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THE HORIZONTAL-BRANCH STARS IN GLOBULAR CLUSTERS. I. THE PERIOD-SHIFT EFFECT, THE LUMINOSITY OF THE HORIZONTAL BRANCH, AND THE AGE-METALLICITY RELATION

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ABSTRACT

Synthetic models of the horizontal branches in globular clusters have been constructed from a new grid of standard horizontal-branch (HB) evolutionary tracks. These models have been used to investigate the period shifts at constant T_{eff} between the RR Lyrae variables in globular clusters of different metallicities and the variation in HB luminosity with [Fe/H]. We find that two effects are responsible for the disagreement between the HB luminosity–[Fe/H] relationship found observationally by Sandage in his investigation of the Oosterhoff effect and that given by calculations of zero-age horizontal-branch (ZAHB) stars. One is that the evolution away from the ZAHB will play a role in the Oosterhoff group II clusters by increasing the mean luminosity and lowering the mean mass of the stars in the instability strip over the values predicted by ZAHB models. This leads to longer periods at a given T_{eff} in the group II clusters than in the group I clusters, as indicated by observations. The second is the realization that Sandage's assumption that light-curve shape (rise time) and amplitude are unique functions of T_{eff} is at odds with the observations of the RR Lyrae variables in the globular cluster ω Cen and of a sample of field variables, which show that both quantities depend on [Fe/H] as well as T_{eff}. When these two effects are taken into account, the observed relationship between period shift and [Fe/H] matches the theoretical model calculations to within the errors. Consequently, there is no reason to hypothesize the existence of a sizable anticorrelation between the abundances of helium and metals or that standard models of HB stars are substantially in error, as had been suggested previously.

The synthetic HB calculations suggest that the observed differences in the mean periods of the *ab* variables and the fraction of *c*-type variables between the two Oosterhoff groups are due to a difference in mean luminosity of the *ab* variables of ~ 0.18 in M_{bol} and to the uneven distribution of variables across the instability strip in the group II clusters.

Our models for nine clusters predict that for a primordial helium abundance of 0.23, the luminosity of the HB varies as $M_V^{RR} \approx 0.17[Fe/H] + 0.82$. This relationship and a similar one for $Y = 0.20$ are compared with other determinations in the literature. When combined with the result of Buonanno, Corsi, and Fusi Pecci (1989) that *in the mean* there is no variation in the difference in V-magnitude between the HB and the main-sequence turnoff (TO), specifically $\Delta V(TO - HB) = 3.59$, these relationships yield a significant age-metallicity relationship for halo globular clusters: 17.0 Gyr at $[Fe/H] = -2.3$ and 12.9 Gyr at $[Fe/H] = -0.8$, for $Y = 0.23$.

Subject headings: clusters: globular — stars: abundances — stars: evolution — stars: horizontal branch — stars: pulsation — stars: RR Lyrae

I. INTRODUCTION

A half-century ago, Oosterhoff (1939) showed that globular clusters can be divided into two groups according to the mean values of the periods of the type *ab* RR Lyrae variables ($\langle P_{ab} \rangle$). Clusters are said to belong to Oosterhoff groups I and II if their values of $\langle P_{ab} \rangle$ are near 0.55 and 0.65 day, respectively. Oosterhoff also discovered that the fraction of RR Lyrae variables that are type *c* is significantly lower in group I ($\sim 20\%$) than in group II ($\sim 50\%$) clusters. Later Arp (1955) discovered that the Oosterhoff effect is related to the metal abundances of the clusters in the sense that the group I clusters are more metal-rich than the group II clusters. More recently, Sandage, Katem, and Sandage (1981) discovered that at the same effective temperature the individual RR Lyrae variables in the group I clusters have shorter periods than the variables in the group II clusters and that the size of this period shift is a function of the metallicities of the clusters (see also Sandage 1981, 1982*a*, *b*).

These phenomena, which we consider the integral parts of the Oosterhoff effect, provide key information on the luminosities of RR Lyrae variables, which in turn are vital to several problems involving the distances to Population II objects. This has been clearly demonstrated by the work of Sandage and his collaborators (Sandage, Katem, and Sandage 1981; Sandage 1981, 1982*a*, *b*), who derived, from the observed period shifts between clusters, a luminosity-metallicity relation with slope $\Delta M_{bol}^{RR}/\Delta[Fe/H] = 0.35$. This relationship was used by Sandage (1982*a*) to fix the distances to globular clusters with main-sequence photometry, for which he then derived ages from the luminosity of the main-sequence turnoff. He found that the ages of the clusters, which span a large range in [Fe/H], are identical to within $\pm 10\%$ and equal to 17 Gyr. The constancy of the ages provided considerable support to Eggen, Lynden-Bell, and Sandage's (1962) hypothesis that the halo of the Galaxy underwent a rapid collapse (duration <1 Gyr; see Sandage 1986).

This work has stimulated numerous investigations into the origin of the Oosterhoff effect and, in particular, into the origin of the period shifts at constant $T_{\rm eff}$. The period shifts observed by Sandage are several times larger than those predicted by standard zero-age horizontal-branch (ZAHB) models, which Sandage (1982*a*) proposed was the consequence of an anticorrelation between the abundances of helium (Y) and metals of the form $\Delta Y = -0.069\Delta[\mathrm{Fe/H}]$. Although an anticorrelation in these quantities is, quoting Sandage (1982*a*), "against intuition" and one of this large size is at odds with other data that suggest that Y is essentially constant (see Buzzoni *et al.* 1983), it remains the most reasonable of the possible explanations that have been explored: CNO enhancements, rotation, and errors in ZAHB and red giant models stemming from incorrect opacities and other input physics (see Caputo, Castellani, and di Gregorio 1983; Renzini 1983; Rood 1984; Sweigart, Renzini, and Tornambè 1987).

The methods used by Sandage and his collaborators to derive the period shifts have also been scrutinized by several groups of investigators (Bingham *et al.* 1984; Caputo 1988; Gratton, Tornambè, and Ortolani 1986; Lee, Demarque, and Zinn 1987, 1988*b*). Caputo (1988) and Lee, Demarque, and Zinn (hereafter LDZ) (1987, 1988*b*) have argued that the true period shifts are substantially smaller than those measured by Sandage (1982*a*), which, with the same analysis, yield a smaller slope for the $M_{\rm bol}^{\rm RR}$–[Fe/H] relation. Since this will produce a significant age–[Fe/H] relationship among the globular clusters (see Zinn 1986), and hence a radically different picture of the formation of the Galactic halo, it is of considerable importance.

In this paper we describe our calculations of synthetic horizontal branches (see § II) and compare them with the observational data on the RR Lyrae variables in several globular clusters (§ III). As briefly reported earlier (LDZ 1987, 1988*b*), our models can reproduce, to within the observational uncertainties, the period shifts without the need of an anticorrelation between Y and [Fe/H]. The resulting $M_{\rm bol}^{\rm RR}$–[Fe/H] relation (§ IV) has approximately one-half the slope of Sandage's relation, in good agreement with recent Baade-Wesselink measurements of field RR Lyrae variables (Cacciari *et al.* 1989; Liu and Janes 1988*a*). Consequently, we find a significant correlation between age and [Fe/H] among the globular clusters that spans 4–5 Gyr over the metallicity range of the Galactic halo (§ V).

II. MODELS OF THE HORIZONTAL BRANCH

a) New Horizontal-Branch Evolutionary Tracks

Our synthetic horizontal-branch (HB) models are based on a new grid of standard (i.e., solar CNO/Fe, no core rotation, and including semiconvection) HB evolutionary tracks (Lee and Demarque 1989), which differ from the most widely used ones (e.g., Sweigart and Gross 1976) in several ways:

1. Most important, the post–ZAHB evolution of the stars was followed until the abundance of helium fell to near zero (i.e., 0.001) in the core. These calculations include, therefore, the last $\sim 10\%$ of the HB lifetime when a star undergoes the final phase of core helium burning. For short periods during this phase, a star experiences composition instabilities in its core (see e.g., Sweigart and Demarque 1973; Castellani *et al.* 1985) that produce small loops in the H-R diagram from its post–ZAHB track. Because very little time is spent on these loops, and because they may not be real, they were removed from the post–ZAHB track before the construction of the synthetic HBs. This should have no effect on our results because none of them depends on the fine details of the evolutionary tracks.

2. The enrichment (ΔY_s) in the envelope helium abundance due to convective dredge-up along the subgiant branch (Faulkner and Iben 1967) was taken into account. Since the amount of ΔY_s ranges from 0.01 to 0.02 (Sweigart and Gross 1978), we will distinguish the main-sequence helium abundance $Y_{\rm MS}$ from the HB envelope helium abundance $Y_{\rm HB}$ (i.e., $Y_{\rm HB} = Y_{\rm MS} + \Delta Y_s$).

3. The mass of the helium core on the ZAHB (M_c) appropriate for the adopted composition was obtained by interpolating among the red giant evolutionary tracks computed by Sweigart and Gross (1978). Fortunately, M_c is only weakly dependent on age, and we have adopted the value of M_c at an age of 15 Gyr for each composition.

4. Finally, fine grids of compositions and total mass (M) have been used, especially chosen for the Galactic globular clusters as follows:

$$Y_{\rm MS} = 0.20,\ 0.23\ ,$$

$$Z = 0.004,\ 0.002,\ 0.001,\ 0.0007,\ 0.0004,\ 0.0002,\ 0.0001\ ,$$

$$M = 0.52(0.04)0.84,\ 0.92\ M_\odot\ ,$$

where the notation $M = M_1(\Delta M)M_2$ means that M varies from M_1 to M_2 in steps of ΔM. A smaller mass increment (0.02 $M_\odot$) was used in some calculations.

We believe that our success at matching the HB morphologies of globular clusters and the properties of their RR Lyrae variables stems in part from these differences. It was found that the inclusion of the final phase of core helium burning is very crucial for the simulations of the properties of the RR Lyrae variables in the Oosterhoff group II clusters (see § III*a*). In addition, the use of find grids of compositions and the adoption of the values of M_c that are consistent with the composition avoided the need for uncertain interpolations in these parameters.

There is excellent agreement between the individual tracks computed by us and recently by Sweigart (1987), who used similar procedures but did not include the final helium exhaustion phase. For the typical case ($Y_{\rm MS} = 0.20$, $Z = 0.001$, $M = 0.64\ M_\odot$, and at the same epoch of post–ZAHB evolution), our tracks are hotter and less luminous by only $\Delta \log T_{\rm eff} \approx 0.02$ and $\Delta \log L \approx 0.02$.

b) Synthetic Models of the Horizontal Branch

Our synthetic HB calculations followed the spirit of the pioneering ones by Rood (1973). For each synthetic HB, the stars were assumed to have a Gaussian distribution in mass [$P(M)$], resulting from variable amounts of mass loss on the red giant branch (RGB):

$$P(M) \propto [M - (\langle M_{\rm HB}\rangle - \Delta M)](M_{\rm RG} - M) \times \exp\left[-(\langle M_{\rm HB}\rangle - M)^2/\sigma^2\right]\ , \quad (1)$$

where $M_{\rm RG}$ is the mass a star would have at the tip of the RGB if it did not lose mass. ΔM is the mean amount of mass loss, which is the difference between $M_{\rm RG}$ and the mean HB mass ($\langle M_{\rm HB}\rangle$), and σ is the mass dispersion. The dispersion in mass due to finite HB lifetime ($<0.002\ M_{\odot}$) was ignored, because it is smaller than the size of the mass bin used in the calculation. The Gaussian distribution was truncated at $M_{\rm RG}$ at the high-mass end and at $\langle M_{\rm HB}\rangle - \Delta M$, which is slightly above the core mass. The ages of the clusters set $M_{\rm RG}$; the ages were estimated from values of the difference in V-magnitude between the HB and the main-sequence turnoff [ΔV(TO − HB)] that are available in the literature. None of our results hinge on the ages (~ 15 Gyr) that were adopted, because the morphology of the synthetic HB depends almost entirely on $\langle M_{\rm HB}\rangle$ and σ appearing in the exponential of the Gaussian. For each cluster, these parameters were varied until a good match was obtained in the V versus $B-V$ diagram between the synthetic and the observed HB. To transform the theoretical calculations to this diagram, we used an updated version of the bolometric corrections and the effective temperature–color transformation given by Ciardullo and Demarque (1979), which for HB stars is nearly identical to the relations determined by Green *et al.* (1989). The synthetic HB for each cluster also includes the observational errors that the observers have estimated are present in their photometry.

For blue edges of the RR Lyrae instability strip, we adopted the results of Tuggle and Iben's (1972) calculations and applied to them the corrections for convection ($+0.011$ in log $T_{\rm eff}$ for $Y_{\rm HB} = 0.20$, and -0.011 for $Y_{\rm HB} = 0.30$), as inferred from Stellingwerf's (1984) recent stellar pulsation calculations that include convection. For most calculations, the width of the strip was taken to be 0.075 in log $T_{\rm eff}$, which yields an excellent

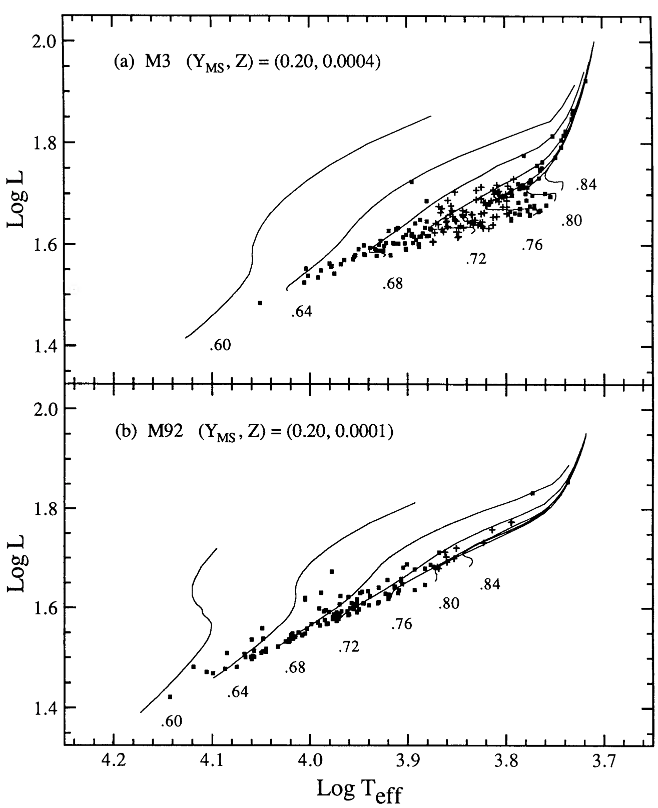

FIG. 1.—Synthetic HB models of M3 (group I) and M92 (group II) with HB evolutionary tracks. Track parameters are ($Y_{\rm MS}$, $Y_{\rm HB}$, Z, M_c) = (0.20, 0.2143, 0.0004, 0.5003) and ($Y_{\rm MS}$, $Y_{\rm HB}$, Z, M_c) = (0.20, 0.2113, 0.0001, 0.5090) for M3 and M92, respectively. RR Lyrae variables are represented by plus signs, and each track is labeled by its total mass in solar units.

