

THE REDSHIFT-DISTANCE AND VELOCITY-DISTANCE LAWS

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Received 1992 May 8; accepted 1992 July 28

ABSTRACT

The distinction between Hubble's linear redshift-distance $z(L)$ law and the linear velocity-distance $V(L)$ law that emerged later is discussed, using first the expanding space paradigm and then the Robertson-Walker metric. The $z(L)$ and $V(L)$ laws are theoretically equivalent only in the limit of small redshifts, and failure to distinguish between the two laws obscures the basic elementary principles of modern cosmology. The linear $V(L)$ law [$V = HL$, where $H(t)$ is the Hubble term] applies quite generally in expanding homogeneous and isotropic cosmological models, and recession velocities can exceed the velocity of light. The $z(L)$ relation in its linear form ($cz = HL$), however, has no theoretical basis and can be used only in the limit of small redshifts. In general, the $z(L)$ relation is nonlinear (with the exception of exponentially expanding spaces) and must be derived separately for each particular model. The general distance-redshift $L(z)$ relation is obtained from the fundamental velocity-redshift relation $V(z) = cH_0 \int dz/H(z)$, where H_0 is the value of the Hubble term at the present epoch. Possible historical reasons for the confusion between the $z(L)$ and $V(L)$ laws, and why both are indiscriminately referred to as Hubble's law, are discussed.

Subject headings: cosmology: theory — galaxies: distances and redshifts

1. INTRODUCTION

Cosmologists generally fail to distinguish between the *redshift-distance law* proposed by Hubble (1929) and the *velocity-distance law* established later on theoretical grounds. Historians (North 1965, 1990; Smith 1979, 1982; and Hetherington 1990) also overlook the subtle but important distinction, and both laws are indiscriminately referred to as Hubble's law. But no general proof exists demonstrating that the two laws are equivalent, and perhaps for that reason many astronomers since the time of Hubble have viewed with reservation the velocity interpretation of extragalactic redshifts. The failure to distinguish between the linear redshift-distance law (an empirical approximation of limited validity) and the velocity-distance law (a theoretical derivation of unlimited validity) leads to confusion¹ and obscuration of the fundamental concepts of modern cosmology. The present treatment describes the two laws (§§ 2 and 3) and attempts to dispel the confusion concerning their equivalence, first by invoking the expanding space paradigm (§ 4) and then by using the Robertson-Walker metric (§ 5). The general velocity-redshift $V(z)$ relation [and hence the general $L(z)$ relation] is derived in § 6, and some particular velocity-redshift expressions are discussed. Possible reasons why confusion still persists concerning the two laws are mentioned in § 7.

¹ Two examples illustrate the confusion. In reviews of observational cosmology, Sandage (1961, 1988) remarks that Hubble's redshift-distance law is valid for all distances measured in Milne's (1933, 1935) world map, where the world map covers homogeneous and isotropic space at an instant of cosmic time. This remark, though true for the velocity-distance law, does not apply to the redshift-distance law. In a review of modern cosmology, Peebles et al. (1991) remark that the velocity-distance law requires relativistic corrections when the velocity approaches the velocity of light (i.e., the distance approaches the Hubble distance). This remark, though true for Hubble's redshift-distance law, does not apply to the velocity-distance law.

2. HUBBLE'S REDSHIFT-DISTANCE LAW

In 1929, Hubble presented the case for a linear relation between the redshift z and distance L of extragalactic nebulae:

$$z = \text{constant} \times L.$$

This linear redshift-distance $z(L)$ relation is usually expressed in the form

$$zc = H_0 L. \quad (1)$$

Here c is the velocity of light; the Hubble term $H(t)$ is everywhere a constant in homogeneous and isotropic space at a common instant of time t ; a zero subscript denotes the present time, $H_0 = H(t_0)$; and L is the distance to a galaxy of redshift z . Such a relation between redshift and distance had previously been suggested (see North 1965; Smith 1979, 1982; Osterbrock 1990), but Hubble's work, incorporating redshifts determined by Slipher and Humason, convinced most astronomers of the dependence of extragalactic redshifts on distance. Interpretation of the redshifts as a recession effect was much less enthusiastic (Zwicky 1929; Macmillan 1932; see Hubble 1937; Smith 1979).

