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MODEL FOR ARTIFICIAL NIGHT-SKY ILLUMINATION

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ABSTRACT

A model has been constructed to allow calculation of the night-sky brightness caused by a city at its center and outside the city, and at arbitrary zenith distances. A circular city of uniform brightness is assumed, with the total brightness proportional to the population. Molecular scattering and aerosol scattering are included, with the amount of aerosols being an adjustable parameter, and different scale heights being adopted for molecules and aerosols. The reflectivity of the ground and the fraction of light radiated above the horizontal are taken as parameters. Applications are given to several cities, to the general population-distance relations, to brightness-distance relations, and to the city center brightness-population relations.

Key words: observatory sites–light pollution

I. Introduction

The astronomical community has become increasingly concerned with the problem of light pollution. There have been many studies of the night-sky brightness at particular observatories, for example at Mount Wilson and Mount Palomar (Turnrose 1974). Some studies have been made of light pollution over wider geographical areas, for example California and Arizona (Walker 1973), Southern Ontario (Berry 1976), and Italy (Bertiau, de Graeve, and Treanor 1973). Berry demonstrated an important relationship between the city center zenith sky brightness and the population of the city. Walker (1977) made a major interpretation of his observations by deriving luminosity-population, brightness-distance, and population-distance relationships. There have been few attempts at constructing models to explain the observations, but both Berry and Bertiau *et al.* made extensive use of an empirical law derived by Treanor (1973). In this paper a model is described which enables the brightness of the night sky to be predicted, and which goes a long way toward explaining the relationships found by Berry and by Walker. We have tried to devise a model which would take into account the main features of the physical situation, but which would at the same time be simple enough for numerical applications to be worked out using a personal computer.

II. The Model

(a) The situation is shown in Figure 1. We idealize the city as a circular area of uniform brightness with center C and radius R, the city lying in a horizontal plane. The observer is situated at O, at a height A above the plane of the city, and at a distance D from the center of the city measured in the plane of the city (BC in Fig. 1). We assume that the city is at a height H above sea level. Real cities depart from this ideal because of the presence of exceptionally bright local areas (such as shopping centers) and streamers (main highways with ribbon development), and in addition they do not have a sharp boundary but merge into the countryside through scattered suburbs. Nevertheless, we think that a uniform circle will give much better results than a point source for observers near the city, and in addition a uniform circle can be used for calculations of the sky brightness seen from within the city. At large distances from a city a point-source approximation may often be adequate, and in such cases our mathematical formulation may be simplified. However, a point-source approximation is not adequate when considering observations at large zenith distances if the line of sight passes close to the city. Caution must be exercised when interpreting the results of model calculations if the observer is close to very bright local areas of light inside a city.

We assume that the city has a population P. We estimated P from 1980 census data; we usually used data from *The World Almanac and Book of Facts* (1983) or from the *Rand-McNally Road Atlas* (1984). We occasionally used census data from an earlier edition of this *Atlas* when we felt that 1970 census data would be more appropriate. We estimated R by an examination of available maps, especially the above *Atlas* and the *Atlas of the World* (1981). For distant cities a rough guess is sufficient. In general we included in P and R a city and its immediate suburbs, and

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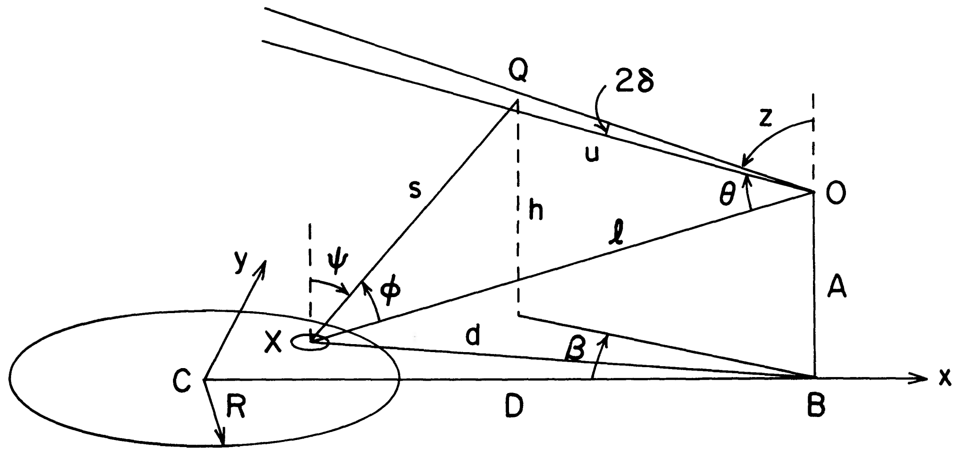

FIG. 1–Scattering of light by a city. Light from a small area at X is scattered at Q and received by the observer O. Q is at a height h above the plane of the city, β is the azimuth of Q, and z is the zenith distance of observation. The observer receives light from a cone of semiangle δ around QO.

we omitted low-population-density outer suburbs (which tend to have low levels of outdoor illumination).

(b) We assume initially that artificial lighting produces an output L lumens per head of the population, so that the total light output is LP lumens. It seems likely that large cities are somewhat brighter, and small cities somewhat fainter, than this formula would indicate. (Large cities have large shopping centers, brightly lit arterial roads, major sports facilities, etc., which are often absent from small towns.) After several trials we adopted the formula $L_0P(P/100{,}000)^{0.1}$ for the light output of a city. The exponent 0.1 is quite uncertain, but seems to give reasonable results. It is trivial in most cases to include or exclude the factor $(P/100{,}000)^{0.1}$ as may be desired. We speak of the brightness as being *enhanced* when we include this factor.

We assume that a fraction F of the light produced by the city is radiated directly into the sky at angles above the horizontal, and that the remainder $(1-F)$ is radiated toward the ground. We assume that the fraction F is radiated with an intensity proportional to ψ^4, where ψ is the zenith distance of an upbound light ray (XQ in Fig. 1), and the constant of proportionality is determined by normalization. The choice of the function ψ^4 is purely arbitrary: it has the required properties, that the intensity of radiation toward the zenith ($\psi = 0$) is essentially zero, and that as ψ approaches 90° (horizontal) the intensity increases rapidly. These properties seem to be true for most street lights and for at least some other forms of outdoor lighting. It is the writer's experience that if one looks downward from a plane approaching a landing at an airport one sees patches of light below street lights, but sees relatively little direct upward radiation until ψ is approaching 90°.

