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DYNAMICAL FRICTION FROM FLUCTUATIONS IN STELLAR DYNAMICAL SYSTEMS

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ABSTRACT

A test particle traveling through a collisionless gravitating background suffers a dissipative drag force known as dynamical friction. As with other dissipative forces, this friction must be related to fluctuations in the underlying medium (fluctuation-dissipation theorem). However, this long recognized aspect of the force did not easily yield to analysis until now, and Chandrasekhar's celebrated formula was obtained by considering momentum exchanges resulting from encounters between a test particle and field particles which were idealized as occurring sequentially. In this paper we return to the underlying basic physics and develop a theory of the interaction of the test particle with the stochastic force of the background. This enables us to derive in a unified way the Chandrasekhar formula for the friction (for the full range of m/M) and the heating of the particle by background fluctuations. This new approach shows that dynamical friction fits into the generic fluctuation-dissipation relation. It also readily suggests the modifications that would be required to treat dynamical friction and heating in an inhomogeneous background, as well as in the presence of initial correlations between background particles.

Subject headings: celestial mechanics, stellar dynamics

1. INTRODUCTION

A particle traveling through a collisionless gravitating background suffers a dissipative drag force known as dynamical friction. This phenomenon, which is of great practical importance in astrophysics [destruction of galaxy's globular clusters (Ostriker & Tremaine 1975), galaxy cannibalism (Tremaine, Ostriker, & Spitzer 1975), etc.] is customarily quantified by Chandrasekhar's dynamical friction formula (see eq. [1.1]), valid for the motion of a single point particle in an infinite homogeneous background. Now, in common with other dissipative forces, dynamical friction must arise in connection with fluctuations in the underlying medium: it must be an aspect of the very general fluctuation-dissipation relation. Indeed we recall that electrical resistance can be expressed in terms of the low-frequency component of voltage fluctuations in the resistor (Nyquist's theorem), and viscous friction is related to the low-frequency component of the stochastic force in the liquid (Reif 1965). This strongly suggests that it should be possible to derive a complete expression for dynamical friction from consideration of the statistics of stochastic forces in a gravitating system.

It seems that this was indeed Chandrasekhar's original aim (Chandrasekhar 1944a, b). He was able to show from statistical considerations that a dynamical friction force is required. But the early attempts to derive a formula for it from statistical mechanics did not yield satisfactory results. Chandrasekhar & Von Neumann (Chandrasekhar 1943, 1944a, b; Chandrasekhar & Von Neumann 1942, 1943) developed in great detail a statistical theory of stochastic motion in gravitational systems under certain simplifying assumptions, but could not derive the frictional effect due to the daunting mathematical complexity of the scheme. Chandrasekhar then devised an alternative *kinetic* approach which views the momentum exchange of the test particle and background as due entirely to successive binary encounters. It is this approach which first yielded the celebrated dynamical friction formula (Chandrasekhar 1943), but it remained unclear whether the summation of successive two-body encounters is not an oversimplified picture. The same uncertainty recurs in Fokker-Planck analyses of dynamical friction (e.g., Rosenbluth, MacDonald, & Judd 1957; Binney & Tremaine 1987) which again calculate the various diffusion coefficients in the binary approximation.

A rebirth of the stochastic approach to dynamical friction (Cohen 1975; Kandrup 1980, 1983) was able to reproduce Chandrasekhar's formula only when the test particle is much massive compared to background particles, and slow by comparison.

A third approach to dynamical friction, borrowed from plasma physics, is the polarization cloud method (Marochnik 1968; Kalnajs 1972; Binney & Tremaine 1987; Tremaine & Weinberg 1984; Bekenstein & Zamir 1990). The linearized collisionless Boltzmann equation is here used to derive the distortion in the background's distribution function due to the perturbation in the gravitational potential induced by the presence of the test particle. The associated wake-shaped distortion in the background's mass density field acts back on the test particle and slows it down. Thus dynamical friction is here viewed as the drag exerted on the test particle by the wake it induces in the field particles. Elegant though this approach may seem, it recovers Chandrasekhar's formula only in the limit of heavy test particle (Kandrup 1983).

A different approach was taken by Gilbert (1968) who developed a theory of collisional relaxation by starting from the Liouville equation, and expanding the dynamics in powers of $1/N$. He has shown that the leading terms of the force on a test particle consist of two parts: a term which is equivalent to the polarization drag, and a term which stands for the effect of the fluctuating gravitational field. Gilbert's approach is broader than the "polarization cloud" method since it produces also an heating mechanism. Although it is of much theoretical importance, this approach has not led to practical calculations such as an explicit formula for dynamical friction, or a clear demonstration of the equipartition process.

The pervasiveness of fluctuation-dissipation relations in physics begs for a consistent treatment of dynamical friction from this point of view. Therefore, we return in this paper to the underlying basic physics, i.e., the statistical mechanics of stochastic forces, and show how the derivation of the dynamical friction formula may be carried out in this framework. Not only does this most basic of all approaches to dynamical friction provides a deeper physical insight, but it readily suggests modifications of the usual description that can cope with an inhomogeneous background, and with the presence of correlations between background particles. Adaptation of the alternative approaches mentioned above to these new problems is not immediately possible. An exception is the polarization cloud method which has been successful in describing friction in circular orbits in spherical systems (e.g., Tremaine & Weinberg 1984; Weinberg 1989, and references therein).

Before going further, let us recall that the friction formula is not the whole story. In a Maxwellian background the formula would read

$$\frac{d\boldsymbol{v}}{dt} = -\frac{4\pi G^2(M+m)\rho \ln \Lambda}{v^3}\left[\operatorname{erf}(x) - \frac{2x}{\sqrt{\pi}} e^{-x^2}\right]\boldsymbol{v} , \tag{1.1}$$

where M is the mass of the test particle traveling through a homogeneous background of particles of mass m and velocity dispersion σ (total mass density ρ) with instantaneous velocity $\boldsymbol{v}$, $x \equiv v/\sqrt{2}\,\sigma$, and $\ln \Lambda$ is the Coulomb logarithm. The tendency of the particle to come to rest eventually, as equation (1.1) would have it, is opposed by fluctuations which force the velocity vector, portrayed by equation (1.1) as retaining its direction, to actually perform a Brownian random walk in velocity space. In fact the background not only drags on the test particle, but simultaneously "heats" it. This heating is necessary if the particle is to ever reach energy equipartition with the background population.