The Hubble term is now numerically expressed in the form $H_0 = 100h \text{ km s}^{-1} \text{ Mpc}^{-1}$, and the value of h currently lies between 0.5 and 1.0. The inverse of the Hubble term corresponds to a time scale $1/H_0 = 10h^{-1} \text{ Gyr}$ and a Hubble distance

$$L_H = c/H_0 \quad (2)$$

of $3h^{-1} \text{ Gpc}$. Equation (1) with equation (2) takes the alternative form

$$z = L/L_H. \quad (3)$$

With the exception of cosmological models in which H is constant in time (see § 6.3), the redshift-distance $z(L)$ law is linear only for small redshifts $z \ll 1$ and short distances $L \ll L_H$. Because the various "distances" (distance at signal reception,

distance at signal emission, distance by apparent size, distance by radar measurement, and luminosity distance) introduce differences of second and higher order in z , equation (1) in its range of validity ($z \ll 1$) holds for most definitions of distance.

3. THE VELOCITY-DISTANCE LAW

Unlike the Hubble law, the origin of the velocity-distance law remains obscure. Friedmann in 1922 and 1924 pioneered “nonstationary” open and closed homogeneous and isotropic cosmological models, and in the following decade these expanding models were developed by Lemaître (1927, 1931) and other cosmologists (e.g., Eddington 1933, Robertson 1933; Tolman 1934). During the formative stages of the expanding space paradigm in the early 1930s it became apparent that expansion must be linear if the universe is homogeneous. Milne’s (1933, 1935) cosmological principle and his emphasis on kinematic symmetries inspired further studies (Robertson 1935, 1936a, 1936b, Walker 1936, 1944) that formally established the Robertson-Walker line element of an expanding homogeneous and isotropic space. This general line element leads immediately to the linear velocity-distance law

$$V = H(t)L, \quad (4)$$

(derived in § 5.1) in which proper distances L are well-defined in a space of uniform curvature at an instant of common (cosmic) time t , and the velocity of recession $V = dL/dt$ of all comoving bodies is relative to the comoving observer. Equations (2) and (4) yield

$$V = cL/L_H, \quad (5)$$

and the recession velocity equals the velocity of light at the Hubble distance L_H . Equation (4) is valid for all distances; if L is infinite, then also V is infinite. Previous to the Robertson-Walker metric, the concept of distance in expanding curved space loomed unclear, and even now a lack of clarity persists owing to emphasis on operational rather than geometrical concepts of distance. The proper distance in equation (4) is of the geometrical tape-measure kind, and the practical problem of its determination in cosmology should not be the reason for rejecting its conceptual utility.

4. THE EXPANDING SPACE PARADIGM

The expanding space paradigm emerged during the formative stages of modern cosmology amidst the controversy concerning the physical meaning of the extragalactic redshifts (North 1965). In an influential paper that enunciated the paradigm, Eddington (1930) said of the galaxies: “it is as though they were embedded in the surface of a rubber balloon which is being steadily inflated.” An expanding rubber surface aptly illustrates some of the properties of curved and dynamic space.² Like the cosmological principle, the expanding space

paradigm serves as a useful idealization enshrined in the Robertson-Walker metric.

Spatial homogeneity and isotropy imply a preferred (universal) space, and the time invariance of homogeneity and isotropy implies a preferred (cosmic) time. In the comoving frame, space is isotropic, receding bodies are at rest, and peculiar velocities have absolute values. (Thus the Sun’s absolute velocity is determined from the dipole anisotropy of the cosmic background radiation.) This picture of expanding and curved space is fully consistent with special relativity locally and general relativity globally (Robertson 1935; Walker 1936).