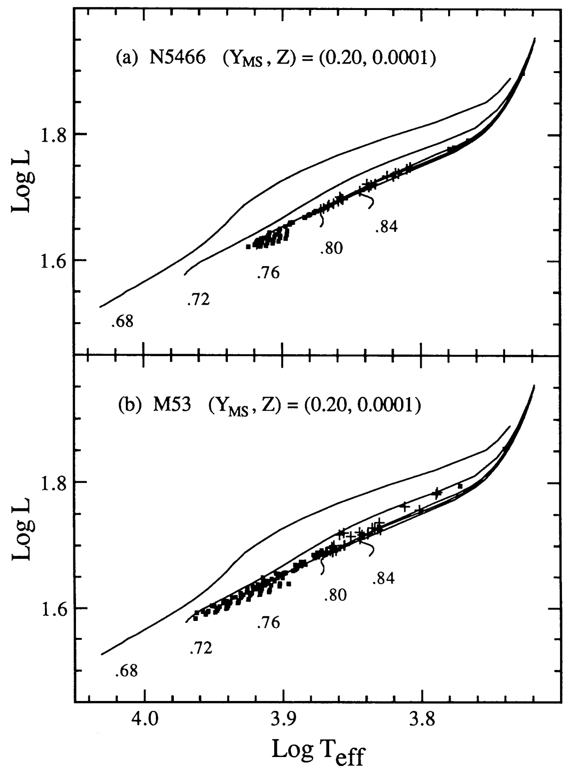

FIG. 2.—Same as Fig. 1, but for two group II clusters, NGC 5466 and M53. ($Y_{\rm MS}$, $Y_{\rm HB}$, Z, M_c) = (0.20, 0.2113, 0.0001, 0.5090).

match between the width in our synthetic HB models and that observed in M3 by Buonanno *et al.* (1986) (see Fig. 3). Since the observed width of the strip is uncertain by $\Delta(B-V) \approx 0.03$ (i.e., Δ log $T_{\rm eff} \approx 0.01$), we have made a separate set of calculations using a width of Δ log $T_{\rm eff} = 0.065$. This width is closer to the frequently used value of 0.060 (Rood 1973) and that predicted by recent calculations of stellar pulsation (i.e., Δ log $T_{\rm eff} \approx$ 0.066 for $Y_{\rm MS} = 0.20$; Stellingwerf 1984). A comparison of these two sets of calculations indicates how our results are sensitive to the adopted width of the instability strip (see § III*c*).

Our method of constructing synthetic HBs is illustrated in Figures 1 and 2, where the model stars and a few of the evolutionary tracks are plotted in the theoretical H-R diagram. The distribution of stars along the evolutionary tracks is obtained by assuming that stars are fed onto the HB from the RGB at a constant rate. Since the HB lifetime $\tau_{\rm HB}$ is only weakly dependent on total mass M, a uniform random number generator is used to give for each HB star the time t elapsed since it began its HB evolution [i.e., $0 < t < \tau_{\rm HB}(M)$]. This time and the evolutionary track appropriate for the star determine its position in the H-R diagram (tracks are interpolated for masses not on the grid). How well this procedure works can be assessed from Figure 3, where our synthetic HBs for M3, NGC 5466, and M92 are compared with the observed ones (see also Lee, Demarque, and Zinn 1988*a*, for clusters M4, M5, NGC 1851, and NGC 6723).

The mean amount of mass loss ΔM required to fit the observations was $\sim 0.15\ M_{\odot}$. However, a significant difference was found between the values of σ needed for the metal-rich clusters (i.e., M3, NGC 6171, NGC 6723, M4, M5, NGC 288, NGC 362, and 47 Tuc) and the metal-poor clusters (i.e., M53, NGC

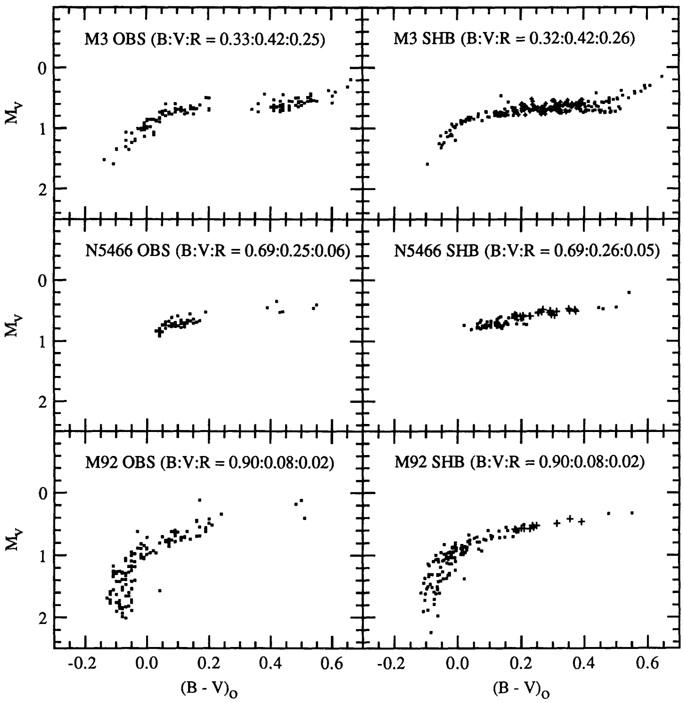

FIG. 3.—Comparison of observations and synthetic HBs. Color-magnitude diagrams and proportions of blue HB, RR Lyrae, and red HB stars ($B:V:R$) are compared for M3, NGC 5466, and M92. RR Lyrae variables are not plotted in the observational diagrams.

5466, M92, and M15). For metal-rich clusters the standard deviation of the mass dispersion (σ_{SD}) was about $\sim 0.03\ M_\odot$ (because the Gaussian distribution is truncated at both ends, the mass dispersion parameter σ in eq. [1] is approximately a factor of 2 larger than this value). The HBs of most metal-poor clusters, e.g., M53 (Cuffey 1965), NGC 5466 (Buonanno, Corsi, and Fusi Pecci 1985), M68 (Alcaino 1977), NGC 5053 (Sandage, Katem, and Johnson 1977), and perhaps NGC 2419 (Christian and Heasley 1988) are characterized by a clump of blue HB stars within a narrow range of color [i.e., $0.00 < (B-V)_0 < 0.20$], some RR Lyrae variables (less than 20% of the total HB population, except M68, where the sample size is small), and a few red HB stars. Our models indicate that the variables and red HB stars in these clusters are all highly evolved stars that were once blue HB stars (see Fig. 2). Since the direction of evolution is from blue to red in these clusters, a spread in color of the HB comes mostly from this effect; hence σ_{SD} required to fit the observations is approximately a factor of 2 smaller than M3 (i.e., $\sigma_{SD} = 0.01$–$0.02\ M_\odot$). The color spread of the HB in M92 is wider than in the other metal-poor clusters (see Fig. 3), and our models give $\sigma_{SD} \approx 0.035\ M_\odot$ for M92, which is similar to our results for M3. Note that the division into metal-rich and metal-poor samples that was made above is equivalent to a division by Oosterhoff period group (group I: M3, NGC 6171, NGC 6723, M4, M5, and NGC 362; group II: M53, NGC 5466, M92, and M15).

The metal-poor, Oosterhoff group II cluster M15 represents a special case. It is well documented (see Buonanno, Corsi, and Fusi Pecci 1985) that its HB has a very large range in color and that its blue HB is divided in two by a statistically significant gap (see Fig. 4*a*). To reproduce the range in color, our models require σ_{SD} to be $\sim 0.06\ M_\odot$, which is roughly twice as large as M92. This ad hoc increase in σ_{SD} fails, however, to reproduce the gap (see Fig. 4*c*), and hence we doubt that the observed long blue tail of the HB simply arises from a large σ_{SD}. On the other hand, the coincidence of the position of the gap in M15 with the blue extreme of M53 and NGC 5466 suggests that the HBs of M15 and these clusters would be identical if it were not for some mechanism working in M15 that produces the blue tail (see Buonanno, Corsi, and Fusi Pecci 1985). If this mechanism does not change the general properties of the RR Lyrae variables in M15, our model for M53 should also represent the variables in M15. Because of these uncertainties on the origin of the blue tail, we will present two alternative models for M15. One with small σ_{SD}, which is identical to M53 but fails to reproduce the blue tail (M15A), and one with large σ_{SD}, which produces a large range in color of the HB but fails to reproduce the gap (M15B) (see Fig. 4).

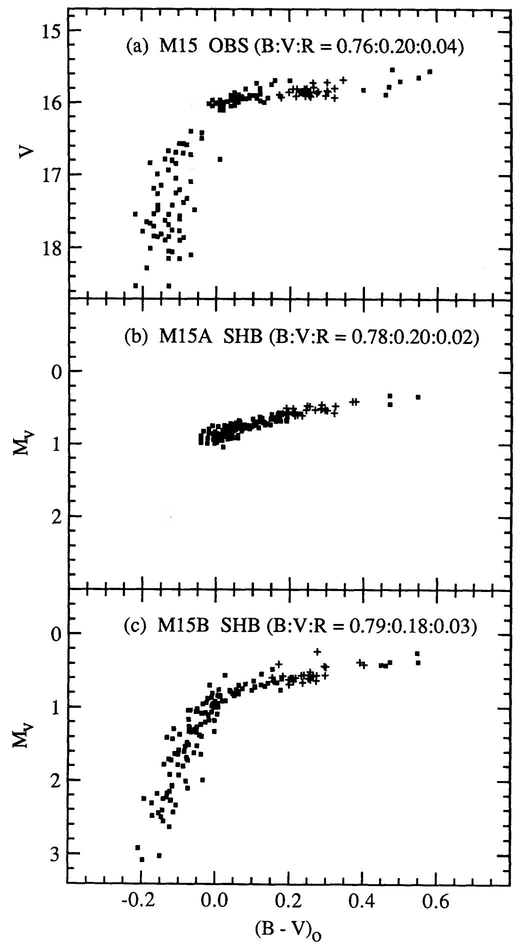

FIG. 4.—Same as Fig. 3, but for M15. Observational diagram in (*a*) is compared with two alternative models in (*b*) and (*c*). Model A, with small mass dispersion, is identical to M53 but fails to reproduce the blue tail. Model B, with large mass dispersion, produces a large range in color but fails to reproduce the gap (see text).

III. COMPARISON WITH OBSERVATIONS

a) Period Shifts in the log *P*–log T_{eff} *Plane*

The observations of Sandage, Katem, and Sandage (1981; see also Sandage 1981 and Bingham *et al.* 1984) established that at the same effective temperature, the RR Lyrae variables

TABLE 1

COMPARISON OF SYNTHETIC HB AND ZAHB AT $\log T_{\rm eff} = 3.830$ ($Y_{\rm MS} = 0.20$)

			Synthetic HB			ZAHB		
Cluster	Group	Z	$\langle \log L \rangle$	$\langle$Mass$\rangle$	$\Delta \log P'_{383}$	$\log L$	Mass	$\Delta \log P_{383}$
M3	I	0.0004	1.661	0.735	0.000	1.626	0.722	0.000
M53	II	0.0001	1.736	0.747	0.044	1.691	0.852	0.005
M92	II	0.0001	1.741	0.746	0.056	1.691	0.852	0.005

in the Oosterhoff group II clusters (e.g., M15) have longer periods than the variables in the group I clusters (e.g., M3). According to the theory of stellar pulsation, the fundamental period (P_f) is a function only of luminosity L, mass M, and effective temperature $T_{\rm eff}$:

$$\log P_f = 0.84 \log L - 0.68 \log M - 3.48 \log T_{\rm eff} + 11.497 , \tag{2}$$

where P is in days and L and M are in solar units (van Albada and Baker 1971). Equation (2) implies that the period shift ($\Delta \log P = \log P_{\rm cluster} - \log P_{\rm M3}$) at a given effective temperature is

$$\Delta \log P = 0.84 \, \Delta \log L - 0.68 \, \Delta \log M . \tag{3}$$

Therefore, the shift toward longer periods of the variables in the Oosterhoff group II clusters compared with the variables in the group I clusters requires that one of the following be true: (1) the RR Lyrae variables in group II clusters are more luminous than those in group I clusters, (2) the masses of the variables in the group II clusters are smaller than those in the group I clusters, or (3) a combination of explanations 1 and 2. Since ZAHB models predict a variation of mass with metallicity that is opposite in sign to that required by explanation 2, Sandage, Katem, and Sandage (1981) and Sandage (1981) proposed that this is offset by a large difference in luminosity between the HBs of the group II and group I clusters (explanation 1). To produce a sufficiently large difference in luminosity, they had to propose that the Oosterhoff group II clusters are more helium-rich than the group I clusters, and hence that Y and Z are anticorrelated. This hypothesis has stimulated a number of investigations of HB stars and the Oosterhoff effect (e.g., Caputo, Castellani, and di Gregorio 1983; Caputo, Cayrel, and Cayrel de Strobel 1983; Caputo *et al.* 1984, 1987; Renzini 1983; Rood 1984; Bingham *et al.* 1984; Gratton, Tornambè, and Ortolani 1986; Caputo 1987; and Sweigart, Renzini, and Tornambè 1987).

