Huygens and Mathematics

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1 Drawings

The drawing in the first figure is by Christiaan Huygens. You may still find some spots quite like it not far from here at ESTEC in Noordwijk. As you see, Huygens was a creditable amateur draftsman. He was also a professional draftsman in as far as his professional work involved drawing many mathematical figures.

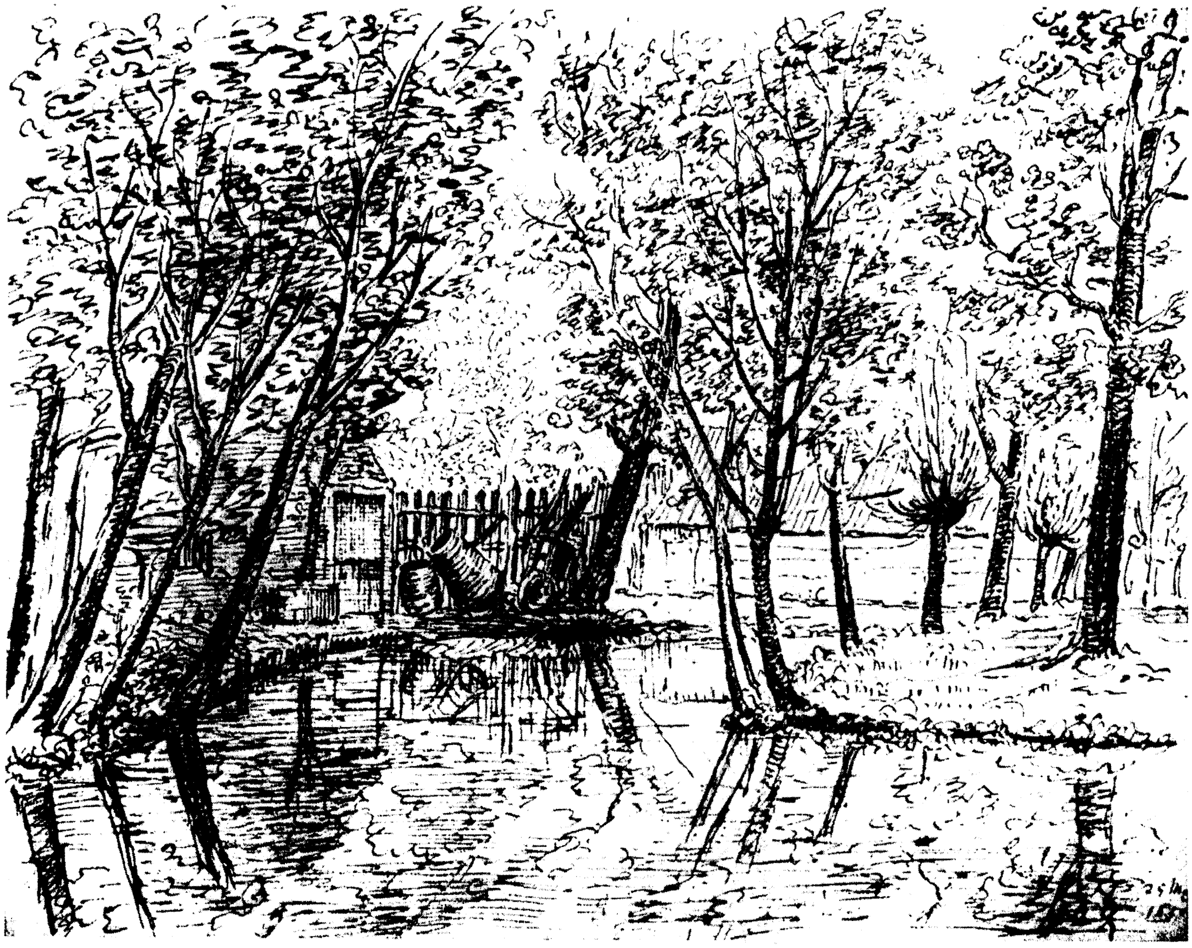

Figure 1: Drawing by Christiaan Huygens, 1657 (O.C. Vol 22, p78-79)

Drawings, especially those appearing in early notes and drafts of arguments, have a special status in the process of mathematical research: they often are the first materialisations of the thoughts in the brain of the mathematician. And even if they are redrawn later, and finally printed, these drawings retain a nearness to mathematical thought which written words and formulas often lack.

With this in mind, I decided to deal with my subject, Huygens and Mathematics, via Huygens' mathematical drawings, and I begin with a very brief, even somewhat hasty tour through the gallery of these drawings and figures.

2 Tour of the gallery

In Figure 2 we have Huygens thinking about rolling.

