1894AJ.....14...49H

THE

ASTRONOMICAL JOURNAL.

No. 319.

VOL. XIV. BOSTON, 1894 JUNE 2. NO. 7.

A SUGGESTION IN THE THEORY OF *MERCURY*,

By A. HALL.

About forty years ago LeVerrier found that the line of apsides of the orbit of *Mercury* is in motion at the rate of 38″ a century more than the known forces will explain. Since then Professor Newcomb has made a new investigation of this question, and from a more extended series of observations has confirmed the result found by LeVerrier. Newcomb's result is a little greater, being 43″ a century. This anomalous motion in the line of apsides is the starting point of LeVerrier's theory of an intra-Mercurial planet, situated nearly in the plane of the orbit of *Mercury*. Several such planets have been supposed to be found; one by Dr. Lescarbault in 1859, with a period of 20 days, and others by Professors Watson and Swift in 1878. These discoveries, however, have not been confirmed to the satisfaction of astronomers. There are also theoretical objections to the introduction of such bodies, since they would disturb the motions of other planets. Some years ago Tisserand applied to this question Weber's theory of electro-dynamics, which would in fact produce a small part of the observed motion, but Weber's theory has, I think, been set aside by recent investigations in physics. Another explanation has been sought in the figure and constitution of the *Sun*, but in this case observations show the *Sun* to be almost exactly a spherical body. Thus, at present, no explanation of this motion remains.

If the Newtonian law of attraction is not a rigorous law of nature, or if it is modified slightly under certain conditions, probably this lack of rigor would become apparent first among the swiftly moving bodies of our solar system, such as our *Moon* and the planet *Mercury*. In his *Principia*, Book I, Newton has given some computations in which he assumes the law of attraction to be not exactly as the inverse second power of the distance. He shows that the perihelia would move under the action of such a central force; and on the other hand the observed fixity of the perihelia is a strong proof of his law of attraction. Laplace in his *Mécanique Céleste*, Book XVI, Chap. IV, has investigated some assumed changes of the law of attraction, and has shown in what terms of the *Moon's* motion these changes would become apparent. In 1873 Bertrand brought forward a more general question than that of Newton, and found an elegant expression for the angle between the smallest radius vector and the greatest in the case of a body moving round a center of force in a closed curve; the attraction depending only on the distance. The central force has the form

$$R = m.r^n$$

m being a constant, and r the radius vector. In the case of a small eccentricity the angle between the minimum and maximum radii is nearly

$$\theta = \frac{\pi}{\sqrt{n+3}}$$

If $n = -2$, we have the Newtonian law; the angle θ is 180°, or the particular radii form a right line, and so far as the action of the central force is concerned the perihelion is fixed. Applying Bertrand's formula to the case of *Mercury* I find, taking Newcomb's value of the motion, or 43″, that the perihelion would move as the observations indicate by taking

$$n = -2.000\,000\,16$$

It will be seen that a very small change in the law of central force would produce the required motion in the line of apsides. The question remains to examine the effect of this change on the other elements of the orbit. If we denote by μ the sum of the masses of *Mercury* and the *Sun* the equations of motion are

$$\left.\begin{aligned} \frac{d^2x}{dt^2} + \frac{\mu x}{r^{3+\Delta}} &= 0 \\ \frac{d^2y}{dt^2} + \frac{\mu y}{r^{3+\Delta}} &= 0 \\ \frac{d^2z}{dt^2} + \frac{\mu z}{r^{3+\Delta}} &= 0 \end{aligned}\right\} \qquad (1)$$

Cross multiplying by z, y, x, and taking the difference

© American Astronomical Society • Provided by the NASA Astrophysics Data System

1894AJ.....14...49H

of the products, we have complete differentials, and integrating,

$$(2)\qquad \begin{cases} \dfrac{zdy-ydz}{dt} = c_1 \\ \dfrac{xdz-zdx}{dt} = c_2 \\ \dfrac{ydx-xdy}{dt} = c_3 \end{cases}$$

c_1, c_2, c_3 are the three known areal constants of integration. It is almost self-evident that the node and inclination are not changed by the introduction of Δ. But this can be shown formally as follows. The rectangular coordinates of a planet in terms of its radius vector, true anomaly, and the constants; ω distance from node to perihelion, Ω the node, and i the inclination, are as follows:

$$x = r[\cos(v+\omega)\cos\Omega - \sin(v+\omega)\sin\Omega\cos i]$$
$$y = r[\cos(v+\omega)\sin\Omega + \sin(v+\omega)\cos\Omega\cos i]$$
$$z = r\sin(v+\omega)\sin i$$

