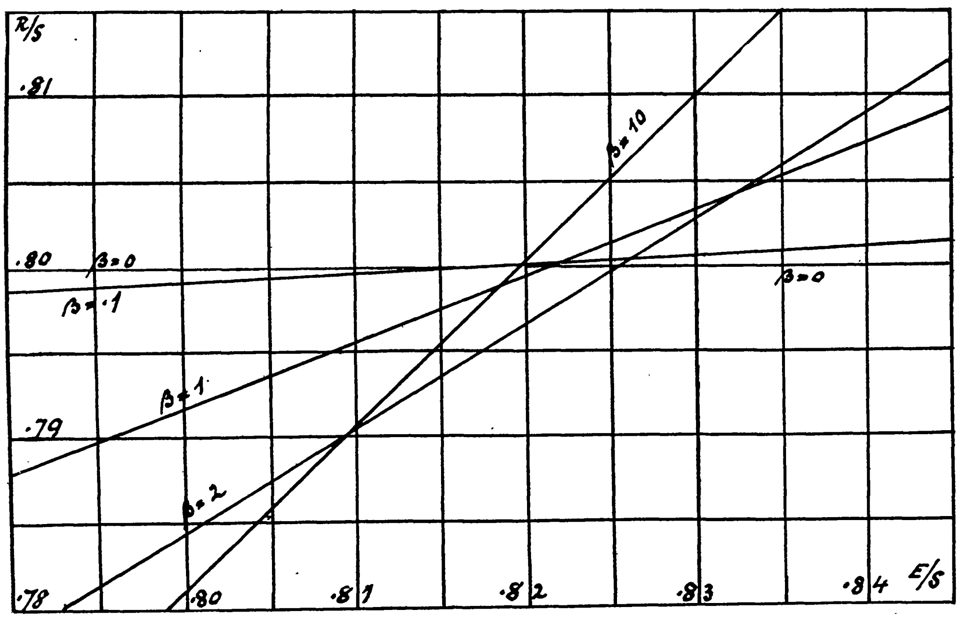

FIG. 3. $st = 0.5$.

this case. Thus when E/S is 0.54, a line defined by $\beta = 10$, shows an increase in brightness over the background $[(R - R_0)/R_0]$ of 5.4 per cent. and a weaker line ($\beta = 1$) gives a deficiency of light of 5.2 per cent.

Table II gives the values of E/S at which the transmitted radiation corresponding to different values of β is equal to that of the transmitted unabsorbed radiation ($\beta = 0$). The numbers given define the temperatures of the absorbing medium at which the transition from the dark to the bright lines takes place.

Table III gives in terms of R_0 the intensities of the radiations when the temperature of the absorbing layer is the same as that of the background, the incident light S being in this case considered to emanate from a black body. The table shows the importance of the effects of scattering on the production of bright line spectra; for, neglecting this scattering, all the numbers would be equal to unity, and we should only obtain the continuous spectrum of the background, the medium not affecting the radiation at all.

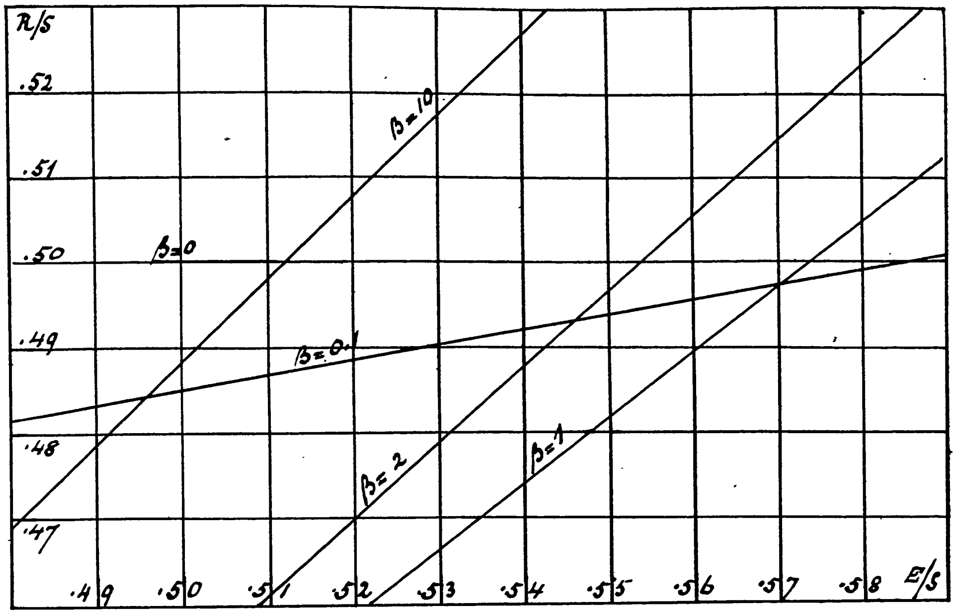

FIG. 4. $st=2$.

