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Non-linear stellar oscillations. Non-radial mode interactions

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Summary. The Hamiltonian particle formalism for non-linear adiabatic *radial* stellar oscillations (Perdang and Blacher, 1982, 1984) is extended to *non-radial* oscillations, using a variational principle of fluid mechanics. There are several advantages of a variational formalism over the direct approach in which the linear modes are inserted into the equations of motion: (a) The Hamiltonian nature of the equations provides theoretical information about the nature of the solutions (existence of chaotic motions) and allows a simple test for numerical integrations (energy conservation). (b) Mass and entropy conservation can be directly built in the formalism; therefore no Lagrange multipliers (i.e. additional unknown fields) are needed for these constraints, and solutions truncated to the lowest non-linear order obey these two conservation laws exactly. (c) As a further conserved quantity, zero-circulation $\left(\text{i.e. } \int_{\mathscr{C}} d\boldsymbol{r} \cdot \boldsymbol{u} = 0\right)$ for all closed curves $\mathscr{C}$ lying on surfaces of constant specific entropy can be incorporated; with this restriction, we not only simplify the structure of the resulting non-linear equations, but we also suppress all stationary components of the motion. (d) The formalism allows for the development of a toroidal velocity component during time evolution; the direct approaches considered so far have discarded toroidal components.

Besides the formalism, various observational facts suggesting non-linear phenomena are reviewed and indirect theoretical arguments pointing towards non-linearities in non-radial stellar oscillations are presented.

Key words: variable stars – stellar oscillations – solar oscillations – non-linear oscillations – chaos

1. Introduction

Until recently, *finite* (i.e. non-linear) non-radial global motions in stars have been viewed as a topic of mathematical speculation rather than of astronomical relevance. While *infinitesimal* non-radial pulsations received a lot of attention – their study preceded the investigation of radial pulsations [Thomson (Lord Kelvin), 1863] –, there has been little observational motivation for exploring the role of non-linear effects. It is the purpose of the first part of this paper to show that this view may require revision on two accounts. On the one hand we present (a) a summary of arguments of an observational character, suggesting non-linear phenomena among the non-radial oscillations (Sect. 2.1), and (b) a collection of indirect theoretical arguments pointing towards hidden non-linearities (Sects. 2.2, 2.3). On the other hand we indicate how the general adiabatic non-radial non-linear stellar motion problem can be tamed by translating it into a Hamiltonian particle formalism (Sect. 2.4). This broad theoretical frame – which in principle lends itself to a direct numerical analysis of *all* (adiabatic) stationary and non-stationary motions – is then reduced to a substantially simpler scheme; the latter is devided to encompass only "pure" oscillations around the radially symmetric equilibrium state, thus suppressing all stationary circulations (Sect. 2.5).

The second part of this paper concentrates on the mathematical derivation of a Hamiltonian particle formalism for "pure" oscillations. Our starting point is a variational principle due to Lynden-Bell and Katz (1981). The main interest of this principle for our purposes is that it allows us to build in constraints on the motion in an intrinsic fashion, without resorting to Lagrange multipliers (Sect. 3). Lagrange multipliers are used in conventional variational schemes of fluid mechanics (see Seliger and Whitham, 1968). The final Hamiltonian equations are found to have a structure more involved than in the purely radial case; in the lowest-order non-linear approximation, our non-radial F-mode interaction problem depends on $F(2F^2+3F+4)/3$ model parameters (some of which vanish as a result of selection rules), while in the radial case we have only $F(F^2+3F+8)/6$ parameters. This in turn suggests that the non-radial non-linear oscillations may exhibit a broader spectrum of different classes of motion than the purely radial stellar oscillations.

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2. Observational and theoretical motivation

2.1. The observational status

Circumstantial observational evidence has been provided for the non-radial nature of the oscillations of several classes of stars. First, the existence of beat phenomena in βCephei stars was considered an argument in favour of non-radial effects (Ledoux, 1951). This hypothesis was supported by the result of Baade's test applied to several βCephei stars (Walker; 1954a, b); in fact, this test seemingly rules out a purely radial character of these pulsations. Second, the detection of period ratios incompatible with the radial period ratios of currently accepted theoretical models in δScuti stars is pointing towards non-radial oscillations (Stobie and Shobbrook, 1976). Third, among the variable white dwarfs of spectral type DA, the multiplicities and lengths of the observed periods (exceeding the probable fundamental radial period) have been regarded as fingerprints of the non-radial nature of the oscillations. This class of variables, known as ZZ Ceti stars, was isolated by McGraw and Robinson (1976) (see the review by Winget and Fontaine, 1982).

These three classes of non-radial pulsators have two features in common. They have a dwarf type structure (no extended envelope), and their oscillation amplitudes are small (with typical light amplitudes in the V range of < 0.1 for βCepheids, 0.002–0.8, and 0.005–0.3 m for δScuti and ZZ Ceti stars, respectively). One may expect that with the development of large telescopes, the improvement of speckle interferometric techniques (see Bates, 1982) and image restoring methods, direct surface observations of these stars will become feasible. Such observations would definitively settle the question of the symmetry of these oscillations.

