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EXAMPLES OF PERIPLEGMATIC ORBITS,

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In the motion of material points it is well known that the determination of the orbits may be considered quite apart from the question what positions upon the orbits the points have at a given time. When the first portion of the problem has been completely investigated, the second is reduced in general to a mere matter of quadratures. GYLDÉN'S later investigations in this line have rendered this division of procedure familiar. Our illustration will be confined to the motion of two points in the same plane.

In this plane, having adopted a pole, let v denote the longitude and r and r' the radii of two orbits in the plane. The line of departure, from which v is measured, may be chosen arbitrarily, but, as r and r' are not in general periodic functions of v, it is not allowable to subtract an integral number of circumferences from the latter, which must be permitted to extend from $-\infty$ to $+\infty$. Then if p and p' are two constants, and we put

$$\frac{p}{r}-1=\rho \quad , \quad \frac{p'}{r'}-1=\rho'$$

the differential equations

$$\frac{d^2\rho}{dv^2}+\rho=0 \quad , \quad \frac{d^2\rho'}{dv^2}+\rho'=0$$

are, as is well known, those of two conics having a focus at the pole. If, more generally, the differential equations are such that they can be written in the form

$$\frac{d^2\rho}{dv^2}=\frac{\partial V}{\partial\rho} \quad , \quad \frac{d^2\rho'}{dv^2}=\frac{\partial V}{\partial\rho'}$$

V may be called the orbital potential. The present discussion will be limited to the case where V does not explicitly involve v. In the foregoing simple case we have

$$V=-\tfrac{1}{2}(\rho^2+\rho'^2)$$

A more general form for this function would be

$$V=f(\rho)+f'(\rho')$$

and then the orbits may be said to be independent of each other, and their determination is evidently a mere matter of quadratures. But, if the differential equations have not this form, nor can be given it through a transformation of variables, the orbits may be said to be *entangled*, it being impossible to determine one of them without the virtual, at least, determination of the other. It is the latter case which demands the employment of LINDSTEDT'S series.

In the simple case adduced V was rational, integral and of two dimensions in ρ and ρ'. In order to construct a very simple case for the application of these series, suppose that V still remains rational and integral, but now involves terms of three dimensions in ρ and ρ'. Were these terms proportional to ρ^3 and ρ'^3, the resulting orbits would be independent, and there would be no occasion for the employment of LINDSTEDT'S series. But let the new terms be proportional to $\rho^2\rho'$ and $\rho\rho'^2$, and the occasion for their use may arise.

Let us suppose that, μ being a constant,

$$2V=-\rho^2-\rho'^2-\mu\rho\rho'(\rho+\rho')$$

Then the differential equations will be

$$\frac{d^2\rho}{dv^2}+\rho+\mu(\rho\rho'+\tfrac{1}{2}\rho'^2)=0$$

$$\frac{d^2\rho'}{dv^2}+\rho'+\mu(\rho\rho'+\tfrac{1}{2}\rho^2)=0$$

It is desirable to limit as far as possible the number of constant parameters appearing in the equations, and that whether they were there originally or have been introduced by integration. In this connection it will be seen that μ is an unnecessary parameter, for it can be got rid of by multiplying both equations by it, and then replacing $\mu\rho$ and $\mu\rho'$ by ρ and ρ'. Thus, representing the radii by the equations

$$r=\frac{\mu p}{\mu+\rho} \quad , \quad r'=\frac{\mu p'}{\mu+\rho'}$$

ρ and ρ' will be determined by the equations

$$\frac{d^2\rho}{dv^2} + \rho + \rho\rho' + \tfrac{1}{2}\rho'^2 = 0$$
$$\frac{d^2\rho'}{dv^2} + \rho' + \rho\rho' + \tfrac{1}{2}\rho^2 = 0$$

which differ from the former only in that μ is replaced by unity.