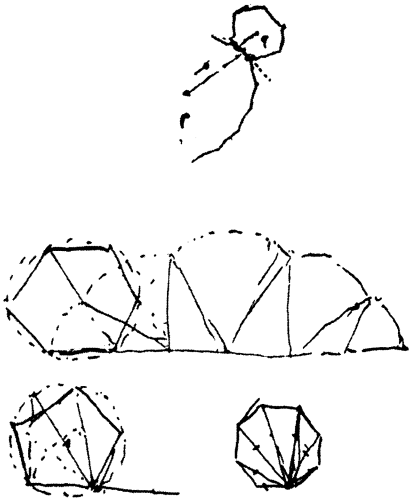

Figure 2: Sketches of rolling figures, 1678 (O.C. Vol 18, p402)

In the middle a hexagon is rolling along a line. He draws a rather bumpy approximation of the process of a circle rolling smoothly: a series of successive turns of the hexagon around a corner. Below a pentagon is rolling, above again a hexagon, now rolling along a curve. Huygens used these sketches to understand the rolling process. Obviously there is a limit process involved: regular polygons with more and more sides are less and less bumpy; real rolling is when the polygons transform into a circle.

The drawings in Figure 3 illustrate a similar approach. They are from the beginning of Huygens' career, when he studied the catenary, the form of a free hanging cord or chain.

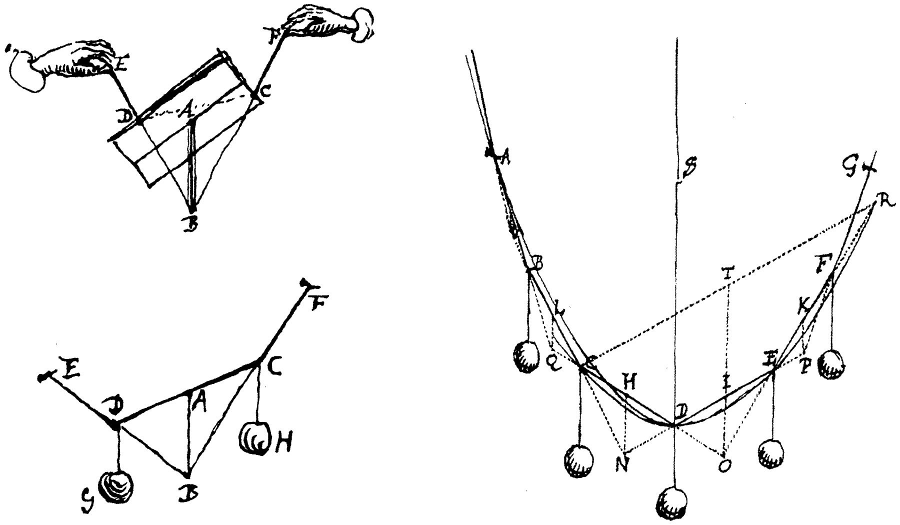

Figure 3: Approximating the catenary, 1646 O.C. Vol 11 pp37-40

Again he uses an approximation. He considers a weightless cord, with equal weights hanging at equal distances. What happens along the successive weights can be exactly determined by statics; the drawing suggests extrapolating this knowledge to the continuous case where the weights are, as it were, spread out all along the chain or the cord. Again, a limit process. In 1646 Huygens managed to prove by such an extrapolation that the catenary could not be a parabola (as Galileo had suggested), but only much later was he able to determine the true form of the curve.

Then another drawing (Figure 4), from October 27th, 1657, and marked (in Greek) *Heureka*, so Huygens had found something. What that was I'll tell later. For now we'll just look at the elements of the drawing.

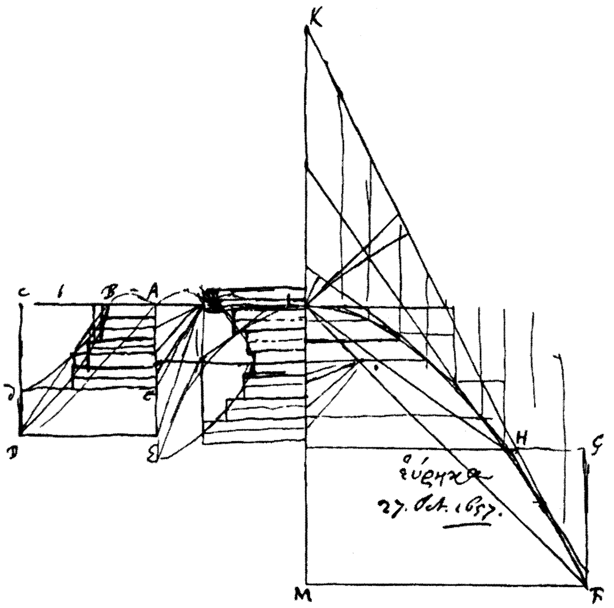

Figure 4: Curves: tangents and areas, 1657 (O.C. Vol 14 p234)

There are curves and axes. Along the curve to the right we see a sequence of tangents. Near the point where they touch the curve they almost coincide with it. The curve is approximated by a polygon of tangent pieces along it.

In the middle there is another curve. Over an area between this curve and the vertical axis narrow strips are drawn; together they form a rectilinear area approximating the area to the right of the curve.