4.1. The Velocity-Distance Law

In expanding homogeneous and isotropic space, let comoving markers A, B, C, ... be equally spaced in a straight line, with a separating distance α between adjacent markers. Homogeneity requires that if B recedes from A at velocity $\alpha H(t)$, then also C simultaneously recedes from B at velocity $\alpha H(t)$, and so on. Hence C recedes from A at velocity $2\alpha H(t)$, D at velocity $3\alpha H(t)$, and so on, thus illustrating the tape-measure nature of proper distance in uniform space, and the essential linearity of the velocity-distance law:

$$V = \text{constant} \times L,$$

where the “constant” is constant in space but not necessarily in time. Unlike the empirical redshift-distance law, this result is valid for all distances L . The “constant” is usually determined by assuming that $V = cz$ for small z , thereby obtaining from the $z(L)$ law of equation (1) the $V(L)$ law of equation (4). If $V(L)$ were nonlinear, the receding markers A, B, C, ... would become unequally spaced, and homogeneity would be destroyed. This simple argument demonstrates that time-invariant homogeneity, as expressed by the cosmological principle (all places are alike at each instant in time), requires a linear velocity-distance law.

The implications of a linear velocity-distance law seem startling: the velocity of recession has no upper limit. Inside a Hubble sphere of radius L_H , the recession velocity of comoving bodies is subluminal ($V < c$), and outside is superluminal ($V > c$). Light emitted toward the observer by a body outside the Hubble sphere travels in space at velocity c , but because space itself recedes superluminally, the light actually recedes. The light may eventually reach the observer, however, if the Hubble sphere expands in the comoving frame, i.e., $dL_H/dt > c$ (Harrison 1991, 1992).

4.2. Wave Stretching

Wave-stretching illustrates an important application of the expanding space paradigm. The rule is that waves, wave trains, and distances between wave packets of radiation in space are progressively stretched and vary in proportion to the scale factor $R(t)$. Let λ_1 and λ_0 be the wavelengths at emission and reception, respectively, and R_1 and R_0 the corresponding values of the scale factor; from the wave-stretching relation $\lambda \propto R$, we have $\lambda_0/\lambda_1 = R_0/R_1$, and the redshift expression $z = (\lambda_0 - \lambda_1)/\lambda_1$ gives Lemaître’s (1927, 1931) important result

$$z = R_0/R_1 - 1. \quad (7)$$

Thus a redshift $z = 1$ means the universe has expanded twofold since the emission of light with this redshift. According to the expanding space paradigm, the progressive stretching of waves is not the familiar Doppler effect (Harrison 1981) and justifies the reservation of earlier workers who hesitated over the appropriateness of a Doppler interpretation.

² Milne revolted against attributing to space, which “by itself has no existence” (Milne 1932), the properties of curvature and expansion, and in protest developed a theory—kinematic relativity—that rejected general relativity (Milne 1933, 1935). Of the expanding space paradigm, he said, “This concept though mathematically significant has by itself no physical content; it is merely the choice of a particular mathematical apparatus for describing and analysing phenomena. An alternative procedure is to choose a static space, as in ordinary physics, and analyse the expansion-phenomena as actual motions in this space” (Milne 1934). Rejection of the expanding space paradigm in favor of Milne’s picture of expansion in fixed Euclidean space contradicts general relativity and leads to the conclusion that the universe possesses a center and an edge. In numerous popular but misleading treatments, Milne’s center is associated with the big bang.

Equation (7) applies to the propagation of time intervals. Let pulses of radiation be emitted by a source of redshift z at time intervals Δt_1 ; the distance $c\Delta t$ that separates the propagating pulses increases steadily in expanding space and the pulses arrive at the receiver at intervals $\Delta t_0 = (1 + z)\Delta t_1$.

5. ROBERTSON-WALKER METRIC

The expanding space paradigm derives from the fundamental assumption of invariant homogeneity that underlies the Robertson-Walker line element

$$ds^2 = dt^2 - R(t)^2[dr^2 + f(r)^2(d\theta^2 + \sin^2\theta d\phi^2)], \quad (8)$$

established formally by Robertson (1935, 1936a, b) and Walker (1936, 1944), and anticipated by Friedmann (1922, 1924), Lemaitre (1927, 1931), Robertson (1929), and Eddington (1930). In this line element, r , θ , ϕ are comoving space coordinates, $f(r) = \sin r$, r , or $\sinh r$ corresponding to curvature-constant values $k = 1, 0$, or -1 , respectively, and $R(t)$ is the scale factor.