These equations have the integral

$$\frac{d\rho^2}{dv^2} + \frac{d\rho'^2}{dv^2} + \rho^2 + \rho'^2 + \rho\rho'(\rho+\rho') = C^2$$

we write C^2 instead of C in order to avoid a radical sign in some of the following relations. When ρ and ρ' are interchanged, the equations remain the same; thus the relation $\rho = \rho'$ constitutes a particular integral of the system of differential equations.

Adopt for exhibiting graphically the simultaneous values of ρ and ρ' (simultaneous with reference to the independent variable v) a system of rectangular coordinates, x exhibiting the value of ρ, and y the value of ρ'. Then the representative point P must lie on the negative side of the curve whose equation is

$$x^2 + y^2 + xy(x+y) - C^2 = 0$$

in order that $\frac{d\rho}{dv}$ and $\frac{d\rho'}{dv}$ may be real. This cubic will have a closed branch surrounding the origin if C^2 falls below a certain limit. It crosses the axes of x and y on both sides of the origin at distances therefrom, equal in all four cases, to C. Its intersections with the right line whose equation is $x+y = 0$, and which bisects two of the angles made by the axes, are also at a distance C from the origin. On the other hand, its intersections with the line bisecting the remaining angles, whose equation is $x-y = 0$, are given by the roots of the equation

$$2x^2 + 2x^3 - C^2 = 0$$

But this cubic cannot have more than one real root unless C^2 does not exceed $\frac{8}{27}$. This is the condition necessary and sufficient that the original cubic should have a closed branch including the origin. As we wish to confine our attention to the case where the radii are restricted to finite limits, we suppose that C fulfils the mentioned condition, and that the representative point P is always within the closed branch.

When x is at a maximum or minimum in the original cubic, the equation

$$2(1+x)y + x^2 = 0$$

must be satisfied. Multiply this by $\frac{1}{2}y$ and subtract the product from the cubic; the result is

$$x^2 + \tfrac{1}{2}x^2y - C^2 = 0$$

But the previous equation yields

$$y = -\tfrac{1}{2}\frac{x^2}{1+x}$$

Hence the quartic

$$x^2 - \tfrac{1}{4}\frac{x^4}{1+x} = C^2$$

by its roots, which immediately embrace 0 between them, furnishes the limits of both the variables ρ and ρ'. However, we are not under the necessity of solving the quartic for the purpose of obtaining these limits; evidently, for C we may substitute a function of another constant rendering the solution easy.

The quartic, in a developed form, is

$$x^4 - 4x^3 - 4x^2 + 4C^2x + 4C^2 = 0$$

To remove the second term from this put $x = z + 1$, and we have

$$z^4 - 10z^2 - 4(4-C^2)z - 7 + 8C^2 = 0$$

We can adopt indeterminates q, q', R, such that the roots of this quartic are

$$\begin{aligned} z_1 &= \sqrt{R} + \sqrt{q+q'\sqrt{R}} \\ z_2 &= -\sqrt{R} + \sqrt{q-q'\sqrt{R}} \\ z_3 &= \sqrt{R} - \sqrt{q+q'\sqrt{R}} \\ z_4 &= -\sqrt{R} - \sqrt{q-q'\sqrt{R}} \end{aligned}$$

Then q, q', R are determined by the equations

$$q+R = 5 \quad , \quad q'R = 4-C^2$$
$$R^3 - 5R^2 + 2(4-C^2)R - \left(\frac{4-C^2}{2}\right)^2 = 0$$

Put, for simplicity, $4 - C^2 = m$, then

$$\begin{aligned} z_1 &= \sqrt{R} + \sqrt{5-R+mR^{-\frac{1}{2}}} \\ z_2 &= -\sqrt{R} + \sqrt{5-R-mR^{-\frac{1}{2}}} \\ z_3 &= \sqrt{R} - \sqrt{5-R+mR^{-\frac{1}{2}}} \\ z_4 &= -\sqrt{R} - \sqrt{5-R-mR^{-\frac{1}{2}}} \end{aligned}$$
$$R^3 - 5R^2 + 2mR - \tfrac{1}{4}m^2 = 0$$

The solution of the last equation, regarding m as the unknown, is

$$m = 4R \pm 2R\sqrt{R-1}$$

whence it follows that

$$C^2 = 4(1-R) \mp 2R\sqrt{R-1}$$

In order that C may be real R should exceed unity, and the cubic in R has always at least one root greater than 1; for, if we make $R = 1$, the left member becomes $-\frac{1}{4}C^4$, while, for $R = +\infty$, the result is $+\infty$.