TABLE II

	$\beta=$	0.1	0.8	1	1.2	2	10
$st=0.5$...	$\frac{0.8-a}{b}$	0.8149	0.8185	0.8213	0.8231	0.8254	0.8196
$st=1$....	$\frac{0.6667-a}{b}$	0.7065	0.7137	0.7150	0.7155	0.7139	0.6831
$st=2$....	$\frac{0.5-a}{b}$	0.5845	0.5776	0.5736	0.5697	0.5539	0.5123

TABLE III

	$\beta=$	0.1	0.8	1	1.2	2	10
$st=0.5$...	$\frac{a+b}{0.8}$	1.011	1.074	1.086	1.098	1.134	1.220
$st=1$....	$\frac{3}{2}(a+b)$	1.042	1.227	1.257	1.283	1.350	1.464
$st=2$....	$2(a+b)$	1.146	1.604	1.658	1.698	1.794	1.952

8. In the problem as it is usually considered, absorption is introduced by having a cooler medium of uniform temperature in front of a hotter one, and the change of temperature is taken to be an abrupt one. But in considering the phenomena of radiation presented to us by celestial bodies, we must bear in mind that no such discontinuous variation in temperature is admissible. It seems therefore desirable to discuss, even if only in a very simple case, the radiation emitted by a gas of continuously varying temperature.

Equation (6) holds when the temperature variation of the medium is such that the radiation of a black body for the particular wave-length considered varies uniformly with x. We write now for the equation defining the temperature of the medium: $E=f-ux$, where f represents the radiation of a black body which is at the temperature of the external surface of the medium, and u being positive, the temperature increases toward the inside. By differentiation of (6) and the application of (3) we then obtain

$$(B-A)=-\frac{2u}{\kappa+s}+K\alpha e^{(\kappa+s)\alpha x}-K_1\alpha e^{-(\kappa+s)\alpha x}$$

Combining this with (6) we find

$$2A=2(f-ux)+\frac{2u}{\kappa+s}+K(1-\alpha)e^{(\kappa+s)\alpha x}+K_1(1+\alpha)e^{-(\kappa+s)\alpha x} \qquad (17)$$

At a certain distance, for which we may put $x=-t$, we may imagine a radiating black surface to be placed, having a temperature equal to that of the medium at that surface. The incident radiation A must therefore here coincide with the radiation E.
This gives

$$A=f+ut \ .$$

If t is very large, we find by applying (17) to this distant layer

$$0=\frac{2u}{\kappa+s}+K_1\alpha e^{(\kappa+s)\alpha t} \ ,$$

which gives for K_1 negligible values when t is sufficiently large.
Putting $x=0$ in (17), we now obtain

$$2R=2f+\frac{2u}{\kappa+s}+K(1-\alpha) \ .$$

If we apply (6) to the case of $x=0$ when $B=0$, $A=R$, $E=f$, we also find, neglecting K_1,

$$R=2f+K \ .$$

Hence, eliminating K,

$$(1+a)R=2af+\frac{2u}{(\kappa+s)} \quad . \tag{18}$$

The character of the radiation is seen from this equation to depend on the relative values of the radiation-gradient and the coefficient of scattering. In order to exemplify the conditions which regulate the nature of the spectrum, draw a plane through the medium at a distance t behind the front surface. Choose t to be such that, owing to scattering and independently of absorption, only 0.8 of the light incident on the layer passes through it. This gives $st=1$. Let $(1+m)f$ be the radiation passing through the plane at a distance t from the front surface. The temperature of the scattering medium varies therefore in such a manner that the radiation of a black body increases in the ratio $(1+m):1$, when $t=1/S$. As the radiation generally is $f-ux$, we have, when $st=1$,