5.1. Velocity-Distance Law

The proper distance of a comoving body of fixed coordinate distance r from a comoving observer is $L = cR(t)r$. Because of the constancy of r , the recession velocity of the comoving body is $dL/dt = c\dot{R}r$ (where $\dot{R} = dR/dt$). The velocity $V = dL/dt$ is $V = (\dot{R}/R)L$, and with $H(t) = \dot{R}/R$ we obtain the velocity-distance law of equation (4).

5.2. Lemaitre's Redshift Law

Lemaitre's redshift equation (7) is usually obtained from the radial null-geodesics ($ds = 0$, $d\theta = 0$, $d\phi = 0$) of the Robertson-Walker line element. Thus $dr = \pm d\tau$, where $d\tau = dt/R$, and the equation to the past light cone (with the minus sign) is

$$r = \tau_0 - \tau. \quad (9)$$

For a body at a fixed comoving coordinate distance r , we are given that $dr = 0$, and an interval of conformal time $d\tau$ is constant and everywhere equal to $d\tau_0$ on the light cone. At emission $d\tau_1$ equals $d\tau_0$ at reception, or $dt_1/R_1 = dt_0/R_0$, and propagated intervals of cosmic time vary with the scale factor $R(t)$. Because $v_0 dt_0 = v_1 dt_1$, and $v_0 \lambda_0 = v_1 \lambda_1$ for frequency ν and wavelength λ , we obtain the wave-stretching result $\lambda_0/\lambda_1 = R_0/R_1$ that gives Lemaitre's equation (7).

The distance of a source at the time when it emits light is

$$l = cR_1(\tau_0 - \tau_1), \quad (10)$$

and the distance to the same source when light arrives is

$$L = cR_0(\tau_0 - \tau_1), \quad (11)$$

and the two distances are related by $L = (1 + z)l$.

6. VELOCITY-REDSHIFT EXPRESSIONS

6.1. Doppler Velocity-Redshift Formula

The Doppler formula

$$V(z) = c(z^2 + 2z)/(z^2 + 2z + 2) \quad (12)$$

is valid for peculiar local motions and not for comoving global motions. We cannot combine the velocity-distance $V(L)$ law of equation (4) and the Doppler $V(z)$ formula of equation (12) to yield a general redshift-distance $z(L)$ relation. Cosmological models have different $V(z)$ relations that are generally quite unlike the Doppler formula except in the limit of small z . Thus,

for $z \ll 1$, Lemaitre's redshift equation (7) yields

$$z \simeq \Delta R/R \simeq L/L_H, \quad (3a)$$

where $\Delta R/R = H_0 \Delta t = H_0 L/c$ for $L = c\Delta t$, and from $V = cL/L_H$ we arrive at the classical approximation $V = cz$.

6.2. The General Redshift-Distance Relation

The distance to a comoving body is

$$L = cR_0 r = cR_0 \int_{t_1}^{t_0} dt/R(t),$$

from equation (9), where the integral is from emission at time t_1 to reception at time t_0 . Using the velocity-distance law $V = H_0 L$ and

$$dt/R = dR/HR^2 = -dz/HR_0,$$

we find

$$V(z) = cH_0 \int_0^z dz'/H(z'), \quad (13)$$

and this, not the Doppler formula of equation (12), is the most general form of the velocity-redshift relation.

6.3. Models of Constant H

Clearly, if H is constant in time, i.e., $H(z) = H_0$, then equation (13) gives

$$V = cz, \quad (14)$$

and this pseudo-classical Doppler expression holds for all values of z in exponentially expanding de Sitter, steady-state, and inflationary spaces. From the $V(L)$ law we see that the linear Hubble law

$$z = L/L_H \quad (3)$$

is true for all z without approximation. The distance to the past light cone at the time of emission, according to equation (10), is

$$l = L_H z/(1 + z), \quad (15)$$

and the light cone at high redshift asymptotically approaches the surface of the Hubble sphere of constant radius L_H .