A fourth type of object undergoing non-radial pulsations is the Sun. So far the Sun is the only star in which non-spherically symmetric variations have been registered directly. Since the first observational evidence of solar pulsations in the early 60's, several classes of global oscillations have been reported. They fall into three period-intervals. (i) In the 5-minute range (3.5–7 min) there are two categories of motion: (a) velocity pulsations of a typical horizontal wave number around 1 Mm (Leighton et al., 1962; Deubner, 1975), and (b) velocity and light variations revealed by whole-disk observations (see, in particular, Grec et al., 1980; Woodard and Hudson, 1983). The velocity amplitudes of the latter are of the order of $10\,\mathrm{cm\,s^{-1}}$, and relative luminosity variations are around 10^{-6}. (ii) In the 10–100 min range fluctuations of the "apparent diameter" have been reported by the SCLERA group (Hill et al., 1976; Brown et al., 1978; Bos and Hill, 1983); the relative amplitude of the diameter variation is now estimated to be $4\,10^{-7}$. (iii) Finally, in the long period range $\gtrsim 2$ h, the challenging periodicity of 160.01 min first detected in Crimea (Severny et al., 1976) and confirmed by several groups (Brooks et al., 1976; Scherrer et al., 1979, 1980) shows a velocity amplitude of 20–$50\,\mathrm{cm\,s^{-1}}$.

Just like the representatives of the first three classes of stars, the Sun has a dwarf structure and its oscillation amplitudes are very small; both the light and velocity amplitudes of the Sun are by a factor of roughly 10^5 smaller than the values typical for the other classes. Incidentally, one may suspect that the Sun does not stand alone among the middle-sized main-sequence stars in showing exceedingly small amplitude variability. A search for low-amplitude global stellar photospheric oscillations was already conducted by Traub et al. in 1978; among the 9 bright stars analysed, some indication for variability was obtained in the case of Procyon only (period around 57 s, velocity amplitude about $25\,\mathrm{m\,s^{-1}}$); notice that Procyon is the only dwarf star among the objects observed by Traub et al. (1978). This search is now being pursued by the Nice group (Gelly et al., 1984). It is likely that a vast amount of low amplitude oscillation data of stars will become available in a few years' time, especially since major improvements of the observation techniques are now being planned [e.g. Connes's (1983) high-sensitivity "accelerometer"].

2.2. Theoretical arguments

The low amplitude level of the non-radial variability in all four classes of stars naturally suggests that a linear oscillation theory should be satisfactory. However, for reasons of approximate resonances among the observed modes in the βCephei stars on the one hand, and for lack of a known vibrational instability of the observable linear oscillation modes on the other hand, several authors investigated non-linear dynamical mode-coupling effects. Two different paths were followed.

(1) For numerical studies of motions of azimuthal symmetry – reducing the realistic 3D problem to a more manageable 2D problem – Deupree (1974) introduced a straightforward non-radial Eulerian finite-difference scheme. The meridian section of the star is divided into cells of uniform angular and arbitrary radial size. The numerical code was applied to a model of a rotating βCepheid, and for comparison also to a model of a giant, namely an RR Lyrae star, in order to discuss the interaction of modes of nearly equal periods. Using as an initial condition an excitation of either the fundamental radial mode or an (approximate) f-mode of degree $l=2$, Deupree (1974) found that a strong coupling between both modes is observed in the rotating βCepheid model; both types of mode acquire similar amplitudes. In contrast, little coupling was observed in the RR Lyrae model. In later work, Deupree's (1974) direct discretization scheme was applied primarily to study the interaction between convection and pulsation (Deupree, 1975, 1976, 1977a, 1978; see also the critique by Toomre, 1982).

(2) Vandakurov (1965, 1967, 1979, 1981), Dziembowski (1982), and Buchler and Regev (1983) adopted mode-coupling schemes essentially similar to the procedures developed by investigators of stellar convection (e.g. Toomre et al., 1976). The main purpose of these formulations was to extract direct information, analytical or almost analytical, on the non-radial oscillatory behaviour of stars under conditions dictated by specific observational circumstances. Vandakurov and Dziembowski's non-linear studies were motivated by a theoretical reason. In some classes of stars a band of non-radial linear modes are vibrationally unstable: the spatial amplitude pattern of these modes, however, is such that they virtually vanish near the surface of the star. Therefore, the variability associated with these unstable modes should be unobservable. However, precisely these stars do exhibit a surface variability. This observation led to the following two tentative non-linear interpretations.

(i) Local stability analysis (Kato, 1966) suggests that the chemically inhomogeneous zone in massive post-main sequence stars produces a vibrational instability in the asymptotic g-modes. Global stability calculations for massive stars ($M > 15\,M_\odot$) confirm this result for g-modes of high degree l (Gabriel and Noels, 1976; see also Shibahashi and Osaki, 1976). These unstable g-modes are trapped in the zone of variable chemical composition, and therefore they cannot lead to directly observable effects. On the other hand, g-modes of low degree ($l=2$) are stable; the latter would be observable, since their amplitudes are large near the surface and in the μ-gradient zone (Ozaki, 1975). The class of

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stellar models that conform to these properties resemble βCephei stars. Therefore, Vandakurov (1979) argued that through non-linear mode-coupling the unstable g-modes might transfer their kinetic energy to stable observable large-scale surface modes. This idea prompted a closer examination of the non-linear mode-interaction problem, in spite of the smallness of the observed amplitudes in βCepheids (Vandakurov, 1979, 1981).

(ii) Cool white dwarf models with hydrogen-helium rich envelopes have been found to have a vibrationally unstable band of g-modes, of degree l in the range of 100–400 (with periods of 5–25 s); the instability is due to the H and He I ionization zones (Dziembowski, 1977); such unstable high l oscillations are manifestly not detectable by standard whole-disk observations. On the other hand, McGraw and Robinson's (1976) class of ZZ Ceti variables, with periods in the 100–1200 s range, reasonably fit this family of theoretical models. Dziembowski (1979) then proposed that a non-linear mode interaction could be responsible for an energy flow from the unstable occluded high modes onto the stable observable low l modes. He initiated a systematic non-linear non-radial mode-coupling analysis (Dziembowski, 1979; and especially 1982), although in the meantime his original motivation has lost much of its weight. In fact, improved DA white dwarf models of masses around $0.6 M_\odot$, with a stratified composition and a hydrogen surface layer of 10^{-14} to $10^{-4} M_\odot$, are found to have unstable g-modes in the observed period range (see the review by Winget and Fontaine, 1982). Dziembowski's (1982) formalism was re-analysed and extended in Buchler and Regev (1983). In particular these authors adapted the two-time method to the non-radial mode-interaction problem in the presence of "slow" non-adiabatic effects and "fast" regular oscillations. Resonances of the form $\omega_2 \simeq 2\omega_1$ and $\omega_3 \simeq \omega_1 + \omega_2$ among the linear frequencies were discussed in detail.