If we make $\sqrt{R-1} = c$, we have

$$C^2 = 2c(1-c)^2$$

If we adopt the right member of this as a substitute for C^2, it is plain that the roots of the quartic will be expressible without the intervention of cubic radicals. While C^2 goes from 0 to $\frac{8}{27}$, c goes from 0 to $\frac{1}{3}$. In terms of c we have

$$x_1 = 1 + \sqrt{1+c^2} + \sqrt{4-c^2+(4-2c)\sqrt{1+c^2}}$$
$$x_2 = 1 - \sqrt{1+c^2} + \sqrt{4-c^2-(4-2c)\sqrt{1+c^2}}$$
$$x_3 = 1 + \sqrt{1+c^2} - \sqrt{4-c^2+(4-2c)\sqrt{1+c^2}}$$
$$x_4 = 1 - \sqrt{1+c^2} - \sqrt{4-c^2-(4-2c)\sqrt{1+c^2}}$$

Then x_4 is evidently the lower limit of the values of ρ and ρ' and x_2 the upper limit of the same. The values of these limits are tabulated below for every 0.01 in c.

Limiting Values of ρ and ρ' as Functions of c.

c	Lower	Upper	c	Lower	Upper	c	Lower	Upper
0.00	0.0000	0.0000	0.12	−0.4529	+0.4385	0.24	−0.5962	+0.5394
0.01	−0.1404	+0.1403	0.13	0.4684	0.4516	0.25	0.6050	0.5435
0.02	0.1972	0.1968	0.14	0.4832	0.4637	0.26	0.6135	0.5470
0.03	0.2399	0.2390	0.15	0.4971	0.4747	0 27	0.6216	0.5500
0.04	0.2751	0.2735	0.16	0.5103	0.4849	0.28	0.6295	0.5526
0.05	0.3056	0.3031	0.17	0.5229	0.4942	0.29	0.6370	0.5546
0.06	0.3326	0.3290	0.18	0.5348	0.5027	0.30	0.6443	0.5562
0.07	0.3569	0.3520	0.19	0.5462	0.5104	0.31	0.6513	0.5574
0.08	0.3791	0.3727	0.20	0.5571	0.5175	0.32	0.6580	0.5581
0.09	0.3996	0.3915	0.21	0.5675	0.5239	0.33	0.6645	0.5585
0.10	0.4186	0.4086	0.22	0.5775	0.5297	$\frac{1}{3}$	$-\frac{2}{3}$	$+\frac{2}{3}(4-\sqrt{10})$
0.11	−0.4363	+0.4242	0.23	−0.5871	+0.5348			

To illustrate the matter let us take a particular case, the radii being represented by the formulas

$$r = \frac{\mu p}{\mu+\rho} \quad , \quad r' = \frac{\mu p'}{\mu+\rho'}$$

suppose that the values of the four constants involved are

$$p = 1 \quad , \quad p' = 2 \quad , \quad \mu = 2 \quad , \quad c = 0.2 \; .$$

The limiting values of r are

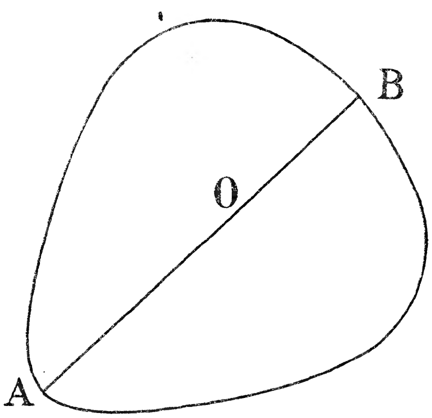
$$r = \frac{2}{2+0.5175} = 0.794 \quad , \quad r = \frac{2}{2-0.5571} = 1.386$$