We assume that of the fraction $1-F$ of light reaching the ground from artificial lights a fraction G is scattered upward with a Lambert distribution, the remaining fraction $(1-G)$ being absorbed. We write

$$I_{\text{up}} = \frac{LP}{2\pi}\{2G(1-F)\cos\psi + 0.554\,F\,\psi^4\} \qquad (1)$$

for the total upward intensity in the direction ψ in lumens sterad $^{-1}$. We replace L by $L_0\,(P/100{,}000)^{0.1}$ if we wish to use the enhanced luminosity. The coefficients 2 and 0.554 are introduced to give the correct normalization when I_{up} is integrated over the upward hemisphere. Ketvirtis (1967, p. 28) quotes 12%–14% for the reflectivity of worn asphalt and 16%–18% for the reflectivity of dirty concrete, with extremes ranging from 10% for new asphalt to 25% for new concrete. Most light is reflected from streets and parking lots: accordingly we decided to adopt G = 15% in all the numerical work described in this paper. (G would need to be increased by about a factor 4 if the ground were snow covered.)

(c) We assume that the atmosphere consists of a mixture of molecules and aerosols. For the molecular component we assume a particle density $N_1(h) = N_m \exp(-c[h + H])$ at height h above the city. The sea level density $N_m = 2.55 \times 10^{19}$ cm^{-3}. We took $c = 0.104$ km^{-1}, based on data in Allen (1973, p. 122). We assume Rayleigh scattering by the molecular atmosphere with an integrated cross section (Goody 1964, p. 414) $\sigma_R = 4.6 \times 10^{-27}$ cm^2 molecule at an assumed wavelength 5500 Å for visual observations. We assume that the atmosphere contains aerosols, of variable amount, but with an exponential distribution

$$N_2(h) = N_a \exp(-ah) \ .$$

N_a is the aerosol particle density at ground level. The reciprocal scale height for aerosols was taken as

$$a = 0.657 + 0.059\,K \qquad (2)$$

in most of our calculations. This is a rough approximation based on data of McClatchey *et al.* (1978, Table 6) and Rabinowich (see Kondratyev 1969, eq. (4.46a)). Other values of a can easily be used by changing the appropriate line in a computer program. We found that larger values of a are needed to represent dense smog layers of small vertical extent, such as occur, for example, in Los Angeles.

We assume an aerosol scattering coefficient $\sigma(\theta) = \sigma_a f(\theta)$, where θ is the angle through which the light is scattered and the scattering function $f(\theta)$ is normalized so that σ_a is the cross section integrated over the whole solid angle. For $f(\theta)$ we used

$$\begin{aligned} 0 \leqslant \theta \leqslant 10^\circ \quad & f(\theta) = 7.0 \exp(-0.2462\, \theta) \\ 10^\circ < \theta \leqslant 90^\circ \quad & f(\theta) = 0.9124 \exp(-0.04245\, \theta) \\ 90^\circ < \theta \leqslant 180^\circ \quad & f(\theta) = 0.02 \ , \end{aligned} \tag{3}$$

with θ in degrees, as an analytical approximation to the curve given by McClatchey *et al.* (1978, Fig. 26 on p. 14–52), and this $f(\theta)$ is normalized. Improved fits could be obtained using more elaborate functions, but we did not consider this worthwhile in view of all the other uncertainties of our model.

(d) We define the parameter K by the equation

$$N_a \sigma_a = 11.11\, K N_m \sigma_R \exp(-cH) \tag{4}$$

so that, apart from the numerical coefficient, K measures the ratio of aerosol $N\sigma$ at ground level to molecular $N\sigma$ at ground level. K is a measure of the relative importance of aerosols and molecules for scattering light. K is the parameter we use as the indicator of the clarity of the atmosphere. The coefficient 11.11 was chosen so that $K = 1$ would correspond to aerosol scattering by fairly clear air at sea level (McClatchey *et al.* 1978, Table 6 on p. 14–14).

(e) We use an approximate correction for double scattering first derived by Treanor (1973) and modified for use in the present work. Higher-order scattering processes are neglected. The required formula is the large bracket expression in Garstang (1984, eq. (4)). We write

$$(DS) = 1 + \frac{N_a \sigma_a \{1 - \exp(-as \cos \psi)\}}{a \cos \psi} + \frac{\gamma N_m \sigma_R \exp(-cH) \{1 - \exp(-cs \cos \psi)\}}{c \cos \psi} \ . \tag{5}$$

Here (see Fig. 1) s denotes the distance along an upward-bound light ray XQ and ψ is the zenith distance of the ray. The last term in formula (5) has been added to make some allowance for double scattering by molecules. The factor γ is not known. It is introduced to allow for the fact that Rayleigh scattering is more nearly spherically distributed than is aerosol scattering. Accordingly larger scattering angles contribute to the double scattering. In addition some scattering angles would require scattering elements below the ground, and are therefore impossible. Most scattering paths are longer than in the aerosol case and are therefore subject to greater extinction. Considering all these factors we guessed that $\gamma = 1/3$ would be a reasonable estimate, and we adopted it. The formula (5) must be used to augment the scattered light reaching Q: for photons reaching O along QO the volume element at Q provides the "last" scattering, and second scattering only occurs between X and Q.

(f) When calculating extinction we replace K by $1.06\, K$ to allow for pure absorption. This is based on data in McClatchey *et al.* (1978, Table 6 on p. 14–14). The extinction along any path can be calculated from our model. For the path QO (Fig. 1) for an observation made at zenith distance z we obtain the extinction factor

$$(EF)_{QO} = \exp(-N_m \sigma_R \exp(-cH) p \sec z) \ , \tag{6}$$

where

$$p = c^{-1} \{\exp(-cA) - \exp(-ch)\} + 11.778\, K a^{-1} \{\exp(-aA) - \exp(-ah)\} \tag{7}$$

and h is the height of Q above the plane of the city. (The numerical coefficient 11.778 is 1.06 times the coefficient in our eq. (4).) $(EF)_{QO}$ is the fractional reduction of the light intensity in traveling from Q to O. A similar formula for the fractional reduction $(EF)_{XQ}$ along the path XO may be obtained by putting $A = 0$ in equation (7) and replacing z in equation (6) by ψ.

(g) We choose axes x and y (Fig. 1) so that CB is the x axis. Light from an element of area $dxdy$ at $X(x,y)$ travels to Q, where it is scattered and some reaches the observer O. Double scattering between X and Q increases the number of photons reaching Q. Extinction takes place between X and Q, and between Q and O. The observer measures the luminous flux arising within a cone of semiangle δ around the direction QO. The lengths u, ℓ, and d and the angles θ and ϕ are defined as shown in Figure 1. The azimuth of OQ measured from BC as zero is denoted by β. Then the geometrical relationships are

$$d^2 = (x - D)^2 + y^2 \tag{8}$$

$$\ell^2 = d^2 + A^2 \tag{9}$$

$$\ell \cos \theta = -(x - D) \sin z \cos \beta + y \sin z \sin \beta - A \cos z \tag{10}$$

$$s^2 = u^2 + \ell^2 - 2u\ell \cos \theta \tag{11}$$

$$\ell = s \cos \phi + u \cos \theta \tag{12}$$

$$h = u \cos z + A \tag{13}$$

$$h = s \cos \psi \ . \tag{14}$$

For a given x, y, u, z, and β these equations allow us to calculate d, ℓ, θ, s, ϕ, h, and ψ.