2.3. *Special motives*

Our particular interest in non-radial oscillations is motivated by two further considerations.

(1) We plan to perform a non-linear investigation of the solar oscillations, paying attention to the 160 min periodicity. In fact, at various occasions it has been conjectured that the latter periodicity might be a manifestation of non-linear interactions among the linear modes (Gough, 1980; Perdang, 1981; Vandakurov, 1981; Kosovichev and Severny, 1983). Moreover, preliminary non-linear *radial* numerical experiments seemingly support the plausibility of non-linear coupling mechanisms: if several adjacent radial modes are interacting, the surface of a star, whose structure is approximated by a standard polytrope, is found to oscillate with a long period of about 2.5 times the fundamental period (besides, of course, oscillations of periodicities fixed by the linear modes). This result is independent of the specific initial conditions; it holds if the interacting linear modes are randomly excited. The long periodicity manifests itself as a clearly defined isolated, generally multiple peak in the power spectrum of the surface motion; it is also directly visible if the surface displacement is plotted against time (Perdang and Blacher, 1984a; Däppen and Perdang, 1984). For more realistic solar experiments, the long period is invariably found to be $\gtrsim 2$ h; by adjusting the model parameters, it could be made to come close to the observed value of 160 min (Perdang and Blacher, 1984b). Although these radial numerical investigations do suggest the relevance of non-linear effects in solar oscillations, they are unsatisfactory for two reasons. In order to exhibit the long periodicity, relative radius amplitudes of the order of one percent are needed in the 6-mode coupling experiments conducted so far; such amplitudes exceed the observed surface amplitudes by a factor of at least 10^4. On the other hand, observed solar oscillations are predominantly non-radial (the actual number of excited linear non-radial modes is estimated to exceed 10^6), and therefore a purely radial calculation cannot have much weight. A priori, we could well imagine that the oscillation energy stored in the radial linear modes might flow onto the non-radial modes; if the existence of the long periodicity were a purely radial manifestation, then the coupling between the radial and non-radial modes could destroy that periodicity altogether. In fact, a straightforward application of Lindstedt's procedure indicates that an increase in the number of excited modes, whether radial or non-radial, typically leads to an increase in the relative power of the long-period peak with respect to the power of the linear peaks (Perdang, 1985). The latter result thus suggests that with *large numbers of coupled modes* the low-amplitude problem could be overcome.

(2) Besides the direct observational motivation there is also a more formal goal. The present paper is a continuation of the radial non-linear mode-interaction work by Perdang and Blacher (1982, 1984a) (henceforth Papers I and II). It thus aims at clarifying and classifying the new features introduced by the non-linear mechanisms operating among the non-radial modes. While the rough classification of the non-steady motions into periodic, quasi-periodic (or multi-periodic), and chaotic oscillations encountered in the radial case continues to hold true in the radial context (see Perdang; 1978, 1983), the characteristics of the non-radial chaotic oscillations are entirely unknown at the moment. For instance in the radial stellar pulsations we have invariability found that the chaotic solutions remain in fact pseudo-periodic or pseudo-multiperiodic over a long time (some hundred cycles); they display one or several approximate periodicities. Such a property makes them almost indistinguishable from strictly periodic or quasi-periodic oscillations observed with a low amplitude resolution. Does this property still hold for non-radial chaos? On the other hand, how does the correlation of the surface oscillation at two different points of the stellar disk evolve in time? One also wishes to know whether, in the case of a large number of coupled non-radial modes, the chaotic state tends to privilege one spatial (horizontal) wave-number range, or whether energy is drained to larger and larger wave numbers.

2.4. *General formalism*

Our aim is to develop a consistent system of equations describing the non-radial non-linear stellar oscillations via a non-linear mode-coupling scheme involving an arbitrary (but finite) number F of linear modes. Previous authors have concentrated on 2 or 3-mode interactions only; moreover, their techniques heavily rely on the *ad hoc* assumption that the oscillations remain regular. Our analysis is concerned with purely non-dissipative motions (see Papers I and II). Under that hypothesis the hydrodynamic equations of motion admit of a variational principle (see Seliger and Whitham, 1968). A single *functional* specifies the adiabatic behaviour of the star completely. In the F-mode coupling approximation, this in turn implies that the nature of the oscillation equations is encoded in a single *function* (a Lagrangian L). To lowest non-linear order, we can readily anticipate the algebraic structure of the Lagrangian. Denoting by $\boldsymbol{q} = (q_1, q_2, q_3, \ldots, q_F)$ the finite set of generalized coordinates [these coordinates being the expansion coefficients of the dependent physical fields in a complete, though truncated, set