Small strips under a curve and small straight tangent segments along a curve; they are perpetually recurring themes in Huygens' drawings; we will see more of them. Huygens saw them as very small, or becoming ever smaller, or infinitely small; I will use the term infinitesimals for these elements. And of course you sense their relation to what we know as differentiation and integration.

Another recurring theme in the drawings is curves. Figure 5 shows an example taken from a letter Huygens wrote in 1694.

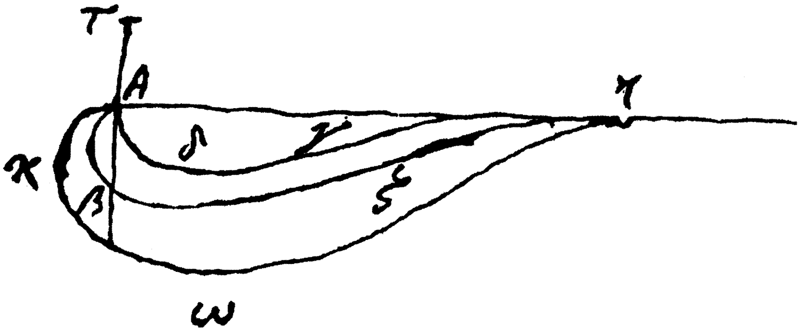

Figure 5: Curves: the 'paracentric isochrone', 1694 (O.C. Vol 10 p668)

Huygens called the three curves 'paracentric isochrones'; they had to do with a complicated problem, actually at the very edge of research at the time, about motion in a vertical plane along curved trajectories.

Another pair of isochronic curves drawn by Huygens is in Figure 6. I show them mainly because I like the spiralling effect.

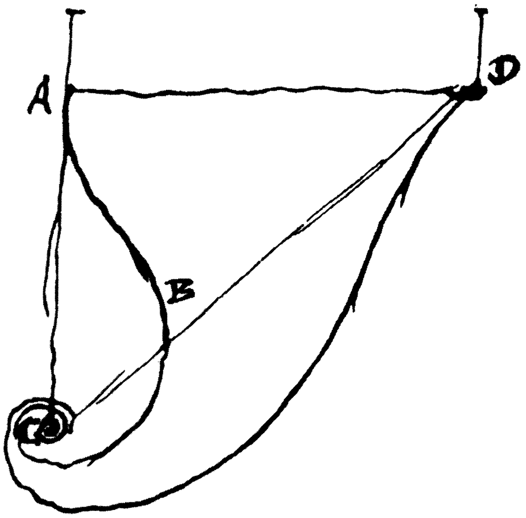

Figure 6: Curves: a spiralling isochrone, 1694 (O.C. Vol 10 p668)

Figure 7 shows a curve whose nature is more easily explained. It concerns what was at the time called an 'inverse tangent problem'.

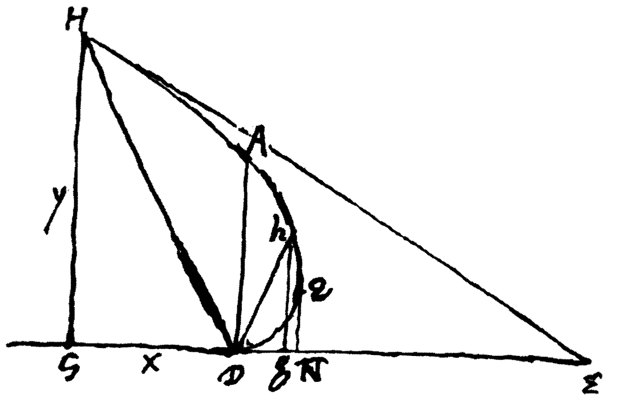

Figure 7: Curves: solution of an 'inverse tangent problem', 1694 (O.C. Vol 10 p475)

The usual tangent problem was: given a curve, determine its tangents. The inverse one was: given a property of tangents, determine a curve whose tangents have that property. Here the property is that at any point *H* on the curve, the subtangent, i.e. the segment along the axis below the tangent, should be equal to the sum of the coordinates *x* and *y* (Huygens takes *x* and *y* positive):

$$GE = x + y$$

You will realise that such problems are equivalent to differential equations. In the case of figure 7 the corresponding differential equation is:

$$\frac{dy}{dx} = \frac{y}{x+y}$$

These inverse tangent problems were difficult, indeed often very difficult.

I noted that the seventeenth-century infinitesimals involved in tangents and areas of curves relate to what soon after became differentiation and integration. Similarly, curves in the seventeenth century had the role that was later taken over by the concept of function. Actually, that transition came later, roughly by the middle of the eighteenth century. For Huygens, curves, not functions, were the natural means to represent mathematical relationships.

Finally three drawings (Figure 8) showing Huygens at work on a curve called the conchoid; it is the one from *A* to *D* in the left-hand drawing, in which Huygens first roughly sketched the curve.