It is well known that the evolution away from the ZAHB produces several of the observed properties of the HBs in globular clusters, e.g., its observed thickness, and several authors have speculated that this evolution may play an important role in the understanding of the Oosterhoff period groups (e.g., Gratton, Tornambè, and Ortolani 1986). Our synthetic HB calculations for $Y_{\rm MS} = 0.20$ do indeed indicate that almost all of the RR Lyrae variables in group II clusters are highly evolved stars from the blue side of the instability strip (see Figs. 1 and 2). Compared to ZAHB stars of the same $T_{\rm eff}$, these stars have smaller masses but larger luminosities (see Table 1). Our calculations for group I clusters like M3 indicate, however, that the masses and luminosities of the RR Lyrae variables are not much different from those of the ZAHB models. Consequently, the variables in M3 and group II clusters have very nearly the same masses, but their difference in luminosity is precisely that required to explain the observed shift in period between the clusters ($\Delta \log P \approx 0.05$).

Table 2 lists the results of our synthetic HB calculations for nine clusters of different metallicities. The clusters are identified by Oosterhoff group in column (2), and column (3) gives the values of [Fe/H] listed by Zinn (1985), which are means of several different measurements shifted to the same metallicity scale (see Zinn and West 1984). The values of the metal abundance that were adopted in our synthetic HB calculations are given in the next column. The mean mass of the HB stars, $\langle M_{\rm HB} \rangle$, and the mean properties of the RR Lyrae variables (assuming $M_{\rm bol} = +4.79$ for the Sun) are listed in the remaining columns. The data in Table 2 are the means of 50 random simulations with the sample size of 1000 HB stars, so that our calculations are not affected by small-number statistics.

TABLE 2

RESULT OF SYNTHETIC HB CALCULATIONS ($Y_{\rm MS} = 0.20$)

Cluster (1)	Group (2)	[Fe/H] (3)	Z (4)	$\langle M_{\rm HB} \rangle$ (5)	$\langle M_{\rm bol}^{\rm RR} \rangle$[a] (6)	$\langle \log L_{\rm RR} \rangle$ (7)	$\langle M_{\rm RR} \rangle$ (8)	$\Delta \log P'_{3.83}$[b] (9)
NGC 6171	I	−0.99	0.002	0.690	0.77	1.608	0.655	−0.010
NGC 6723	I	−1.09	0.002	0.656	0.77	1.607	0.651	−0.010
M4	I	−1.28	0.001	0.694	0.71	1.630	0.683	−0.005
M5	I	−1.40	0.0007	0.671	0.69	1.641	0.697	−0.002
M3	I	−1.66	0.0004	0.727	0.64	1.661	0.733	0.000
M53	II	−2.04	0.0001	0.744	0.49	1.719	0.745	+0.044
M15A[c]	II	−2.15	0.0001	0.744	0.49	1.719	0.745	+0.044
M15B	II	−2.15	0.0001	0.705	0.51	1.711	0.778	+0.030
NGC 5466	II	−2.22	0.0001	0.762	0.50	1.714	0.763	+0.035
M92	II	−2.24	0.0001	0.703	0.47	1.729	0.744	+0.056

[a] Assuming $M_{\rm bol} = +4.79$ for the Sun.
[b] $\Delta \log P' = \log P'_{\rm cluster} - \log P'_{\rm M3}$.
[c] M15A is identical to M53 (see text).

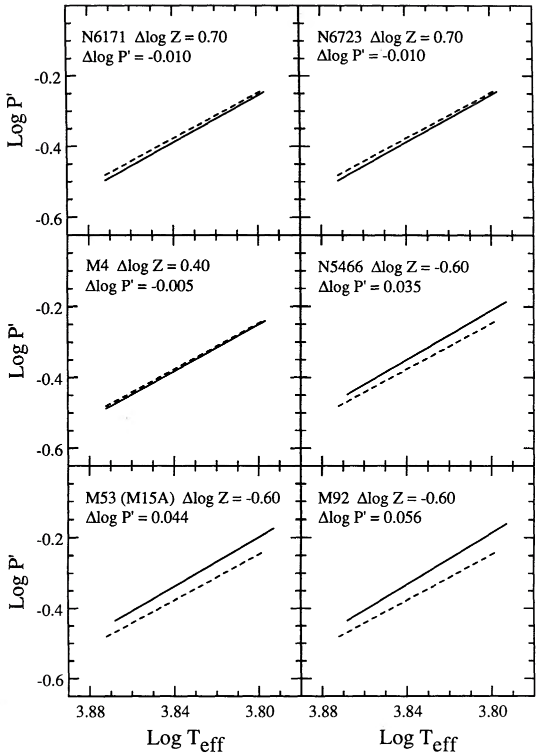

FIG. 5.—Theoretical log P'–log $T_{\rm eff}$ relations for the RR Lyrae variables in six clusters based on the synthetic HB models ($Y_{\rm MS} = 0.20$). The synthetic M3 relation is reproduced in each diagram by a dashed line, and the solid line is the fit to the model stars for each cluster.

The period shifts for the synthetic HBs at log $T_{\rm eff} = 3.830$ (Δ log P'_{383}), which is near the center of the instability strip, were found from the theoretical log P'–log $T_{\rm eff}$ diagrams in Figure 5, where the synthetic M3 relation is reproduced in each diagram by the dashed line and the solid line is the fit to the model stars for each cluster (the model stars have very little scatter, as shown in Fig. 6). It is important to note that the period shifts measured in these diagrams are nearly independent of the choice of log $T_{\rm eff}$. Following van Albada and Baker (1971), we have removed the intrinsic scatter in the log P–log $T_{\rm eff}$ diagram due to the spread in luminosity among the RR Lyrae variables in each cluster by calculating the "reduced period" (P') from the equation log P' = log $P + 0.336(M_{\rm bol} - \langle M_{\rm bol}\rangle)$, where $M_{\rm bol}$ is the absolute bolometric magnitude and $\langle M_{\rm bol}\rangle$ is the mean magnitude of all variables in the cluster. We have considered only the fundamental periods of the variables because the theoretical transition edges between type *ab* and type *c* are very uncertain. The results of model M15A are identical to the ones for M53 because of the assumption we made in § II*b*. If we increase the mass dispersion to explain the observed blue tail (i.e., model M15B), the period shift decreases to 0.030 (see Table 2). It is important to emphasize that our calculations have not assumed a $Y-Z$ anticorrelation and that they are based on standard assumptions regarding opacity and other input physics.

The most direct way to see whether our results are correct is to compare the period shifts given by our calculations with the observations of the variables in the log P'–log $T_{\rm eff}$ diagram which Sandage (1981) and Sandage, Katem, and Sandage (1981) have used in their analyses of the Oosterhoff period groups. In Figure 7 this diagram is plotted for the variables in

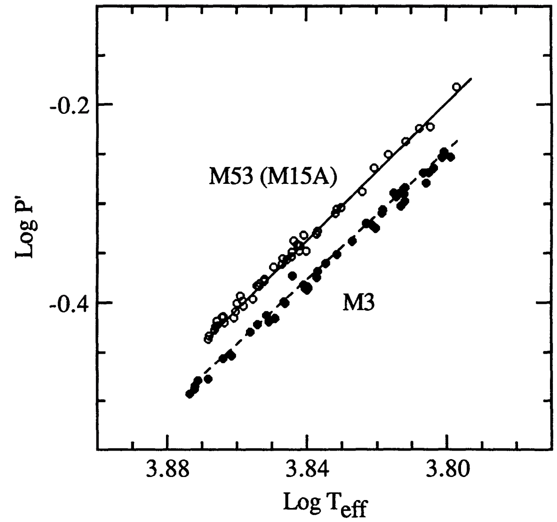

FIG. 6.—Theoretical log P'–log $T_{\rm eff}$ relations for the RR Lyrae variables in M3 and M53 (M15A). The lines are least-squares fits to the means of 50 calculations. The points are the results obtained by one calculation ($Y_{\rm MS}$ = 0.20).

the archetypal group I and group II clusters M3 and M15 ([Fe/H] = -1.66 and -2.15, respectively; Zinn 1985). For this figure and subsequent ones, we have transformed the observed first harmonic periods of the type *c* variables to fun-

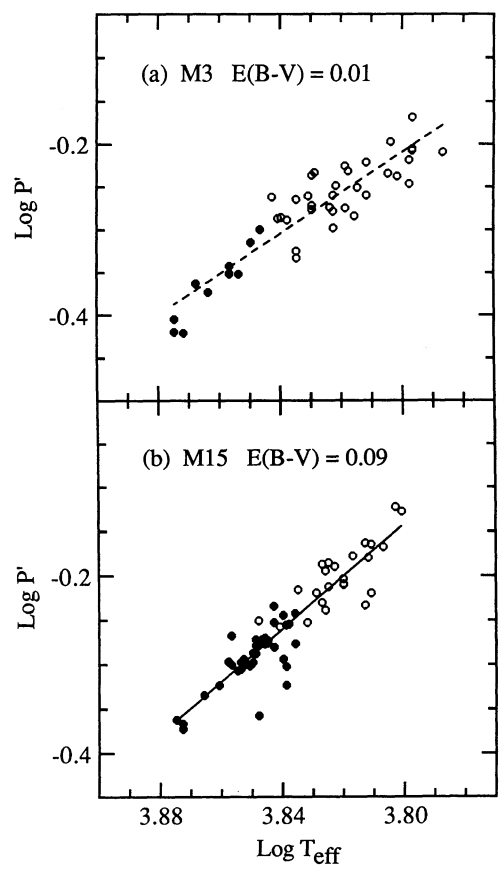

FIG. 7.—Observational log P'–log $T_{\rm eff}$ relations for the RR Lyrae variables in M3 and M15. The best fits are represented by dashed and solid lines for M3 and M15, respectively. The filled and open circles are for type *c* (fundamentalized periods) and type *ab* variables, respectively.

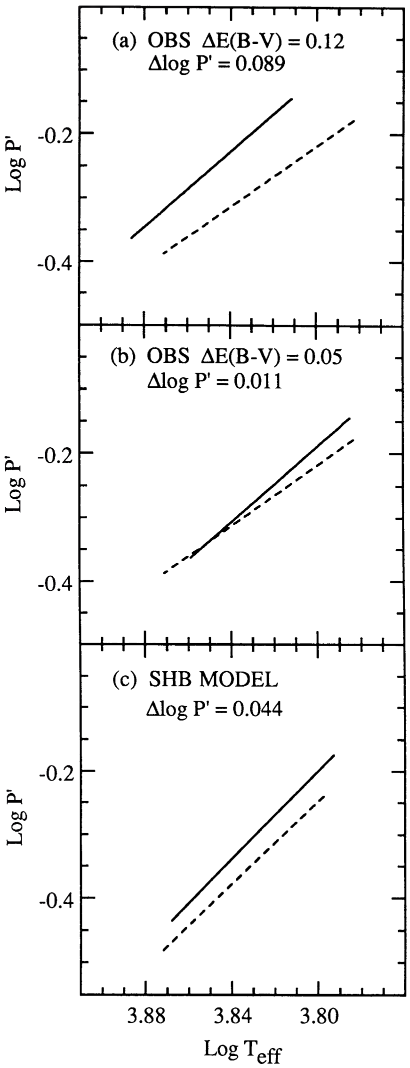

FIG. 8.—Comparison of observations and synthetic HB models in the log P'–log $T_{\rm eff}$ diagram. The solid and dashed lines are the best fits for M15 and M3, respectively. (*a*, *b*) Two extreme cases for the difference in reddening [$\Delta E(B-V)$] between M15 and M3 [$E(B-V)=0.00$]. (*c*) Synthetic HB model (M15A and M3) relation. $\Delta E(B-V) \approx 0.08$ would produce a good match between the observations and the models ($Y_{\rm MS}=0.20$).

damental periods by adding 0.125 to their logarithms (Bingham *et al.* 1984), so that the resulting log P'–log $T_{\rm eff}$ relationships can be directly compared with those given by our synthetic HB models. For the M3 and M15 data and for all other $(B-V)$ data discussed in this paper, we have used the equilibrium color defined by Bingham *et al.* (1984), $(B-V)_{\rm eq} = \frac{2}{3}\langle B-V\rangle + \frac{1}{3}(\langle B\rangle - \langle V\rangle)$, and the $(B-V)$–$T_{\rm eff}$ transformation of Green *et al.* (1989) in the calculations of $T_{\rm eff}$. The photometry of the M3 and M15 variables was taken from Sandage (1981) and Bingham *et al.* (1984), respectively. The mean lines shown in Figure 7 have been replotted in Figure 8, under two extreme cases, as judged from the measurements of $E(B-V)$ by Sandage (1969), Zinn (1980), and Reed, Hesser, and Shawl (1988), for the difference in reddening, $\Delta E(B-V) = E(B-V)_{\rm M15} - E(B-V)_{\rm M3}$. Figure 8 illustrates that an intermediate value for $\Delta E(B-V)$, $\sim$0.08, would produce good agreement with our calculations.

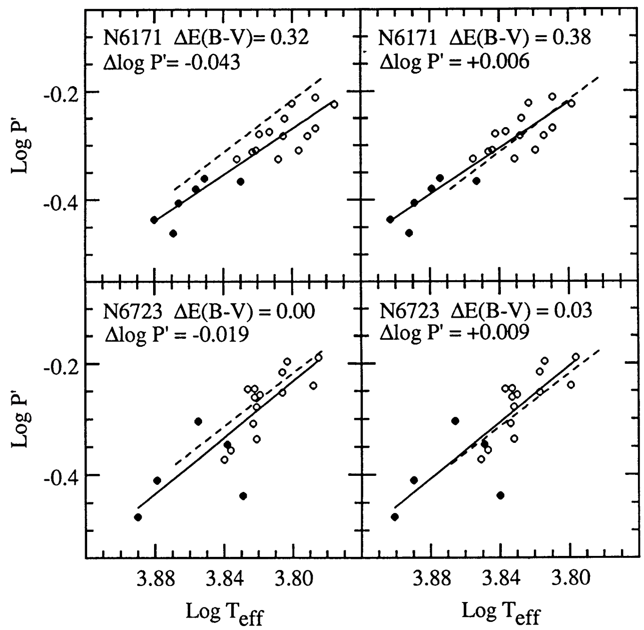

FIG. 9.—Plots of log P' vs. log $T_{\rm eff}$ for the RR Lyrae variables in NGC 6171 and NGC 6723 under two different values for the reddening. The observed M3 [$E(B-V)=0.00$] relation is reproduced in each diagram by a dashed line, and the solid line is the fit to the stars in each cluster. The filled and open circles depict type *c* and type *ab* variables, respectively.