$$f+ut=(1+m)f \ ,$$

or

$$ut=mf \ .$$

If in the second term of (18) we multiply numerator and denominator by t and substitute $st=1$, $\kappa t=\kappa/s=\beta$, we find

$$R=2f\left(\frac{a}{1+a}+\frac{m}{(1+\beta)(1+a)}\right) \ ,$$

$$=2f\left[\frac{a}{1+a}+m(1-a)\right] \tag{19}$$

It remains to discuss this equation. For $\beta=0$,

$$R_0=2mf$$

and this may be taken to be the intensity of the continuous background of the spectrum. If the radiative power of any homogeneous radiation is very large, so that β is very large,

$$R_c=f \ .$$

The lines with large radiative power appear therefore bright or dark according as m is less or greater than one-half.

Differentiating (19) with respect to a,

$$\frac{dR}{da}=2f\left[\frac{1}{(1+a)^2}-m\right] \ ,$$

we see that as a increases from zero, the radiation increases or diminishes according as m is smaller or greater than one. A turning-point is reached when

$$m=\frac{1}{(1+a)^2} ,$$

or

$$a=\frac{1}{\sqrt{m}}-1 \tag{20}$$

As a is necessarily positive and smaller than one, this turning-point has a meaning applicable to our problem only when m is greater than one-quarter and smaller than unity. A maximum of radiation is reached in this case, and for values of the coefficient of absorption greater than those defined by (20) the radiation diminishes again. The radiation reaches the same value it has for $a=0$, when

$$R=R_0 ,$$

or

$$m=\frac{a}{1+a}+m(1-a) ,$$

or

$$m=\frac{1}{1+a} .$$

As a is a positive fraction, it follows that in order that $R=R_0$ for a second value of a, it is necessary that m should be greater than one-half.

When there is a maximum, its value is easily found to be

$$R=2f[m+(1-\sqrt{m})^2] .$$

We may summarize our results thus:

Case I: $m<\frac{1}{4}$.—With increasing coefficient of absorption, the radiation increases. All homogeneous vibrations appear as bright lines. The brightness of the background ($\kappa=0$) is given by

$$R_0=2mf ;$$

that of the brightest line

$$R_c=f .$$

Case II: $\frac{1}{4}<m<\frac{1}{2}$.—With increasing coefficient of absorption the radiation increases and reaches a maximum when

$$a=\frac{1}{\sqrt{m}}-1 .$$

For greater values of α, the radiation diminishes. All lines are bright, but the lines with the greatest coefficient of absorption are not the brightest. Thus when $m=\frac{1}{2}$, the maximum radiation takes place when $\alpha=0.414$ ($\beta=0.207$), and gives an intensity of 1.172 f, while for infinite values of κ the intensity is f.

Case III: $\frac{1}{2}<m<1$.—The intensity rises to a maximum as in Case II, then diminishes until, when $\alpha=\frac{1-m}{m}$, the radiation has the

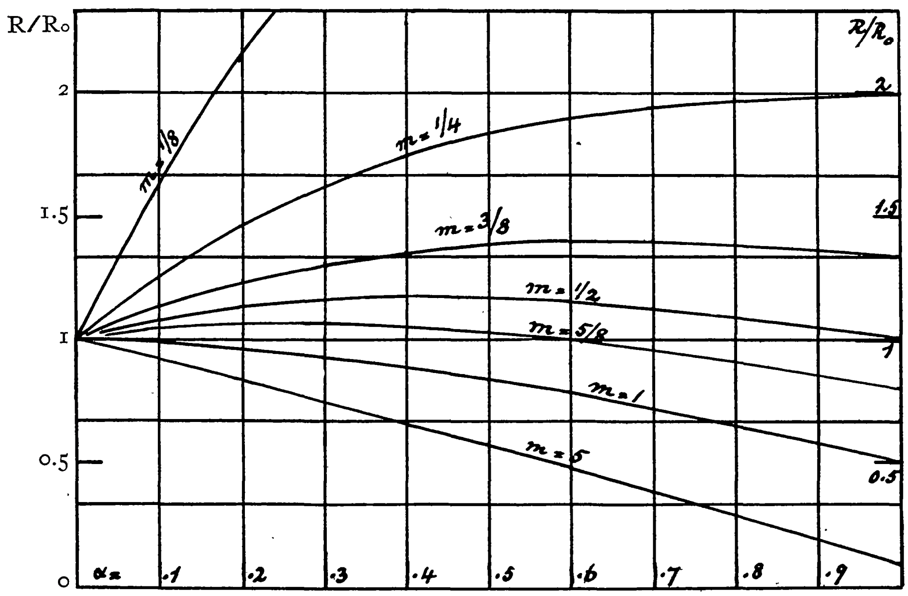

FIG. 5.

same value as when $\alpha=0$. After this point it diminishes. Homogeneous radiations will in this case appear as bright lines when $\alpha>\frac{1-m}{m}$ and as dark lines for greater values of α.