In Chandrasekhar's kinetic approach the drag comes from momentum exchange in direction parallel to the velocity; in his original paper (Chandrasekhar 1943) the momentum exchange in the normal directions is assumed to sum to zero. Actually, the normal components of velocity cannot remain zero in face of the fluctuations. The departure of these components from zero together with the fluctuation of the parallel velocity component is what generates the heating (Rosenbluth et al. 1957). The distinction between the treatment of heating and drag in the kinetic approach is noteworthy. In the stochastic approach here developed both effects are treated together as two sides of the same coin.

2. ENERGY EQUATION

A test particle in a stellar system is subject to a many-body force. If the system is in steady state, the force may be separated into a smooth part—derived from the potential Φ of the mean field of the system—and a fluctuating or stochastic force $\boldsymbol{F}(\boldsymbol{r}, t)$ arising mainly, but not exclusively, from near neighbors. This alluded separation can be made unique by demanding that the statistical (ensemble) average of $\boldsymbol{F}$ vanishes: $\langle \boldsymbol{F} \rangle = 0$. We shall be more precise later about the kind of ensemble to be considered. If the test particle's mass is M, we may write its equation of motion as

$$\frac{d\boldsymbol{v}}{dt} = -\nabla\Phi + M^{-1}\boldsymbol{F}(\boldsymbol{r}(t), t) , \tag{2.1}$$

where Φ satisfies Poisson's equation

$$\nabla^2\Phi = 4\pi G m n(\boldsymbol{r}) , \tag{2.2}$$

where m is a field particle's mass and $n(\boldsymbol{r})$ is the smoothed field particle density.

The force $\boldsymbol{F}$ turns out to be weak as compared with $-M\nabla\Phi$, the smoothed force. This because we expect $|\nabla\Phi| = O(GNmR^{-2})$ where N is the (large) number of particles making up the background, and R is its typical dimension (we assume the background *not* to be exactly spherical about the point in question). Now $|\boldsymbol{F}| = O(GMmn^{2/3})$ (Kandrup 1980). Since $N = O(nR^3)$, it is plain that

$$\frac{|\boldsymbol{F}|}{M|\nabla\Phi|} = O(N^{-1/3}) . \tag{2.3}$$

Likewise $\boldsymbol{F}$ is rapidly fluctuating. The briefest time scale of fluctuation, τ, is set by the motion of the nearest neighbors at distance $n^{-1/3}$ so that $\tau = O(n^{-1/3}\sigma^{-1})$. This has to be compared with the time scale T over which $\boldsymbol{v}$ changes significantly due to the smooth force; clearly $T \sim |\boldsymbol{v}|/|\nabla\Phi|$. Our previous remarks, together with the estimate $\sigma^2 \sim GNmR^{-1}$ from the virial theorem give

$$\frac{\tau}{T} \sim \frac{\sigma}{v} O(N^{-1/3}) \tag{2.4}$$

It is plain that unless v has become very small—and the tendency to equipartition is against this—the stochastic force is indeed rapidly fluctuating. We may thus conclude that in a large system $\boldsymbol{F}$ may be treated as weak and rapidly fluctuating.

So far we have said nothing about the permitted dimension of M. For our later arguments it will be useful to assume that the potential well about the test particle is shallow, namely

$$GMn^{1/3} \ll \sigma^2 . \tag{2.5}$$

This also means that the test particle perturbs the field particles only slightly, except those rare ones which approach it much nearer than $n^{-1/3}$.

Let us define the energy of the test particle as

$$E = M(\tfrac{1}{2}\boldsymbol{v}^2 + \Phi) . \tag{2.6}$$

Since the smooth potential Φ does not include all forces on the particle, this energy is not conserved. From equation (2.1) it follows that

$$dE/dt = \boldsymbol{F} \cdot \boldsymbol{v} . \tag{2.7}$$

According to the theory of Brownian motion (Reif 1965), frictional energy loss arises from an equation like (2.7) because $\boldsymbol{v}$ and $\boldsymbol{F}$ get correlated in the course of time as a result of the influence of $\boldsymbol{F}$ on the particle's dynamics. Evidently if the particle has entered the background at time $t = -\infty$, integration of equation (2.1) gives

$$\boldsymbol{v}(t) = \boldsymbol{v}(-\infty) + M^{-1} \int_{-\infty}^{t} \boldsymbol{F}[\boldsymbol{r}(t'), t']dt' - \int_{-\infty}^{t} \nabla\Phi[\boldsymbol{r}(t')]dt' , \tag{2.8}$$

which makes the correlation explicit.

We are interested in the expectation value $\langle \boldsymbol{F} \cdot \boldsymbol{v} \rangle$ which contains the dissipative energy loss. But use of $\boldsymbol{v}(t)$ from equation (2.8) in practice means we need to know the motion of the particle over a long time in order to perform the integrals. Of course fluctuations throw the particle off the deterministic path after some time making this procedure impractical. What is more, we have the feeling that the dissipation must come about from correlation over brief times, so that to follow the motion over long periods must be superfluous anyway. Then why not imagine that the particle is injected into the background at an earlier (but not infinitely earlier) time? Of course this procedure would entail transient effects generated by a sudden appearance of a test particle whose effect is hard to assess. If so, why not start the integrals in equation (2.8) from some time just prior to t? This is fine provided it is realized that the initial value of $\boldsymbol{v}$ at such time is not really known; it must, according to equation (2.1) itself, contain a stochastic contribution from the integral of $\boldsymbol{F}$. We may obtain the required information about the initial value by invoking conservation of momentum.