We can now write an expression for the luminous flux λ received at O by a telescope of area w from within the cone of semiangle δ

$$\lambda = \iiint I_{up} s^{-2} (dxdy / \pi R^2)(EF)_{XQ}(DS) \pi \delta^2 u^2 du \times \{N_1(h) \sigma_R 3(1 + \cos^2[\theta + \phi])/(16\pi) + N_2(h) \sigma_a f(\theta + \phi)\} \times (EF)_{QO}(w/u^2) \ . \tag{15}$$

Here $I_{up}\, s^{-2}\,(dxdy/\pi R^2)$ is the flux per unit area falling on the scattering volume $\pi\delta^2 u^2 du$ at Q from the area $dxdy$ of the city. The expression in { } is the scattering cross section per unit volume per unit solid angle. w/u^2 is the solid angle of the telescope as seen from Q. $(EF)_{XQ}$ and $(EF)_{QO}$ are the extinction factors and (DS) is the double scattering correction.

We denote the sky brightness by b in lamberts. Then the brightness is b/π lumens cm^{-2} sterad^{-1}. If we imagine a radiating surface of area $\pi\delta^2 u^2$ at Q we can write the flux received as

$$\lambda = (b/\pi)\pi\delta^2 u^2 (w/u^2) = b\,\delta^2 w \quad . \qquad (16)$$

We use equation (16) to eliminate $\lambda/\delta^2 w$ from equation (15), insert the expressions for $N_1(h)$ and $N_2(h)$, and introduce K by using equation (4). We obtain

$$\begin{aligned} b = \pi N_m \sigma_R \exp(-cH) \iint (dxdy/\pi R^2) \int_0^\infty du \\ \times I_{up}\, s^{-2} (EF)_{XQ}\, (EF)_{QO}\, (DS) \\ \times \{\exp(-ch) 3(1 + \cos^2[\theta + \phi])/(16\pi) \\ + \exp(-ah)\, 11.11\, Kf(\theta + \phi)\} \quad . \qquad (17) \end{aligned}$$

This is the basic equation for our model. The double integration over x and y is over the area of the city.

We wrote several computer programs to evaluate b from equation (17). One program calculates b for observations by an observer O situated at the city center. Another program calculates the brightness in the vertical plane through the city center and an observer outside the city, at any zenith distance toward or away from the city. This program is sufficient for many purposes, including the calculation of the zenith brightness for an observer caused by several distant cities. These two programs were used for the calculations reported later in this paper.

In preparation for future work we also wrote programs to calculate the brightness for any azimuth for an observer outside a circular city and for a point source. We assumed a flat earth for all the calculations reported in the present paper. When carrying out the integration over $dxdy$ for a circular city we divided the city up into a finite number of areas. For city-center calculations we used 30 areas, reduced to 15 by symmetry for the actual calculations. For observations from outside the city we used a seven-area approximation in the general case (Abramowitz and Stegun (1964), middle of the left-hand column on p. 892), reduced to four areas by symmetry in the case when we take the azimuth $\beta = 0$. We omit the integration over $dxdy$, omit the factor $dxdy/\pi R^2$, and put $x = y = 0$ for a point-source program. All our calculations were performed on an Apple computer.

(h) There still remains the estimation of the brightness of the night-sky natural background, which must be added to the artificial component to obtain the total night-sky brightness.

We assume a background natural sky brightness $V =$ 21.9 mag sec^{-2} at Junipero Serra Peak, altitude 1787 m, based on Walker (1970, 1973). This is a magnitude inside the atmosphere. We make a rough allowance for the brightness of the natural sky background with increasing zenith distance z by including a factor $(1-0.96 \sin^2 z)^{-1/2}$ based on a simple van Rhijn layer approximation: we intend to refine this approximation in future work. This gives the brightness outside the lower scattering atmosphere. The brightness as seen by an observer is calculated by applying the appropriate extinction factor along the line of sight to the outside brightness. We neglected scattering effects other than extinction, and we also neglected the effect of ground albedo. If included, these effects would increase the background sky brightness by a few percent for moderate zenith distances, the factor increasing to perhaps 30% for zenith distances of 80°. For the extinction of the night-sky background we put $h =$ infinity in equation (7) and replaced sec z in equation (6) by the air mass. We used the formula

$$\text{air mass} = \sec z - g(\sec z - 1)^2 \qquad (18)$$

when calculating extinction of stars. This is based on Snell and Heiser (1968) with $g = 0.010$ chosen to reproduce their Table I, column 3 and Allen (1973, p. 125) to better than 0.1 air mass at zenith distance 85°. For larger zenith distances we used the tabular values in Allen.

(i) We assume the relation

$$b = 34.08 \exp(20.7233 - 0.92104\, V) \qquad (19)$$

between the visual (V) magnitude of the sky in magnitudes per square second and its brightness (b) expressed in nanolamberts (nL). This equation is based on the conversion given by Allen (1973, p. 26). Another unit which is sometimes used is S_{10}, the number of tenth-magnitude stars per square degree seen through clear unit air mass which are equivalent to the observed brightness. From Allen's conversion we obtain 1 $S_{10} = 0.22$ nL. The brightness may also be expressed in terms of photons by the relation 1 nL $= 1.31 \times 10^6$ photons

$$(\lambda 5550)\ \sec^{-1}\ \text{cm}^{-2}\ \text{sterad}^{-1}.$$

(j) We use formulae given by Weaver (1947) to calculate the limiting magnitudes of stars which can just be seen by the naked eye, and we apply our calculated extinction corrections to determine the stellar magnitudes outside the atmosphere. Expressed in natural logarithms the limiting magnitudes are

$$\begin{aligned} V = 7.930 - 2.171 \ln(1 + 0.1122\, b^{1/2}) \quad b \leq 1479\ \text{nL} \\ V = 4.305 - 2.171 \ln(1 + 0.001122\, b^{1/2}) \\ b > 1479\ \text{nL} \quad . \qquad (20) \end{aligned}$$

We wish to apply these formulae to bright skies (inside cities) as well as to the relatively dark skies at observatories. There seem to have been few attempts to check these formulae. We made such a check for bright stars seen against a correspondingly bright background by calculating (Garstang 1985) the visibility of Sirius and Canopus near sunset and comparing with observations by Henshaw (1984). Weaver's formulae give excellent results.

III. Vertical Extinction and Horizontal Visibility

Our assumption of exponentially decreasing molecule and aerosol densities allows us to obtain simple formulae for vertical extinction and horizontal visibility. We find that the extinction between the observer and the zenith is Δm magnitudes, where

$$\Delta m = 1.0857\, N_m \sigma_R \exp(-cH) \times \{c^{-1} \exp(-cA) + 11.778\, Ka^{-1} \exp(-aA)\} \ . \qquad (21)$$

From this equation we calculate that for an observer at sea level ($H = A = 0$) the vertical extinction is 0.23 mag if $K = 0.5$ and 0.33 mag if $K = 1.0$. The optical depth, τ, of the atmosphere above the observer is given simply by $\tau = \Delta m/1.0857$.