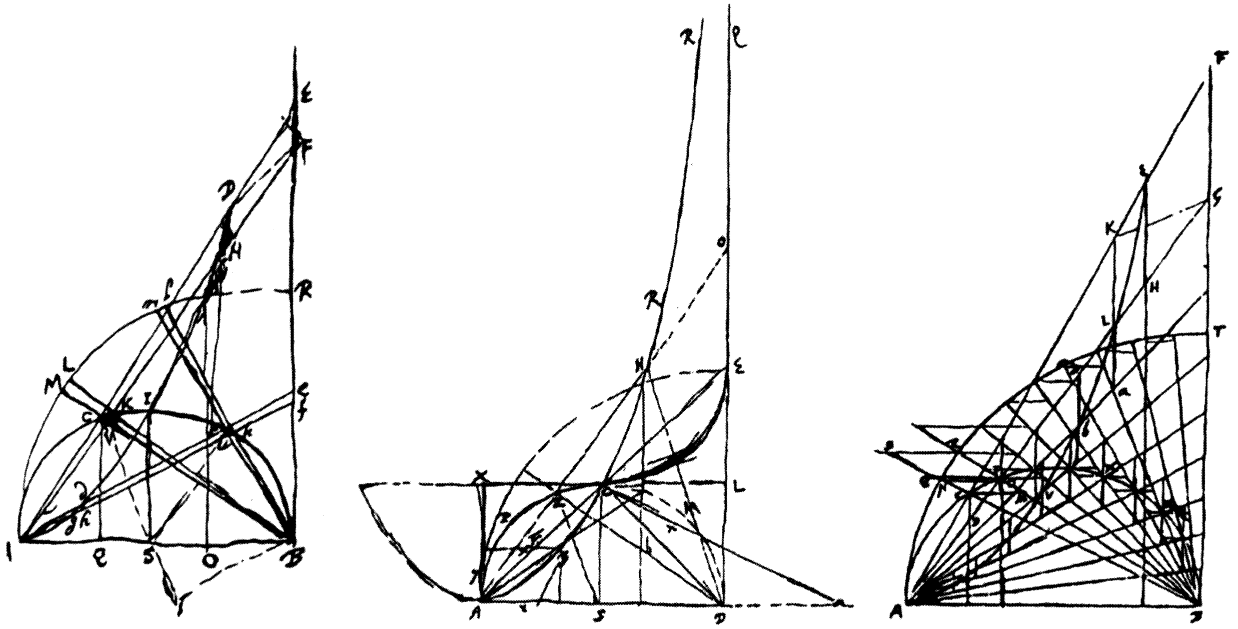

Figure 8: At work on the conchoid, 1657 (O.C. Vol 14 pp309-311)

You note the infinitesimals he was interested in here: they are the small triangular strips. In the middle drawing he added some details and apparently decided that the drawing was still too sketchy for clarity about the infinitesimals, so for the right-hand drawing Huygens turned to tools of the trade, ruler and compass, to get a better result.

3 Seeing through the drawings

So far some glimpses from the gallery of Huygens' mathematical drawings. How did they function in Huygens' research?

Obviously they helped him, first of all to order complex spatial information. But they also showed him something that is not on them. He could, as it were, see through the drawings to what cannot be represented in a static drawing, notably motion and the infinitely small. He could see motion of objects along curves, and he could see limits when rolling polygons turned into a rolling circle and when curves temporarily took the form of a polygon of tangent lines.

I shall now turn to a few examples in which Huygens used his drawings in this way to represent the unrepresentable. I divide them according to the following three themes: infinitesimals and limits, motion, and the modelling of processes of movement and change.

3.1 Infinitesimals and limits

For the infinitesimals and limits I return to a drawing shown earlier (Figure 9), the one with the '*heureka*', which I used as an illustration of infinitesimals, the small tangent parts along a curve and the small strips approximating the area under a curve. The curve to the right in the drawing is a parabola; the one in the middle is a hyperbola.

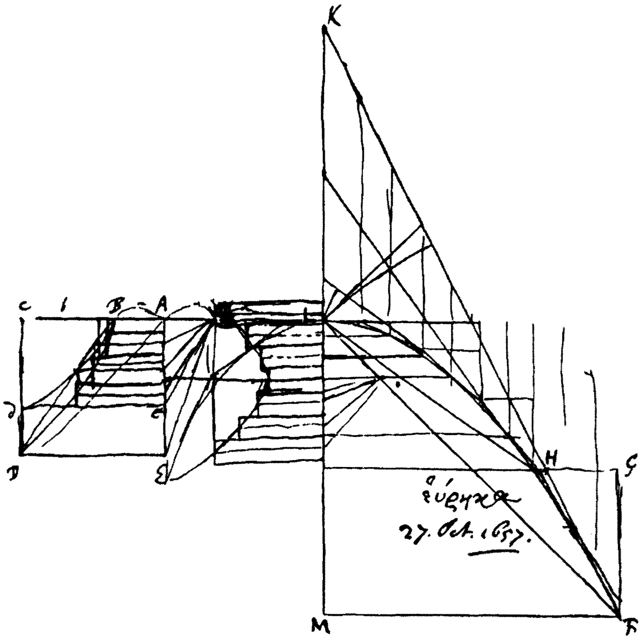

Figure 9: Arc lengths and areas, 1657 (O.C. Vol 14 p234)

What Huygens found – "*heureka!*" – was a relationship between two problems that were famously difficult at the time. The one was to determine the arc length of a parabola between two given points on it: the so-called 'rectification of the parabola'. The other problem was to determine the area under a hyperbola between two given ordinates: the 'quadrature of the hyperbola'. Around 1657, when Huygens made the drawing, a few mathematicians had seen that the quadrature of the hyperbola depended on logarithms.