If at $t = -\infty$ the background as a whole was at rest, and at some time later—call it $t = 0$—the test particle's velocity was $\boldsymbol{v}_0 \equiv \boldsymbol{v}(t = 0)$, it is plain, by conservation of momentum, that

$$\boldsymbol{v}_0 - \boldsymbol{v}(-\infty) = -\frac{m}{M} \sum_{i=1}^{N} [\boldsymbol{u}_i(0) - \boldsymbol{u}_i(-\infty)] = -\frac{m}{M} \sum_{i=1}^{N} \boldsymbol{u}_i(0) , \tag{2.9}$$

where $\boldsymbol{u}_i(0)$ is the velocity of the ith field particle at $t = 0$, and the rightmost equality is due to the assumption that the center of mass of the background particles was initially at rest. This exact equation relates the stochastic unknown $\boldsymbol{v}_0$ to $\boldsymbol{v}(-\infty)$, which may be regarded as known, via knowledge of the $\boldsymbol{u}_i$ which are subject to some probability distribution. To the extent that the distribution is known, we obtain a distribution for $\boldsymbol{v}_0$. While continuing to use the formal result (2.8), we shall bring equation (2.9) to bear on the problem soon.

Calculating dE/dt from equation (2.8) at the time $t = \delta t$ briefly *after* $t = 0$, ensemble averaging, and interchanging time integral and statistical average, we have

$$\langle dE/dt \rangle_{\delta t} = \langle \boldsymbol{F}(\delta t) \cdot \boldsymbol{v}(-\infty) \rangle + M^{-1} \int_{-\infty}^{\delta t} \langle \boldsymbol{F}[\boldsymbol{r}(\delta t), \delta t] \cdot \boldsymbol{F}[\boldsymbol{r}(t'), t'] \rangle dt' - \int_{-\infty}^{\delta t} \langle \boldsymbol{F}[\boldsymbol{r}(\delta t), \delta t] \cdot \nabla\Phi[\boldsymbol{r}(t')] \rangle dt' . \tag{2.10}$$

We shall now argue that the second contribution to $\langle dE/dt \rangle_{\delta t}$ represents the heating due to fluctuations while the first stands for the energy loss or drag.

3. THE HEATING TERM

3.1. *The Autocorrelation Function*

The tensor $\langle \boldsymbol{F}[\boldsymbol{r}(t), t]\boldsymbol{F}[\boldsymbol{r}(t'), t'] \rangle$, whose trace appears in equation (2.10), can only depend on t and t' through the difference $s \equiv t' - t$ because of the assumed stationary nature of the medium. We thus define the *correlation tensor* by

$$\overleftrightarrow{C}(\boldsymbol{r}, \boldsymbol{r}', s) \equiv \langle \boldsymbol{F}(\boldsymbol{r}', t + s)\boldsymbol{F}(\boldsymbol{r}, t) \rangle . \tag{3.1.1}$$

A time translation $t \to t - s$ in (3.1.1) should not change the correlation function. However, formally the two $\boldsymbol{F}$-values exchange roles and $s \to -s$. Therefore, in explicit component notation we have the identity (Reif 1965; Landau & Lifshitz 1980) $C_{ab}(\boldsymbol{r}, \boldsymbol{r}', s) = C_{ba}(\boldsymbol{r}', \boldsymbol{r}, -s)$. Time reversal is a good symmetry in gravitational systems, and must leave the correlation tensor (as seen by a moving test particle) unchanged. Imagine that the test particle travels from $\boldsymbol{r}$ to $\boldsymbol{r}'$ in time s. Under time reversal of the system the test particle moves from $\boldsymbol{r}'$ to $\boldsymbol{r}$. We thus have $C_{ab}(\boldsymbol{r}, \boldsymbol{r}', s) = C_{ab}(\boldsymbol{r}', \boldsymbol{r}, -s)$. Combining the two results we learn that $\overleftrightarrow{C}$ is a symmetric tensor, and even in s.

Suppose that in the first integral in equation (2.10), prior to taking the dot product, we set $t' \to \delta t - s$. We can cast it into the form

$$\int_{-\infty}^{\delta t} \langle \boldsymbol{F}[\boldsymbol{r}(\delta t), \delta t]\boldsymbol{F}[\boldsymbol{r}(t'), t'] \rangle dt' = \int_{-\infty}^{0} \overleftrightarrow{C}[\boldsymbol{r}(0), \boldsymbol{r}(s), s]ds . \tag{3.1.2}$$

Use of the last identity, together with time translational invariance, shows that the integral of $\overleftrightarrow{C}$ from 0 to ∞ has exactly the same value, and so we may write our desired integral as half the integral of $\overleftrightarrow{C}$ from $-\infty$ to ∞. Thus

$$\int_{-\infty}^{\delta t} \langle \boldsymbol{F}[\boldsymbol{r}(\delta t), \delta t] \cdot \boldsymbol{F}[\boldsymbol{r}(t'), t'] \rangle dt' = \tfrac{1}{2} \int_{-\infty}^{\infty} \mathrm{Tr}\, \overleftrightarrow{C}[\boldsymbol{r}(0), \boldsymbol{r}(s), s]ds , \tag{3.1.3}$$

where Tr $\overleftrightarrow{C}$ denotes the trace of $\overleftrightarrow{C}$.

Now the Wiener-Khintchine theorem (Reif 1965) states that the Fourier transform of an autocorrelation function equals the power spectrum of the quantity being autocorrelated. The integral in (3.1.3) is just the sum of the Fourier transforms of the diagonal components of $\overleftrightarrow{C}$ at vanishing frequency, so that each term must be positive. Therefore, the second term in equation (2.10) is responsible for a *gain* in the energy of the test particle as a result of fluctuations. It is a *heating* term. We must stress the naturalness with which the heating term appears in the fluctuation approach; by contrast, the polarization cloud approach is not very well suited to bringing this term out, and is not discussed in developments of this method (Marochnik 1968; Kalnajs 1972; Bekenstein & Zamir 1990).

Let us define a tensor L as the temporal integration of $\overleftrightarrow{C}$ over all s. The heating term is just proportional to Tr $\overleftrightarrow{L}$. Under the approximation that the test particle moves uniformly with constant velocity $\boldsymbol{v}_0$

$$\overleftrightarrow{L}(\boldsymbol{v}_0) \equiv \int_{-\infty}^{\infty} \overleftrightarrow{C}(\boldsymbol{r}, \boldsymbol{r} + \boldsymbol{v}_0 s, s)ds \equiv \int_{-\infty}^{\infty} \langle \boldsymbol{F}(\boldsymbol{r} + \boldsymbol{v}_0 s, s)\boldsymbol{F}(\boldsymbol{r}, 0)\rangle ds \, . \tag{3.1.4}$$

There is no point in taking into account here the deviations from this uniform path due to $\boldsymbol{F}$ because $\overleftrightarrow{C}$ already contains two $\boldsymbol{F}$-values and any corrections of the kind considered would be small because $\boldsymbol{F}$ is small. There is equally no point in accounting for deviations due to the smooth force since this acts over times long compared to the time over which $\overleftrightarrow{C}$ is nonvanishing (a few times τ, see eq. [2.3]). Hence the approximation envisioned here is a good one.