and those of r' double these

$$r' = 1.589 \quad , \quad r' = 2.772$$

Here the upper limit of r is less than the lower limit of r'; hence the orbits have no point in common, and do not interfere with each other. We shall call this the quality of noninterference. It will be seen at once that the values of p, p', μ, c can be varied through a considerable range without the failure of this quality. But here is evidently an opportunity to apply Lindstedt's series in integrating the differential equations determining ρ and ρ'. Thus the applicability of these series does not imply dynamical instability in the motions which can take place upon the two orbits.

The form of the cubic circumscribing the values of ρ and ρ' for the special case noted above, where $C^2 = 0.256$, is shown in the adjacent figure (the scale is two inches to the unit). O is the origin, and the right line AOB, passing through that point and bisecting the angle between the axes of coordinates, is the path of the representative point P for the case where $\rho' = \rho$, and the solution of the differential equations is a periodic one. It may be noted that this point in general never attains the closed branch of the cubic curve, as this cannot happen unless the values $\frac{d\rho}{dv} = 0$, $\frac{d\rho'}{dv} = 0$ are simultaneous.*



It is interesting to know whether the orbits are periplegmatic in the sense of Gyldén. With his notation we should have

$$\frac{d^2\frac{\rho}{\mu}}{dv^2} + \frac{\rho}{\mu} = -\frac{1}{\mu}(\rho\rho'+\tfrac{1}{2}\rho'^2) = P$$

$$\frac{d^2\frac{\rho'}{\mu}}{dv^2} + \frac{\rho'}{\mu} = -\frac{1}{\mu}(\rho\rho'+\tfrac{1}{2}\rho^2) = P'$$

*The infinite branch is not given in the diagram, as it is useless for our purposes. The curve is species 67, and is shown in Fig. 71 of Newton's *Enumeratio linearum tertii ordinis*, printed at the end of Dr. Samuel Clarke's Latin translation of Newton's Optics.

For the quality in question P and P' must not fall below -1. As the greatest value of $\rho\rho'+\frac{1}{2}\rho'^2$ or $\rho\rho'+\frac{1}{2}\rho^2$ is $\frac{2}{3}$, if μ exceeds this, the orbits will be periplegmatic.

The treatment of the differential equations is, in general, easier if we make the linear transformation

$$u = \tfrac{1}{2}(\rho+\rho') \quad , \quad s = \tfrac{1}{2}(\rho-\rho')$$

They then take the form

$$\frac{d^2u}{dv^2} + u + \tfrac{3}{2}u^2 - \tfrac{1}{2}s^2 = 0$$

$$\frac{d^2s}{dv^2} + s - us \qquad = 0$$

The radii of the orbits are represented by the equations

$$r = \frac{\mu p}{\mu+u+s} \quad , \quad r' = \frac{\mu p'}{\mu+u-s}$$

The integral, in terms of the new variables, is

$$\frac{du^2}{dv^2} + \frac{ds^2}{dv^2} + u^2 + s^2 + u^3 - us^2 = \tfrac{1}{2}C^2$$

The adoption of the solution $s = 0$, satisfying the equations, leads directly to a periodic solution of them. In this case we have the single differential equation

$$\frac{du^2}{dv^2} = \tfrac{1}{2}C^2 - u^2 - u^3$$

to be integrated. Make the substitution

$$u = g + g'\cos 2\psi$$

g and g' being constants; then

$$4g'^2(1-\cos^2 2\psi)\frac{d\psi^2}{dv^2} = \tfrac{1}{2}C^2-(g+g'\cos 2\psi)^2-(g+g'\cos 2\psi)^3$$

