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E. J. STONE, M.A., F.R.S., President, in the Chair.

John Wilson Appleton, 12 Amberley Street, Toxteth Park, Liverpool; and

The Hon. Cecil Duncombe, Nawton Grange, Nawton, North Yorkshire,

were balloted for, and duly elected Fellows of the Society.

On Newton's Solution of Kepler's Problem. By Professor J. C. Adams, M.A., F.R.S.

Of all the methods which have been proposed for the solution of this problem, that which leads most rapidly to a result having any required degree of precision may be briefly explained as follows:—

The equation to be solved by successive approximations is

$$x - e \sin x = z,$$

where z is the known mean anomaly, e the eccentricity, and x the eccentric anomaly to be determined.

Suppose x_0 to be an approximate value of x, found whether by estimation, by graphical construction, or by a previous rough calculation, and let

$$x_0 - e \sin x_0 = z_0.$$

Then if

$$\delta x_0 = \frac{z - z_0}{1 - e \cos x_0},$$

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and

$$x' = x_0 + \delta x_0,$$

x' will be a much more approximate value of x than x_0.

Similarly, if we put

$$x' - e \sin x' = z',$$

and if

$$\delta x' = \frac{z - z'}{1 - e \cos x'}$$

and

$$x'' = x' + \delta x',$$

x'' will be a much more approximate value of x than x'; and so on, to any required degree of approximation.

If the error of the assumed value x_0 be supposed to be of the order i, when e is taken as a small quantity of the first order, then the error of the value x' will be of the order $2i + 1 = i'$ suppose, similarly the error of the value x'' will be of the order $2i' + 1 = 4i + 3$, and so on, so that the order of the error is more than doubled at each successive approximation.

The above explains the immense advantage of this process over the use of series proceeding according to powers of e, when great precision is required in the result; since, in this latter method, the addition of a new term only increases the order of the error by unity.

The degree of rapidity of the approximation may be still further increased by the following slight modification of the above process.

Starting, as before, with the value x_0, and calling $z - z_0 = \delta z_0$, we should obtain a much more accurate value than before of the correction δx_0 to be applied to x_0, by putting

$$\delta x_0 = \frac{z - z_0}{1 - e \cos(x_0 + \frac{1}{2}\delta x_0)} = \frac{\delta z_0}{1 - e \cos(x_0 + \frac{1}{2}\delta x_0)}.$$

Now, e being supposed to be small, δz_0 is an approximate value of δx_0, and may be written for it in the small term in the denominator.

Hence, if we put

$$\delta x_0 = \frac{\delta z_0}{1 - e \cos(x_0 + \frac{1}{2}\delta z_0)},$$

$$x' = x_0 + \delta x_0,$$

x' will be a nearer approximation to the true value of x than was obtained before by the corresponding operation.

Similarly, if

$$x' - e \sin x' = z',$$

and

$$z - z' = \delta z',$$

and if

$$\delta x' = \frac{\delta z'}{1 - e \cos (x' + \frac{1}{2} \delta z')},$$

then

$$x'' = x' + \delta x'$$

will be the next approximate value of x, and the process may be continued as far as we please.

If the error of x_0 be of the order i, that of x' will now be of the order $2i+2$, that of x'' will be of the order $2(2i+2)+2 = 4i+6$, and so on, so that the degree of rapidity of the approximation is still greater than before.

If we chose to take the mean anomaly itself as the first approximate value of the eccentric anomaly—that is, if we put

$$x_0 = z,$$

we should have

$$z_0 = z - e \sin z,$$

and the value of δx_0 given by the first method would be

$$\delta x_0 = \frac{e \sin z}{1 - e \cos z},$$

while that given by the second and more accurate method would be

$$\delta x_0 = \frac{e \sin z}{1 - e \cos (z + \frac{1}{2} e \sin z)},$$

and the error of $x' = x_0 + \delta x_0$ would be of the 3rd order in the former case, and of the 4th order in the latter.

In practice, however, a much nearer first approximate value of x may be always found by inspection, and of course the smaller the error of this value is, the more rapid will be the rate of the subsequent approximations.

The methods above explained have been long known. The first method is given at p. 41 of Thomas Simpson's "Essays on several Subjects in Speculative and Mixed Mathematics," published in 1740; and Gauss' method given at pp. 10–12 of the "Theoria Motus," published in 1809, is essentially the same.

The second method, or rather the modification of the first, is given by Cagnoli in his "Trigonométrie," at pp. 377, 378 of the first edition, published in 1786, and at pp. 418–420 of the second edition, published in 1808.

Now, my object in the present note is to point out that the first method explained above is exactly equivalent to that given by Newton in the "Principia," at pp. 101, 102 of the second edition, and at pp. 109, 110 of the third edition, when Newton's expressions are put into the modern analytical form.

None of the subsequent authors, however, mentions this method as being Newton's, the unusual form in which Newton's solution is given having, no doubt, caused them to overlook it.