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of linear eigenfunctions (Sect. 3)], we can write the Lagrangian in the form

$$L=\varepsilon^2 L^{(2)}(\boldsymbol{q},\dot{\boldsymbol{q}})+\varepsilon^3 L^{(3)}(\boldsymbol{q},\dot{\boldsymbol{q}})+O(\varepsilon^4), \tag{1}$$

where ε is a book-keeping parameter of the expansions. The component $\varepsilon^2 L^{(2)}(\boldsymbol{q},\dot{\boldsymbol{q}})$ describes the linear oscillations, and therefore reduces to the form

$$L^{(2)}=\frac{1}{2}\sum_{j=1}^{F}\dot{q}_j^2-\frac{1}{2}\sum_{j=1}^{F}\omega_j^2 q_j^2, \tag{2}$$

where ω_j, $j=1,2,\ldots,F$, is the frequency of the mode labelled j. There is a basic difference with the radial oscillation problem already at this stage: In a consistent non-radial treatment the complete set of linear modes should involve spheroidal and toroidal modes; the latter are all associated with neutral frequencies ($\omega=0$). In the radial problem, however, all frequencies are non-zero. The component $\varepsilon^3 L^{(3)}(\boldsymbol{q},\dot{\boldsymbol{q}})$, takes care of the lowest order non-linear coupling between the linear modes. Its most general form is a homogeneous cubic multinomial in $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$. We can specify this form further by observing that the Lagrangian L is the difference between the kinetic energy K and the potential energy V (gravitational plus thermal energy) involved in the oscillations. Since the potential energy is independent of the velocity, the algebraic expression of V in the mode-coupling scheme is a function of $\boldsymbol{q}$ alone; the generalized velocities $\dot{\boldsymbol{q}}$ enter the Lagrangian (1) through the kinetic energy K only. The latter has the following form

$$K=\frac{1}{2}\int_{\mathscr{V}(t)} d^3 r\,\varrho\,\boldsymbol{u}^2. \tag{3}$$

[$\mathscr{V}(t)$, volume of the star; ϱ, density; $\boldsymbol{u}$, hydrodynamic velocity field]. In the case of purely radial oscillations, we can adopt the invariant mass $m(r)$ contained in a sphere of radius r as the independent positional variable; with the generalized coordinates $\boldsymbol{q}$, defined as the expansion coefficients of the displacement δr, the full kinetic energy reduces to the diagonal quadratic form

$$K=\varepsilon^2\frac{1}{2}\sum_{j=1}^{F}\dot{q}_j^2. \tag{4}$$

In the general case of non-radial non-linear oscillations, such a reduction of the algebraic form of the kinetic energy does not seem to be feasible. However, since the density variations can be expressed in the generalized coordinates only, the structure of the kinetic energy is given by

$$K=\varepsilon^2\frac{1}{2}\sum_{j=1}^{F}\dot{q}_j^2+\varepsilon^3\sum_{j,k,l=1}^{F}K_{jkl}q_j\dot{q}_k\dot{q}_l+O(\varepsilon^4). \tag{5}$$

The diagonal quadratic form of the lowest order component of the kinetic energy can be achieved through a convenient definition of the generalized coordinates. The $O(\varepsilon^3)$ correction is linear in $\boldsymbol{q}$ and quadratic in the generalized velocities $\dot{\boldsymbol{q}}$. It follows that the coupling Lagrangian takes the form

$$L^{(3)}=\sum_{j,k,l=1}^{F}K_{jkl}q_j\dot{q}_k\dot{q}_l-\sum_{j,k,l=1}^{F}V_{jkl}q_jq_kq_l. \tag{6}$$

The coefficient matrix V_{jkl} describing the coupling through gravitational and thermal effects, is symmetric in its three subscripts; it contains therefore $F(F+1)(F+2)/6$ independent elements. The coefficient matrix K_{jkl} produces a kinetic coupling between the modes; it is symmetric in k and l and therefore involves $F^2(F+1)/2$ independent elements. Hence, to order ε^3, we anticipate that non-linear, non-radial F-mode oscillations are completely described by $F(2F^2+3F+4)/3$ global model coefficients, namely the F linear frequencies, and the $F(F+1)(2F+1)/3$ coupling parameters.

2.5. *Particularization of the general method*

The general formalism sketched above enables us, in principle, to deal with (a) non-radial equilibrium states, (b) stationary motions and (c) general non-stationary motions ("oscillations") on top of a non-radial stationary background. Logically, a systematic analysis of the latter motions should then be preceded by an investigation of the stationary motions, which in turn should be preceded by a study of the possible non-radial equilibrium states. Theoretically, very little in known about subproblems (a) and (b).

(a) Regarding non-radial equilibrium states, the only outstanding result applicable to fairly realistic stellar models is a general symmetry property due to Wavre (1932) and Lichtenstein (1933): Any equilibrium configuration of a self-gravitating rotating body has a symmetry plane σ_h normal to the angular momentum axis, under the hypothesis that the surfaces of constant density be regular and monotonically nested.

More precise results are available only in the framework of academic stellar models. The *homogeneous* (density ϱ space-independent) *self-gravitating configuration* has given rise to a large body of literature, mainly in mathematical circles. The classical examples of the non-spherically symmetric equilibria are the Maclaurin spheroids, of dihedral symmetry $D_{h\infty}$[1], the Jacobi ellipsoids, of dihedral symmetry D_{h2}, branching off from the Maclaurin sequence for an angular momentum parameter $J^2=0.384$ (in units $GMa_1a_2a_3$, G gravitational constant, M mass, and a_i, $i=1,2,3$, the principal axes of the ellipsoid), and the Poincaré pear-shaped configurations of symmetry C_{v2}, bifurcating from the Jacobi sequence at $J^2=0.632$ (see Chandrasekhar, 1969). Less widely known is a more recent result by Constantinescu et al. (1979). By using group-theoretical methods, these authors show that in addition to the Jacobi bifurcations an infinity of configurations of symmetries D_{hm}, $m=3,4,5,\ldots$, are branching off the Maclaurin sequence. It is likely that this sequence of non-radial Constantinescu-Michel-Radicati equilibrium configurations (in a uniformly rotating reference frame) also survives for realistic stars. These examples illustrate that non-radial equilibria are possible, on condition that a non-radial mechanism (the angular momentum in the previous examples) is operating to react against the radial gravitational pull.