Equation (21) may be used to check the extinction at observatories. In most of the present paper we adopt a standard model with $K = 0.5$. Then for Mount Palomar ($H = 0$, $A = 1.87$ km) we calculate $\Delta m = 0.13$ mag, which may be compared with an observed value $\Delta m = 0.15$ mag (Hayes and Latham 1975). For Lick Observatory ($H = 0$, $A = 1.28$ km) we calculate $\Delta m = 0.15$ mag, an observed value (Hayes 1970) being $\Delta m = 0.18$ mag. The agreement is satisfactory.

It is conventional in elementary meteorology (see, e.g., Middleton 1952, pp. 4, 63, 94) to adopt as the visibility the distance Δx at which a black object would show a brightness (due to scattered light between the observer and the object) of 0.98 of the brightness of the background horizon behind the object, this being referred to as a contrast of 0.02. If we write the extinction law as

$$I = I_0 \exp(-\eta x) \text{ then } \Delta x = 3.91\, \eta^{-1} \ .$$

We calculate η from our model, and obtain

$$\Delta x = 3.91\, N_m^{-1} \sigma_R^{-1} (1 + 11.778\, K)^{-1} \exp(cH) \ . \qquad (22)$$

At sea level $K = 0.5$ gives $\Delta x = 48$ km, $K = 1$ gives $\Delta x = 26$ km, and $K = 2$ gives $\Delta x = 14$ km. In heavily polluted areas our model should be modified by increasing K and increasing a; this combination of changes gives a better representation of a smog layer of small vertical extent but high horizontal opacity, and it would be even better to replace our exponential aerosol distribution by a uniform density layer.

IV. Brightness-Zenith Distance Relation

As a first application we calculated the variation of the brightness of the sky caused by Denver ($P = 1{,}300{,}000$, $R = 15$ km) as seen from the distance of Boulder ($D = 40$ km) at various zenith distances in the vertical plane containing the observer and the center of Denver, and ignoring the lights of Boulder. The results for $K = 0.5$, $F = 15\%$, are shown in Figure 2. The rapid increase in brightness as one looks toward Denver is clearly seen. Of some importance in connection with the interpretation of sky-brightness observations at dark sites is the fact that the sky brightness

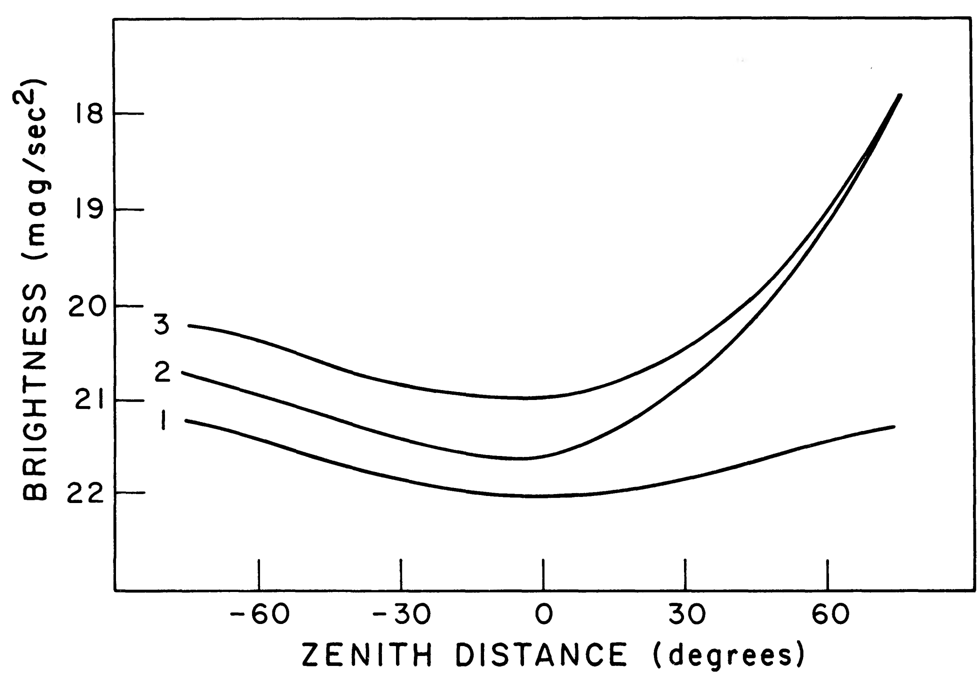

FIG. 2–Sky brightness due to Denver as a function of zenith distance as seen from a distance of 40 km in the vertical plane containing the observer and the center of Denver. Curve 1: sky background. Curve 2: Denver only. Curve 3: Denver and sky background. Negative zenith distances are away from Denver. We used $L_0 = 1000$ lumens per head, $K = 0.5$, $F = 15\%$.

increases with increasing zenith distance in the direction away from Denver. (This is in addition to the increasing brightness of the natural sky background.) This phenomenon was pointed out by Berry (1976); it has received little discussion in the literature.

V. Brightness-Distance Relation

Another simple application of our model is to the brightness caused by a city at various distances of the observer, with a fixed zenith distance. In Figure 3 we give results for Denver under clear ($K = 0.5$) and slightly hazy ($K = 2.0$) conditions: only the brightness caused by the city is shown, the sky background being omitted. When the sky is slightly hazy the zenith brightness inside the city is increased, and the zenith brightness far from the city is decreased. The former result is the direct result of increased scattering; the latter result is because the increased absorption outweighs the increased scattering. The relationships in Figure 3 are not quite linear, even for large distances, so that a power-law relation between b and D is not exact. If we write $b = CPD^{\alpha}$ we find that α becomes more negative for increasing D, and that α is more negative for larger K.