The most metal-rich clusters with adequate data for a comparison in the log P'–log $T_{\rm eff}$ diagram are NGC 6171 and NGC 6723 ([Fe/H] = -0.99 and -1.09, respectively; Zinn 1985). The reddening of NGC 6171 is relatively large, and there is considerable scatter among the values of $E(B-V)$ that have been published for this cluster (see Table 2 in Smith and Hesser 1986). The top two panels of Figure 9 show the period shifts obtained under two assumptions for the difference in reddening between NGC 6171 and M3 (photometry from Sandage 1981). Our models give $\Delta \log P' = -0.010$, which would be the same as the observed shift if $\Delta E(B-V) = 0.36$, a value near the middle of the scatter for NGC 6171. All measurements have indicated that the reddening of NGC 6723 is small and approximately the same as that of M3 (see Smith and Hesser 1986). In the lower panels of Figure 9, we have plotted the data for NGC 6723 (photometry from Menzies 1974) under two assumptions for $\Delta E(B-V)$. The observed period shift would be the same as that given by our models (-0.010) if $\Delta E(B-V) = 0.01$.

Sandage (1981) plotted log P'–log $T_{\rm eff}$ diagrams for M3, M15, and NGC 6171, assuming $E(B-V) = 0.00$, 0.08, and 0.28, respectively, from which he obtained period shifts of 0.070 and -0.060 (our sign convention) for M15 and NGC 6171, respectively. Since these values and those for other clusters were very similar to the period shifts that he obtained by plotting log P' against the amplitude in blue light (A_B) and against the fraction of the period spent on the rising branch of the light curve ($\Delta\phi_{\rm rise}$), he concluded that A_B and $\Delta\phi_{\rm rise}$ were unique functions of $T_{\rm eff}$, and hence the log P'–A_B and log P'–$\Delta\phi_{\rm rise}$ diagrams, which are unaffected by reddening, could be substituted for the log P'–log $T_{\rm eff}$ diagram. Unfortunately, he did not plot a log P'–log $T_{\rm eff}$ diagram for NGC 6723. In Sandage (1982*a*) he lists the period shift of -0.045 (our sign convention) that he obtained from the log P'–$\Delta\phi_{\rm rise}$ diagram. In order to produce this period shift with the data plotted in Figure 9, one would have to adopt $\Delta E(B-V) = -0.03$, which we believe is unlikely because it suggests that M3 (galactic latitude $b = 79°$) is more heavily reddened than NGC 6723 ($b = -17°$).

Recently, Sandage (1990) has presented a new analysis of the period shifts between clusters which involves the measurements of the position of the ZAHB in the log P–log $T_{\rm eff}$ diagram, which he defines as the lower envelope of the scatter of the variables (see his paper for a more detailed description). From the ZAHB positions in 10 clusters, Sandage (1990) obtained $\Delta \log P/\Delta[\mathrm{Fe/H}] = -0.12$. We have not compared our models with these results, for the following reasons. First, our models suggest that in group II clusters, including M15 and M92, which were analyzed by Sandage (1990), very few if any of the variables are near their ZAHB locations, and hence for these clusters Sandage's analysis is likely to attribute to the ZAHB a larger period shift relative to M3 than it actually has. Second, even in clusters that are expected to have variables near the ZAHB, its position in the log P–log $T_{\rm eff}$ diagram is much less well determined than the mean position of the variables, because, being the lower boundary of a distribution, it is influenced more by the observational errors in the photometry and by statistical fluctuations stemming from the finite samples of variables. Consequently, we have judged the success of our models on how well they can reproduce the mean positions of the cluster variables in the log P'–log $T_{\rm eff}$ diagram (Fig. 5).

The comparisons in Figures 8 and 9 suggest that the period shifts predicted by our synthetic HB models are consistent, considering the uncertainties in cluster reddening, with the available observations in the log P'–log $T_{\rm eff}$ diagram, at both ends of the metallicity range of the clusters containing RR Lyrae variables. Given these uncertainties, however, this diagram is at most a weak test of the models (see also Caputo 1987), and other tests must be examined.

b) Variation in Period Shift with [Fe/H]

To derive the variation in period shift with [Fe/H], Sandage (1982*a*) made the assumption that the period shifts in the log P'–$\Delta\phi_{\rm rise}$ and log P'–A_B diagrams were equivalent to the shifts in the log P'–log $T_{\rm eff}$ diagram. *This would be true if* $\Delta\phi_{\rm rise}$ *and* A_B *were the same functions of* $T_{\rm eff}$ *for all metallicities.* From the plots of log P' against $\Delta\phi_{\rm rise}$, Sandage (1982*a*) has argued that there is a linear relationship between period shift and [Fe/H] over the whole range in [Fe/H] of the clusters that contain RR Lyrae variables (i.e., $\Delta \log P' = -0.116\Delta[\mathrm{Fe/H}]$, using our sign convention; $-2.3 < [\mathrm{Fe/H}] < -0.8$). In Figure 10 this slope is compared with the one predicted by our synthetic HB calculations, where the crosses are the results of our calculations (from Table 2), and the solid line is a linear least-squares fit to these data ($\Delta \log P'/\Delta \log Z = -0.043$).[1] The short-dashed line has the same slope as Sandage's (1982*a*) relationship between $\Delta \log P'$ and [Fe/H]. While our synthetic models of HB, which include evolution away from the ZAHB, produce a much steeper slope compared with the one predicted by ZAHB models ($\Delta \log P/\Delta[\mathrm{Fe/H}] = -0.003$; Sweigart, Renzini, and Tornambè 1987), this is not enough to reproduce Sandage's relationship between period shift and [Fe/H].

A similar result would be obtained using period shifts that have been obtained from the period-amplitude relation. But, as

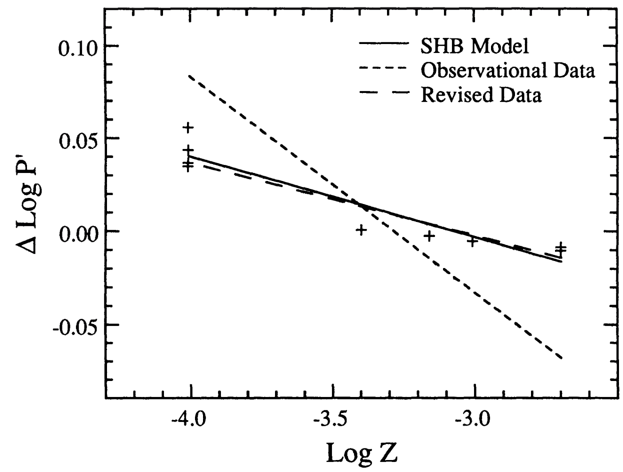

FIG. 10.—Comparison of the relationships between $\Delta \log P'$ and [Fe/H]. The short-dashed line is Sandage's (1982*a*) empirical relationship ($\Delta \log P'/\Delta[\mathrm{Fe/H}] = -0.116$). The long-dashed line is the relationship obtained once Sandage's relation is corrected for the dependence of rise time on [Fe/H] ($\Delta \log P'/\Delta[\mathrm{Fe/H}] = -0.041$). The solid line is a least-squares fit ($\Delta \log P'/\Delta[\mathrm{Fe/H}] = -0.043$; $Y_{\rm MS} = 0.20$) to the results of our synthetic HB calculations, which are plotted as plus signs.

noted by Sandage (1982*a*) and discussed in more detail by Gratton, Tornambè, and Ortolani (1986), the $\Delta \log P$–[Fe/H] relation that is obtained using amplitude in place of $\Delta\phi_{\rm rise}$ has considerably more scatter.

We find that the disagreement between our results and Sandage's can be traced to his assumption that $\Delta\phi_{\rm rise}$ and A_B are unique functions of $T_{\rm eff}$ for all metallicities. The papers by Bingham *et al.* (1984), Lub (1987), and Caputo (1988), and our earlier papers (LDZ 1987, 1988*b*), have noted inconsistencies with this assumption. To investigate the dependence of $\Delta\phi_{\rm rise}$ and A_B on [Fe/H], one needs a large sample of RR Lyrae variables, spanning a large range in [Fe/H], for which precise light curves and measurements of mean $T_{\rm eff}$ exist. The variables in ω Cen, which span a large range in [Fe/H] (Butler, Dickens, and Epps 1978), are one such sample whose mean values of $T_{\rm eff}$ can be inferred straightforwardly from their mean $B-V$ colors because they are all reddened by the same amount. In Figure 11 we have plotted $\Delta\phi_{\rm rise}$ and A_B against $T_{\rm eff}$ for three groups of type *ab* variables in ω Cen that differ in mean [Fe/H] (photometry from Sandage 1981). One can see in these diagrams that $\Delta\phi_{\rm rise}$ and A_B are not the same functions of $T_{\rm eff}$ for all metallicities (see also Caputo 1988, who has independently arrived at a similar conclusion).

To be sure that this effect is not simply another anomaly of the stellar population of ω Cen, we have made a similar analysis of a sample of field variables. The field RR Lyrae variables observed by Lub (1977, 1979) in the Walraven photometric system are a sufficiently large sample whose values of reddening, mean values of $T_{\rm eff}$, and metallicities have been measured in a uniform way from different combinations of the Walraven colors. From this sample, we selected 59 type *ab* variables whose light curves were sufficiently well observed that we could estimate both $\Delta\phi_{\rm rise}$ and A_B. Lub (1977) has estimated the mean values of $T_{\rm eff}$ of these stars from a comparison of their dereddened colors with Kurucz's (1975) calculations of line-blanketed model atmospheres. Values of [Fe/H] have been estimated from the $\Delta[B-L]$ index of the Walraven system (Lub 1979). Very similar results would be obtained if

[1] Even though the least-squares fit gives a slope $\Delta \log P'/\Delta \log Z = -0.043$, it is worth noting that among the Oosterhoff group I clusters the period shift is only weakly dependent on cluster metallicity.

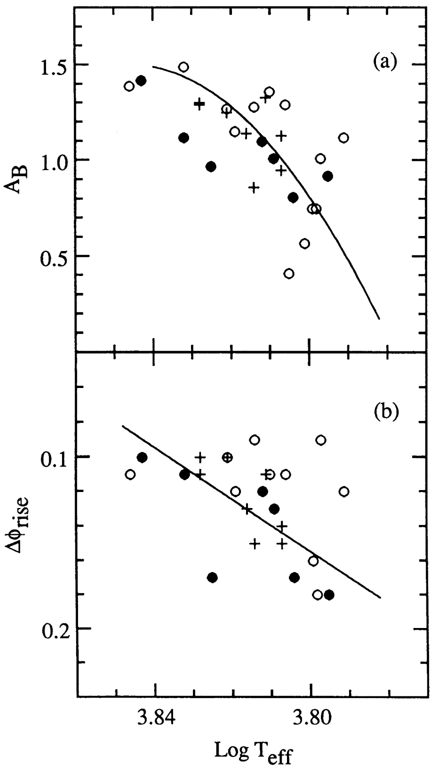

FIG. 11.—Relationships (*a*) between A_B and log T_{eff} and (*b*) between rise time and log T_{eff}, for type *ab* RR Lyrae variables in ω Cen. The M3 relations are represented by solid lines in each diagram. Filled circles, plus signs, and open circles depict the variables in the metal-rich ($\langle[Fe/H]\rangle = -0.76$), intermediate ($\langle[Fe/H]\rangle = -1.31$), and metal-poor ($\langle[Fe/H]\rangle = -1.82$) groups, respectively.

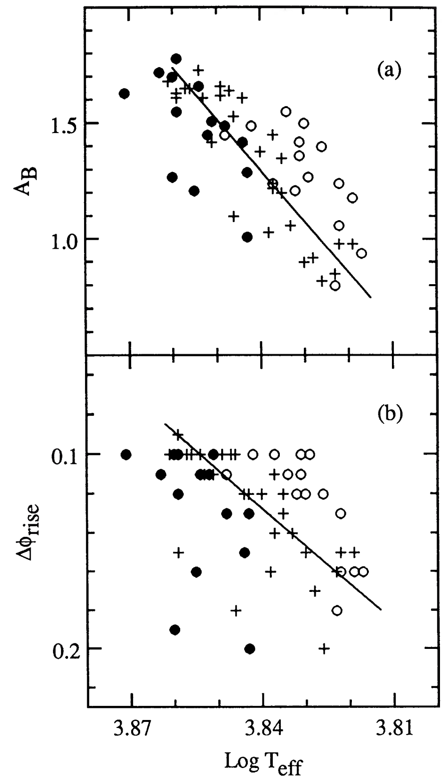

FIG. 12.—Same as Fig. 11, but for the sample of field RR Lyrae variables observed by Lub (1977, 1979). Mean relations are represented by solid lines. $\langle[Fe/H]\rangle = -0.76$, -1.40, and -1.84, for metal-rich (*filled circles*), intermediate (*plus signs*), and metal-poor (*open circles*) groups, respectively.

the spectroscopic ΔS index was used instead. The dependences of $\Delta\phi_{rise}$ and A_B on [Fe/H] are illustrated in Figure 12 for this sample of field variables, where one sees more clearly than in Figure 11 that these quantities depend on [Fe/H] as well as on T_{eff}. The dependence of A_B on [Fe/H] was noted previously by Lub (1987).

From Figures 11 and 12, the differences in T_{eff} at constant $\Delta\phi_{rise}$ and at constant A_B have been obtained, and they are plotted against [Fe/H] in Figure 13. The values of $\Delta \log T_{eff}/\Delta[Fe/H]$ at constant $\Delta\phi_{rise}$ and at constant A_B that have been derived from these diagrams are listed in Table 3. These data show that $\Delta\phi_{rise}$ and A_B are clearly functions of [Fe/H] as well as T_{eff}, and the similarities between the two samples suggest that this does not depend on a particular photometric system and color-T_{eff} transformation.

The significance of this effect is illustrated in Figure 10. To adjust the slope of Sandage's relationship for the dependence of $\Delta\phi_{rise}$ on [Fe/H], we have adopted the mean of the values of $\Delta \log T_{eff}/\Delta[Fe/H]$ given by the ω Cen and field samples (i.e., 0.022). Since, at constant mass and luminosity, $\Delta \log P = -3.48\Delta \log T_{eff}$ (see eq. [2]), this value of $\Delta \log T_{eff}/\Delta[Fe/H]$ will by itself produce a relationship between period shift and [Fe/H] with slope $\Delta \log P/\Delta[Fe/H] = -0.075 \pm 0.015$. Subtracting this from the slope of Sandage's relation ($\Delta \log P'/\Delta[Fe/H] = -0.116 \pm 0.018$), we obtain an *estimate* of the slope of the true relationship between period shift and [Fe/H] (-0.041 ± 0.023), and the long-dashed line in Figure 10 has this slope.