If we multiply x by $\sin\Omega\sin i$, y by $-\cos\Omega\sin i$, and z by $\cos i$, and add the products, we have

$$x\sin\Omega\sin i - y\cos\Omega\sin i + z\cos i = 0$$

Because the velocities x', y', z', along the axes are linear quantities, we have likewise

$$x'\sin\Omega\sin i - y'\cos\Omega\sin i + z'\cos i = 0$$

These two equations give

$$\sin\Omega\tan i = \frac{yz'-zy'}{xy'-yx'} = \frac{c_1}{c_3}$$
$$\cos\Omega\tan i = \frac{xz'-zx'}{xy'-yx'} = -\frac{c_2}{c_3}$$

These equations determine the node and inclination, which are therefore constant. If we multiply the equations of motion by $2dx$, $2dy$, $2dz$, add the products and integrate, then

$$(3)\qquad \frac{dx^2+dy^2+dz^2}{dt^2} - \frac{2\mu}{(1+\Delta)r^{1+\Delta}} + h = 0$$

h being the constant of integration. If we add the squares of equations (2) we find, after a little reduction, and putting $c_1^2+c_2^2+c_3^2 = k^2$.

$$(4)\qquad \frac{r^2(dx^2+dy^2+dz^2)}{dt^2} - \frac{r^2dr^2}{dt^2} = k^2$$

Eliminating the square of the velocity between equations (3) and (4), we have

$$(5)\qquad dt = \frac{rdr\,(1+\Delta)^{1/2}}{\sqrt{2\mu r^{1-\Delta}-(1+\Delta)r^2h-(1+\Delta).k^2}}$$

The two expressions for the element of the curve in space, in rectangular and polar coordinates, give

$$dx^2+dy^2+dz^2 = dr^2+r^2dv^2$$

and from equation (4)

$$dv = \frac{k.dt}{r^2} \qquad (6)$$

Substituting from (5) the value of dt,

$$dv = \frac{k(1+\Delta)^{1/2}.\,dr}{r\sqrt{2\mu r^{1-\Delta}-(1+\Delta)r^2h-(1+\Delta)k^2}} \qquad (7)$$

This equation shows that the maximum and minimum values of r are determined by the equation

$$r^2 - \frac{2\mu.\,r^{1-\Delta}}{(1+\Delta)h} + \frac{k^2}{h} = 0$$

Since Δ is small, let us put $\mu = \mu' r^{\Delta}$, and the equation becomes

$$r^2 - \frac{2\mu' r}{(1+\Delta)h} + \frac{k^2}{h} = 0$$

If the roots of this equation are $a(1+e)$, and $a(1-e)$, then

$$a = \frac{\mu'}{(1+\Delta)h}: \qquad a^2(1-e^2) = \frac{k^2}{h}$$

and inversely

$$h = \frac{\mu'}{(1+\Delta)a}: \qquad k = \sqrt{\frac{\mu' a(1-e^2)}{1+\Delta}}$$

With these values of h and k equation (7) gives

$$dv = \frac{\sqrt{a(1-e^2)}.\,dr}{r\sqrt{2r-\dfrac{r^2}{a}-a(1-e^2)}} \qquad (8)$$

The value of dv is in the common form, and the integral is

$$r = \frac{a(1-e^2)}{1+e\cos(v-\omega)} \qquad (9)$$

This is the polar equation of an ellipse, the pole being at the focus, and ω being the constant of integration. Equation (5) gives

$$dt = \frac{(1+\Delta)^{1/2}.\,rdr}{\sqrt{\mu'}.\sqrt{2r-\dfrac{r^2}{a}-a(1-e^2)}}$$

or

$$dt = \left(\frac{(1+\Delta)a}{\mu'}\right)^{1/2}.\ \frac{rdr}{\sqrt{a^2e^2-(a-r)^2}}$$

If we assume $r=a(1-e\cos u)$ from which $a-r=ae\cos u$, then

$$dt = \left(\frac{1+\Delta}{\mu'}\right)^{1/2}.\ a^{3/2}(1-e\cos u).\,du$$

and integration gives

$$t+T = \left(\frac{1+\Delta}{\mu'}\right)^{1/2}.\ a^{3/2}(u-e\sin u) \qquad (10)$$

where T is the constant of integration. Equation (10) completes the solution. The relation between the eccentric anomaly u, and v, the true anomaly, will be found by equating the two values of the radius vector.