Let us first denote the distribution function (DF) that we start with by $f_*(\boldsymbol{r}, \boldsymbol{u})$. The corresponding particle density is

$$n(\boldsymbol{r}) = \int f_*(\boldsymbol{r}, \boldsymbol{u})d^3u \, , \tag{3.1.5}$$

so that $\int n \, d^3r = N$, the total number of particles in the system. Plainly the stochastic force acting on the particle of mass M is

$$\boldsymbol{F}(\boldsymbol{r}, t) = -GMm\nabla\left[\sum_{j=1}^{N} \frac{1}{|\boldsymbol{r} - \boldsymbol{r}_j(t)|} - \int \frac{n(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} d^3r'\right], \tag{3.1.6}$$

where the integral is just Φ from Poisson's equation with the smoothed density $n(\boldsymbol{r})$ as source. Expressing $|\boldsymbol{r} - \boldsymbol{r}_j(t)|^{-1}$, $|\boldsymbol{r} - \boldsymbol{r}'|^{-1}$ and $n(\boldsymbol{r})$ as Fourier integrals with respect to $\boldsymbol{r}$, applying the convolution theorem, and carrying out the gradient we find

$$\boldsymbol{F}(\boldsymbol{r}, t) = -i\frac{GMm}{2\pi^2}\int\left[\sum_{j=1}^{N} \exp\left[-i\boldsymbol{k} \cdot \boldsymbol{r}_j(t)\right] - n_{\boldsymbol{k}}\right]\frac{\boldsymbol{k} \exp(i\boldsymbol{k} \cdot \boldsymbol{r})d^3k}{k^2} , \tag{3.1.7}$$

with

$$n_{\boldsymbol{k}} \equiv \int n(\boldsymbol{r}) \exp(-i\boldsymbol{k} \cdot \boldsymbol{r})d^3r \, . \tag{3.1.8}$$

Now

$$\langle \exp(-i\boldsymbol{k} \cdot \boldsymbol{r}_j)\rangle_* \equiv N^{-1}\int \exp(-i\boldsymbol{k} \cdot \boldsymbol{r}_j)f_*(\boldsymbol{r}_j, \boldsymbol{u}_j)d^3u_j d^3r_j = N^{-1}n_{\boldsymbol{k}} \, , \tag{3.1.9}$$

where the first equality is a definition, and the second follows from equations (3.1.5) and (3.1.8). Therefore, $\langle \boldsymbol{F}(\boldsymbol{r}, t)\rangle_* = 0$ which verifies that the expression (3.1.6) correctly gives the stochastic force.

It follows from equation (3.1.1) that

$$\overleftrightarrow{C}(\boldsymbol{r}, \boldsymbol{r}', s) = -\frac{G^2M^2m^2}{4\pi^4}\int\left[\sum_{i;j=1}^{N} E_{\boldsymbol{kl}}(s) - n_{\boldsymbol{k}} n_{\boldsymbol{l}}\right]\frac{\boldsymbol{kl} \exp\{i(\boldsymbol{k} \cdot \boldsymbol{r} + \boldsymbol{l} \cdot \boldsymbol{r}')\}d^3k \, d^3l}{k^2l^2} , \tag{3.1.10}$$

where we shall interpret $\boldsymbol{r}' = \boldsymbol{r} + \boldsymbol{v}_0 s$ and

$$E_{\boldsymbol{kl}}(s) \equiv \langle \exp\{-i[\boldsymbol{k} \cdot \boldsymbol{r}_j(t) + \boldsymbol{l} \cdot \boldsymbol{r}_i(t + s)]\}\rangle_* \, . \tag{3.1.11}$$

We have here suppressed the i and j indices because, as will become clear, only the case $i = j$ gives something new. In the general spirit of this work we shall ignore correlations between field particles. Thus the average in equation (3.1.11) factors whenever $i \neq j$. By equation (3.1.9) this factored average equals $N^{-2}n_{\boldsymbol{k}} n_{\boldsymbol{l}}$; there are $N(N-1)$ such equal terms. In the case $i = j$ the average does not factor; however, it has the same value for all j. Therefore,

$$\overleftrightarrow{C} = -\frac{G^2M^2m^2}{4\pi^4}\int [NE_{\boldsymbol{kl}}(s) - N^{-1}n_{\boldsymbol{k}} n_{\boldsymbol{l}}] \frac{\boldsymbol{kl} \exp\{i[\boldsymbol{k} \cdot \boldsymbol{r} + \boldsymbol{l} \cdot \boldsymbol{r}']\}d^3k \, d^3l}{k^2l^2} . \tag{3.1.12}$$

Let us now calculate $E_{\boldsymbol{kl}}(s)$ for $i = j$ under the assumption that the *field* particles move inertially: $\boldsymbol{r}_j(t + s) = \boldsymbol{r}_j(t) + \boldsymbol{u}_j(t)s$. This neglect of field particle accelerations requires justification on various counts. The effect of the smoothed potential Φ is negligible at the same level as it is for the test particle, basically because it acts over a long time scale. T. The effect of the potential of the test particle itself may be neglected on account of assumption (2.5). Then there are the encounters of the field particles with one another. This scattering destroys the correlation of each field particle's orbit with itself, which means that for sufficiently long s its $E_{\boldsymbol{kl}}(s)$ factors again even for $i = j$. In that case the term $NE_{\boldsymbol{kl}}(s)$ in equation (3.1.12) is just $N^{-1}n_{\boldsymbol{k}} n_{\boldsymbol{l}}$ so that $\overleftrightarrow{C}$ vanishes. Thus whenever a particle is contributing to $\overleftrightarrow{C}$, it can be regarded as moving freely. In summary, for s small enough for $\overleftrightarrow{C}$ to be nonvanishing, each particle contributes

$$E_{kl}(s) = N^{-1} \int \exp\,[-i(\boldsymbol{k} + \boldsymbol{l}) \cdot \boldsymbol{r}_j] \exp\,(-i\boldsymbol{l} \cdot \boldsymbol{u}_j s) f_*(\boldsymbol{r}_j, \boldsymbol{u}_j) d^3u_j\, d^3r_j\,. \tag{3.1.13}$$