Case IV: $m>1$.—The intensity diminishes continuously with increasing coefficient of absorption. All homogeneous lines appear as dark lines. In Fig. 5 I have drawn the curves which give the intensity of radiation in terms of R_0. The abscissæ represent α and the ordinates:

$$\frac{\alpha}{m(1+\alpha)}+(1-\alpha) .$$

Table IV gives the corresponding values of α and β.

TABLE IV

α	β	α	β
0.1	0.0101	0.6	0.5625
0.2	0.0417	0.7	0.9608
0.3	0.0989	0.8	1.7778
0.4	0.1905	0.9	4.2632
0.5	0.3333	1.0	∞

In applying the results obtained, it should be remembered that m defines the "radiation-gradient," which depends not only on the "temperature-gradient," but also on the wave-length. The same increase in temperature will cause a greater radiation-gradient at the violet end than at the red end of the spectrum, and at comparatively low temperatures the radiation-gradient may in the violet be enormously larger than the temperature-gradient. Hence we conclude that with moderate increases of temperature toward the inside of a gas, the lines of a spectrum which have a shorter wave-length are much less likely to be bright than those of longer wave-length.

9. It may help the reader to draw the proper inferences from the preceding results if an elementary demonstration is given showing how bright line spectra are formed in an infinitely thick layer of incandescent gas, which, when the temperature is uniform, should, according to the ordinary theory, give a spectrum identical with that of a black body at the same temperature. In Fig. 6 let an observer look from E at a mass of gas giving out two homogeneous radiations λ_1 and λ_2, differing very largely in their emissive and absorptive powers κ_1, and κ_2 The wave-length λ_1 being that for which the emissive power is great, will chiefly be due to the radiation of a thin layer $L_1K_1K_2L_2$, because waves of the same wave-length coming from behind will be strongly absorbed by it. On the other hand, the vibrations of small emissive, and therefore small absorptive power will be due to the radiation of a much thicker layer. Neglecting in the first instance the loss of light due to scattering, we may draw H_1H_2 so that the layer $L_1H_1H_2L_2$ sends out a total intensity of waves, λ_2 representing the same fraction of the radiation of a black body of the same temperatures as does the layer $L_1K_1K_2L_2$ for

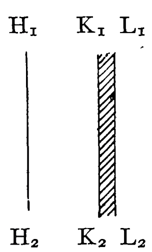

FIG. 6.

the wave-length λ_1. It is well known that whatever be the emissive powers, an indefinite increase in thickness will ultimately give the radiation of a black body. Now, introduce scattering in addition to the absorption. The wave-length λ_1 will not be much affected by scattering, as the light which leaves the gas has only traversed a small thickness of it. On the other hand, the wave-length for which the emissive and absorptive powers are small, being due to the radiation of a thick layer, will be much more weakened by the light scattered, and returned backward. Hence while, in spite of the scattering, λ_1 still shows an intensity nearly equal to that of the black body, the intensity of λ_2 is less. Consequently the radiation will be the stronger, the stronger the emissive power, and hence the gas gives a bright-line spectrum.

In this reasoning it has been supposed that the scattering is of the nature of that due to small bodies and is not a phenomenon primarily dependent on absorption. In the latter case it might be argued that the scattering might be stronger for the wave-length λ_1 in the same proportion as the emissive power is stronger. It is quite possible that a portion of the molecular scattering is selective in character, and, so far as this portion is concerned, our investigation does not apply, without more detailed consideration.