Huygens noticed that the small tangent pieces along the parabola are equal to the corresponding strips under the hyperbola. To see that requires an intimate familiarity with the properties of both curves. Huygens concluded that the sum of all the tangent pieces along the parabola is equal to the sum of the strips under the hyperbola. In the limit, when the corresponding pieces and strips are 'infinitely small', the sums become equal to the arc length of the parabola and the area under the hyperbola, respectively. Hence the two problems were strictly related: if the quadrature of the hyperbola was found, then the rectification of the parabola was found as well, and *vice versa*. And thus Huygens had found that for determining the lengths of parabolic arcs one needed logarithms in the same way as for the quadrature of the hyperbola.

The drawing, then, illustrates how Huygens used a sketch of curves and infinitesimals to see and understand the limit processes involved in measuring curvilinear lengths and areas.

It is instructive to compare this visual understanding of rectification with the modern, analytic, standard formula for the arc length of a curve with equation $y = f(x)$:

$$s = \int \sqrt{1 + (\frac{dy}{dx})^2}\, dx$$

Huygens' drawing, as it were, carries the proof that this formula indeed provides the arc lengths, as well as the fact that in the case of the parabola the function to be integrated is a hyperbola. Both the proof and the fact are implied in the formula, but they are much less visible than in the drawing.

3.2 Motion

Infinitesimals, such as in the previous example, occur in Huygens' work especially in connection with motion and dynamics. My second example is about a special kind of motion, namely the unrolling or 'evolution' of curves. Figure 10, taken from Huygens' book on pendulum motion from 1673, illustrates the process. The pendulum consists of a weight *P*, connected via a thread to a fixed point *K*, in a vertical plane in which two curved strips (of metal, for instance) *KM* and *KI* are fixed.

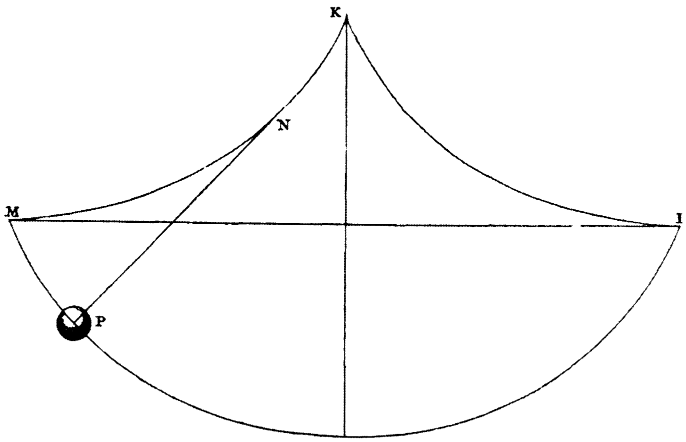

Figure 10: Unrolling a curve and the radius of curvature, 1673 (O.C. Vol 18 p105)

At rest, the weight hangs vertically under the point *K* and the thread is straight. If the weight is moved to the left, the thread will wind up along the curve *MK*; when the weight is at *P*, as drawn in the figure, the thread is partly straight (the part *PN*) and partly wound up along the curve (the part *NK*). If the weight is released from position *M* it will swing down, pass the lowest point, and move up towards *I*, and then return along the same path to *M*, then back down again, and so on. During this motion the thread first unwinds from the curve *MK* and then winds up along *KI*, and then winds off *KI* again and so on. Huygens was fascinated by this process of threads winding, or rolling up or from curves. In the case illustrated in the figure the two curves *KM* and *KI* are symmetrically placed halves of a special curve called the 'cycloid'; in that case the path *MPI* of the weight turns out to be a full cycloid. This phenomenon was crucial in Huygens' theory of oscillation. But the process of unrolling can be generalised to apply for any curve *MK* producing 'evolutes' of *KM* such as the curve described by *P*. Huygens derived various properties of curves and their evolutes, such as the fact that the curvature of the evolute at *P* is equal to the curvature of a circle with centre *N* and radius *NP*. This length is therefore called the 'radius of curvature' of the evolute at *P*.

Figure 11 shows some of the drawings through which Huygens 'saw' the process of unrolling along curves with varying curvatures, a process involving infinitely small line segments along the curve and even doubly infinitely small ones perpendicular to the curve.