3.2. *Homogeneous Distribution*

Henceforth in this paper we shall be concerned with a homogeneous system, meaning one in which Φ may be regarded as fairly constant over a sizable region. The consistency of this view with restriction (2.2) will be examined in a subsequent paper. With Φ constant, the DF of a *homogeneous stationary* background is usually a function only of $\frac{1}{2}m\boldsymbol{u}^2$. We denote it by $f_*(u)$. Then equation (3.1.13) reduces to

$$E_{kl}(s) = (2\pi)^3 N^{-1} \int \delta(\boldsymbol{k} + \boldsymbol{l}) \exp\,(-is\boldsymbol{l} \cdot \boldsymbol{u}) f_*(u) d^3u\,. \tag{3.2.1}$$

Since $n_{\boldsymbol{k}}$ is now proportional to $\delta(\boldsymbol{k})$, the second term in the square brackets of equation (3.1.12) vanishes by virtue of the factor $\boldsymbol{k}$. Substituting equation (3.2.1) in equation (3.1.12), and integrating over $\boldsymbol{l}$ (which results in the replacement $\boldsymbol{l} \to -\boldsymbol{k}$) gives

$$\overleftrightarrow{C}(\boldsymbol{r}, \boldsymbol{r}', s) = \frac{2G^2M^2m^2}{\pi} \int \frac{\boldsymbol{k}\boldsymbol{k} \exp\,[i\boldsymbol{k} \cdot (\boldsymbol{r} - \boldsymbol{r}')] d^3k}{k^4} \int f_*(u) \exp\,(is\boldsymbol{k} \cdot \boldsymbol{u}) d^3u\,. \tag{3.2.2}$$

Since $\boldsymbol{r}' - \boldsymbol{r} = \boldsymbol{v}_0\, s$, $\overleftrightarrow{C}$ depends only on $\boldsymbol{v}_0$ and $f_*(\boldsymbol{u})$. In view of equation (3.1.4) we have after interchange of the integrals

$$\overleftrightarrow{L}(\boldsymbol{v}_0, f_*) = \frac{2G^2M^2m^2}{\pi} \int f_*(u) d^3u \int \frac{\boldsymbol{k}\boldsymbol{k}\, d^3k}{k^4} \int_{-\infty}^{\infty} ds \exp\,[is\boldsymbol{k} \cdot (\boldsymbol{u} - \boldsymbol{v}_0)]\,. \tag{3.2.3}$$

To calculate Tr $\overleftrightarrow{L}$ we first contract $\boldsymbol{k}\boldsymbol{k}$, and integrate over s to get $2\pi\,\delta[(\boldsymbol{u} - \boldsymbol{v}_0) \cdot \boldsymbol{k}]$. Taking the z-axis along $\boldsymbol{u} - \boldsymbol{v}_0$ we get

$$\mathrm{Tr}\, \overleftrightarrow{L}(\boldsymbol{v}_0, f_*) = 4G^2M^2m^2 \int \frac{f_*(u)}{|\boldsymbol{v}_0 - \boldsymbol{u}|} d^3u \int \frac{d^2k_\perp\, dk_z\, \delta(k_z)}{k_\perp^2 + k_z^2}\,, \tag{3.2.4}$$

where $\boldsymbol{k}_\perp$ is the two-dimensional projection of $\boldsymbol{k}$ onto the plane normal to $\boldsymbol{v}_0 - \boldsymbol{u}$. The integral over k-space reduces to a two-dimensional one and gives $2\pi \ln (k_{\max}/k_{\min})$ with $k_{\max}^{-1}$ ($k_{\min}^{-1}$) denoting the smallest (largest) length scale at which the treatment here makes sense. The integral over $\boldsymbol{u}$ is the well-known Rosenbluth potential (Rosenbluth et al. 1957; Binney & Tremaine 1987). For an isotropic velocity distribution the heating term (compare eqs. [2.10] and [3.1.3]) becomes

$$\left\langle \frac{dE}{dt} \right\rangle_h = \frac{1}{2} M^{-1}\, \mathrm{Tr}\, \overleftrightarrow{L}(\boldsymbol{v}_0, f_*) = 4\pi G^2 M m^2 \ln \Lambda \int \frac{f_*(u)}{|\boldsymbol{v}_0 - \boldsymbol{u}|} d^3u\,, \tag{3.2.5}$$

which is the traditional result (Rosenbluth et al. 1957; Binney & Tremaine 1987).

4. THE COOLING TERM

What about the first term in equation (2.10)? At first sight we might be tempted to equate it to zero since we have started from $\langle \boldsymbol{F} \rangle = 0$. This would mean there is no drag on the moving particle. This "paradox" has often been commented upon. For example, Kandrup (1980) blamed it on the description of $\boldsymbol{F}$ as a space quantity instead of a phase-space quantity. We shall here rather adhere to the more plausible viewpoint of condensed matter physics which focuses on the modification of the ensemble determining the statistics of $\boldsymbol{F}$ arising from the motion itself. In other words, the probabilities $P[\boldsymbol{F}(t)]$ used to define the averages are evidently related to the distribution function (DF) for the field population. Originally this is $f_*(\boldsymbol{r}, \boldsymbol{u}) = f_*(\frac{1}{2}\boldsymbol{u}^2 + \Phi)$. It defines a distribution $P[\boldsymbol{F}(t)]$ such that $\langle \boldsymbol{F} \rangle = 0$. We denote this kind of average $\langle \ldots \rangle_*$. As a result of the motion of the test particle, energy is conveyed to the background, so that $P[\boldsymbol{F}(t)]$ will change resulting in $\langle \boldsymbol{F} \rangle \neq 0$. The change in $\langle \boldsymbol{F} \rangle$ may be calculated by the following modification of the argument (Reif 1965) employed in the case of strongly interacting particles, i.e., a liquid.