6.4. Friedmann Models

For the Friedmann models of zero cosmological constant,

$$(HR)^2(\Omega - 1) = k, \quad (16)$$

we have

$$\Omega H^2 R^3 = \Omega_0 H_0^2 R_0^3, \quad (17)$$

at zero pressure, where Ω is the density parameter. Hence

$$H(z) = H_0(1 + z)(1 + z\Omega_0)^{1/2}, \quad (18)$$

and on substituting this result in equation (13) and integrating, we find

$$\begin{aligned} \frac{V}{c} &= (\Omega_0 - 1)^{-1/2} \\ &\times \sin^{-1} \left(\frac{2(\Omega_0 - 1)^{1/2}}{\Omega_0^2(1 + z)} \{z\Omega_0 + (\Omega_0 - 2)[(1 + z\Omega_0)^{1/2} - 1]\} \right), \end{aligned} \quad (19)$$

and in the limit of small z this yields $V = cz$, as expected.

6.5. Power-Law Models

For models in which $R \propto t^n$, with n constant in the range $1 < n < 1$, we find

$$H(z) = H_0(1+z)^{1/n}, \quad (20)$$

and hence

$$V(z) = c[n/(1-n)][1 - (1+z)^{(n-1)/n}]. \quad (21)$$

The nonrelativistic (relativistic) Einstein-de Sitter model corresponds to $n = \frac{2}{3}$ ($n = \frac{1}{2}$). Comparison of equation (12) with equations (14), (19), and (21) shows that the expansion redshift is not a simple Doppler effect.

7. DISCUSSION

In the literature, both the redshift-distance $z(L)$ and the velocity-distance $V(L)$ relations are referred to as Hubble's law. Strictly, in a homogeneously expanding universe, the linear $z(L)$ relation advanced by Hubble is the true Hubble law, valid only for small redshifts, whereas the linear $V(L)$ relation is valid for all geometric distances L . Because of the curious custom of referring to both the $z(L)$ and $V(L)$ relations as Hubble's law, the $z(L)$ relation has acquired an undeserved validity that properly belongs to the $V(L)$ relation.

The habit of converting redshifts into radial velocities by means of the Doppler approximation $V = cz$, though convenient astronomically, has undoubtedly caused much of the confusion surrounding the $z(L)$ and $V(L)$ relations. Cosmologists in the late 1920s and early 1930s were greatly influenced by the properties of the de Sitter metric (Ellis 1989, 1990), that in its exponentially expanding form yields the expression $V = cz$ for all z . This simple pseudo-Doppler expression interchanges z and V in the $z(L)$ and $V(L)$ laws, and the two laws become equivalent, justifying their joint reference as Hubble's law in the case of this metric. But incorrect application of the pseudo-Doppler expression in other metrics has fostered the

belief that the $z(L)$ and $V(L)$ laws are generally identical. Expansion redshifts and Doppler redshifts are in fact physically distinct except in Milne's theory of kinematic relativity that rejects general relativity and the expanding space paradigm.

Newtonian cosmology provides a lucid description of the velocity-distance law in Euclidean space (Milne 1934; McCrea & Milne 1934), and the popularity of this treatment (Bondi 1952) has perhaps contributed to the belief that a linear velocity-distance law applies only to nonrelativistic recession velocities. The unlimited recession velocities of the velocity-distance law required by invariant homogeneity are fully consistent with general relativity.

From a purist point of view one cannot but deplore the expression *big bang*, "loaded with inappropriate connotations" (McVittie 1974), which conjures up a false picture of a bounded universe exploding from a center in space. In modern cosmology, the universe does not expand in space, but consists of expanding space,³ and this correct picture leads naturally to a distinction between the redshift-distance and the velocity-distance laws.

In all expanding homogeneous and isotropic cosmological models, the linear velocity-distance law is the fundamental relation, valid for all distances, and the linear redshift-distance law is only an approximate relation, valid for small redshifts and distances that are small compared with the Hubble distance.