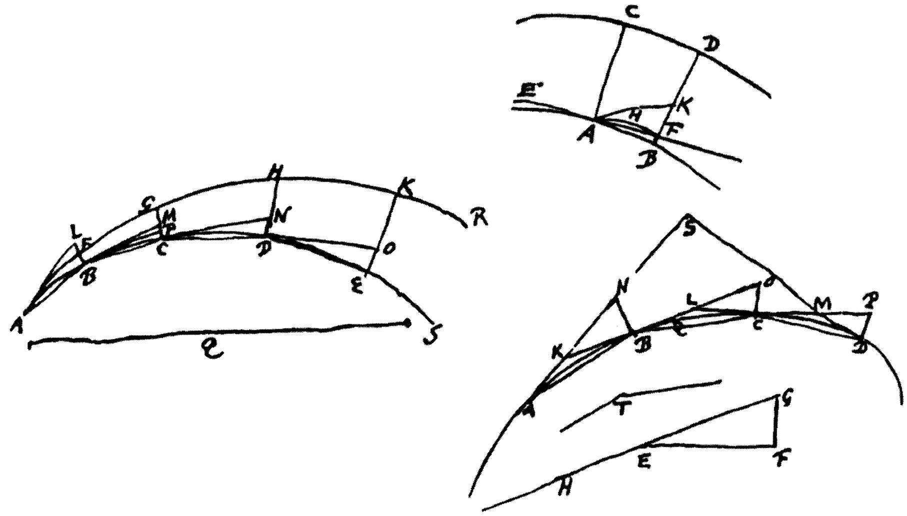

Figure 11: Evolutes and second-order infinitesimals, 1659 (O.C. Vol 14 pp400-402)

In the drawing to the left (the other two are variants or details of it) we recognise the tangent pieces *AL*, *BM*, *CN*, *DO*, etc., touching the curve *AS*. They are infinitesimals in the sense that in the limit, when the arc *AS* is divided in more and more (infinitely many) pieces, their number becomes (is) infinite, and the sum of their lengths becomes (is) equal to the total length of the arc *AS*. Now consider the small sides *BL*, *CM*, *DN*, *EO*, etc. of the triangles *ABL*, *BCM CDB*, *DEO* etc. They are perpendicular to the curve. In the limit process these perpendiculars will of course become zero, but the drawing suggests that they will also become very (infinitely) small with respect to *AB*, *BC*, etc. along the curve, which themselves also become infinitely small. Huygens made precise what this meant: unlike the 'first order' infinitesimals *AB*, *BC*, etc. which become zero but whose sum becomes equal to a finite value (namely the length of the curve), these perpendiculars are 'second order' infinitesimals; they will become zero and their sum will become zero as well. Huygens even provided an explicit proof of this phenomenon, which formed the basis of his further theory of the evolutes of curves.

Again it is instructive to compare Huygens' infinitesimal geometric arguments based on drawings with a modern formula for one of his results. Let ρ be the radius of curvature of a curve $y = f(x)$. Then

$$\rho = \frac{\frac{d^2y}{dx^2}}{(1+(\frac{dy}{dx})^2)^{\frac{3}{2}}}$$

One notes that the formula implies the same ingredients as Huygens' drawings: the tangents to the given curve (the derivative $\frac{dy}{dx}$), and the second order infinitesimals (the second order derivative $\frac{d^2y}{dx^2}$).

3.3 Models

Before I turn to my third and last example of Huygens' use of drawings, I owe the reader a remark about the mathematical technicalities in my discussion of Huygens' drawings (the example below has even more). I am aware that I may well lose some readers for the good reason of lack of time for, or affinity with, the details of the material. I hope however that the text can still be used as a guideline in taking some time to look at the drawings, note their charm and aesthetics, and imagine Huygens making them and pondering natural phenomena by means of the art of scientific drawing. These aspects, I feel, are in fact more important than the technical mathematical details.

The remaining example concerns the motion of a body, falling under the influence of gravity through a medium with resistance proportional to the velocity of the moving body. Figure 12 shows Huygens' drawing in which he incorporated the four variables involved in the process, velocity, acceleration, time and resistance, as well as their mutual relations.

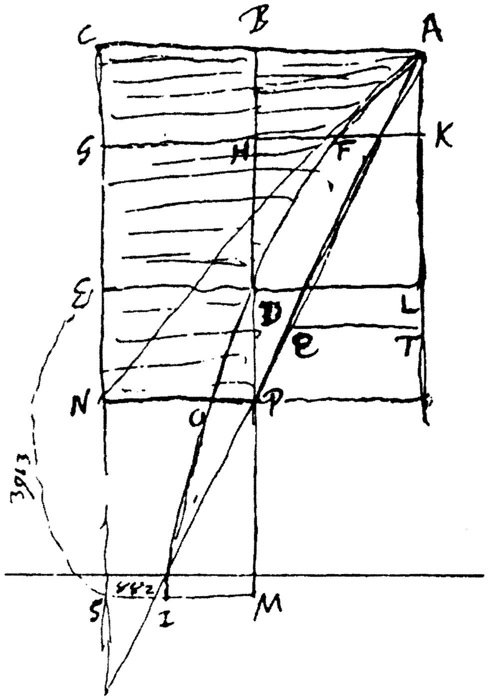

Figure 12: Huygens' geometrical model for fall in a medium with resistance proportional to velocity, 1668 (O.C. Vol 19 p102)

I will use the letters *v*, *a*, *t*, and *r* respectively for these, but note that Huygens did not use these letters in his drawing. His drawing served the function of a 'mathematical model', be it that at present we expect such a model to consist of a set of formulas giving the equations and/or differential equations, which describe the process. Huygens' model was a geometrical one.