(b) Regarding adiabatic stationary motions, our theoretical information is even more limited. Among the academic models,

1 A configuration is said to have dihedral symmetry D_{hm} if it is invariant (a) under rotations of an angle $2\pi/m$ about an axis C_m (the angular momentum axis J), (b) under rotations of an angle π about an axis Λ_2 normal to C_m (there exist m such binary rotation axes if m is odd, and $m/2$ binary axes if m is even), and (c) under reflection on a "horizontal" plane σ_h normal to C_m and through the axes Λ_2. In particular $D_{h\infty}$ means that the axis C_∞ is a symmetry axis for rotation of arbitrary angle θ. Symmetry C_{vm} indicates that the configuration is invariant (a) under a rotation of angle $2\pi/m$ about an axis C_m and (b) under reflections on a "vertical" plane σ_v through C_m (depending on whether m is odd or even, there are m or $m/2$ such planes)

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Dirichlet's classical problem, followed up by Dedekind and especially by Riemann (see Chandrasekhar, 1969), has been explored most thoroughly. Dirichlet raised the question of defining the conditions under which a velocity field, which is a linear function in the coordinates, leads to an ellipsoidal configuration in a homogeneous self-gravitating body described in an absolute reference frame. Riemann showed that the general answer is a motion field made up of a uniform rotation and a uniform vorticity in the rotating frame of reference, with the rotation and vorticity axes lying in a symmetry plane of the ellipsoid; if both axes coincide or if the vorticity vanishes, then the axis of the motion becomes a symmetry axis of the figure. The sequences with Riemann-type circulation branch off the Maclaurin sequence precisely at the transition point towards the Jacobi ellipsoids.

A general survey of the allowed circulations, even in the simple homogeneous model, is still lacking. However, the following argument suggests that there should exist an infinity of stationary velocity fields of different symmetries (see Perdang, 1984a). In the absence of any motion, the equilibrium structure of a star belongs to the full spherical symmetry K_h (any axis through the centre of mass O is a C_∞ axis, and any plane σ_h through O is a reflection plane). The linearized equations around this state admit of an infinity of neutral modes ($\omega_{\text{tor}}=0$), associated with infinitesimal toroidal velocity fields (see Sect. 3). The non-linear dynamics enables each of these modes to build up a finite steady circulation. In the presence of rotation, the infinite degeneracy of the modified toroidal modes is lifted, the associated frequencies $\omega_{\text{tor}}(J^2)$ becoming functions of the angular momentum parameter J. We conjecture that for specific finite critical values of J one of the frequencies $\omega_{\text{tor}}(J^2)$ vanishes; the corresponding velocity eigenfunction then has a well defined rank m. Each such critical J then specifies a bifurcation towards a steady state with a stationary velocity field, the symmetry of the configuration being $D_{h|m|}$. Since the Constantinescu-Michel-Radicati relative equilibrium figures precisely arise through the same mechanism ($\omega_{\text{tor}}=0$ for a well defined rank m), it seems likely that the bifurcations towards the latter coincide with the bifurcations towards the stationary-circulation configurations. In the simplest case, namely the transition from the Maclaurin spheroid towards the Jacobi ellipsoid (no velocity field in the rotating frame), and to the Dedekind ellipsoid (uniform vorticity along the smallest axis), this property is in fact well known (Chandrasekhar, 1969). Again, since our discussion does not rely on the particular assumption of homogeneity of the configuration, we are led to conjecture that this class of circulations survives in realistic stars.

Incidentally we should call attention to a further class of stationary velocity fields not captured by the Lagrangian (1). Following von Zeipel's (1924) classical result, rigid body rotation is incompatible with radiative equilibrium; it has been argued that a meridian circulation activated by a temperature gradient over the isobaric shells is then required to secure steady state solutions of the stellar equations [see Tassoul (1978) for a detailed discussion]. The precise nature of the latter velocity field, however, remains a matter of debate (Tassoul and Tassoul; 1982, 1983).

Besides the missing *theoretical* information on the structure of non-radial equilibrium states and stationary velocity fields in stars, there is, at the moment little direct *observational* motivation for studying asymmetries, or stationary motions more involved than pure rotations. In fact, even in the Sun, observational evidence for such motions seems to remain problematic (Gilman, 1974). Among the other stars, the disk of Betelgeuze alone has been reasonably resolved so far to suggest non-radial surface features (star spots) (Lynds et al., 1976; McDonnell and Bates, 1979; Murdin and Allen, 1979). Direct observation of surface motions is not likely to become feasible before the advent of the large telescopes planned for the next decade.

Owing to the lack of a satisfying theoretical understanding of stationary circulations as well as to a lack of direct observational urge, we have been led to devise a formalism that enables us to filter out circulations altogether from the equations of non-linear stellar motions: Our final variational principle has the property of generating "pure" oscillations around a spherically symmetric equilibrium state. We anticipate that our approach can however be extended to include circulations of simple topology (see Lynden-Bell and Katz, 1981; Katz and Lynden-Bell, 1982).