TABLE 3

VALUES OF $\Delta \log T_{eff}/\Delta[Fe/H]$

Sample	At Constant $\Delta\phi_{rise}$	At Constant A_B
ω Cen	0.019 ± 0.007	0.011 ± 0.004
Field	0.024 ± 0.005	0.015 ± 0.002

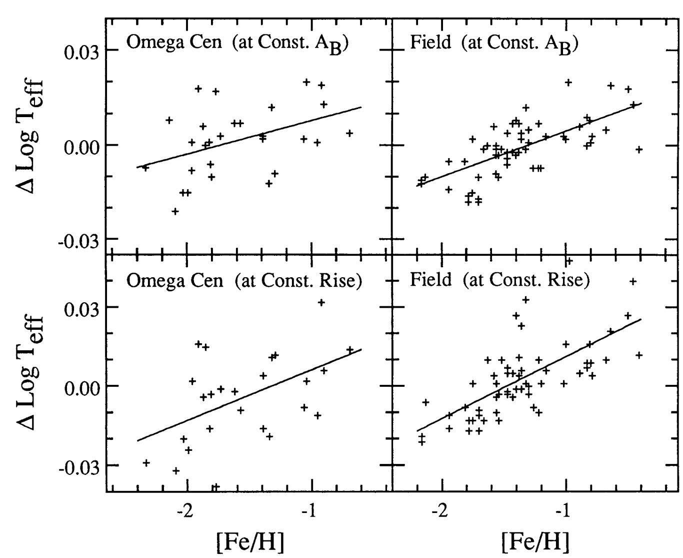

FIG. 13.—Differences in log T_{eff} at constant A_B and rise time, as functions of [Fe/H] for the variables in ω Cen and for the sample of field variables.

Figure 10 illustrates that once Sandage's relationship is corrected for the dependence of $\Delta\phi_{\rm rise}$ on [Fe/H], it matches the model calculations to within the errors of the original relationship and the correction. Thus, there is no reason to believe that the standard picture of the HB is substantially in error or that the abundances of helium and [Fe/H] are anticorrelated (as proposed by Sandage 1982*a*). More important for the dating of globular clusters, there is now no evidence from the analysis of the period shifts for the steep dependence of RR Lyrae luminosity on [Fe/H] that Sandage derived from the slope of his $\Delta \log P$–[Fe/H] relation.

The results presented in Table 2 are based on the HB evolutionary tracks for $Y_{\rm MS} = 0.20$ ($Y_{\rm HB} \approx 0.22$). When the helium abundance is low (i.e., $Y_{\rm HB} \leq 0.22$), metal-poor HB stars ($Z < 0.0002$) within the instability strip evolve only redward. However, at higher helium abundance, metal-poor stars evolve first blueward and then later redward. These blueward loops become more prominent as Y increases (see, e.g., Sweigart 1987), and in synthetic HB models based on helium-rich tracks (i.e., $Y_{\rm HB} \geq 0.27$), most variables in group II clusters are no longer evolved stars from the blue side of the strip but are on tracks that originated in the instability strip. They are more massive and less luminous than the variables that evolved from the blue side of the strip. Therefore, increasing Y decreases the period shift between M3 and Oosterhoff group II clusters, mainly because it increases the difference in mass between the variables in these clusters. Our calculations for $Y_{\rm MS} = 0.23$ ($Y_{\rm HB} \approx 0.25$) indicate that the increase in Y by ~ 0.03 decreases the slope of $\Delta \log P/\Delta[{\rm Fe/H}]$ by ~ 0.016 (i.e., $\Delta \log P'/\Delta[{\rm Fe/H}] \approx -0.027$ for $Y_{\rm MS} = 0.23$). This value is still consistent with the observations, considering the uncertainties in the observed relationship between $\Delta \log P'$ and [Fe/H].

Another check on the model HBs is provided by a comparison, in absolute terms, of the periods at a given $T_{\rm eff}$. In this case, we find that our models based on $Y_{\rm MS} = 0.20$ systematically underestimate the observed periods of the variables in groups I and group II clusters by small amounts (i.e., $\Delta \log P' = \log P'_{\rm obs} - \log P'_{\rm model} \approx 0.05$ at $\log T_{\rm eff} = 3.380$). Because the luminosity of the HB and hence the periods of the variables depend critically on the size of the helium core mass (M_c), only a small increase in $M_c \sim 0.03\ M_\odot$, is required to remove this discrepancy. This is not unreasonable because the calculations of M_c are uncertain by at least this amount and because M_c will be larger if there is some core rotation in these stars (Mengel and Gross 1976). Because the luminosity of the HB and the periods are also very sensitive to Y, our synthetic HB models for $Y_{\rm MS} = 0.23$ provide a nearly perfect match to the observed periods (e.g., at $\log T_{\rm eff} = 3.830$, $\Delta \log P' \approx 0.01$), and a revision of M_c is not needed. We doubt that $Y_{\rm MS}$ can be as large as 0.28–0.30 because then the model variables will have periods that are too long, unless the rise in HB luminosity with increased Y is offset by a substantial decrease in M_c. Thus, to match both the periods and the period shifts between clusters, our models favor values of $Y_{\rm MS}$ in the range 0.20–0.23, in good agreement with the value 0.23 ± 0.02 obtained by Buzzoni *et al.* (1983) from applying the R method to 15 globular clusters.

c) The Oosterhoff Period Groups

To be completely successful, our models must also explain the observation by Oosterhoff that the clusters divide into two groups according to $\langle P_{ab} \rangle$ and the fraction of type c variables [$f_c = n_c/(n_{ab} + n_c)$]. Unfortunately, the calculation of these quantities for our models is not straightforward because they depend on the location in the H-R diagram of the transition edge between the fundamental and the first harmonic modes and whether or not stars experience a hysteresis in their pulsation mode as they evolve across the instability strip. These questions have been investigated numerous times using stellar pulsation calculations, but without a clear consensus being reached among investigators. Without a firm theory of the transition edge, we proceed by first considering whether our models reproduce the $T_{\rm eff}$ distributions of the variables in M3 and M15. We then consider two, possibly extreme, cases for the transition edge to see whether our models have the potential of explaining the observed behaviors of $\langle P_{ab} \rangle$ and f_c.

Our models indicate that in the vicinity of the instability strip the direction of evolution of HB stars is predominantly blueward in group I clusters and that the RR Lyrae variables in these clusters should be distributed more or less evenly in $T_{\rm eff}$ across the strip. In contrast, our models show that all, or nearly all, of the variables in group II clusters are evolving redward from ZAHB positions beyond the blue boundary of the strip (see Figs. 1 and 2). This causes an uneven distribution in $T_{\rm eff}$ because the evolution speeds up as a star evolves further from its ZAHB position. The group II clusters are predicted, therefore, to have higher incidences of high $T_{\rm eff}$ variables than group I clusters, which is consistent with their observed larger values of f_c. In Figure 14 the $T_{\rm eff}$ distributions of the RR Lyrae variables in M3 and M15 are compared with the ones given by our models, which include the effects of observational errors of ± 0.01 in $\log T_{\rm eff}$. The similarities between the observed and calculated distributions suggest that our models would approximately reproduce the observed values of f_c, if we knew where to place the transition edge in $T_{\rm eff}$.

To obtain estimates of f_c and $\langle P_{ab} \rangle$ from our models, we consider two cases for the transition edge. For case 1 we assume that a switch in pulsation mode occurs abruptly (i.e., no hysteresis) at the fundamental blue edge. The position of this edge and the first harmonic blue edge in the H-R diagram and their dependences on the masses of the stars were taken from Tuggle and Iben (1972). Since these calculations did not include the effects of convection, we have applied to them small corrections for convection described in § II*b*. The width of the

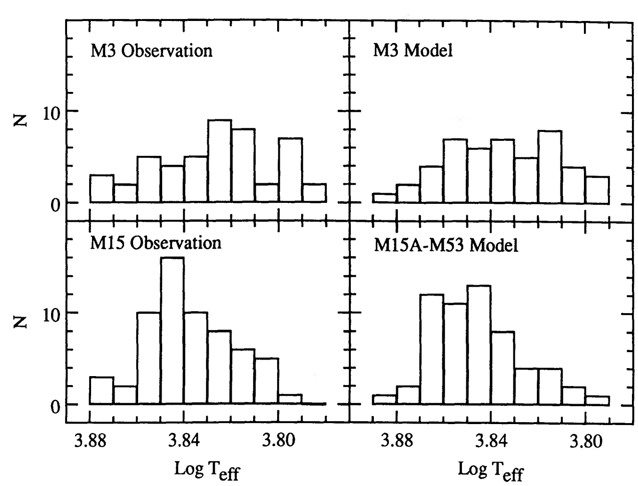

FIG. 14.—$T_{\rm eff}$ distributions of the RR Lyrae variables in M3 and M15. The observed histograms are compared with the model calculations for M3 and M15A–M53. The effects of observational errors in $\log T_{\rm eff}$ are included in the model calculations.

instability strip was taken to be $\Delta \log T_{\rm eff} = 0.065$. For case 2 we have adopted the same blue edges and strip width as in case 1, but have also adopted Stellingwerf's (1975) result that there exists a zone in the H-R diagram, approximately 300 K wide, extending from the fundamental blue edge to the red, where either the fundamental mode or the first harmonic is preferred, depending on the star's mode of pulsation as it entered this "either-or" zone. This is the hysteresis mechanism that van Albada and Baker (1973) proposed was the explanation of the Oosterhoff effect. It cannot be the whole explanation, because it fails to explain the period shifts at constant $T_{\rm eff}$ (Sandage, Katem, and Sandage 1981). The recent stellar pulsation calculations that include convection (Deupree 1977; Stellingwerf 1984) suggest that the red edge of the first harmonic mode occurs at a larger $T_{\rm eff}$ than the red edge of the fundamental mode, and that therefore there exists a region near the low-$T_{\rm eff}$ boundary of the strip where only the fundamental mode can be excited. Hence it appears unlikely that the either-or zone extends, as was once thought (e.g., Cox 1980), from the fundamental blue edge to the low-$T_{\rm eff}$ boundary of the strip. One of the convective models calculated by Stellingwerf (1984) showed signs of hysteresis; hence there is evidence that the effect was not an artifact of the earlier radiative models. However, to our knowledge, the size of the either-or zone has not been explored using the recent methods that incorporate convection.

The results obtained with our two assumptions for the transition zone are presented in Figure 15, where our model calculations for $Y_{\rm MS} = 0.20$ are compared with the observed values of $\Delta \log \langle P_{ab} \rangle$ and f_c (data from Castellani and Quarta 1987) for the nine clusters considered in this paper (see Table 2). Since, as noted above, our models yield periods that are slightly smaller than the observed values, the comparison is made between values of $\Delta \log \langle P_{ab} \rangle$, which is defined as $\log \langle P_{ab} \rangle_{\rm cluster} - \log \langle P_{ab} \rangle_{\rm M3}$, rather than values of $\log \langle P_{ab} \rangle$. What matters is the behavior of $\log \langle P_{ab} \rangle$ as a function of [Fe/H], which is independent of its normalization.

Figure 15 shows that for either case 1 (no hysteresis) or case 2 (hysteresis) our models correctly predict that there is only a small decrease in $\Delta \log \langle P_{ab} \rangle$ with increasing [Fe/H] among the clusters more metal-rich than [Fe/H] $= -1.7$ (the group I clusters). There is, however, a big difference between the two cases for the more metal-poor group II clusters. Our models predict values of $\Delta \log \langle P_{ab} \rangle$ that are clearly too small if no hysteresis is assumed and values that are too large if the either-or zone is ~ 300 K wide. A similar behavior is seen in the calculations of f_c, but in this case the assumption of hysteresis produces better agreement among the group I clusters. We estimate from these comparisons that our models would produce a satisfactory match to the observations if the either-or zone is roughly 120 K wide. It is important to note that this

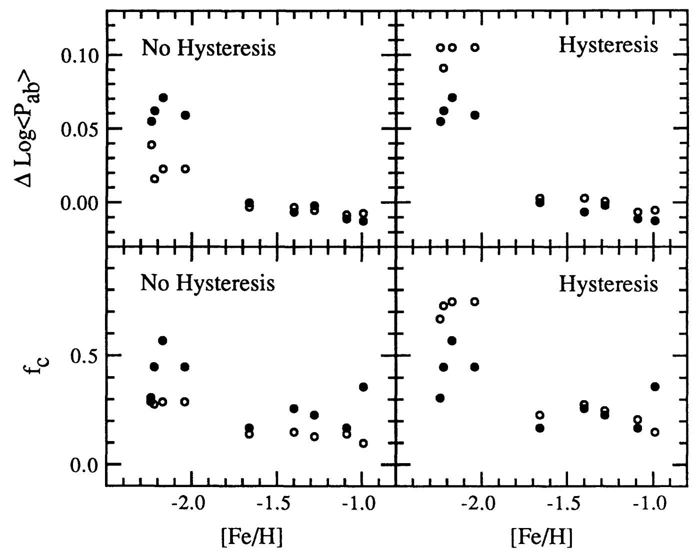

FIG. 15.—Comparison of observations and synthetic HBs for nine clusters. The model calculations (*open circles*) for $\Delta \log \langle P_{ab} \rangle$ and f_c, obtained under two assumptions for the transition zone (see text), are compared with the observed values (*filled circles*).

does not depend on the existence of hysteresis, for the same result would be obtained if, for some unknown reason, the transition edge in the group II clusters occurred at roughly 120 K lower $T_{\rm eff}$ than we adopted in the case of no hysteresis. Our models for $Y_{\rm MS} = 0.23$ yield very similar results, and in this case a good match is obtained if the either-or zone is ~ 80 K wide.

If, as suggested by our calculations, the either-or zone is only ~ 100 K wide, it will be difficult to detect observationally by measuring the overlap in $T_{\rm eff}$ between the type *c* and type *ab* variables in a cluster. Most, if not all, observations of $T_{\rm eff}$ overlaps can be attributed to the observational errors in the mean colors of the stars and/or the way these colors have been defined (see Bingham *et al.* 1984).