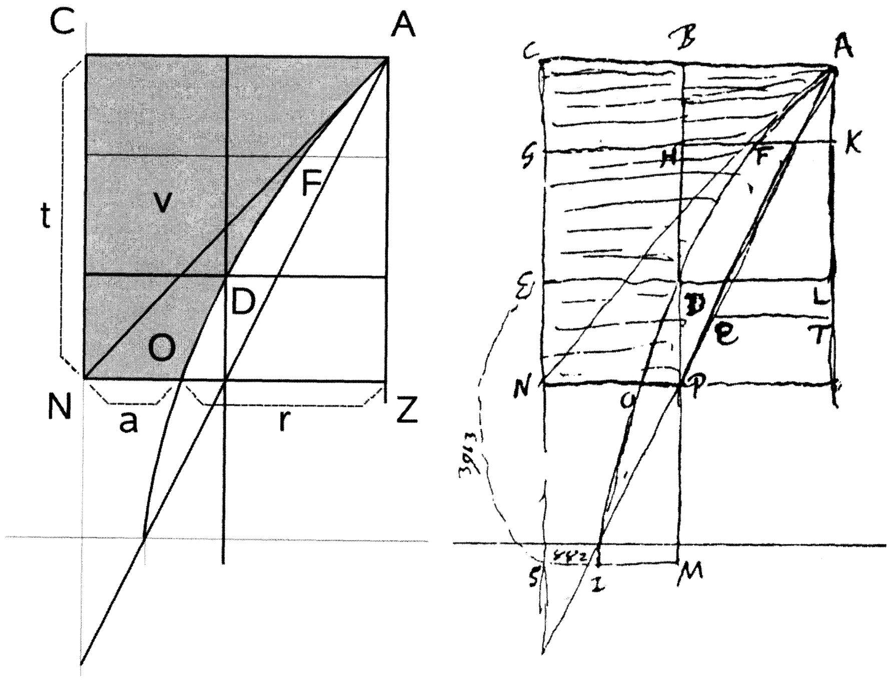

Figure 13: Fall in a medium with resistance, the variables redrawn

In Figure 13 I have indicated the elements of his drawing corresponding to the four variables mentioned (I have added the letter *Z* for a point which in Huygens' drawing was not lettered):

- time t is represented along a vertical axis AZ (or equivalently along CN)
- velocity v by an area under an as yet unknown curve $AFDO$ with respect to the axes CA and CN
- acceleration a by the ordinate NO of the unknown curve (whereby the relation $a = \frac{dv}{dt}$ is incorporated in the drawing)
- resistance r turns out to be represented by the segment OZ

The problem, then, is to determine the nature of the curve *AFDO* from the given that the resistance r is proportional to the velocity v.

Here is how Huygens argued on the basis of his drawing: If there were no resistance the velocity would be proportional to the time, according to Galileo's law of fall. Thus the area v would be proportional to the time t, which implies that the curve from A coincides with the axis AZ. We conclude that, because there is resistance, the unknown curve must extend from A to the left of the axis, and that CA represents the acceleration if resistance is absent, that is the gravitational acceleration (modern: g). Moreover the curve cannot extend to the left of the axis CS because then the acceleration would be negative and the body would rise again. Thus the geometrical model directly provides a global insight in the process of fall with resistance.

Then Huygens incorporates the given that the resistance is proportional to the velocity. NO represents the acceleration of the body, which is the sum of the gravitational acceleration represented by CA and the (negative) acceleration caused by the resistance. Thus the resistance is represented by $CA - NO$, that is, by OZ. Hence the curve has the property that the difference $CA - NO$ between any of its ordinates and the first ordinate CA is proportional to the area between these two ordinates. Note that the argument until now corresponds to the derivation of the differential equation $\frac{dv}{dt} = g - \beta v$ from the Newtonian law $F = m \times a$ and the given proportionality $r \propto \beta v$. (The correspondence, however, is less straightforward than it may seem because the drawing models proportionalities rather than equalities.)

But a differential equation is no solution of the problem it describes; it has to be solved. Similarly Huygens' result about the unknown curve is not the answer to which curve it is. He did determine the curve however, because in earlier studies he had encountered a curve with the same property, namely the 'Logarithmica', which was the seventeenth-century name of what now is called the exponential curve with equation $y = e^x$. Huygens' solution corresponds to the solution $v(t) = \frac{g}{\beta}\left(1 - e^{-\beta t}\right)$ of the differential equation above.