I am indebted to W. H. McCrea for earlier discussions, and to P. J. E. Peebles, A. R. Sandage, and D. N. Schramm for recent comments on aspects of this work.

³ "The theory of [general] relativity brought the insight that space and time are not merely the stage on which the piece is produced, but are themselves actors playing an essential part in the plot" (de Sitter 1931).

REFERENCES

- Bondi, H. 1952, *Cosmology* (Cambridge: Cambridge Univ. Press)
 de Sitter, W. 1931, *Scientia*, 49, 1
 Eddington, A. S. 1930, *MNRAS*, 90, 668
 ———. 1933, *The Expanding Universe* (Cambridge: Cambridge Univ. Press)
 Ellis, G. F. R. 1989, in *Einstein and the History of General Relativity*, ed. D. Howard & J. Stachel (Boston: Birkhauser)
 ———. 1990, in *Modern Cosmology in Retrospect*, ed. B. Bertotti et al. (Cambridge: Cambridge Univ. Press), 97
 Friedmann, A. 1922, *Z. Phys.*, 10, 377
 ———. 1924, *Z. Phys.*, 21, 326
 Harrison, E. R. 1981, *Cosmology: The Science of the Universe* (New York: Cambridge Univ. Press)
 ———. 1991, *ApJ*, 383, 60
 ———. 1992, *ApJ*, in press
 Hetherington, N. S. 1990, *Am. Sci.*, 78, 142
 Hubble, E. 1929, *Proc. Natl. Acad. Sci.*, 15, 168
 ———. 1936, *The Realm of the Nebula* (New Haven: Yale Univ. Press)
 ———. 1937, *The Observational Approach to Cosmology* (Oxford: Clarendon)
 Lemaitre, G. E. 1927, *Ann. Soc. Sci. de Bruxelles*, 47A, 49
 ———. 1931, *MNRAS*, 91, 483
 Macmillan, W. D. 1932, *Nature*, 129, 93
 McCrea, W. H., & Milne, E. A. 1934, *Quart. J. Math. Oxford Ser.*, 5, 73
 McVittie, G. C. 1974, *QJRAS*, 15, 246
 Milne, E. A. 1932, *Nature*, 130, 9
 ———. 1933, *Z. Astrophys.*, 6, 1
 Milne, E. A. 1934, *Quart. J. Math. Oxford Ser.*, 5, 64
 ———. 1935, *Relativity, Gravitation and World Structure* (Oxford: Clarendon)
 North, J. D. 1965, *The Measure of the Universe* (Oxford: Clarendon)
 ———. 1990, in *Modern Cosmology in Retrospect*, ed. B. Bertotti et al. (Cambridge: Cambridge Univ. Press), 11
 Osterbrock, D. E. 1990, in *Modern Cosmology in Retrospect*, ed. B. Bertotti et al. (Cambridge: Cambridge Univ. Press), 247
 Peebles, P. J. E., Schramm, D. N., Turner, E. L., & Kron, R. G. 1991, *Nature*, 352, 769
 Robertson, H. P. 1929, *Proc. Natl. Acad. Sci.*, 15, 822
 ———. 1933, *Rev. Mod. Phys.*, 5, 62
 ———. 1935, *ApJ*, 82, 284
 ———. 1936, *ApJ*, 83, 187
 ———. 1936b, *ApJ*, 83, 257
 Sandage, A. R. 1961, *ApJ*, 133, 355
 ———. 1988, *ARA&A*, 26, 561
 Smith, R. W. 1979, *J. Hist. Astron.*, 10, 133
 ———. 1982, *The Expanding Universe: Astronomy's "Great Debate," 1900–1931* (Cambridge: Cambridge Univ. Press)
 Tolman, R. C. 1934, *Relativity, Thermodynamics and Cosmology* (Oxford: Clarendon)
 Walker, A. G. 1936, *Proc. Lond. Math. Soc.*, (2) 42, 90
 ———. 1944, *J. Lond. Math. Soc.*, 19, 219
 Zwicky, F. 1929, *Proc. Natl. Acad. Sci.*, 15, 773