To investigate which stellar parameters cause the difference in $\log \langle P_{ab} \rangle$ between the Oosterhoff groups, we have used the observed values of f_c to fix the transition edges in our models. In Table 4 the observed properties of the RR Lyrae variables in Oosterhoff group II clusters with 20 or more variables are compared with M3, which is the group I cluster that has particularly well-determined values because of its large number of variables (data from Castellani and Quarta 1987). NGC 2419 is not included in Table 4 because its value of f_c is very unusual for a group II cluster. Since $\log \langle P_{ab} \rangle$ and $\log \langle P_f \rangle$, log (mean fundamental + fundamentalized type *c*), are strongly affected by statistical fluctuations in the distribution of stars within the

TABLE 4

OBSERVATIONS OF M3 AND OOSTERHOFF GROUP II CLUSTERS WITH 20 OR MORE VARIABLES

Cluster	Group	f_c	$\log \langle P_{ab} \rangle$[a]	$\log \langle P_f \rangle$[a]	$\Delta \log \langle P_{ab} \rangle$[b]	$\Delta \log \langle P_f \rangle$[b]
M3	I	0.17	-0.259 ± 0.006	-0.276 ± 0.005	...	...
M68	II	0.62	-0.205 ± 0.019	-0.259 ± 0.017	0.054 ± 0.020	0.017 ± 0.018
M53	II	0.45	-0.200 ± 0.020	-0.250 ± 0.018	0.059 ± 0.021	0.026 ± 0.019
NGC 5466	II	0.45	-0.197 ± 0.026	-0.270 ± 0.023	0.062 ± 0.027	0.006 ± 0.024
M15	II	0.57	-0.188 ± 0.016	-0.250 ± 0.013	0.071 ± 0.017	0.026 ± 0.014
Mean (group II)	...	0.52	-0.198 ± 0.012	-0.257 ± 0.010	0.062 ± 0.013	0.019 ± 0.011

[a] The uncertainties in these quantities are *estimates* based on simulations, *not* observed values.
[b] $\Delta \log \langle P \rangle = \log \langle P \rangle_{\rm cluster} - \log \langle P \rangle_{\rm M3}$.

TABLE 5

SYNTHETIC HB CALCULATIONS FOR M3 AND THE OOSTERHOFF GROUP II CLUSTER WITH THE WIDTH OF THE INSTABILITY STRIP $\Delta \log T_{\rm eff} = 0.075$

Cluster	$\langle \log L \rangle_{\rm RR}$	$\langle M \rangle_{\rm RR}$	$\langle \log T_{\rm eff} \rangle_{\rm RR}$	$\log \langle P_f \rangle_{\rm RR}$	$\langle \log L \rangle_{ab}$	$\langle M \rangle_{ab}$	$\langle \log T_{\rm eff} \rangle_{ab}$	$\log \langle P \rangle_{ab}$	$\langle \log T_{\rm eff} \rangle_{\rm tr}$
				$Y_{\rm MS} = 0.20$					
M3	1.661	0.733	3.835	−0.352	1.665	0.736	3.828	−0.328	3.860
Group II[a]	1.719	0.745	3.844	−0.339	1.737	0.747	3.828	−0.272	3.850
Group II − M3	0.058	0.012	0.009	0.013	0.072	0.011	0.000	0.056	−0.010
$\Delta \log P$[b]	0.049	−0.005	−0.031	...	0.060	−0.004	0.000	...	...
				$Y_{\rm MS} = 0.23$					
M3	1.718	0.750	3.385	−0.311	1.721	0.753	3.828	−0.288	3.858
Group II[a]	1.773	0.793	3.846	−0.319	1.800	0.791	3.829	−0.240	3.853
Group II − M3	0.055	0.043	0.011	−0.008	0.079	0.038	0.001	0.048	−0.005
$\Delta \log P$[b]	0.046	−0.016	−0.038	...	0.066	−0.015	−0.003	...	...

[a] Model M15A (or M53).
[b] The difference in log P produced by the difference group II − M3.

instability strip, we have minimized these effects by calculating the mean values obtained from 50 random simulations, each with a sample size of 1000 HB stars. Similar calculations with sample sizes appropriate for the clusters provided estimates of the statistical fluctuations to be expected in the observed values (see Table 4). To calculate $\log \langle P_{ab} \rangle$ for our models for M3 and a typical group II cluster (model M15A or M53), we first ordered the stars in the instability strip by increasing $T_{\rm eff}$ and then divided them into two groups, type *ab* and type *c*, at the value of $T_{\rm eff}$ that would yield $f_c = 0.17$ and 0.52 for M3 and the group II cluster, respectively. The log $T_{\rm eff}$ where this transition occurs ($\langle \log T_{\rm eff} \rangle_{\rm tr}$) is listed in Tables 5 and 6. These values suggest, as have the calculations with hysteresis, that only small differences of 80–160 K in the transition $T_{\rm eff}$ between the Oosterhoff groups are required to explain the differences in f_c and $\log \langle P_{ab} \rangle$. Tables 5 and 6 list, in addition, the values obtained for the mean luminosity ($\langle \log L \rangle$), mean mass ($\langle M \rangle$), mean temperature ($\langle \log T_{\rm eff} \rangle$), and mean period ($\log \langle P \rangle$) of all variables and, separately, the type *ab* variables (denoted by subscripts RR and *ab*, respectively).[2] These calculations employed the same blue edges for the instability strip, but assumed strip widths of $\Delta \log T_{\rm eff} = 0.075$ (Table 5) and 0.065 (Table 6). The data in the rows denoted II-M3 show that the differences in $\log \langle P_{ab} \rangle$ between the models for the group II cluster and M3 are very similar to the observed differences (Table 4). The data in the row denoted $\Delta \log P$ show that this difference is due mainly to the difference in luminosity between the type *ab* variables in the two model clusters and that the small differences in $\langle M \rangle_{ab}$ and $\langle \log T_{\rm eff} \rangle_{ab}$ have only minor effects. This is not the case for $\log \langle P_f \rangle_{\rm RR}$, because the nonuniform distribution in $T_{\rm eff}$ across the strip in the group II clusters produces a significant difference in $\langle \log T_{\rm eff} \rangle_{\rm RR}$ between these clusters and M3, which when transformed to $\Delta \log P$ nearly cancels that produced by the difference in luminosity. This explains why $\Delta \log \langle P_f \rangle$ is observed to be smaller than $\Delta \log \langle P_{ab} \rangle$ (see Table 4).

While our calculations for both $Y_{\rm MS} = 0.20$ and $Y_{\rm MS} = 0.23$ yield values of $\Delta \log \langle P_{ab} \rangle$ that are consistent with the observed values, our calculations for $Y_{\rm MS} = 0.23$ yield values of $\Delta \log \langle P_f \rangle$ that appear to be too small. Additional calculations show that this disappears if the positions of the blue edge or the red edge of the instability strip are shifted in $T_{\rm eff}$ by small amounts. While the suggestion that the difference in $\langle P_{ab} \rangle$ between the Oosterhoff groups is due primarily to a difference in HB luminosity is not new, we believe that our models, which also match the observed HB morphologies of the clusters, are the first ones to show how this is a consequence of standard HB evolution, without the need of added hypotheses, such as the Y–[Fe/H] anticorrelation proposed by Sandage (1982*a*, 1990).

While our models can explain, within the present limitations of pulsation theory, the difference in $\log \langle P_{ab} \rangle$ and f_c between the two Oosterhoff groups, it is not obvious that they will reproduce the observed dichotomy in $\langle P_{ab} \rangle$ and not produce a continuum of values, including ones intermediate between the two groups. This is a sufficiently involved question that we postpone our discussion of it to Paper III of this series. To anticipate the results presented there, we find that intermediate values of $\log \langle P_{ab} \rangle$ can be produced in principle, but only by clusters within a very narrow range of [Fe/H] and HB morphology. It is then understandable that the ~50 clusters in the Galaxy containing sufficient numbers of RR Lyrae variables fall into one or the other Oosterhoff group.

d) Masses of the RR Lyrae Variables

Our results differ in important respects from the analyses of the double-mode RR Lyrae variables (type *d*) found recently in the globular clusters M15, M3, M68, and IC 4499 (see Cox, Hodson, and Clancy 1983; Clement *et al.* 1986; Cox 1987). The masses of these stars, which in the H-R diagram lie near the boundary of the type *c* and type *ab* regions, have been estimated from comparisons of plots of the ratio of their first harmonic to fundamental periods against their first harmonic

[2] Our models systematically underestimate the mean periods, $\log \langle P_{ab} \rangle$ and $\log \langle P_f \rangle$, by substantial amounts. However, it is important to note that the absolute values of $\log \langle P_{ab} \rangle$ and $\log \langle P_f \rangle$ are sensitive to the adopted blue edge of the instability strip (or to the width of the strip). The location of blue edges is sensitive to the treatment of convection theory, the choice of surface boundary conditions, and the choice of opacity (see, e.g., Iben 1971). Thus, the discrepancy between our models and the observations could be removed by decreasing the temperature of the blue edge within the uncertainty (by $\Delta \log T_{\rm eff} = 0.01$–$0.02$). This suggests that the corrections for convection at blue edges adopted in our calculations (see § II*b*) are partly responsible for the discrepancy. Runs of our synthetic HB code with different values for the blue edge and the width of the instability strip indicate, however, the relative differences in mean periods (i.e., $\Delta \log \langle P_{ab} \rangle$ and $\Delta \log \langle P_f \rangle$) are less affected by the uncertainties in these values, and, more important, since the blue edges and the width of the strip are not sensitive functions of metallicity, neither $\Delta \log P/\Delta[{\rm Fe/H}]$ nor $\Delta \log L/\Delta[{\rm Fe/H}]$ depends on the small changes in the location of the blue edges or the width of the strip.

TABLE 6

SYNTHETIC HB CALCULATIONS FOR M3 AND THE OOSTERHOFF GROUP II CLUSTER WITH THE WIDTH OF THE INSTABILITY STRIP $\Delta \log T_{\rm eff} = 0.065$

Cluster	$\langle \log L \rangle_{\rm RR}$	$\langle M \rangle_{\rm RR}$	$\langle \log T_{\rm eff} \rangle_{\rm RR}$	$\log \langle P_f \rangle_{\rm RR}$	$\langle \log L \rangle_{ab}$	$\langle M \rangle_{ab}$	$\langle \log T_{\rm eff} \rangle_{ab}$	$\log \langle P \rangle_{ab}$	$\langle \log T_{\rm eff} \rangle_{\rm tr}$
				$Y_{\rm MS} = 0.20$					
M3	1.657	0.729	3.840	−0.371	1.662	0.732	3.834	−0.350	3.861
Group II[a]	1.717	0.746	3.847	−0.352	1.733	0.747	3.832	−0.289	3.851
Group II − M3	0.060	0.017	0.007	0.019	0.071	0.015	−0.002	0.061	−0.010
$\Delta \log P$[b]	0.050	−0.007	−0.024	...	0.060	−0.006	0.007	...	...
				$Y_{\rm MS} = 0.23$					
M3	1.714	0.746	3.840	−0.330	1.718	0.749	3.834	−0.310	3.860
Group II[a]	1.771	0.791	3.849	−0.330	1.796	0.789	3.833	−0.256	3.855
Group II − M3	0.057	0.045	0.009	0.000	0.078	0.040	−0.001	0.054	−0.005
$\Delta \log P$[b]	0.048	−0.017	−0.031	...	0.066	−0.015	0.003	...	...

[a] Model M15A (or M53).
[b] The difference in log P produced by the difference group II − M3.

periods (the Petersen diagram) with the relations given by stellar pulsation calculations. These analyses have shown that while there is little variation in mass among the type *d* variables in any one cluster, there is a significant difference between the masses obtained for the variables in the group II cluster M15 and the group I clusters M3 and IC 4499, i.e., 0.65 $M_\odot$ and 0.54 $M_\odot$, respectively. Our models suggest that, near the transition temperatures (i.e., $\langle \log T_{\rm eff} \rangle_{\rm tr} \pm 0.003$) listed in Tables 5 and 6, the mean masses of the variables are ~ 0.716 and ~ 0.744 for $Y_{\rm MS} = 0.20$ (~ 0.735 and ~ 0.792 for $Y_{\rm MS} = 0.23$) for M3 and the group II cluster, respectively. These values are significantly larger and the difference in mass between the two groups is smaller than those obtained from the analyses of the Petersen diagram. As Bingham *et al.* (1984) and Clement *et al.* (1986) have pointed out, HB stars that have the masses given by the stellar pulsation analyses of the type *d* variables and the compositions of these globular clusters will spend, according to the standard HB evolutionary theory, nearly their whole HB lives to the blue side of the instability strip and will enter it briefly if at all. The discrepancy involving the difference in mass between the variables in the two Oosterhoff groups can be partially reconciled if the double-mode pulsation is caused by some form of mode switching as a star evolves redward across the instability strip. According to our synthetic HB model for M3 ($Y_{\rm MS} = 0.20$; see Fig. 1), stars with masses as low as 0.66 $M_\odot$ cross the instability strip on short time scales as they evolve to the asymptotic giant branch. These stars are less massive by about 0.08 $M_\odot$ than what our model suggests for the masses of the type *d* variables in M15. The difficulty with this scenario is that while it correctly predicts that few type *d* variables should be found in M3, it fails to account for their high incidence in IC 4499. These difficulties with the type *d* masses are reminiscent of the results obtained for the double-mode classical Cepheids, which show that the masses derived from the Petersen diagram are systematically lower than those indicated by the standard stellar evolution calculations.