Let g and g' be so chosen that the right member of this, equated to zero, may have the two roots $\cos 2\psi = \pm 1$. Then g and g' are determined by the equations

$$\tfrac{1}{2}C^2 - (g+g')^2 - (g+g')^3 = 0$$

$$\tfrac{1}{2}C^2 - (g-g')^2 - (g-g')^3 = 0$$

or by

$$\tfrac{1}{2}C^2 - g^2 - g'^2 - g^3 - 3gg'^2 = 0$$

$$2g + 3g^2 + g'^2 = 0$$

If we divide both members of the last differential equation by $1 - \cos^2 2\psi$ the result is

$$4g'^2\frac{d\psi^2}{dv^2} = \tfrac{1}{2}C^2 - g^2 - g^3 + g'^3\cos 2\psi$$

But, eliminating C^2, this becomes

$$4\frac{d\psi^2}{dv^2} = 1 + 3g + g'\cos 2\psi = 1 + 3g + g' - 2g'\sin^2\psi$$

If we put $\sqrt{3}\,g' = \sin\theta$, then will $3g = \cos\theta - 1$, and

$$\frac{d\psi^2}{dv^2} = \frac{1}{2\sqrt{3}}\left[\sin(\theta+60°) - \sin\theta\sin^2\psi\right]$$

If next

$$k^2 = \frac{\sin\theta}{\sin(\theta+60°)}$$

we have

$$\frac{d\psi^2}{dv^2} = \tfrac{1}{4}\frac{1}{\sqrt{1-k^2+k^4}}(1-k^2\sin^2\psi)$$

and to u may be given the form

$$u = \tfrac{1}{3}\left(\frac{1+k^2}{\sqrt{1-k^2+k^4}} - 1\right) - \frac{k^2}{\sqrt{1-k^2+k^4}}\sin^2\psi$$

It will be seen that k takes the place of the arbitrary constant C^2 which is attached to the integral. In the Gudermannian notation for elliptic functions, putting m for $\dfrac{1}{2\sqrt[4]{1-k^2+k^4}}$, and c being an arbitrary constant,

$$\sin\psi = sn(mv+c) = sn\,x$$

and

$$u = \frac{1}{\sqrt{1-k^2+k^4}}\left(\frac{1+k^2}{3} - \tfrac{1}{3}\sqrt{1-k^2+k^4} - k^2\,sn^2x\right)$$

The value of C is of interest; we have

$$\tfrac{1}{2}C^2 = g^2+g^3+g'^2(1+3g)$$

$$= \tfrac{1}{27}(3-6\cos\theta+3\cos^2\theta-1+3\cos\theta-3\cos^2\theta+\cos^8\theta) + \tfrac{1}{3}(1-\cos^2\theta)\cos\theta$$

$$= \tfrac{2}{27}(1+3\cos\theta-4\cos^3\theta) = \tfrac{2}{27}(1-\cos 3\theta)$$

$$C^2 = \tfrac{8}{27}\sin^2\tfrac{3}{2}\theta$$

If C^2 is wanted in terms of k we have

$$C^2 = \tfrac{4}{27}\left(1 - \frac{1-\frac{3}{2}k^2-\frac{3}{2}k^4+k^6}{(1-k^2+k^4)^{\frac{3}{2}}}\right)$$

The argument on which u depends is

$$\frac{\pi}{2K}\frac{1}{\sqrt[4]{1-k^2+k^4}}v + c$$

where K, as usual, denotes the period of the elliptic integral; or, it is

$$\frac{1}{\sqrt[4]{1-k^2+k^4}}\,\frac{1}{1+(\frac{1}{2})^2k^2+(\frac{1\cdot 3}{2\cdot 4})^2k^4+(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6})^2k^6+\ldots}v + c$$

$$= (1-\tfrac{15}{64}k^4-\tfrac{15}{64}k^6-\tfrac{5621}{16384}k^8+\tfrac{393}{8192}k^{10}+\ldots)v+c$$

It is to be noted that the square of k is absent from the latter expression, hence this parameter must become quite a large fraction before a marked difference results in the period.