3. Variational formulation. Choice of the independent variables and the dependent fields

We start out from the general variational principle describing adiabatic motions about the equilibrium of a self-gravitating configuration in an inertial frame [see, e.g., Chapter 10 of Ledoux (1958) or Chapter 15 of Serrin (1959)]

$$\delta \int_{t_1}^{t_2} L(t)\, dt = 0\,. \tag{7}$$

The Lagrangian $L(t)$ is here the difference between the total kinetic energy K [Eq. (3)] and the total potential energy V (thermal and gravitational) of the fluid with respect to the equilibrium configuration,

$$V(t) = \int_{\mathscr{V}(t)} d^3r\, \varrho(\boldsymbol{r},t)\, U(s(\boldsymbol{r},t),\, \varrho(\boldsymbol{r},t)) + \frac{1}{2}\int_{\mathscr{V}(t)} d^3r\, \varrho(\boldsymbol{r},t)\, \Phi(\{\varrho(\boldsymbol{r},t)\},\boldsymbol{r},t)\,. \tag{8}$$

$\mathscr{V}(t)$ again denotes the time dependent volume of the star, d^3r is the volume element, U and Φ are the specific internal and gravitational energy respectively. In the system of natural thermodynamic variables, U is a function of specific entropy s and specific volume $v=1/\varrho$, i.e. $U=U(s,v)$. The gravitational potential is given by

$$\Phi(\{\varrho(\boldsymbol{r},\mathrm{t})\},\boldsymbol{r},\mathrm{t}) = -G\int_{\mathscr{V}(t)} d^3r'\, \varrho(\boldsymbol{r}',t)\cdot\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\,. \tag{9}$$

This variational principle lends itself to a straightforward investigation of the *radial* non-linear pulsations of a star (see Paper I). In the radial case, the time-dependent domain of integration is in fact transformed into a time-independent domain through the use of the mass $m(r)$ contained in a sphere of radius r as the independent variable. This choice of the independent variable then builds mass conservation directly into the variational principle. As the free dependent field, the radial position r (together with its time derivative r), or equivalently, the finite radial displacement $\delta r(m,t)$ from the equilibrium configuration proves a convenient variable.

In the case of *non-radial* pulsations, we likewise seek a system of space variables in which the time-dependent domain transforms into a time-independent domain. Again, this choice should be such as to incorporate a major conservation law. Due to the three-dimensional nature of the motions, the most general free dependent variables are now three scalar fields, which we could, in principle, take as the three components of the finite non-radial displacement $\delta\boldsymbol{r}$ measured from the equilibrium position. However, as argued above (Sect. 2.5), we are not concerned with the

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most general non-radial motions, but only with those corresponding to "pure" oscillations. To generate the latter, we require that the Helmholtz-Kelvin circulation

$$C(\mathscr{C}) = \int_{\mathscr{C}} d\boldsymbol{r} \cdot \boldsymbol{u} \tag{10}$$

(hereinafter simply "circulation") vanishes over any closed path $\mathscr{C}$ on an invariant entropy surface. We shall show below that this constraint eliminates stationary motions. In the linear approximation it also eliminates the toroidal components in the velocity and displacement fields.

3.1. The space variables (s, θ, ϕ)

We shall provisionally assume that at any time the specific entropy is monotonically increasing outwards. This condition is essentially obeyed in stars in stable radiative equilibrium, at least in the absence of a marked space-dependent chemical composition. More precisely, we mean by monotonically increasing that all surfaces of constant entropy have spherical topology without critical points, and that for any two entropy surfaces $s(\boldsymbol{r}, t) = s_1$ and $s(\boldsymbol{r}, t) = s_2$, with $s_1 < s_2$, the surface s_1 remains nested in s_2 at all times t. We should immediately observe that in realistic stars (convective zones, composition gradients), the requirement of a monotonically increasing entropy may not be satisfied. The general case is dealt with in Appendix 3, where we show that the formalism remains unaltered.

Suppose we start with spherical coordinates (r, θ, ϕ) and time t as independent variables. Since entropy is assumed to be monotonic, we can introduce a new system of independent variables, (s, θ, ϕ) and t, defined by $s = s(r, \theta, \phi; t)$, with the remaining variables θ, ϕ, and t being the same as in the original system. These new coordinates have the following two properties.

(1) Integrals over the total time-dependent volume of the star transform into integrals over a constant domain of integration. If Q is a specific stellar property (e.g. kinetic energy per unit mass, thermal energy per unit mass...) then the corresponding global property (total stellar kinetic energy, total thermal energy...) becomes

$$\int_{\mathscr{V}(t)} d^3 r \varrho Q = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \int_{s_c}^{s_s} ds \left\{ [r(s, \theta, \phi; t)]^2 \left(\frac{\partial r}{\partial s}\right)_{\theta, \phi; t} \cdot \varrho \right\} Q. \tag{11}$$

In this expression s_c and s_s are the specific entropy at the centre and the surface of the star, respectively; $r(s, \theta, \phi; t)$ is the inverse function of $s(r, \theta, \phi; t)$ at fixed θ, ϕ, and t. The limits of integration s_c and s_s are independent of time because of entropy conservation.

(2) By using entropy as the independent variable, entropy conservation is automatically guaranteed. Thus in the non-radial problem, the new system of space coordinates (s, θ, ϕ) is the analogue of the coordinate m of the radial problem.

3.2. The free fields δr and $\delta \mu$

An arbitrary (sufficiently smooth) vector field $\boldsymbol{A}(\boldsymbol{r}, t)$ defined over the star may be represented locally in the Clebsch-type form

$$\boldsymbol{A}(\boldsymbol{r}, t) = \nabla a(\boldsymbol{r}, t) + b(\boldsymbol{r}, t)\, \nabla c(\boldsymbol{r}, t), \tag{12}$$

where a, b, c are three (possibly multi-valued) scalar functions. Incidentally this form was first introduced by Monge (1784) (see Erickson, 1960). We adopt this representation for the velocity field $\boldsymbol{u}(r, t)$ of the disturbed equilibrium state of the star. The circulation of the velocity around a closed loop $\mathscr{C}(t)$ moving with the fluid is then given by

$$C[\mathscr{C}(t)] = \int_{\mathscr{C}(t)} d\boldsymbol{r} \cdot \boldsymbol{u} = \int_{\mathscr{C}(t)} d\boldsymbol{r} \cdot \nabla a(\boldsymbol{r}, t) + \int_{\mathscr{C}(t)} d\boldsymbol{r} \cdot b(\boldsymbol{r}, t)\, \nabla c(\boldsymbol{r}, t). \tag{13}$$

The following properties can be proved:

(1) If the circulation of the velocity $\boldsymbol{u}$ over any loop on a surface of constant s vanishes, then the most general Clebsch representation (12) involves only two arbitrary fields, a and b, with c becoming a function of b and s.