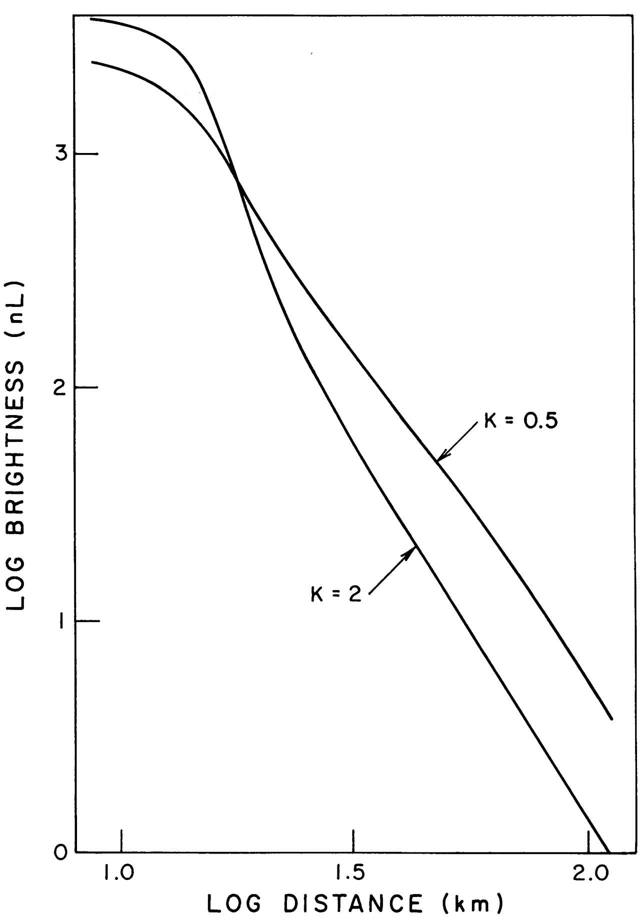

FIG. 3–Sky brightness at the zenith due to Denver as a function of the distance of the observer from Denver. The brightness is in nanolamberts, and refers to the city without sky background. We used $L = 1500$ lumens per head, $F = 15\%$, and $K = 0.5$ and $K = 2.0$. Note that for a slightly hazy ($K = 2.0$) sky the zenith brightness inside the city ($R = 15$ km) is increased and the brightness far from the city is decreased.

No observations are at present available to test the calculations just described. A crucial test of our model is whether it predicts the relationship between brightness and distance observed by Walker (1977) for Salinas ($P = 68600$). He measured the sky brightness at distances from Salinas ranging from 8.0 km to 35.7 km, and he estimated the brightness at 98.7 km. At each place of observation he measured the sky brightness at a zenith distance of 45° in the direction of Salinas and at a distance of 45° in the darkest part of the sky away from Salinas. He stated (and the present writer agrees) that his most accurate results are given by the ratio

$$Q = \frac{b(\text{Salinas at } +45°) - b(\text{Salinas at } -45°)}{b(\text{sky background only at } +45°)} \ . \qquad (23)$$

He tabulates his values of Q as a function of D, for visual magnitudes, in the second column (headed V(1)) of his Table III. For comparison we have calculated Q for $K = 0.25$, 0.50, and 0.75, and for $F = 5\%$, 10%, and 15%, as a function of D ranging from 8 km to 100 km. We assumed $L = 1500$ lumens per head as a trial value and we took $R = 3.5$ km. We fitted our calculated $(Q-D)$ relationships to Walker's observations. Our best fit was obtained for $L_0 = 986$ lumens per head, $K = 0.43$, and $F = 11\%$.

We also used a *standard model* with $L_0 = 1000$, $K = 0.5$, and $F = 10\%$: the fit obtained was very nearly as good as our best fit. We show in Figure 4 Walker's observations and the curve calculated for our standard model. The agreement is remarkably good. The average slope of the curve between $D = 10$ km and $D = 35$ km is about -2.3. This result provides an interpretation of the $Q \propto D^{-2.5}$ law which Walker suggested as providing an excellent representation of his observations.

Another check is possible using observations of King City, California ($P = 4320$) by Walker (1977). Our results are given in Table I. Although the agreement between calculation and observation is not perfect we believe it is acceptable, bearing in mind that Walker's values were obtained by extrapolation from measurements at smaller distances, and that our model may not use the correct value of the luminous output of the city.

In connection with prospective observatory-site surveys it is of interest to give approximate formulae for the brightness of a city at very large distances. We have done calculations of the brightness as a function of distance for various populations, and fitted our results to a formula of the form $b = CPD^{\alpha}$. The index α was determined as the slope of the graph of log b against log D for a b value of

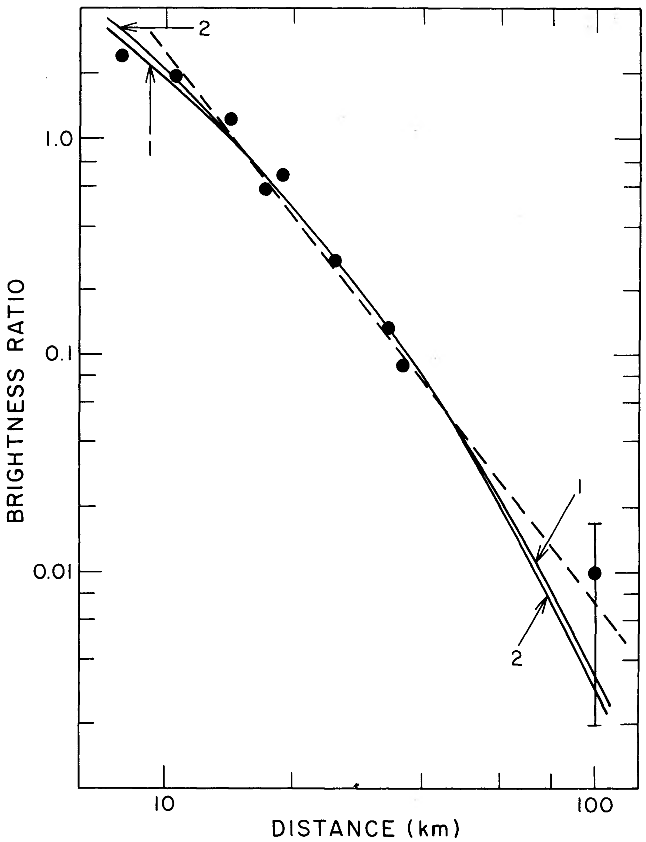

FIG. 4—Variation with distance from the city of the ratio Q of sky brightness due to Salinas. Q is defined in equation (23). Curve 1: L_0 = 986 lumens per head, K = 0.43, F = 11%. Curve 2: L_0 = 1000 lumens per head, K = 0.5, F = 10%. Dots are observations by Walker (1977). The dashed line is the relation $Q \propto D^{-2.5}$.

TABLE I

Brightness of King City

	Q	D (km)
Calculated, K = 0.5	0.038	8.0
Calculated, K = 1.0	0.046	9.5
Observed, Walker (1977)	0.08	10.0

Q calculated at D = 16.1 km, observed Q extrapolated by Walker. The tabulated D is the distance at which the sky brightness at a zenith distance of +45° towards King City exceeds the natural background at the same zenith distance by 0.20 magnitude.

0.01 times the natural background. The coefficient C was determined to fit the calculated b values as well as possible over a range of brightness from about 0.05 to 0.005 natural. The results are listed in Table II. Linear interpolation of α and log C as functions of log P is adequate; a more elaborate treatment is not justified. For brightness above about 0.1 times natural detailed model calculations are recommended.

TABLE II

Brightness-Distance Relation*

for Large Distances

log P	log C	α
3.0	-1.49	-1.90
3.5	-1.12	-2.18
4.0	-0.68	-2.46
4.5	-0.31	-2.66
5.0	0.12	-2.88
5.5	0.60	-3.10
6.0	1.41	-3.46
6.5	2.35	-3.84

*Standard model, sea level, K = 0.5, zenith observations, b = CPD^{α}, for b about 0.01 of natural background.