10. I may, in conclusion, briefly indicate the bearing of the results obtained on some problems of astrophysics. It has been shown that a spectrum of bright lines may be given by a mass of luminous gas, even if that gas is of great thickness. There is therefore no difficulty in explaining the existence of stars giving bright lines. The essential criterion which separates the bright-line emission from the dark-line absorption lies in the temperature-gradient of the luminous gas. If the increase of temperature toward the inside of a star is small, bright lines will appear, while the absorption spectra observed in the majority of cases accompany a more rapid variation of temperature. The temperature-gradient is chiefly regulated by the gravitational force, and a star in the early stages of condensation will therefore be in the condition most favorable for the bright-line emission. If the light is but feebly absorbed, so that we can look into considerable depths of the star, it may be possible that the outer regions contribute bright lines, while the hotter inner portions show absorption lines.

The possibility of the simultaneous presence of bright and dark lines of the same element, e. g., of hydrogen, is also strongly indicated by our theory. The matter has already been discussed in sections 7 and 8. The conditions under which the equations of sec. 8 have been deduced are more likely to apply to stars than the conditions of the problem as discussed in sec. 7, and as pointed out at the end of sec. 8, the lines of smaller frequency are those most liable to appear as bright lines. This agrees with the observed facts. The simultaneous appearance of bright lines of smaller and dark lines of greater frequency, has however, also been observed in cases where it is difficult to imagine that scattering plays an important part. I refer to Professor Hale's observations[1] on the spark-spectra observed in liquids. The proper explanation of these and similar observations suggests itself at once, if it is considered that the essential part of the effect of scattering lies in the diminution of the intensity of the continuous background. This diminution is not called for when the body giving the continuous spectrum has not infinitely great thickness and radiates with an intensity less than that belonging to a black body. Putting $s=0$ in (12), we obtain the ordinary equation for the radiation of a body sending out light of intensity S, which before reaching the observers traverses a body which is at a temperature for which the completely black radiation is E, viz.:

$$R=E+(S-E)e^{-\kappa t} .$$

For $\kappa=0$, we have

$$R_0=S .$$

Hence the question of brightness or darkness for a particular wavelength depends on the sign of the quantity

$$R-R_0=(E-S)(1-e^{-\kappa t}) ,$$

and this depends entirely on the question as to whether $E-S$ is positive or negative. If S is the radiation due to a black body at a higher temperature than that of the absorbing body, $E-S$ is necessarily negative and an absorption line will appear. If the radiation S is not that of a black body, but, e. g., a radiation reduced by the same quantity in the red and blue, or even reduced in the same ratio, the peculiar dependence of the radiation curve on temperature and wave-

[1] *Astrophysical Journal*, **15**, 227, 1902.

length shows that $E-S$ which is now positive when the temperature of the two bodies is equal, keeps positive longer with a diminishing temperature of the absorbing layer when the wave-lengths are long than when they are short. I need not enter more fully into this question, because Professor Kayser[1] has fully discussed it in giving what is practically an identical explanation. On applying Professor Kayser's explanation to the case of stars, we meet, however, with the very serious difficulty that we are obliged to consider the radiation of the continuous spectrum which serves as background to be less than that of a black body, which, on the views hitherto held, could not be the case when the radiating body has a great thickness. The consideration of the effect of scattering as explained by the present investigation removes the difficulty. I differ from Kayser in so far that he considers the existence of bright lines in stars to be an indication of high temperature. The small temperature-gradient seems, on the contrary, to argue more in favor of relatively low temperatures. Discussion on these and other connected matters is difficult, however, owing to our ignorance of the relative values of the coefficient of emission κ for different elements, and for different lines of the same element. We do not even know whether in a series such as that formed by the hydrogen lines κ increases or diminishes toward the root of the series.

The appearance of bright hydrogen lines covered by the dark calcium absorption, as presented by the spectrum of *Mira Ceti*, presents no difficulty according to the views of the present investigation. It only implies that the interior of the star has a temperature-gradient insufficient to reverse the hydrogen lines, and an outer atmosphere containing cooler calcium vapor. I consider it indeed as quite possible that, if we could remove the outer layers of the solar atmosphere, we should obtain a spectrum of bright lines.