An expression in terms of the nome q may be preferred. The period has the equivalent

$$\frac{2K\sqrt{k'}}{\pi}\sqrt[4]{1+\frac{k^4}{k'^2}}$$

where $k' = \sqrt{1-k^2}$. But the first factor has the value

$$\frac{2K\sqrt{k'}}{\pi} = 1+4\left[-\frac{q^2}{1+q^2}+\frac{q^6}{1+q^4}-\frac{q^{12}}{1+q^6}+\frac{q^{20}}{1+q^8}-\ldots\right]$$

and the second can be derived from

$$\frac{k}{\sqrt{k'}} = 4\sqrt{q}\,\frac{[1+q^2+q^6+q^{12}+\ldots]^2}{[1+2q^4+2q^{16}+\ldots]^2-[2q+2q^9+2q^{25}+\ldots]^2}$$

The series for the period or its reciprocal in powers of q is tardily convergent, and it seems better to retain the foregoing expressions where the law of progression is obvious.

If we put $k = \sin\eta$, q may be derived by tentation from the equation

$$\frac{\sin^2 \frac{1}{2}\eta}{(1+\sqrt{\cos\eta})^2} = \frac{q+q^9+q^{25}+\ldots}{1+2q^4+2q^{16}+\ldots}$$

When $k = 1$, the numerator and denominator of the second member become divergent series, but the proper value of q, in this case, is unity. K may be derived from

$$\sqrt{\frac{2K}{\pi}} = 1+2q+2q^4+2q^9+2q^{16}+\ldots$$

To have u expressed as a periodic function of its argument substitute for the transcendental function $cn^2 x$ its equivalent

$$\frac{2\pi^2}{k^2K^2}\left[\frac{q}{1-q^2}-\frac{2q^2}{1-q^4}+\frac{3q^3}{1-q^6}-\ldots + \frac{q}{1-q^2}\cos\left(\frac{\pi}{K}x\right)+\frac{2q^2}{1-q^4}\cos\left(2\,\frac{\pi}{K}x\right) + \frac{3q^3}{1-q^6}\cos\left(3\,\frac{\pi}{K}x\right)+\ldots\right]^*$$

There is still another linear transformation of the differential equations worthy of notice. In order to remove from the potential the terms of three dimensions which are products, let us put

$$\rho = u+hu' \quad , \quad \rho' = u'+hu$$

where h is either of the complex cube roots of unity, or such that

$$h^2+h+1 = 0$$

Then

$$\frac{1}{2}\,\frac{d\rho^2+d\rho'^2}{dv^2} = -\frac{1}{2}\,h\,\frac{du^2-4du\,du'+du'^2}{dv^2}$$

$$V = \tfrac{1}{2}\,h\,(u^2-4uu'+u'^2)+\tfrac{1}{2}\,(u^3+u'^3)$$

Hence it is seen that the differential equations take the form

$$\left[\frac{d^2}{dv^2}+1\right](2u'-u) = \tfrac{3}{2}\,h^2u^2$$

$$\left[\frac{d^2}{dv^2}+1\right](2u-u') = \tfrac{3}{2}\,h^2u'^2$$

or, if, as a symbol of operation, we put

$$\mathsf{D} = \tfrac{2}{3}\,h\left[\frac{d^2}{dv^2}+1\right]$$

the simple form

$$\mathsf{D}\,[2u'-u] = u^2 \quad , \quad \mathsf{D}\,[2u-u'] = u'^2$$

* For these formulas in elliptic functions consult Broch, *Traité Elémentaire des Fonctions Elliptiques*, p. 207, Eq. (124); pp. 210–211, Eqs. (5) and (6); p. 210, Eq. (3); p. 172, Eq. (17).

Thus, if from the double of one of the dependent variables we subtract the other, and on the remainder operate with D, the result is the same as if we squared the latter variable. Simple as are these equations, no completely satisfactory general expressions of the unknowns for an infinite range in longitude have been found.