In the first edition of the "Principia" a modification of the method is given which was, I have no doubt, intended by Newton to be equivalent to the second method given above; but by some inadvertence, instead of the denominator of $\delta x'$ being

$$1 - e \cos(x' + \tfrac{1}{2}\delta z'),$$

when expressed in the above notation, he takes it to be what is equivalent to

$$1 - e \cos(x' + \tfrac{1}{2} e \sin x'),$$

which is only true for the first approximation when x_0 is taken $= z$.

In the second and third editions this error is corrected, but Newton contents himself with the more simple expression given by the first method.

We need not be surprised that Newton should have employed this method of solving the transcendental equation

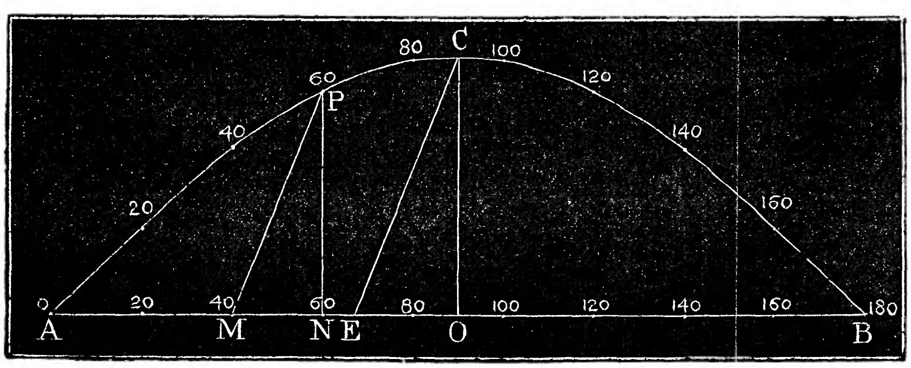
$$x - e \sin x = z,$$

since the method is identical in principle with his well-known method of approximation to the roots of algebraic equations.

For convenience of calculation, the approximate values x_0, x', x'', &c., should be so chosen that their sines may be taken directly from the tables without interpolation; and, since each approximation is independent of the preceding ones, this may always be done if x' be taken equal, not to $x_0 + \delta x_0$ itself, but to the angle nearest to $x_0 + \delta x_0$ which is contained in the tables, and if similarly x'' be taken equal to the tabular angle which is nearest to $x' + \delta x'$, and so on. In the first approximation it will be amply sufficient to use 5-figure logarithms, but in the subsequent ones tables with a larger number of decimal places should be employed.

A first approximate value of the eccentric anomaly corresponding to any given mean anomaly may be found by a very simple graphical construction, provided we have traced, once for all, a curve in which the ordinates are proportional to the sines of the angles represented on any given scale by the abscissæ.

This curve is commonly called "the curve of sines." It will be sufficient to trace the portion of the curve for which the ordinates are positive.



Let A O B be the line of abscissæ, and let A O be taken equal to O B, and let each of them be divided into 90 equal parts representing degrees of angle. Let A N be any abscissa representing the angle x, and let the corresponding ordinate N P $= c \sin x$; then the greatest ordinate will be O C $= c$, corresponding to the abscissa A O.

Suppose the curve line A P C B to be divided into 180 parts which correspond to equal divisions on the line of abscissæ A N O B.

Then if E be taken in A O so that E O $= e \times 57{\cdot}296$ divisions, or if A E $= 90 - e \times 57{\cdot}296$ divisions, and if C E be joined and P M be drawn parallel to it through P meeting the line of abscissæ in M, then A M will represent the mean anomaly corresponding to the eccentric anomaly represented by A N.

For, since the triangles P M N, C E O are similar,

$$\frac{\mathrm{MN}}{\mathrm{EO}} = \frac{\mathrm{PN}}{\mathrm{CO}} = \sin x,$$

and therefore M N $=$ E O $\sin x = 57{\cdot}296\ (e \sin x)$.

Hence M N represents the number of degrees in $x - z$, and therefore A M represents the mean anomaly z.

Conversely, if A M represents any given mean anomaly, then if M P be drawn parallel to E C, it will cut the curve in the point P corresponding to the eccentric anomaly.

By the employment of a parallel ruler we may find the eccentric anomaly corresponding to any given mean anomaly, or conversely, without actually drawing a line. For if we lay an edge of the ruler across the points E C and then make a parallel edge to pass through the point M it will cut the curve in the point P required.

Thus we may always find a first approximate value of the eccentric anomaly, without making repeated trials, whether the eccentricity be large or small.

I described this graphical method of solving Kepler's problem at the Birmingham meeting of the British Association in 1849. It is referred to in a paper by Mr. Proctor in vol. xxxiii. of the *Monthly Notices*, p. 390.

The construction is so simple that it has probably been proposed before, though I have nowhere met with it.

Note on Professor Zenger's solution of the same problem given in Number 9 of the last volume of the "Monthly Notices."

The only peculiarity in this solution is in the mode of obtaining the first approximate value employed. The subsequent approximations are carried on by means of the first method given above. Professor Zenger's process may be represented in a slightly different form as follows:—

We have

$$x - z = e \sin x,$$