For the RR Lyrae variables there are at least two possible explanations for this discrepancy. One is that oxygen is systematically enhanced relative to iron (see Buonanno, Corsi, and Fusi Pecci 1989), as suggested by some recent observations of field stars (see, e.g., Bond and Luck 1987 and references therein). When the oxygen abundance is increased, the increased energy output of the hydrogen shell, together with the increase in the metal opacity, forces the envelope of a HB model to expand to a large radius. This leads to less massive and slightly fainter HB models at a given effective temperature (Rood and Seitzer 1981; VandenBerg 1985; Sweigart, Renzini, and Tornambè 1987; Lee and Demarque 1989). These models suggest that [O/Fe] $\approx +0.7$ may produce masses that are consistent with the ones derived from the Petersen diagram. This value is, however, approximately twice as large as the one expected from the observations (i.e., [O/Fe] $\approx +0.4$). The other possibility is that the masses derived from the Petersen diagram are uncertain by significant amounts. Kovács (1985) has shown that the Petersen diagram is sensitive to the details of the stellar pulsation code, the opacities used in the calculations, and the adopted metal abundance. Cox, Hodson, and Clancy (1983) found 0.65 $M_\odot$ for the type *d* variables in M15, whereas Kovács obtained 0.7–0.75 $M_\odot$ for the same stars and adopted composition. We note in addition that the variables in M15 have been compared with pulsation calculations for the same relatively metal-rich composition as the variables in M3 and IC 4499, even though M15 is undoubtedly more metal-poor by a large factor, and that these calculations have also assumed $Y = 0.30$, which may be too large for globular clusters (Buzzoni *et al.* 1983). The calculations by Kovács (1985) indicate that the adoption of a lower metal abundance for M15 would reduce the difference in mass between it and the group I clusters, but by how much is not known because a Petersen diagram for the appropriate composition has not been calculated. The Petersen diagram shown by Cox (1987) suggests that the adoption of $Y = 0.20$ would lead to larger masses for at least the M15 variables. Thus, some upward revision of the masses of the type *d* variables may be warranted. Furthermore, on the basis of comparisons of his results with those of Cox, Hodson, and Clancy (1983), Kovács (1985) concluded that the uncertainties in the pulsation code and opacities may cause errors of 10%–20% in the masses given by the Petersen diagram. Independently, Clement *et al.* (1986) have put the uncertainty at 15% or more. Systematic errors of this magnitude could explain most of the discrepancies between our results and those of Cox, Hodson, and Clancy (1983) and Clement *et al.* (1986).

To summarize this section, we find that our synthetic HB models, which are based on standard calculations of HB evolution and constant Y, can reproduce, to the precisions of existing observations, the period shifts in the $\log P'$–$\log T_{\rm eff}$ diagram and also the slope of Sandage's (1982*a*) relationship between $\Delta \log P'$ and [Fe/H] once it is corrected for the dependence of $\Delta\phi_{rise}$ on [Fe/H] as well as T_{eff}. Our models suggest that the Oosterhoff effect is produced by the period shifts stemming from the luminosity–[Fe/H] relationship, the uneven distribution of stars across the instability strip in the group II clusters, and a small amount of hysteresis (or, equivalently, a small difference in transition T_{eff} between the Oosterhoff groups).

IV. VARIATION OF HORIZONTAL-BRANCH LUMINOSITY WITH [Fe/H]

Our synthetic HB models for nine clusters considered in this paper (see Table 2) predict that the mean absolute magnitude of the RR Lyrae variables is a linear function of log Z (see Fig. 16) according to the following equations:

$$M_{\rm bol}^{\rm RR} = 0.22[{\rm Fe/H}] + 0.99 \,, \tag{4}$$

$$M_V^{\rm RR} = 0.19[{\rm Fe/H}] + 1.00 \tag{5}$$

for $Y_{MS} = 0.20$, and

$$M_{\rm bol}^{\rm RR} = 0.20[{\rm Fe/H}] + 0.81 \,, \tag{6}$$

$$M_V^{\rm RR} = 0.17[{\rm Fe/H}] + 0.82 \tag{7}$$

for $Y_{MS} = 0.23$. To derive these equations, we have assumed that $[Fe/H] = -1.66$ corresponds to $Z = 0.0004$ and have adopted $M_{bol} = +4.79$ for the Sun and the bolometric corrections that Buser and Kurucz (1978) calculated and then normalized to the empirical values obtained by Code *et al.* (1976). It is important to note that the choice of the bolometric corrections affects the zero points (hereafter β) of the M_V relations, but not the slopes (α). If we adopt the bolometric corrections that have been used in the construction of the revised Yale isochrones (see Green 1988), the values of β become 0.96 and 0.78 for $Y_{MS} = 0.20$ and 0.23, respectively. The α-values of our relations are slightly different from those given by ZAHB calculations (i.e., $\Delta M_{\rm bol}^{\rm RR}/\Delta[{\rm Fe/H}] = 0.18$; Sweigart, Renzini, and Tornambè 1987) because the majority of the variables in the Oosterhoff group II clusters have evolved far from their ZAHB positions. If [O/Fe] is $\sim +0.4$ in Population II stars, the values of α and β may need to be reduced by, very roughly, ~ 0.02 (i.e., the luminosity of the HB is slightly fainter at a given effective temperature), as one can infer from Figure 1 of VandenBerg (1985).

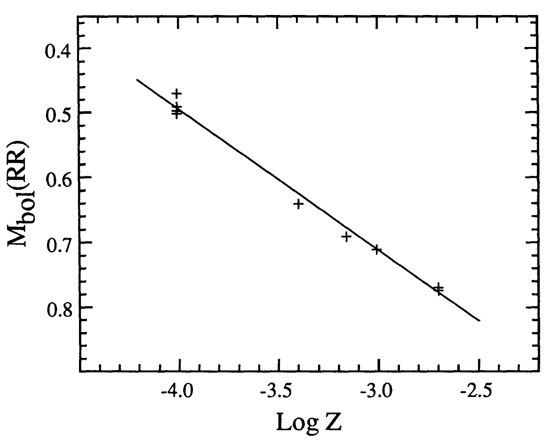

FIG. 16.—HB luminosity–log Z relationship predicted by the synthetic HB models for nine clusters ($Y_{MS} = 0.20$). A least-squares fit yields $M_{\rm bol}^{\rm RR} \approx 0.22$[Fe/H] + 0.99 (assuming $[Fe/H] = -1.66$ for $Z = 0.0004$, and $M_{bol} = +4.79$ for the Sun).

As the above relations illustrate, the choice of Y_{MS} has a bigger effect on β than on α. The value of α is sensitive to how the mass of the helium core (M_c) on the ZAHB varies with Z, while β is sensitive to this and to the value M_c that is adopted for a particular value of Z. Sweigart, Renzini, and Tornambè (1987) have shown that while uncertainties in the input physics significantly affect the size of M_c in red giant models, the partial derivative of M_c with respect to Z is nearly unaffected. Hence, we have much more confidence in our values for α than in those for β. Note that if ΔV(TO − HB), the difference in luminosity between the turnoff and the mean level of HB at the color of the instability strip, is in the mean constant for all metallicities, as indicated by observations, then the value of α dictates whether or not there is an age-metallicity relation among the globular clusters (see Zinn 1986 and below), while α and β together set the ages of the oldest clusters and hence the minimum age of the universe.

There are several recent investigations that have also considered the variation of the luminosity of the RR Lyrae variables with [Fe/H], and it is important to compare these results with ours. A very valuable reference in this regard is the paper by Buonanno, Corsi, and Fusi Pecci (1989), which discusses in detail the arguments for and against α being ~ 0.35, as found by Sandage (1982*a*) in his analysis of the Oosterhoff effect. In our discussion, we will distinguish between evidence for period shifts among variables of the same T_{eff} and more direct evidence for a particular value of α.

The investigations by Lub (1977, 1987), Sandage (1982*b*), and Kemper (1982) have established that period shifts, much like the ones observed between globular clusters, exist among field RR Lyrae variables of different [Fe/H]. However, the data for the field stars scatter more than the cluster data in the log P–log T_{eff} diagram because a correction for the luminosity width of the HB cannot be applied to the field variables, and hence for them log P' cannot be calculated. While Kemper's (1982) sample of field variables clearly shows that period shifts exist, their size cannot be estimated with confidence, given the scatter in the log P–log T_{eff} diagram. Sandage's (1982*b*) analysis showed that the field and the cluster variables have the same relationship between $\Delta \log P$ and [Fe/H] when A_B is used as an estimator of T_{eff}. Since we have shown here that the period shifts of the clusters are compatible with $\alpha \approx 0.20$, it follows that the data for the field variables are too. In his recent reanalysis of the Walraven photometry of a large sample of field RR Lyrae variables, Lub (1987) obtained from his measurements of T_{eff} and the period–mean density relation of van Albada and Baker (1971) $M_V^{\rm RR} = 0.20[{\rm Fe/H}] - 2.03 \log M + 0.37$, where M is the mass of the variables. This equation yields $\alpha = 0.20$, if $\Delta \log M/\Delta[{\rm Fe/H}] = 0$, which, however, is not predicted by theory. To illustrate the size of the disagreement, we have used Lub's equations (1) and (2) to calculate the period shift at constant T_{eff} given by his analysis. The result is $\Delta \log P/\Delta[{\rm Fe/H}] = -0.084$, which is stepper than the slopes given by our synthetic HB models (-0.043 and -0.027 for $Y_{MS} = 0.20$ and 0.23, respectively), but shallower than Sandage's (1982*a*) value of -0.116. Sandage (1990) has recently reanalyzed Lub's (1977) earlier data and has found a slope of -0.09.

These results are not necessarily evidence that our models are incorrect. Lub's sample contains few stars as metal-poor as M15, but many that are somewhat more metal-rich and comparable to ω Cen in abundance (see Fig. 5 of Lub 1987). It seems likely that at least some stars of this second group are similar to the variables in ω Cen and in other globular clusters of the same [Fe/H]. The majority of the clusters of this abundance ($-2 <$ [Fe/H] < -1.6 on the scale of Zinn 1985) have very blue HBs (see Fig. 3 of LDZ 1988*b* and Zinn 1986), and this is particularly true at the galactocentric distances <10 kpc. With only two exceptions, M3 and NGC 4147, the clusters in this metallicity range that contain RR Lyrae variables are classified Oosterhoff group II on the basis of their values of $\langle P_{ab} \rangle$ and f_c (i.e., ω Cen and NGC 4833, 5286, 5634, 5986, 6333, 6656, 6809, 7089; see Castellani and Quarta 1987). These clusters have not received as much attention as the more metal-poor group II clusters, because in general they contain fewer variables. Our synthetic HB models for these very blue HB clusters (to be described in detail in Paper III) show that their variables are highly evolved stars from the blue HB and, consequently, that these variables are shifted in period relative to M3 variables of the same $T_{\rm eff}$ by approximately the same amounts as the variables in M15. One prediction of our models is, therefore, that the relationships between [Fe/H] and period shift and the related quantity log $M^{0.81}/L$, which was plotted by Lub (1987) and by Sandage (1990), will become nonlinear with decreasing [Fe/H] at approximately [Fe/H] $= -1.6$. There is some evidence for this effect in Figure 5 of Lub (1987) and Figures 6 and 8 of Sandage (1990). We suggest that the disagreements between the slopes, $\Delta \log P/\Delta$[Fe/H], obtained by us and by the analyses of Lub's data can be traced to (*a*) the fact that our analysis in § III does not include blue HB clusters in the range $-2 <$ [Fe/H] < -1.6, whereas Lub's sample of field variables contains many stars in this range, and (*b*) the likelihood that the log P–[Fe/H] relation is nonlinear, hence fits of straight lines to different data sets may produce significantly different slopes.

Besides Sandage's interpretation of the period-shift effect, the strongest piece of evidence that Buonanno, Corsi, and Fusi Pecci (1989) quote in favor of $\alpha \approx 0.35$ is their own result of $\alpha = 0.37 \pm 0.14$, which they obtained by fitting 19 globular cluster main sequences to one another via the "theoretician's route to distances." In this procedure, the theoretical isochrones of VandenBerg and Bell (1985) were used to estimate the differences in main-sequence color stemming from the differences in [Fe/H] among the clusters. While the resulting value of α is essentially the same as the one found by Sandage, Buonanno *et al.* noted that the difference between this value and the one given by ZAHB calculations (~ 0.18) is not statistically significant. More recently, King, Demarque, and Green (1988) have obtained values of $M_V^{\rm RR}$ by fitting the main sequences of 10 globular clusters to the revised Yale isochrones (Green, Demarque, and King 1987) for $Y_{\rm MS} = 0.24$. They obtained $M_V^{\rm RR} = 0.20$[Fe/H] $+ 0.84$, which agrees with our results for both α and β. Although no error bars were given, it seems reasonable to conclude from this investigation and from that of Buonanno *et al.* that the results of the theoretician's route to distances are compatible with $\alpha \approx 0.20$. The method of fitting globular cluster main sequences to subdwarfs of known parallax, the "observer's route to distances," is beset with uncertainties that preclude a definite statement about α (see Buonanno, Corsi, and Fusi Pecci 1989).

In support of $\alpha \approx 0.35$, Buonanno, Corsi, and Fusi Pecci (1989) also mention the P-L relation for Mira variables in globular clusters (Menzies and Whitelock 1985), the better agreement that Frenk and White (1982) obtained for the Sun's galactocentric distance from separate analyses of the spatial distributions of the metal-rich and the metal-poor globular clusters if they adopted a large value for α (discussed also by Feast 1987), and the preliminary results that Buonanno *et al.* (1985) obtained for the variation in HB luminosity with [Fe/H] among the globular clusters in the Fornax dwarf spheroidal galaxy. In our opinion, none of these results carries much weight. The argument regarding the Miras depends on how much the zero point of their P-L relation varies with [Fe/H]. Some variation is definitely required because without it the Mira P-L relation yields $\alpha \approx 0.54$ (Menzies and Whitelock 1985), which is incompatible with all other measurements. More zero-point variation is required if $\alpha = 0.20$ rather than 0.35, but to our knowledge this cannot be ruled out by existing observations (see Feast and Whitelock 1987). The recent work of Armandroff (1988) on the metal-rich globular clusters has shown that the previous estimates of the distance moduli of several of them, including five of the 11 metal-rich clusters in Frenk and White's (1982) analysis, are substantially in error. A detailed reanalysis is clearly needed before any conclusion about α can be drawn from the spatial distributions of these clusters. The globular clusters in Fornax may provide a definitive value for α once, as Buonanno *et al.* (1985) conclude, a color-magnitude diagram reaching the HB is constructed for the most metal-rich cluster and the diagrams for the other clusters are improved.