In applying Lindstedt's series to the integration of these equations we should assume

$$u = \Sigma_{i,\,i'}\,A_{i,\,i'}\,\epsilon^{(ik+i'k')\,v} \quad , \quad u' = \Sigma_{i,\,i'}\,A'_{i,\,i'}\,\epsilon^{(ik+i'k')\,v}$$

where the A and A' are constants as well as k and k', and i and i' are integers reaching from $-\infty$ to $+\infty$. The substitution of these values in the equations shows that A, A', k, k' must satisfy, for each combination i, i' the conditions

$$[(ik+i'k')^2+1]\,(2A'_{i,\,i'}-A_{i,\,i'}) = \tfrac{3}{2}\,h^2\,\Sigma_{j,\,j'}\,A_{i-j,\,i'-j'}\,A_{j,\,j'}$$

$$[(ik+i'k')^2+1]\,(2A_{i,\,i'}-A'_{i,\,i'}) = \tfrac{3}{2}\,h^2\,\Sigma_{j,\,j'}\,A'_{i-j,\,i'-j'}\,A_{j,\,j'}$$

These equations should suffice for determining the A and A' as well as k and k' in terms of the four arbitrary constants introduced by the integration. But two of these constants are involved in the expressions only through addition to the two elementary arguments kv and $k'v$; thus the mentioned quantities involve only two arbitrary parameters. Since u and u' as periodic functions of v involve only cosines, we have the conditions

$$A_{-i,\,-i'} = A_{i,\,i'} \quad , \quad A'_{-i,\,-i'} = A'_{i,\,i'}$$

If, besides k and k', either of the two groups of coefficients A and A' is known, the other is deducible.

The differential equations may be reduced to a system in which all are of the first order; employing for this purpose those in terms of the variables u and s, the closed curve enveloping the area in which the differential coefficients are real has the equation

$$s^2 = \frac{\frac{1}{2}\,C^2-u^2-u^3}{1-u}$$

The maximum value of $|u|$ is then $\frac{2}{3}$, and the maximum of $|s|$ corresponds to the value of u given by the smaller positive root of

$$u^3-u^2-u+\tfrac{1}{4}\,C^2 = 0$$

Thus, if we put

$$\tfrac{1}{4}\,C^2 = c'\,(1+c'-c'^2)$$

it will be found that

$$|s| = \sqrt{2c'+3c'^2}$$

And if $C^2 = \frac{8}{27}$ we shall have approximately $|s| = 0.392$

In place of the two variables u and s we employ the four u, y, e', l' such that

$$\frac{du}{dv} = -y \quad , \quad s = e'\cos l' \quad , \quad \frac{ds}{dv} = -e'\sin l'$$

The integral equation will then be expressed in the form

$$y^2 + u^2 + u^3 + e'^2(1-u\cos^2 l') = \tfrac{1}{2} C^2$$

which gives

$$\tfrac{1}{2} e'^2 = \tfrac{1}{2}\,\frac{\tfrac{1}{2} C^2 - y^2 - u^2 - u^3}{1-u\cos^2 l'} = W$$

And the differential equations are

$$\frac{du}{dv} = -y$$

$$\frac{dy}{dv} = u + \tfrac{3}{2} u^2 - \tfrac{1}{2} e'^2 \cos^2 l'$$

$$\frac{d(e'\cos l')}{dv} = -e'\sin l'$$

$$\frac{d(e'\sin l')}{dv} = (1-u)\,e'\cos l'$$

The third and fourth are equivalent to

$$\frac{d\,.\log e'}{dv} = -\tfrac{1}{2} u \sin 2l' \quad , \quad \frac{dl'}{dv} = 1 - u\cos^2 l'$$

From the latter it is plain that l' and v advance together; thus l' will serve equally well as v for the independent variable. By division and elimination of e'^2, the first and second equations become

$$\frac{du}{dl'} = -\frac{y}{1-u\cos^2 l'}$$

$$\frac{dy}{dl'} = \frac{u+\tfrac{3}{2}u^2}{1-u\cos^2 l'} - \tfrac{1}{2}\,\frac{\tfrac{1}{2}C^2-u^2-u^3}{(1-u\cos^2 l')^2}\cos^2 l'$$

or, as they may be written

$$\frac{du}{dl'} = \frac{\partial W}{\partial y} \quad , \quad \frac{dy}{dl'} = -\frac{\partial W}{\partial u}$$