VI. Population-Distance Relation

In his papers Walker (1970, 1973, 1977) derived a relation between the population of a city and the distance at which the brightness of the sky is increased by 0.2 magnitude over the natural background at a zenith distance of 45° in the direction toward the city. We have calculated such a relation for a number of sets of parameters in our model, and compared our theoretical relations with the best observations of Walker (1977, Fig. 3 and Table IV). We find that our models with enhanced luminosity give better fits than our models with luminosity simply proportional to the population. We find that L_0 = 1000 lumens per head gives a good fit; the value of L_0 could be increased or decreased somewhat without the fits deteriorating significantly. We show in Figure 5 our theoretical relations for K = 0.5 and K = 1.0 with L_0 = 1000 lumens per head and F = 10%. K = 0.5 is the better fit. The excellent fit provides additional evidence that our standard model gives a good representation of the observations. Walker showed two points for Los Angeles as observed from Palomar, one using the Los Angeles City population and the other using the Los Angeles County population. Figure 5 clearly shows that one should include the whole metropolitan area in estimating the population of a city.

VII. City-Center Zenith Brightness-Population Relation

Our model can be applied to the sky brightness as seen from a city center. There are few observational data available for comparison with calculations. It is therefore of particular significance that Berry (1976) presented the results of a cooperative program of sky brightness mea-

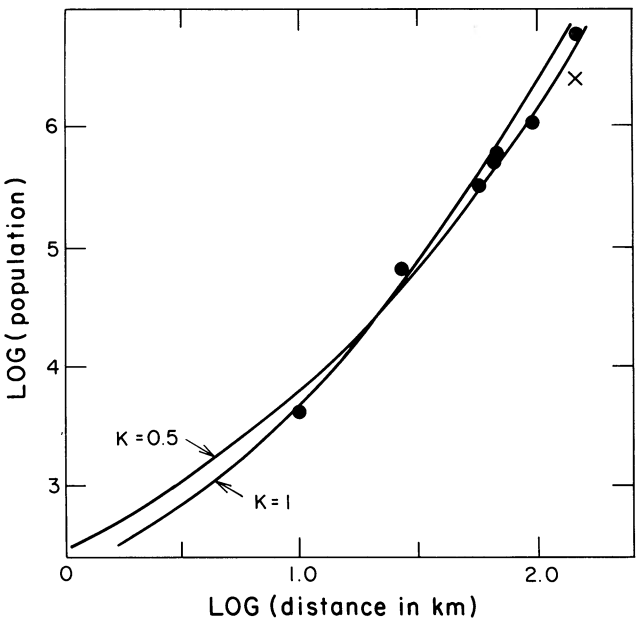

FIG. 5–Population-distance relation at sea level for an increase of sky brightness of 0.20 magnitude above the background at a zenith distance of +45° toward a city. L_0 = 1000, F = 10%. The cross denotes that the Los Angeles City population was used, the point above the cross used the Los Angeles County population. The dots are observations by Walker (1977).

surements in Southern Ontario made by members of the Toronto Center of the Royal Astronomical Society of Canada which included a series of measurements of the zenith brightness in the center of a dozen cities of various populations. Berry indicated that a law in which the brightness is proportional to the square root of the population fits the observations very well. He did suggest, however, that the size of the cities might be partly responsible for the observed behavior, in that the light from the outer parts of a large city would be weakened in the course of propagation to the city center. We have calculated the zenith brightness for the centers of cities of various populations, using $K = 0.5$ and $F = 10\%$. Our results are shown in Figure 6, together with observations reported by Berry. The fit is improved if we take $L_0 =$ 1380 lumens per head instead of our standard model value. The fit is not too bad, given the scatter in the observations. We tried changing certain parameters: our curve for $K = 0.5$, $F = 15\%$, and $L_0 = 1120$ lumens per head is also shown in Figure 6. We conclude that our model with brightness proportional to $P^{1.1}$ (includes enhancement factor) does account for the major features of the relation found by Berry and it is not necessary to postulate a square-root law. The scatter of the observations is rather large: it does not seem profitable to try to

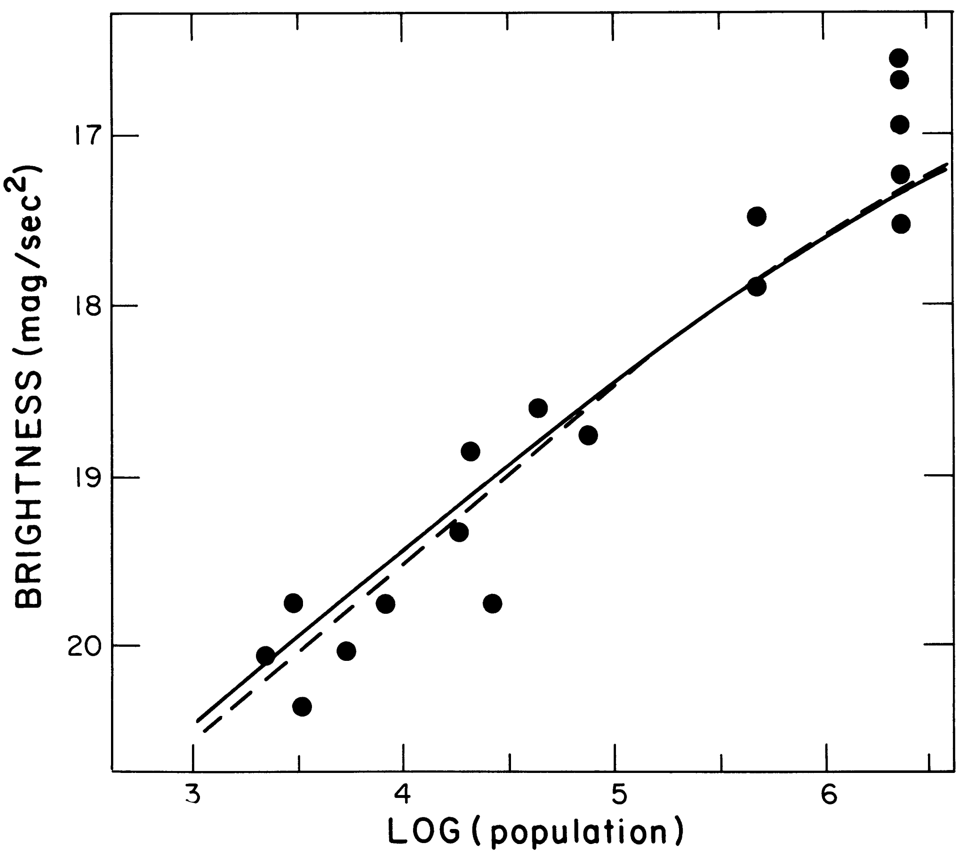

FIG. 6–City-center zenith brightness for 12 Ontario cities as a function of population. Continuous line: $K = 0.5$, $F = 10\%$, $L_0 = 1380$ lumens per head. Dashed line: $K = 0.5$, $F = 15\%$, $L_0 = 1120$ lumens per head. Dots: observations by Berry (1976).

get better fits by arbitrarily changing some of our parameters. There is some indication that for Toronto (the five points with the largest population in Fig. 6) our model predicts too low a brightness. Some parameter changes which we might envisage to raise the sky brightness of Toronto would raise the brightness of smaller cities and make the overall agreement worse.