All probabilities to be discussed here are conditional ones, namely given $\boldsymbol{v}_0$. In the interval $(-\infty, \delta t)$, the particle *gives* energy to the field. This gets distributed among the field particles with consequent distortion of the DF. If we normalize DF by

$$\int f(\boldsymbol{r}, \boldsymbol{u}) d\Gamma = N\,, \tag{4.1}$$

where N is the number of field particles and $d\Gamma = d^3r\, d^3p$ is the one-particle phase space element, and imagine that a particle at phase space position Γ_i has been "kicked" to position Γ_i', the distortion may be expressed as

$$\int_{\Gamma \approx \Gamma_i'} \delta f(\boldsymbol{r}, \boldsymbol{u}) d\Gamma = 1 \qquad \int_{\Gamma \approx \Gamma_i} \delta f(\boldsymbol{r}, \boldsymbol{u}) d\Gamma = -1\,, \tag{4.2}$$

which implies that

$$\int_{\text{all } \Gamma} \delta f(\boldsymbol{r}, \boldsymbol{u}) d\Gamma = 0\,, \tag{4.3}$$

in agreement with particle conservation.

How does the said distortion modify the probabilities $P[\boldsymbol{F}(t)]$? We shall suppose that these are modified in direct proportion to the change in the number of states available to the whole system as a result of the distortion. There is a well known relation between the number of states and the (information-theoretic) entropy. Namely, the entropy of the system,

$$S = S_* - \int f \ln f \, d\Gamma \,, \tag{4.4}$$

can be interpreted as the logarithm of the number of microstates available to that system. The arbitrary constant S_* expresses the well known arbitrariness of the zero point of the entropy in classical physics; this will not hinder us, as we shall only be concerned with differences of entropy. How does S change when the DF goes from f_* to $f = f_* + \delta f$? Evidently

$$\delta S = -\int \delta f \ln f_* \, d\Gamma - \int \delta f \, d\Gamma \,. \tag{4.5}$$

By equation (4.3) the second integral vanishes. Introducing equation (4.2) we get

$$\delta S = -\sum_{i=1}^{N} \ln f_*(\Gamma_i') + \sum_{i=1}^{N} \ln f_*(\Gamma_i) \,. \tag{4.6}$$

Therefore, the total number of background states, exp (S), has changed by a factor

$$K \equiv \exp(\delta S) = \prod_{i=1}^{N} \frac{f_*(\Gamma_i)}{f_*(\Gamma_i')} \,. \tag{4.7}$$

This is precisely the factor by which the probability $P[\boldsymbol{F}(t)]$ has been augmented.

Most encounters between particles are weak ones. Thus, as in Fokker-Plank analyses, we may ignore the rare close encounters. As a consequence, when field particle i which energy ϵ_i interacts with the test mass, it experiences a change in energy which in most cases satisfies $\delta\epsilon_i \ll \epsilon_i$. We may, therefore, expand equation (4.6) as

$$\delta S \approx \sum_{i}^{N} \frac{1}{f_*} \frac{\partial f_*}{\partial \epsilon}\bigg|_{\epsilon_i} \delta\epsilon_i \,. \tag{4.8}$$

For a Maxwellian DF, equation (4.7) with (4.8) turns out to be proportional to exp $(\Sigma\, \delta\epsilon_i)$, a statistical factor well known from treatments of the modification of the background distribution in condensed matter (Reif 1965). More generally, since we assume that the $\delta\epsilon_i$ are small, the argument in the exponential in equation (4.7) is small; expanding we have

$$K = \exp\left[\sum_{j}^{N} \frac{1}{f_*} \frac{\partial f_*}{\partial \epsilon}\bigg|_{\epsilon_j} \delta\epsilon_j\right] \approx 1 + \sum_{j}^{N} \frac{1}{f_*} \frac{\partial f_*}{\partial \epsilon}\bigg|_{\epsilon_j} \int_{-\infty}^{\delta t} \boldsymbol{f}_j[\boldsymbol{r}_j(t'), t'] \cdot \boldsymbol{u}_j(t') dt' \,, \tag{4.9}$$

where $\boldsymbol{f}_j$ is the force exerted *on* the jth particle by the test mass M. In the expression for $\delta\epsilon_i$ we ignored the contribution from interaction of field particles with themselves since these energy changes have nothing to do with the presence of the test particle, and thus should sum to zero in equation (4.8). We notice that K fluctuates via its dependence on $\boldsymbol{f}_j$. Thus to compute an average, we first multiply the relevant quantity by K, and then average with respect to the *unperturbed* $P[\boldsymbol{F}(t)]$. In this way we bring to bear the statistical weight of the states made accessible by the injection of energy.

Denoting the cooling rate by W we thus have ($\boldsymbol{F} = -\Sigma \boldsymbol{f}_i$)

$$W \equiv \langle \boldsymbol{F}(\delta t) \cdot \boldsymbol{v}(-\infty)\rangle = -\sum_{i}^{N} \langle \boldsymbol{f}_i(\delta t) \cdot \boldsymbol{v}(-\infty)\rangle_* - \sum_{j}^{N} \left\langle \frac{1}{f_*(\epsilon_j)} \frac{\partial f_*}{\partial \epsilon}\bigg|_{\epsilon_j} \int_{-\infty}^{\delta t} \boldsymbol{f}_j(t') \cdot \boldsymbol{u}_j(t') dt' \sum_{i}^{N} \boldsymbol{f}_i(\delta t) \cdot \left[\boldsymbol{v}_0 + \frac{m}{M} \sum_{k}^{N} \boldsymbol{u}_k(0)\right]\right\rangle_* \,, \tag{4.10}$$

where the subscript "*" denotes averaging in the unperturbed ensemble, and $\boldsymbol{v}(-\infty)$ in the second term was replaced with the help of equation (2.9).