(2) The latter representation is equivalent to choosing

$$\boldsymbol{u}(r, t) = \nabla a(\boldsymbol{r}, t) + b(\boldsymbol{r}, t)\, \nabla s(\boldsymbol{r}, t), \tag{14}$$

where a and b are arbitrary scalar fields, except for the condition that a is single-valued. By equivalence of the two triplets of Monge potentials, $(a', b', c' = c'(s, b'))$ and (a, b, s), we mean that given the first set $(a', b', c'(s, b'))$ of potentials describing a velocity field $\boldsymbol{u}$, we can find a second set (a, b, s) describing exactly the same velocity field.

(3) If the surfaces $s = \text{const.}$ are smooth and topologically equivalent to spheres, if $\boldsymbol{u}$ is a smooth velocity field tangent to the surfaces $s = \text{const.}$, and if the circulation of $\boldsymbol{u}$ vanishes over any closed loop of the surfaces $s = \text{const.}$, then the velocity field $\boldsymbol{u}$ vanishes identically. It follows in fact from the construction given in Lynden-Bell and Katz (1981) that a *stationary* velocity field, under the condition of adiabaticity, is tangent to the surfaces of constant entropy. Therefore there are non-zero stationary velocity fields whose circulation vanishes over all closed loops lying on surfaces of constant entropy. In the linear approximation this property is obvious, because representation (14) reduces to a purely spheroidal vector field (only the radially symmetric equilibrium part s_0 of the entropy enters in this approximation). But only toroidal velocity components produce stationary motions, while spheroidal components describe oscillations in dynamically stable stars.

Essentially we could use the components of the velocity field as the free field variables of the variational formulation. In the absence of further constraints besides mass and entropy conservation, we then would have three free fields. With the additional constraint of zero circulation along closed loops on surfaces of constant specific entropy, only *two* scalar fields remain free. We found it convenient to choose not the fields entering expression (14), but two other related independent scalar fields, one being the radial displacement field and the other a field describing the stratification of matter.

In analogy with the problem of radial oscillations, we introduce as one free field the *radial displacement*

$$\delta r(s, \theta, \phi; t) = r(s, \theta, \phi; t) - r_0(s), \tag{15}$$

where $r(s, \theta, \phi; t)$ has been defined above and $r_0(s)$ is the distance of a surface of constant entropy s from the centre in the radially symmetric equilibrium configuration.

The second free field is chosen to incorporate mass conservation. The integrals (3), (8), and (9) of the variational principle all involve, when expressed in the independent coordinates (s, θ, ϕ),

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the same quantity, shown between curly brackets in Eq. (11). This leads us to introduce a new field variable

$$\mu(s,\theta,\phi;t)=4\pi\,[r(s,\theta,\phi;t)]^2\cdot\left(\frac{\partial r}{\partial s}\right)_{\theta,\phi;t}\cdot\varrho(s,\theta,\phi;t). \tag{16}$$

In terms of this field, we can write the mass $m(s_1,s_2)$ contained in the entropy shell specified by $s_1<s<s_2$ as

$$m(s_1,s_2)=\frac{1}{4\pi}\int_0^{2\pi}d\phi\int_0^{\pi}d\theta\,\sin\theta\int_{s_1}^{s_2}ds\,\mu(s,\theta,\phi;t). \tag{17}$$

μ represents therefore 4π times the mass per unit specific entropy and unit solid angle. In a radially symmetric configuration, μ becomes the mass per unit specific entropy. Since μ specifies the stratification of matter in the star, we shall refer to this field as the *stratification field.* Let $\mu_0(s)$ be the radially symmetric stratification of the equilibrium of the star. Let $\mu(s,\theta,\phi;t)$ be the stratification field in the disturbed star. Then conservation of mass requires that for any entropy shell $s_1<s_2$ we have

$$\begin{aligned} m(s_1,s_2)&=\int_{s_1}^{s_2}ds\,\mu_0(s)\\ &=\frac{1}{4\pi}\int_0^{2\pi}d\phi\int_0^{\pi}d\theta\,\sin\theta\int_{s_1}^{s_2}ds\,\mu(s,\theta,\phi;t)=\text{const}. \end{aligned} \tag{18}$$

If we now introduce the stratification disturbance

$$\delta\mu(s,\theta,\phi;t)=\mu(s,\theta,\phi;t)-\mu_0(s), \tag{19}$$

we can express mass conservation by the condition

$$\frac{1}{4\pi}\int_0^{2\pi}d\phi\int_0^{\pi}d\theta\,\sin\theta\int_{s_1}^{s_2}ds\,\delta\mu(s,\theta,\phi;t)=0. \tag{20}$$

It remains to be proved (see Appendix 1) that the velocity field generated by an arbitrary choice of the two fields δr and $\delta\mu$ can be written in the form of Eq. (14).