This brings me to the second consideration suggested by the previous investigation. The prominent part played by the H and K lines of the solar spectrum in stellar atmospheres may be, to a great extent, due to the high values of the coefficient κ. The experiments of Sir William and Lady Huggins show conclusively that when calcium gas is rendered luminous by the electric discharge under

[1] *Astrophysical Journal*, **14**, 313, 1901.

conditions under which the H and K lines can appear, they are most persistent and are seen even when only very minute quantities of the substance are present. We are justified in concluding from these experiments that the emissive power of H and K is very great. The same may be true of other lines characteristic of spark-spectra, and the appearance of these lines in the stellar spectra must therefore be treated with some caution. If a star in its process of condensation increases the temperature-gradient of its outer layers, those lines will first make their appearance as dark lines which have high values of κ. But I must defer the fuller discussion of this matter to another occasion.

The effect of scattering on the intensity of the continuous spectrum of what we call the photosphere of a star may be considerable. When the radiating layer of a gas is sufficiently cool to admit of the presence of particles of solid or liquid matter, of dimensions large compared with molecular dimensions, the reduction in luminosity would take place fairly equally throughout the range of the visible spectrum. There would consequently be no great alterations in the relative intensities of red or blue, and we could obtain a correct idea of the temperature of the radiating body by a thermal comparison of the intensity of radiation in different parts of the spectrum. But when the scattering is molecular, it is sixteen times as large for the extreme visible violet as for the extreme visible red. Consequently the radiation emitted by a mass of gas under these conditions would show the violet considerably weakened as compared with the red. This opens out the possibility that with increasing temperature the violet portion of the continuous spectrum of a star may diminish in intensity as compared with the red end. As will presently appear, we possess some independent evidence that the photosphere emits less violet light than it should do if it were a black surface, but a closer experimental investigation is necessary before this can be definitely established.

I consider that for this purpose the careful investigation of the distribution of intensity in the solar spectrum is a matter of urgent importance. It would be necessary, however, for a satisfactory solution of the problem to measure the intensities everywhere in the intervals between Fraunhofer lines, or, at any rate, to select portions of the spectrum where no prominent Fraunhofer lines are situated.

Unless this is done, we risk that the violet portion of the spectrum should show too small an intensity, merely because it contains a greater number of Fraunhofer lines.

The loss of light by scattering in the solar atmosphere renders it possible for bright lines to appear which are due to vibrations in the front layers, though the temperature at these layers may be less than that which supplies the continuous spectrum. If the scattering is insufficient, its effect may be to obliterate the Fraunhofer line without entirely converting it into a bright line. It is necessary, however, that some kind of law should exist as to which of the Fraunhofer lines are obliterated. The two striking facts to be explained are the absence of the ultra-violet portion of the hydrogen series and the absence of the helium lines. In both cases the lines appear with considerable brilliancy in the so-called chromosphere, and in the flash-spectrum observed at the beginning and end of total eclipses. With regard to the hydrogen series, observations on stellar spectra and laboratory experiments which have already been quoted, would have led us to expect that the ultra-violet lines would be more easily reversible than the less refrangible lines. If the Sun forms an exception, it seems to indicate that the violet part of the continuous spectrum is reduced in intensity relatively to the red portion, and that this reduction is not a mere temperature effect.

This consideration strengthens to some extent the idea that the comparative weakness of the ultra-violet radiation in solar stars is not due to a diminution of temperature. As already mentioned, molecular scattering in the photospheric region might account for the comparative poverty of the more rapid vibrations.

The behavior of helium cannot be due to the same cause, as none of its lines have been seen reversed in the solar spectrum. The correct explanation is, I believe, to be found in this case in the great height to which helium is found to rise above the photospheric layer. The previous investigation has shown that a bright line is more likely to appear when the product of the coefficient of scattering and the thickness of the absorbing layer is large. This may be caused either by a great coefficient of scattering or by a great thickness of the absorbing layer. It is true that this reasoning should apply equally to hydrogen and the metals which rise as high as helium, and I believe that it does apply. The absence of some of the hydrogen lines in the solar spectrum has already been noted. That the red and blue lines can be seen is no doubt a consequence of the fact that hydrogen exists in much greater quantities than helium, for it should be noted that the helium lines are not bright, but only insufficiently dark to be observed.

This comparative weakness of some Fraunhofer lines which are very prominent in the flash-spectrum, and are probably due to the high temperature of the portion of the solar disk emitting the correspondent radiation, has been commented upon by Mr. Evershed, whose explanation I consider in the main to be correct. Although a further discussion of some points of detail may be desirable, the matter is independent of scattering, and lies therefore outside the range of this communication.

Victoria University,
Manchester, Eng.