One possibility for improving the agreement between theory and observation in the special case of Toronto is to use a rectangular area instead of a circular area for the city. This is a much better approximation for this case, Toronto being extended along the shore of Lake Ontario. We wrote a modified computer program to integrate over a rectangular area, which we took to be 48 km along the lake and 16 km perpendicular to the lake. We took our point of observation as 3 km inland from the center of the lake front. Results for our standard model are shown in Figure 7, which also shows observations by Berry (1976) at various zenith distances. The agreement is not bad considering that no fitting was involved in the calculations, but we would like to obtain improved agreement.

The disagreement between observation and calculation is in the sense that the predicted brightness toward Lake Ontario (looking over the downtown area) is too low and the brightness away from the lake is too high. It seemed that an improvement might be obtained if we increased the luminous output of the downtown area. Accordingly we modified our computer program to assign double the average luminosity to an area 2 km × 2 km at the city center bordering on the lake front, and renormalizing the calculation to give the same total luminous output as our standard model. Finally, we attempted to improve the agreement still further by adjusting L. Our final result is shown in Figure 7. Our final L = 1304 lumens per head.

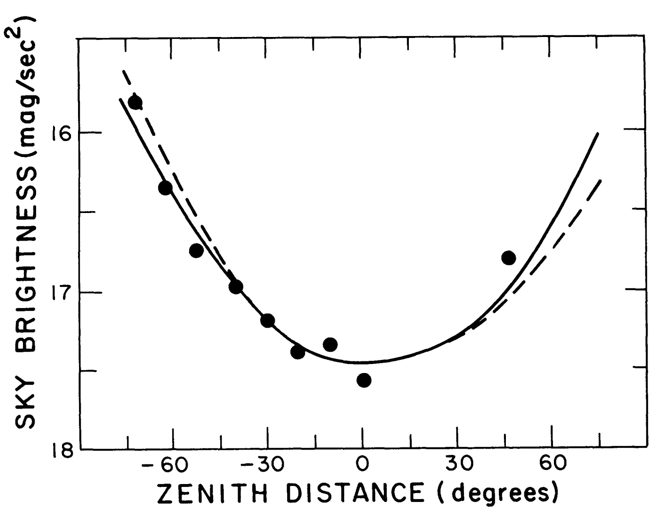

FIG. 7–Sky brightness along the meridian through downtown Toronto. Zenith distance is positive toward Lake Ontario. The dashed line is our standard model with L_0 = 1000 lumens per head. The continuous line is for L = 1304 lumens per head, with doubled downtown brightness. The dots are observations by Berry (1976).

(This includes the enhancement factor, and would correspond to L_0 = 950 lumens per head.) The agreement between theory and observation is significantly improved.

We have not attempted to improve the model any further. It is clear that at the accuracy we have attained the presence of major local-light perturbations (such as a bright downtown area) has become of significance, and further progress with simple models is probably elusive. The situation is much worse for Los Angeles, for which a calculation yields a brightness much fainter than an observed value (kindly communicated to me by Dr. G. Reaves) obtained at the University of Southern California.

VIII. More Elaborate Applications of Our Model

Our model can be applied to a number of more elaborate situations. One is to consider two cities, and calculate the sky brightness as seen from the center of one city at various zenith distances in the vertical plane containing the centers of the two cities. This is a case in which we can simply calculate the brightnesses for the two cities separately and add the results. We have performed such calculations for Boulder (P = 77,000; R = 4 km) and Denver (P = 1,300,000; R = 15 km; D = 40 km). Our results are shown in Figure 8 for our standard model with K = 0.5 and with K = 2.0. The calculations clearly show

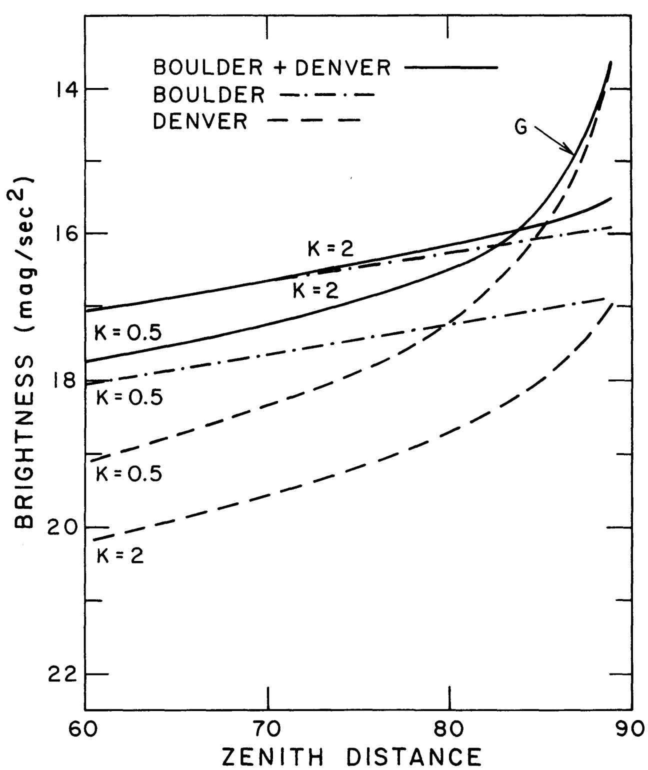

FIG. 8–Sky brightness seen from center of Boulder toward Denver as a function of zenith distance. G denotes the "Denver glow", which is bright on very clear (K = 0.5) nights.

that at smaller zenith distances the sky is brighter on hazy nights. At larger zenith distances (> 84° in the present calculations) on very clear nights ($K = 0.5$) the sky becomes very bright in the direction of Denver. This "Denver glow" is striking to the eye, as it is for other cities. The presence of this glow is a sensitive test for very clear nights, as well as being a test for the presence of artificial illumination as suggested by Walker (1970). At present no reliable observations are available to check our calculations. A rough measurement by the present writer, made after midnight at the Sommers-Bausch Observatory in Boulder, gave 17.3 mag sec^{-2} at a zenith distance of 80° toward Denver. Our calculations for $K = 0.5$ give 16.8 mag sec^{-2}. Part of the discrepancy may be due to the location of the observatory some distance to the south of the Boulder city center, and part may be due to the reduction of outdoor lighting in the late evening.