The strongest observational evidence in favor of $\alpha < 0.20$ is provided by three recent investigations of field RR Lyrae variables using the Baade-Wesselink method. In the past, this method was plagued by unphysical phase lags between the angular diameters obtained from the color and radial velocity curves. The studies by Jones, Carney, and Latham (1988), Cacciari *et al.* (1989), and Liu and Janes (1988*a*) discuss how this problem can be avoided by using the $V-K$ or $V-I$ color indices instead of $B-V$ or some other blue color. The data for the seven variables studied by Jones, Carney, and Latham (1988) yield $M_V^{\rm RR} = 0.06(\pm 0.04)$[Fe/H] $+ 0.91(\pm 0.05)$. The data for the six variables measured by Cacciari, Clementini, and Buser (1989) and Cacciari *et al.* (1989) yield $M_V^{\rm RR} = 0.11(\pm 0.03)$[Fe/H] $+ 0.92(\pm 0.04)$, and Liu and Janes (1988*b*) obtained $M_V^{\rm RR} = 0.19(\pm 0.03)$[Fe/H] $+ 1.05(\pm 0.03)$ from their sample of 13 variables. The quoted errors are those given by fits of straight lines to the data points; the true uncertainties are undoubtedly larger. Nonetheless, as noted by these authors, the Baade-Wesselink results are clearly at odds with $\alpha > 0.30$. In order to increase the sample size, Cacciari, Clementini, and Buser (1989) have combined their measurements with those of others (not including the more recent ones of Liu and Janes 1988*a*). With this (albeit heterogeneous) sample of Baade-Wesselink measurements for 19 field variables and four variables in the globular cluster M5, Cacciari and coworkers found $M_V^{\rm RR} = 0.17$[Fe/H] $+ 1.0$, which is in excellent agreement, in terms of both α and β, with the independent study of Liu and Janes and our theoretical calculations. A very similar result was obtained by Liu and Janes (1988*b*): $M_V^{\rm RR} = 0.16$ (± 0.02)[Fe/H] $+ 1.02(\pm 0.03)$ from combining their results with those of Jones, Carney, and Latham (1988).

The analysis by Walker and Mack (1986) of the space distribution of the RR Lyrae variables in Baade's window also suggests that $\alpha < 0.20$. Although the [Fe/H] measurements are crude (σ([Fe/H]) ≈ 0.4), their sample of 31 variables shows little variation in mean B-magnitude in spite of a large variation in [Fe/H]. This suggests that, as expected, most of the variables lie at nearly the same distance from the Sun and that there is not a steep dependence of $\langle B \rangle$ on [Fe/H]. Walker and Mack made a maximum-likelihood solution for α, including the estimated error in [Fe/H] and $\sigma(\langle B \rangle) \approx 0.15$, which yielded $\alpha = 0.05 \pm 0.08$, and they concluded that this was strong evidence against $\alpha \approx 0.35$. The difference between our result ($\alpha \approx 0.18$) and theirs is only 1.6 σ, and hence possibly not significant.

There is some evidence in favor of $\alpha \approx 0$, the most stringent of which appears to be the observation by Butler, Dickens, and Epps (1978) that there is no variation in the mean of the apparent magnitudes of the RR Lyrae variables in ω Cen, in spite of their more than 1 dex variation in [Fe/H]. This and the lack of a period-shift–[Fe/H] relation among these variables was recognized by Sandage (1981, 1982*a*) as a possible contradiction to his explanation of the Oosterhoff effect. Although our models predict roughly one-half the luminosity variation and roughly one-third the period shift that Sandage suggested, the ω Cen variables still pose a serious problem that we will address in detail later by constructing synthetic HB models that include the [Fe/H] distribution observed in ω Cen. For now, we note that Gratton, Tornambè, and Ortolani (1986) have suggested that the absence of the luminosity variation may be explained by conventional HB theory if the majority of the variables are in post–ZAHB phases of evolution and that this is consistent with the HB morphology of the cluster.

The recent measurements of the absolute magnitudes of field RR Lyrae variables by the method of statistical parallax (Strugnell, Reid, and Murray 1986; Hawley *et al.* 1986; Barnes and Hawley 1986), which employed very similar techniques and data bases, found no evidence for a variation of M_V with [Fe/H], but the significance of this result has been debated. Strugnell, Reid, and Murray (1986) concluded that their results are inconsistent with Sandage's slope of $\alpha \approx 0.35$, while Hawley *et al.* (1986) and Barnes and Hawley (1986) were more cautious and concluded that more data are needed before a definite conclusion about the variation with [Fe/H] can be made. It appears, then, that the more shallow slope given by our models is consistent with these results. These investigations also provide valuable information on the zero point. If the Sturch (1966) reddening system is adopted, the results of Strugnell, Reid, and Murray (1986) and Barnes and Hawley (1986) are $\langle M_V \rangle = 0.74 \pm 0.13$ and $\langle M_V \rangle = 0.68 \pm 0.14$, respectively. The adoption of the H I reddening system of Burstein and Heiles (1982) produces values that are 0.10 mag fainter. The mean [Fe/H] of these nearly identical samples of variables is -1.10. For this value, our models give $M_V = 0.79$ ($Y_{MS} = 0.20$) and 0.63 ($Y_{MS} = 0.23$), in reasonable agreement with the results obtained with the Sturch reddenings.

In conclusion, we believe that there is now considerable evidence that α is significantly less than 0.30, and probably not far from the values given by our theoretical calculations. The value of β is more uncertain, but the reasonable agreement between our models and the results obtained via the Baade-Wesselink method, statistical parallax, and the fitting of globular cluster main sequences to isochrones (King, Demarque, and Green 1988) is encouraging.

V. IMPLICATIONS FOR GLOBULAR CLUSTER AGES

The relationship between the mean HB luminosity and [Fe/H] predicted by our models has implications for the chronology of the formation of the Galaxy, which we will illustrate by adopting the method of cluster dating that makes use of ΔV(TO − HB). This method is not affected by the uncertainties in reddening and the mixing length used in convection theory, which may make it superior to the direct comparison of color-magnitude diagrams with theoretically computed isochrones (see Iben and Renzini 1984; Sandage 1986). Sandage (1982*a*) and more recently Buonanno, Corsi, and Fusi Pecci (1989) have argued that ΔV(TO − HB) is constant, to within the observational errors ($\sim$0.15 mag), over the metallicity range of the clusters with well-determined turnoffs ($-2.3 <$ [Fe/H] < -0.7). For the purpose of our illustration, we have adopted this result, specifically $\Delta M_{\rm bol}$(TO − HB) ≈ 3.50 [i.e., ΔV(TO − HB) ≈ 3.59], and the $M_{\rm bol}^{\rm RR}$–[Fe/H] relations predicted by our models (eqs. [4] and [6]), to derive the luminosity of the turnoff. Since Buonanno, Corsi, and Fusi Pecci (1989) defined ΔV(TO − HB) as the difference in luminosity between the turnoff and the ZAHB at the color of the instability strip, while our equations (4)–(7) represent the mean magnitude for the HB and not the ZAHB luminosity, we have adjusted their value of ΔV(TO − HB) = 3.55 by adding 0.04 mag. The ages derived from the $M_{\rm bol}$(TO)–age relationships predicted by the revised Yale isochrones (Green, Demarque, and King 1987), which assume a solar mix of the elements, are listed in Table 7 for $Y_{MS} = 0.23$ and 0.20. It is important to note that the ages in Table 7 are not affected by the uncertainties in the normalization of the bolometric corrections (BC), but depend only on the adopted difference in BC between the TO and the HB at the color of the instability strip [i.e., Δ(BC) = $\text{BC}_{\rm TO} - (\text{BC})_{\rm HB} \approx -0.09$], which is nearly independent of [Fe/H] (Green *et al.* 1989; Buser and Kurucz 1978).

TABLE 7

AGES OF HALO GLOBULAR CLUSTERS[a,b]

Y_{MS}	[Fe/H] −2.3	−1.6	−0.8
0.23.........	17.0	15.8	12.9
0.20.........	20.0	18.4	15.0

[a] Assuming ΔV(TO − HB) = 3.59 for all metallicities and [O/Fe] = 0.0.
[b] In units of Gyr.

To illustrate that the ages derived by the ΔV(TO − HB) method are not sensitive to the choice of isochrones, we have also derived ages using equation (4) of Buonanno, Corsi, and Fusi Pecci (1989), which is an analytical fit to the isochrones calculated by VandenBerg and Bell (1985). For $Y_{MS} = 0.23$, $M_{\rm bol}^{\odot} = +4.72$ (to be consistent with the value adopted by VandenBerg and Bell 1985), and [Fe/H] = −2.3, −1.6, and −0.8, we obtain ages of 17.2, 15.4, and 13.6 Gyr, respectively. These values are nearly identical to the ones obtained from the revised Yale isochrones (see Table 7).

Recent determinations of the O to Fe ratio in field stars (see Bond and Luck 1987 and references therein) suggest that [O/Fe] is approximately constant at +0.4 over the range [Fe/H] = −2.3 to −0.7 and then declines linearly to zero as [Fe/H] approaches zero. If [O/Fe] = +0.4 in the globular clusters, then the calculations of VandenBerg (1985) indicate that both sets of ages in Table 7 should be systematically reduced by $\sim$1.8 Gyr. The ranges in ages are predicted to remain essentially unchanged (see VandenBerg 1985). It is not clear, however, that the clusters have these O enhancements. Pilachowski (1988), for example, finds [O/Fe] $\approx +0.2$ for the very metal-poor cluster M92 (see also Gratton and Ortolani 1989).

Globular clusters that are more metal-rich than [Fe/H] = −0.8 contain very few if any RR Lyrae variables because their HBs lie redward of the instability strip. Since the luminosity given by an M_V–[Fe/H] relation for RR Lyrae variables may be systematically too faint for a stubby red HB by ~0.1–0.2 mag (see LDZ 1987), some extra care must be taken when dating these clusters with the ΔV(TO − HB) method. The best-studied metal-rich cluster is 47 Tuc, for which Hesser *et al.* (1987) have constructed an excellent color-magnitude diagram. Their data yield 3.60 ± 0.10 for the difference between the mean V-magnitude of the red HB and the turnoff. Unlike the case of the metal-poor clusters containing RR Lyrae variables, the difference in BC between the turnoff and the red HB is substantially larger (Δ(BC) ≈ +0.12; Green *et al.* 1989) because the red HB is significantly cooler than the turnoff; hence we estimate $\Delta M_{\rm bol}$(TO − HB) ≈ 3.72 for 47 Tuc. Our models for the red HB of 47 Tuc predict mean $M_{\rm bol}$ values of 0.49 and 0.41 for $Y_{\rm MS} = 0.20$ and 0.23, respectively. Adopting these values, $\Delta M_{\rm bol}$(TO − HB) = 3.72, and [Fe/H] = −0.7 (Zinn 1985), we find $M_{\rm bol} = 4.22$ and 4.12 for the turnoff and ages of 13.2 and 11.8 from the revised Yale isochrones. At these relatively metal-rich compositions, the choice of opacity table used in the construction of the stellar models does have some effect on their luminosities. The results presented above are based on models that incorporated the opacity tables of Cox and Stewart (1970). We estimate that if we had used instead the latest Los Alamos tables (Huebner *et al.* 1977), the age obtained for 47 Tuc would be slightly greater (≤1 Gyr; see Demarque *et al.* 1988). Finally, if [O/Fe] = 0.4, these ages will need to be reduced by ~1.8 Gyr. In spite of these uncertainties, it is difficult to increase the age of 47 Tuc beyond 15 Gyr ($Y_{\rm MS} = 0.20$); hence it appears to be substantially younger than the most metal-poor clusters.

The ages presented above and in Table 7 suggest that there is a significant age–[Fe/H] relationship over the range of the halo globular clusters ([Fe/H] < −0.8, Zinn 1985) and that one cluster, 47 Tuc, of the thick disk population (Zinn 1985; Armandroff 1989) is significantly younger than the mean age of the halo. These results depend, of course, on the validity of our HB luminosity–[Fe/H] relationship and the constancy of ΔV(TO − HB). If one accepts our relationship, uniform ages for all clusters can be obtained only if $d[\Delta V(\mathrm{TO-HB})]/d[\mathrm{Fe/H}] \approx 0.22$. Since Buonanno, Corsi, and Fusi Pecci (1989) obtained -0.01 ± 0.08 for this derivative, this can be ruled out. Essentially the same age for all clusters is obtained with the HB luminosity–[Fe/H] relationships derived by Sandage (1982*a*) and by Buonanno, Corsi, and Fusi Pecci (1989), but these appear to be at odds with several measurements of the luminosities of RR Lyrae variables (see § IV), and to be understood with present theory they require the existence of a large anticorrelation between Y and [Fe/H] that appears to be in the wrong sense and at odds with observational data.

Several recent investigations of the ages of globular clusters that have used the method of fitting the cluster main sequences to theoretical isochrones (King, Demarque, and Green 1988; Stetson and Harris 1988) have also found that metal-rich clusters are younger than metal-poor ones by several gigayears. Other investigations of clusters of approximately the same [Fe/H] but very different HB morphologies (i.e., Pal 12; NGC 288, 362, 1261; M5) have also found substantial variations in age (LDZ 1988*b*; Gratton and Ortolani 1988; King, Demarque, and Green 1988; Stetson *et al.* 1989; Bolte 1989). In this case, age correlates well with HB morphology in the sense that the clusters with blue HBs are older than those with red HBs. This has provided a strong case that age is the infamous second parameter that in addition to [Fe/H] controls HB morphology. In a preliminary report (LDZ 1988*b*), we showed that there was reasonable agreement between the ages derived from the turnoffs of M5, NGC 288, and NGC 362 and those predicted by our synthetic HB models. This is extended in Paper II of this series, which also discusses, much more thoroughly than before, the uncertainties in these comparisons.

The mounting evidence that there are large variations in age among the halo globular clusters—some correlated with [Fe/H] and some correlated with HB morphology—provides considerable support, we believe, for a picture of galaxy formation like that of Searle and Zinn (1978) in which the halo was built up over several gigayears out of the destruction fragments, each having its own history of chemical enrichment. An essential part of their argument was that age is the second parameter, from which it followed that there are large age variations in the halo and a systematic decrease in mean age and an increase in age dispersion with increasing galactocentric distance (see Zinn 1986 for a review). The Searle and Zinn picture is also supported by several investigations of the kinematics of globular clusters and stars that show that there is very little, if any, variation in kinematical properties over the metallicity range of the halo (Norris 1986; Carney 1988; Zinn 1988; Norris and Ryan 1989). The papers by Sandage and Fouts (1986) and Sandage (1986) present an opposing point of view that favors the Eggen, Lynden-Bell, and Sandage (1962) picture in which the halo formed rapidly (<1 Gyr) and increased in metal abundance as it collapsed and spun up. Soon, one or the other of these very different pictures may be decisively ruled out. Then real progress will have been made toward deciphering the fossil record of the formation of the Galaxy.

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