Hence the solution is simple, but it must be tried by comparison with observations. Taking the mass of *Mercury*

© American Astronomical Society • Provided by the NASA Astrophysics Data System

$= \frac{1}{5000000}$, the mean distance $a = 0.387\,098\,8$, and computing the value of μ' from the value of a, I find

$$\log.\left(\frac{1+\Delta}{\mu'}\right)^{1/2} = 9.999\,999\,958\,4$$

The factor, therefore, by which equation (10) differs from the corresponding one under the Newtonian law, is so nearly unity that it seems possible such a change may be permitted by the observations.

The question may be raised whether the phenomenon which is sought to be explained is a real one. LeVerrier says: "The necessity of an increase in the secular motion of the perihelion of *Mercury* results exclusively from observations of the transits of the planet over the disk of the *Sun*. The exactitude of these observations is beyond doubt." Professor Newcomb has used the same kind of observations extended over a longer time. For my own part I cannot doubt the correctness of the increase of the secular motion found by these astronomers. Still, it would be well to have such an important result confirmed by meridian observations, or by referring the position of the planet to known stars.

1894 *May* 5.

SUPPLEMENT TO SECOND CATALOGUE OF VARIABLE STARS,

By S. C. Chandler.

This Supplement to the Second Catalogue (*A.J.* 300) is arranged in three tables, like the Supplement (*A.J.* 216) to the First Catalogue (*A.J.* 179–80). These tables conform strictly to the explanations in the preface to the Second Catalogue. The Catalogue and Supplement combined comprise 279 known variables, and furnish a full statement of our knowledge up to date, so far as is possible in so succinct a form. The elements of the Catalogue have been revised, and those for the newer stars have been supplied where possible, although some of the latter are rude.

There are signs of improvement in the positions of new variables currently announced, thus removing one source of annoyance and confusion to observers. The very moderate degree of accuracy required for this purpose is easy of attainment even by the most poorly equipped amateur. Perhaps it is not too much to hope that the coarse standards of precision of a round tenth of a minute in right-ascension and round minute in declination, will in future disappear.

There is urgent need that some observer should take up observation of variables visible only in the Southern hemisphere, and especially the independent confirmation of the stars found at the Boyden Observatory at Arequipa, and by Roberts in South Africa. The latter seems to have had unexampled success in finding, within a short time, a large number of variables which seem to belong exclusively to one interesting type, and he has assigned their periods with apparent exactness. It would be of great value if he would publish the details of the extremely interesting work he is so zealously prosecuting; as well as his methods of observation, and determination of elements.

The following stars are to be struck out of the list of "Unconfirmed Stars" in the Second Catalogue, since they have been confirmed, and will be found in Table I on the next page: — (1805), (2170), (6943), (7351), (7450), (7457), (7458), (8116); also (7784) which had already been transferred in the Second Catalogue (7783 *RU Cygni*). I am indebted to Prof. Kreutz for the note of this oversight. The following stars may also now be struck out of the same list, the first proving to be a mistaken announcement by the discoverer, and the others appearing to be constant, after careful watch by trustworthy observers: — (1220), (1948), (2305), (7020), (7280), (8100), (8499), (8617). Since the Catalogue was published Yendell has carefully observed all these objects, and his authority seems sufficient, in addition to Sawyer's and my own observations, to warrant their excision from the list.

The following are additional notes to some of the stars in the "Unconfirmed List" of the Second Catalogue.

(691) — *Persei*. Hartwig has assigned the letter *V* to this star. I abstained from applying any letter in the Catalogue, from suspicion of error in the original announcement of variability. Yendell, who has given it strict and continuous attention, is of the same opinion. The letter *V* has therefore been given to another star (980 *V Persei* in Table I). If there should hereafter prove to be a variable in or near the place of (691), it may receive, when confirmed, the next letter then unappropriated. This seems the best way out of the embarrassment, and affords a new example of the need of a conservative avoidance of precipitancy in assigning a definitive notation.

(1279) — *Camelopardalis*. The narrow extremes of brightness yet observed in this extraordinarily red and difficult star, do not yet exclude possibility of mistake. Every experienced observer is aware of the uncertainty of estimates in such cases, and there is no reason to think that photographic images can be relied upon any better. Mr. Yendell's counsel has been followed, by assigning no letter as yet. There would be equally good ground for lettering a large number of other red stars showing uncertain variations of the same unimportant type. These are common in the heavens. The result would be confusion beyond remedy.

(1801) — *Camelopardalis*. Correct position, (1855) $4^h\ 55^m\ 21^s$, $+68°\ 28'.2$.

(7238) — *Cygni*. There is still conflict between the observations of Espin, Deichmüller, Hartwig, Yendell, and myself.

© American Astronomical Society • Provided by the NASA Astrophysics Data System