A second situation in which our model can be applied is to the calculation of the zenith brightness seen by an observer when a number of cities contribute to the sky brightness. We give in Table III results for Boulder for our standard model with $K = 0.5$ and with $K = 2.0$, taking into account Boulder, Denver, and five smaller cities. We have also included the parking lights of a liquor store, which happens to be our largest, and most aggravating, local disturbing source of light; it is sad to note that this single source (at a distance of 650 m) produces scattered light in the zenith comparable to the natural sky brightness. (It was to avoid this disturbance, which is in the direction of Denver, that we made our rough measurement of the Denver glow after midnight, when the lights had been turned off.)

TABLE III

Sky Brightness in Boulder

Source	Brightness (nL) K = 0.5	Brightness (nL) K = 2
Boulder	670	1274
Denver	57	21
Longmont	5	2
Broomfield	4	2
Lafayette	2	2
Louisville	2	2
Gunbarrel	3	3
Night sky	56	45
Liquor store	37	79
Total	836	1430
Total (mag/sec^2)	19.0	18.4
Sky only (mag/sec^2)	22.0	22.2

Calculated for the zenith, L_0 = 1000 lumens per head, F = 10%. City center has been assumed for Boulder. Liquor store brightness calculated for Sommers-Bausch Observatory.

We attempted to reproduce the zenith brightness at Palermo, Ontario, a small town 40 km west of Toronto. We performed calculations, using our standard model with $K = 0.5$, on the zenith brightness at Palermo caused by Toronto and 34 other cities. Our results are given in Table IV. Toronto is the largest source of light. Oakville, Burlington, and Hamilton make major contributions, and Palermo itself is fifth in importance. The contribution of Palermo is perhaps the least well determined of all because of the somewhat scattered population in the Palermo area, and the corresponding uncertainties in fixing a population and radius for our calculation (we used $P = 1000$, $R = 2$ km). The overall agreement is satisfactory. Improved agreement might be obtained if it were possible to include additional cities and areas.

Our final application concerns calculations of the limiting naked-eye stellar magnitude. We performed calculations for Boulder and used Weaver's formulae given in equations (20) above. Our results are given in Table V; they refer to the zenith and $L_0 = 1000$ lumens per head. We also give the averages of observations made on many evenings by the present author. The agreement is excellent, but it must be said that the limiting magnitudes are not very sensitive to the brightness of the sky, so the agreement is not a stringent test of our model.

More complex applications of our model can be made. For example, we can calculate the zenith brightness in

TABLE IV

Sky Brightness at Palermo, Ontario

City	Brightness (nL)	Notes
Toronto	121	1
Oakville	61	
Burlington	63	
Hamilton	51	
Brampton	9	
Milton	5	
28 other cities	14	2
Palermo	16	3
Sky background	55	
Total (nL)	395	
Total (mag/sec^2)	19.84	
Observed {nL	466	4
Observed {mag/sec^2	19.66	

1. Includes Toronto City, York, North York, East York, Scarborough, Etobicoke and Mississauga, treated as a single city.
2. Twenty-eight other cities treated individually or in groups, and the results added. The cities ranged from Kitchener to Buffalo.
3. Based on population and radius estimates supplied by Dr. R. W. Nicholls.
4. By Berry (1976).

TABLE V

Limiting Naked-Eye Magnitude for Boulder

K \ F	5%	10%	15%
	Calculated		
0.5	4.7	4.6	4.5
1	4.5	4.3	4.2
2	4.1	3.9	3.8
4	3.5	3.5	3.5
	Observed		
Exceptionally clear, averted vision			4.7
Clear, normal vision			4.3
Slightly hazy (10 km visibility)			3.7

Table for zenith, L_0 = 1000 lumens per head.

TABLE VI

Sky Brightness at Junipero Serra Peak

Source	Brightness (nL)	Notes
Salinas	0.6	
Monterey and 4 others	0.6	1
King City	0.2	
San Francisco Bay Area	2.3	2
Fresno	0.3	
Greenfield and 4 others	0.3	3
Total without background	4.3	
Observed	57.6	4
True background	53.3	5

1. Monterey, Seaside, Pacific Grove, Marina and Carmel.
2. San Francisco, Contra Costa, Alameda, San Mateo and Santa Clara Counties.
3. Greenfield, Soledad, Gonzales, Paso Robles and Atascadero.
4. Walker (1973), equivalent of 21.93 mag/sec^2.
5. Difference of previous two lines.

side a city, but not at its center, by dividing the city into a series of suitably chosen zones, applying our model to each zone, and adding the results. We used this method to calculate the brightness of Denver at D = 10 km and D = 15 km so that we could extend Figure 3 above closer to the city center.

IX. Junipero Serra Peak

In our calculations we assumed a natural sky background V = 21.9 mag sec^{-2} for Junipero Serra Peak. This peak has become a standard for dark-sky comparisons, so we might ask whether we can improve on our value. We have calculated the artificial brightness at the zenith using our standard model (L_0 = 1000 lumens per head, K = 0.5, F = 10%) for a number of cities and counties, with the results given in Table VI. Walker (1973) gives data indicating that 21.93 mag sec^{-2} is the best observed brightness. Our results suggest that 22.01 mag sec^{-2} (the equivalent of 53.3 nL) would be the faintest possible estimate of the natural background. This figure is uncertain to the extent that we have not allowed for the possible and likely presence of some fog and smog in the San Francisco Bay Area; the figure might have to be revised by 0.02 or 0.03 magnitude brighter to allow for this possibility. 21.99 ($\pm$0.02) mag sec^{-2} (equivalent to 54.5 $\pm$ 1.0 nL) would be our best guess.

X. General Remarks

We have made many additional calculations using our model. Three matters seem of some general interest.

(a) Our calculations demonstrate the value of reducing the amount of light radiated upward by shielding street lamps and other lighting whenever it is practicable. For zenith observations at 40 km from Denver, with our standard model, the brightness due to Denver would be reduced from 57 nL to 36 nL if F were reduced from 10% to 5%, the total sky brightness would be reduced from 21.20 mag sec^{-2} to 21.42 mag sec^{-2}, and the limiting magnitude for naked-eye observation changed from 6.03 to 6.14, leading to an increase of 11% in the number of stars visible to the eye.

(b) We performed a few calculations for Denver with and without the correction for double scattering. Double scattering increases the city center brightness by about 20% (K = 0.5), and by a factor 2 at a distance of 100 km from the city. The effect is worth including in our model.

(c) We examined the importance of allowing for the altitude of a city above sea level. At high-altitude city centers the brightness is reduced because of the reduced atmospheric density. At moderate distances from a large city the brightness is reduced for the same reason. At large distances the brightness is reduced only slightly, or may be increased: this is because reduced atmospheric absorption tends to win over reduced scattering for large path lengths.

XI. Conclusion

We have shown that a relatively simple model can make night-sky brightness predictions which are in satisfactory agreement with observations in most cases when the observer is outside the city, and for city centers for all but the largest cities. Our model can be applied to many existing observatories and prospective sites for observatories. We hope to present results for these cases in a future paper.

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