These equations may be still further varied by putting

$$u = e\cos l \quad , \quad y = e\sin l$$

Then if

$$W = \tfrac{1}{2}\,\frac{\tfrac{1}{2}C^2 - e^2 - e^3\cos^3 l}{1-e\cos l\cos^2 l'}$$

we have

$$\frac{d\,.\tfrac{1}{2}e^2}{dl'} = \frac{\partial W}{\partial l} \quad , \quad \frac{dl}{dl'} = -\frac{\partial W}{\partial\,.\tfrac{1}{2}e^2}$$

After u and y or e and l have been determined in terms of l' through the integration of these equations, v can be found by a quadrature from

$$\frac{dv}{dl'} = \frac{1}{1-u\cos^2 l'}$$

and thence, by inversion, l' in terms of v, and thus the problem completely solved.

W can be developed in an infinite series of the form

$$\Sigma_{i,i'}\, A_{i,i'} \cos[il + 2i'l']$$

For putting

$$\beta = \frac{2-u-2\sqrt{1-u}}{u}$$

we have

$$W = \frac{\tfrac{1}{2}C^2 - y^2 - u^2 - u^3}{\sqrt{1-u}}\left[\tfrac{1}{2} + \beta\cos 2l' + \beta^2\cos 4l' + \beta^3\cos 6l' + \ldots\right]$$

From this u and y may be eliminated by substituting their values in terms of e and l.

The integrals of the two differential equations may then be approximated to by a series of Delaunay transformations, as the function W is quite similar to Delaunay's R in the lunar theory. The only noteworthy differences being that here there are only two unknowns in place of Delaunay's six, and only one constant parameter C instead of Delaunay's three n', e', a'.

We may give here Delaunay's rule for making a transformation. If we have integrated the differential equations (L is put for $\tfrac{1}{2}e^2$)

$$\frac{dL}{dl'} = \frac{\partial W}{\partial l} \quad , \quad \frac{dl}{dl'} = -\frac{\partial W}{\partial L}$$

when W is limited to the terms involving one argument $il + i'l'$ (the constant term is included) and have found in this manner (θ designating the argument)

$$\theta = \theta_0(l'+c) + \theta_1\sin\theta_0(l'+c) + \theta_2\sin 2\theta_0(l'+c) + \theta_3\sin 3\theta_0(l'+c) + \ldots$$

$$L = L_0 + L_1\cos\theta_0(l'+c) + L_2\cos 2\theta_0(l'+c) + L_3\cos 3\theta_0(l'+c) + \ldots$$

c being a constant, and $\theta_0, \theta_1, \theta_2, \ldots\ldots L_0, L_1, L_2, \ldots\ldots$ being known functions of another constant (e_0 for instance), we can replace

$$L \text{ by } L_0 + L_1\cos(il+i'l') + L_2\cos 2(il+i'l') + \ldots.$$

$$l \text{ by } l + \frac{\theta_1}{i}\sin(il+i'l') + \frac{\theta_2}{i}\sin 2(il+i'l') + \ldots.$$

and we shall have, for determining the new variables e_0, l, precisely the same equations

$$\frac{dL}{dl'} = \frac{\partial W}{\partial l} \quad , \quad \frac{dl}{dl'} = -\frac{\partial W}{\partial L}$$

provided, first, that we put for W the function obtained by making the preceding substitutions in the old function W (complete) augmented by the quantity

$$-\frac{i'}{i}(L-L_0) + \frac{i'}{2i}(\theta_1 L_1 + 2\theta_2 L_2 + 3\theta_3 L_3 + \ldots)$$

second, that we regard the new variables L as connected with e_0 by the relation

$$L = L_0 + \tfrac{1}{2}(\theta_1 L_1 + 2\theta_2 L_2 + 3\theta_3 L_3 + \ldots)$$