The first summation, deriving from the zeroth-order term of K, must vanish since at the level of the unperturbed DF there should be no correlations between the initial test particle velocity and the current forces, i.e., $\langle \boldsymbol{f}_i(\delta t) \boldsymbol{v}(-\infty)\rangle_* = 0$. Moreover, since we are ignoring field particle correlations, we are left only with terms in the second average for which $i = j = k$. Thus the second multiple summation term reduces to a sum of N identical terms:

$$W = -N \left\langle \frac{1}{f_*(\epsilon)} \frac{\partial f_*}{\partial \epsilon} \int_{-\infty}^{\delta t} \boldsymbol{f}(t') \cdot \boldsymbol{u}(t') dt' \, \boldsymbol{f}(\delta t) \cdot \left[\boldsymbol{v}_0 + \frac{m}{M} \boldsymbol{u}(0)\right]\right\rangle_* \,. \tag{4.11}$$

As usual the average $\langle \dots \rangle_*$ is performed by integrating over phase space with weight $f_*(\epsilon)$, and dividing by N for normalization. We thus obtain

$$W = -\int d\Gamma \frac{\partial f_*}{\partial \epsilon} \boldsymbol{u}(0) \cdot \int_{-\infty}^{\delta t} \boldsymbol{f}(t') \boldsymbol{f}(\delta t) dt' \cdot \left[\boldsymbol{v}_0 + \frac{m}{M} \boldsymbol{u}(0)\right] \,. \tag{4.12}$$

By having taken $\boldsymbol{u}(t')$ out of integral and replaced it by $\boldsymbol{u}(0)$, we have neglected terms of $O(\nabla\Phi\, \delta t)$ which are small compared to $\boldsymbol{u}(0)$ by our assumptions on Φ (§ 2). We have also neglected a term of order $m^{-1} \int \boldsymbol{f}(t') dt'$; such a term generates a product of three $\boldsymbol{f}$-values in the average which is of the same order as terms already neglected by making the approximation in equation (4.9). Again, because we assume a weak stochastic force, we are entitled to neglect these three-point correlation functions as compared to $\overleftrightarrow{C}$.

The force exerted on a background particle which started at x with velocity u at $t = 0$ is

$$f(t') = \frac{GMm(r + v_0 t' - x - ut')}{|r + v_0 t' - x - ut'|^3} = -\frac{iGMm}{2\pi^2}\int \frac{k \exp\{ik \cdot (r + v_0 t' - x - ut')\}d^3k}{k^2} . \tag{4.13}$$

Suppose we first integrate the product $f(t')f(\delta t)$ over x, and then integrate over the Fourier index l in a way similar to that used to obtain equation (3.2.2); we get

$$\int d^3x f(t')f(\delta t) = \frac{2G^2M^2m^2}{\pi}\int \frac{kk \exp\{ik \cdot (v_0 - u)(\delta t - t')\}d^3k}{k^4} . \tag{4.14}$$

According to equation (4.11), we have to integrate this result over t'. If we extend the integral to $t' = \infty$ and correct by a factor of $\frac{1}{2}$ we get a factor $2\pi\delta[(u - v_0) \cdot k]$. Thus taking the z-axis along $u - v_0$, equation (4.12) reads

$$W = 2G^2M^2m^2 \int \frac{\partial f_*}{\partial \epsilon} \frac{d^3u}{|v_0 - u|} \int \frac{k_\perp \cdot u\, k_\perp \cdot [v_0 + (m/M)u]d^2k_\perp}{k_\perp^4} , \tag{4.15}$$

where $k_\perp$ is now a two-dimensional vector perpendicular to $u - v_0$. Thus we may replace the rightmost u in the integrand by v_0. Since $u\partial f_*/\partial\epsilon = m^{-1}\partial f_*/\partial u$, an integration over $k_\perp$-space gives

$$W = 2\pi G^2M^2m\left(1 + \frac{m}{M}\right)\ln\Lambda \int \frac{\partial f_*}{\partial u} \cdot \frac{v_\perp}{|v_0 - u|}\, d^3u , \tag{4.16}$$

where $v_\perp$ is the component of v_0 perpendicular to $v_0 - u$, and $\ln\Lambda$ has the same meaning as in equation (3.2.5). Writing $v_\perp$ explicitly as

$$v_\perp = v_0 \cdot \left[1 - \frac{UU}{U^2}\right] , \tag{4.17}$$

where $U \equiv v_0 - u$, $U \equiv |U|$, 1 is a unit tensor, and using the identity

$$\frac{\partial}{\partial u} \cdot \left[\frac{1}{U} - \frac{UU}{U^3}\right] = \frac{2U}{U^3} , \tag{4.18}$$

we obtain after integrating equation (4.16) by parts

$$W = -4\pi G^2M^2m\left(1 + \frac{m}{M}\right)\ln\Lambda \int d^3u f_* \frac{(v_0 - u) \cdot v_0}{|v_0 - u|^3} . \tag{4.19}$$

Since W is negative, it indeed represents a cooling. It coincides with the rate predicted by Chandraskehar's dynamical friction formula (Binney & Tremaine 1987).

5. TOTAL RATE OF ENERGY CHANGE

5.1. *The Total Energy-Loss Rate*

The heating and cooling terms (3.2.5) and (4.19) coincide with those obtained by the method of diffusion coefficients (Binney & Tremaine 1987; eq. [8.66]). The total rate of energy change of a particle traveling with *speed* $v \equiv |v_0|$ through a background with isotropic DF $f(u)$ is

$$\frac{dE}{dt}(v) = 16\pi^2G^2M^2m\ln\Lambda\left[\beta\int_v^\infty f(u)u\,du - v^{-1}\int_0^v f(u)u^2\,du\right] , \tag{5.1.1}$$

where $\beta \equiv m/M$. Substituting in equation (5.1.1) a Maxwellian DF

$$f(u) = \frac{n_0}{(2\pi\sigma)^{3/2}} \exp\left(-\frac{u^2}{2\sigma^2}\right) , \tag{5.1.2}$$

using the relation $dE = Mv\,dv$, and defining $x \equiv v/\sqrt{2}\,\sigma$, we obtain

$$\frac{dv}{dt} = -\frac{4\pi G^2M\rho\ln\Lambda}{v^2}\left[\operatorname{erf}(x) - \frac{2(1 + \beta)xe^{-x^2}}{\sqrt{\pi}}\right] , \tag{5.1.3}$$

where we assume, as always, that all background particles have mass m. This formula is displayed for various values of β in Figure 1.