Finally, Figure 14 illustrates how Huygens could adapt his geometrical model with the four variable quantities involved in fall with resistance, to other assumptions about the relation between the resistance and the velocity.

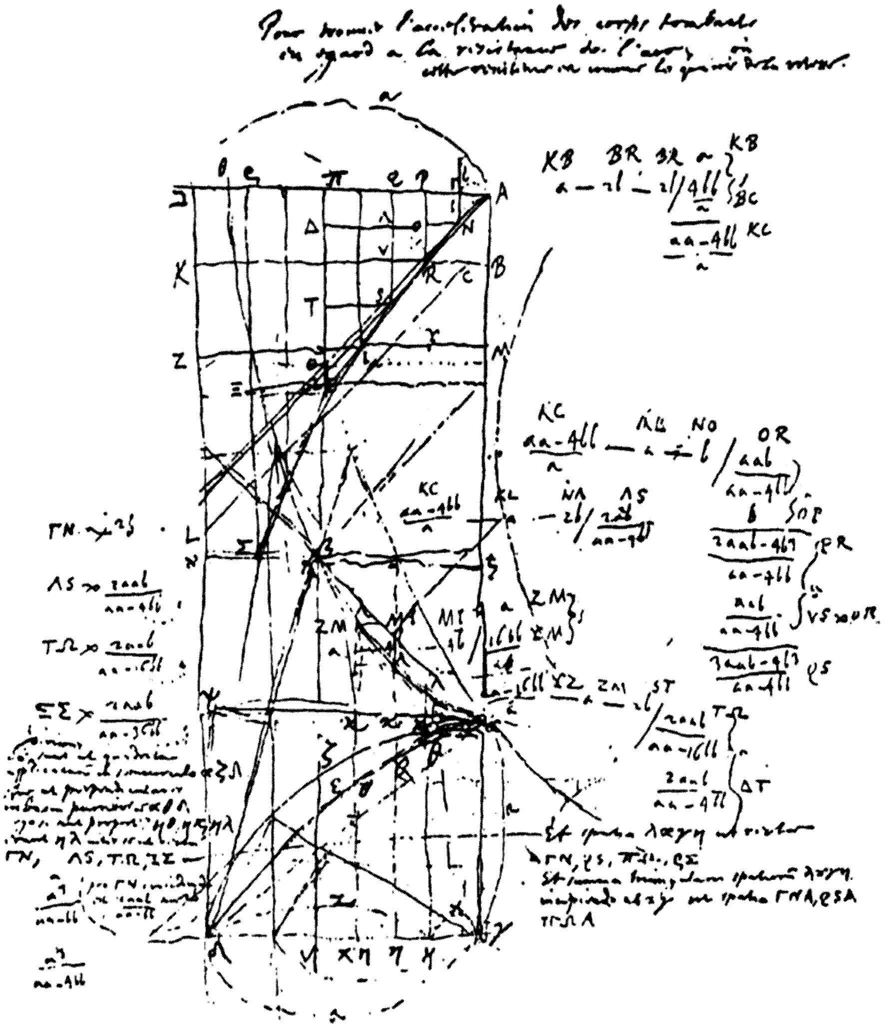

Figure 14: Fall in medium with resistance proportional to the square of the velocity, 1668 (O.C. Vol 19 p159)

In this case he assumed the resistance to be proportional to the square of the velocity, and succeeded in determining the required curve. Thus the drawing illustrates the power of geometrical modelling in the hands of the master who pioneered this approach.

4 Conclusion

After this brief survey, how to characterise Huygens' mathematics? It was geometrical infinitesimal analysis of curves and of motion. As to inspiration and imagery it was inseparable from mechanics; in style it was pure mathematics. It was geometrical because it was essentially dependent on drawings for handling infinitesimals, limit processes, and motion.

Huygens brought this kind of mathematics to great heights. But this mathematics passed. The next generations changed the style: the drawings were replaced by formulas; the infinitesimal lines and strips were replaced by differential quotients $\frac{dy}{dx}$ and integrals $\int ydx$; drawing figures was replaced by manipulation of formulas.

Newton and Leibniz set this transformation in motion. Huygens was the grand master of the previous style. In the long run, this style could not compete with the new, formula-based, differential calculus in solving the problems that confronted mathematicians and mechanicists.

So Huygens was no longer the solution, and, as the saying goes, if you're not part of the solution, you're part of the problem. Something like this has indeed happened to him. Historians of science and modern scientists often experience Huygens' mathematics as problematical and they sometimes see the stylistic aspect of his mathematics as a deplorable detour from how it should have been. This is understandable, because his mathematics is indeed difficult; it takes time, and lack of time is a valid excuse for a historian to take a short cut in the telling. But the idea that Huygens took a detour is nonsense. Geometrical analysis and physics was an essential and necessary phase in the development of mathematics.

Huygens' mathematics, then, was, and still is, authentic, brilliant mathematics; it can speak to us (given time) and bring us near to a powerful mode of thinking, which we may value and enjoy as much as any modern mathematical achievements.