Throughout the entire derivation leading to equation (5.1.3) nothing was assumed about β. Thus this formula adequately describes the modification of the motion of a light particle in a sea of heavier ones as that of a heavy particle in a background of lights. In the limit $\beta \to 0$, Chandrasekhar's formula (eq. [1.1]) is fully recovered; this is the formula used by most workers in analyzing the effects of dynamical friction in extragalactic systems. But for $\beta \neq 0$ equation (5.1.3) gives different results. As shown in Figure 1, the dv/dt experienced by the test particle changes sign when v goes through the value $\sigma\sqrt{(2\beta)}$. This reflects the competition between the dissipative mechanism and the heating by the field particles. This type of behavior was obviously expected,

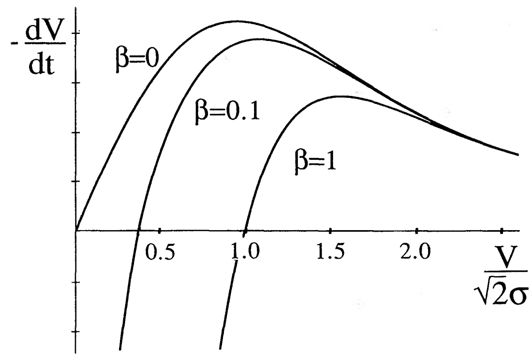

FIG. 1.—Deceleration of a test particle traveling in an ambient background (eq. [5.1.3]). Results are shown for three ratios of test to field particles masses β. In the limit $\beta \to 0$ Chandrasekhar's dynamical friction formula is fully recovered.

and there are many conceivable extragalactic situations where it may be important so that use of equation (5.1.3) is preferable to that of the plain Chandrasekhar drag formula. We now show that equation (5.1.3) is consistent with the equipartition of energy principle.

5.2. *Consistency with Equipartition*

Let us perform the following thought experiment: We are given an infinite homogeneous distribution of equal mass particles. At a certain moment an arbitrarily chosen ensemble of particles from that background are painted green. Obviously, the green population can be chosen in such a way that it will not satisfy the equipartition principle internally, or with respect to the rest of the system.

Each particle changes its velocity in a stochastic way due to the fluctuating forces. Since fluctuations never vanish, even in a relaxed system, the velocity of each particle will *never* settle down to a constant value, *regardless of the particle's velocity*. Each green particle performs a Brownian motion in velocity space in such a way that the *average* of dE/dt of all green particles with the same velocity satisfies equation (5.1.1). This means that even if all green particles had initially the same velocity, their distribution function in velocity space, $g(\boldsymbol{v})$, will get broader with time. We expect that after a sufficiently long period, $g(\boldsymbol{v}, t)$ will reach equipartition with the background population which is described by $f_*(\boldsymbol{v})$. This should also hold if the chosen particles differ not just in color but also in mass.

Let us denote a green particle mass by M. The rate of change of the total energy E_{tot} of the green particles' population is

$$\frac{dE_{\text{tot}}}{dt} = \int \frac{dE}{dt}(v)\, g(\boldsymbol{v}, t) d^3 v \,, \tag{5.2.1}$$

where $dE/dt(v)$ refers to a single green particle; see equation (5.1.1). Equipartition would require that the above expression vanish. Is this so? Suppose $f_*(\boldsymbol{v})$ is a Maxwellian DF with velocity dispersion σ and particle mass m, while $g(\boldsymbol{v}, t)$ tends towards a Maxwellian with velocity dispersion s and particle mass M. Substituting these in equation (5.2.1), and integrating over velocity space, we easily see that the following relation *must* hold if the integral in equation (5.2.1) is to vanish:

$$\frac{s^2}{\sigma^2} = \frac{m}{M} \,. \tag{5.2.2}$$

But this is precisely the statement of equipartition.

By contrast, if dE/dt is calculated from Chandrasekhar's drag formula, equation (1.1), and substituted in equation (5.2.1), then it is easy to verify that dE_{tot}/dt is a negative quantity, and can never vanish.

6. SUMMARY AND CONCLUSIONS

Equation (5.1.1) for the total rate of change of the energy is also obtained in the analysis via diffusion coefficients of systems governed by an inverse-square law force (Rosenbluth et al. 1957; Binney & Tremaine 1987). This indicates that summation of the effects of successive two-body encounters is not a bad model in an homogeneous background. It is, however, unclear whether the same will be true for an inhomogeneous medium, since describing an encounter by Keplerian orbits in the presence of external potential is inadequate.

The diffusion coefficients approach ascribes the drag to transfers of momentum to the test particle along the line of motion, and the heating effect to momentum transfer in all directions (both kinds of diffusion coefficients are involved and not only by transfer normal to the motion). The corresponding points in our approach are that all components of $\overleftrightarrow{L}$ are involved democratically in the heating term in equation (3.2.5), while the energy loss term involves a projection of the appropriate tensor along $\langle \boldsymbol{v}_0 \rangle$, as in equation (4.12). One advantage of the fluctuation approach's treatment of heating is that in *slow motion* through a homogeneous background, only one parameter, Tr $\overleftrightarrow{L}$, shows up, instead of two diffusion coefficients, $D(\Delta v_{\parallel})$ and $D(\Delta v_{\perp}^2) = 2D(\Delta v_{\parallel}^2)$.

As mentioned in § 1, the polarization drag method is known to recover the correct result for the drag only for $M \gg m$. This is not surprising since it studies the response of a system to the perturbation associated with the presence of a *test* particle, and this is an adequate description only for $M \gg m$. Furthermore, the approach considers the test particle as following a specified trajectory.

While this is a reasonable approximation for motion of a heavy particle in a sea of light ones, it neglects an important part of the physics when $M \leq m$. By contrast, our analysis takes into account stochastic behavior of the test particle in an implicit and natural way by relating its dynamics to that of the entire distribution of field particles via conservation of momentum; see equation (2.9).

Dynamical friction has been derived here from the interaction of a test particle with the stochastic force of the background, under the assumptions of negligible gravitational potential field, absence of initial correlations between field particles, and an isotropic velocity distribution for the field population. This approach shows that dynamical friction fits into the generic fluctuation-dissipation relation. Thus the old challenge going back to the early work of Chandrasekhar has been answered. On a more practical level, the approach here developed has two virtues. First, contrary to the other approaches mentioned earlier, it produces a unified picture of the energy dissipation and heating of particles in gravitating systems for the *full* range of m/M. Second, it readily suggests the modifications in the formalism that would be required to treat dynamical friction and heating in an inhomogeneous background (Maoz 1992), as well as in the presence of initial correlations between background particles.

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