A remark on the linear adiabatic stellar oscillations is in order. The variational principle (7) restricted to velocities of the form (14) generates the conventional oscillation equation for spheroidal infinitesimal displacements (see Appendix 1, Sect. 4). The family of neutral toroidal modes is filtered out [for a discussion of the latter see, for instance, Aizenman and Smeyers (1977)]. We require additionally that the star be dynamically stable, so that all eigenvalues ω_{nl}^2 [$(2l+1)$-degenerate as a result of spherical symmetry] associated with the spheroidal modes are strictly positive. Let $\delta r_{nlm}(s,\theta,\phi)$ and $\delta\mu_{nlm}(s,\theta,\phi)$ be the radial displacements and stratification field components of the eigenfunctions belonging to ω_{nl}. As usual, n is here the radial order, and l and m are the degree and azimuthal order of the spherical harmonics entering the eigenfunction. We found it convenient to choose a *real* set of spherical harmonics (defined in Appendix 2). The set of all eigenfunctions being complete, the non-linear time-dependent fields can be expanded in the eigenfunctions of the linearized problem in a way analogous to the Woltjer series of the radial case (Woltjer, 1935; Paper I):

$$\begin{pmatrix}\delta r(s,\theta,\phi;t)\\ \delta\mu(s,\theta,\phi;t)\end{pmatrix}=\sum_{nlm}q_{nlm}(t)\begin{pmatrix}\delta r_{nlm}(s,\theta,\phi)\\ \delta\mu_{nlm}(s,\theta,\phi)\end{pmatrix}, \tag{21}$$

$q_{nlm}(t)$ being time-dependent expansion coefficients. We read off from representation (21) that mass conservation is rigorously incorporated in our formalism. If we substitute the stratification component of (21) into the conservation requirement (20), we obtain

$$\frac{1}{4\pi}\sum_{nlm}q_{nlm}(t)\int_0^{2\pi}d\phi\int_0^{\pi}d\theta\,\sin\theta\int_{s_1}^{s_2}ds\,\delta\mu_{nlm}(s,\theta,\phi)=0. \tag{22}$$

But the integral over non-radial components $\delta\mu_{nlm}$, $l\neq 0$, vanishes identically. Moreover $\delta\mu_{n00}\equiv 0$, for all radial orders, as transpires from a direct calculation; physically the linear eigenfunctions must in fact obey mass conservation. Therefore, relation (22) is seen to be satisfied *exactly* for *any* stratification field disturbance.

In a finite truncation of the generalized Woltjer-type expansion (21), we have the following set of generalized coordinates

$$q_{nlm}(t),\quad n=0,1,2,\ldots N;\quad l=0,1,2,\ldots L;\quad m=-L,\ldots 0,1,\ldots L. \tag{23}$$

The number of degrees of freedom of our truncated non-radial stellar oscillation problem is thus

$$F=(N+1)(L+1)^2. \tag{24}$$

If $L=0$, we recover the usual radial expansion, since the stratification component $\delta\mu$ vanishes identically.

A remark about the index n is in order. Here, we regard this integer merely as a book-keeping label of the modes and not as a parameter explicitly related to the modal structure of the eigenfunctions. In this way acoustic and gravity modes – if we want to include the latter in a numerical treatment – are just distinguished by different values of n. Of course the finite truncation of the Woltjer series (21) can be carried out following a scheme different from (23); it may prove convenient to choose different numbers N of n-labels for different degrees l; except for a change in the number of degrees of freedom (24) such a choice would not alter the technique of this paper.

To avoid notational complications through a flurry of subscripts, we shall use, in the remainder of this paper, a single (Greek) subscript in expressions such as (21, 22, 23). The generalized coordinates $q_{nlm}(t)$ will thus be written $q_\alpha(t)$.

4. Expansion of the total energy of the star

Assuming smoothness of the energy components with respect to the fields δr and $\delta\mu$, we can expand the kinetic and potential energy in the generalized coordinates $q_\alpha(t)$. To keep track of the order of the expansion, we use, as in Papers I and II, the book-keeping parameter ε, which will be set equal to 1 at the end of the calculations. Thus we rewrite Eqs. (15) and (19) in the form

$$\begin{pmatrix}r(s,\theta,\phi;t)\\ \mu(s,\theta,\phi;t)\end{pmatrix}=\begin{pmatrix}r_0(s)\\ \mu_0(s)\end{pmatrix}+\varepsilon\begin{pmatrix}\delta r(s,\theta,\phi;t)\\ \delta\mu(s,\theta,\phi;t)\end{pmatrix}. \tag{25}$$

To generate the lowest order non-linear equations of motion, the energy components must be expanded up to ε^3. In an order of increasing complexity these energies are the internal (or thermal) energy, the gravitational energy, and the kinetic energy. Only ε^2 and ε^3 terms need to be considered; the zero order term representing the equilibrium energy is independent of the generalized coordinates; the first order term must vanish for stability reasons (see Paper I). The ε^2 terms in the total Lagrangian lead to the linearized equations of motion, while the lowest non-linear contribution to the equations of motion arises from the ε^3 terms in the Lagrangian.

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We insert the Woltjer expansion (21) into Eq. (25), substitute the result into Eqs. (3, 8, 9), expand in a Taylor series in ε, and obtain a Lagrangian in the form (1, 2, 5, 6)

$$L=\varepsilon^2\left(\frac{1}{2}\sum_\alpha \dot{q}_\alpha^2-\frac{1}{2}\sum_\alpha \omega_\alpha^2 q_\alpha^2\right)+$$

$$+\varepsilon^3\left(\sum_{\alpha\beta\gamma} K_{\alpha\beta\gamma}q_\alpha\dot{q}_\beta\dot{q}_\gamma-\sum_{\alpha\beta\gamma} V_{\alpha\beta\gamma}q_\alpha q_\beta q_\gamma\right). \tag{26}$$

In this expression, the coefficients $K_{\alpha\beta\gamma}$ are symmetric in (β,γ) and $V_{\alpha\beta\gamma}$ in (α,β,γ).

4.1. Total potential energy

We split the total potential energy up into a thermal and gravitational part

$$V_{\alpha\beta\gamma}=V^{\text{th}}_{\alpha\beta\gamma}+V^{\text{gr}}_{\alpha\beta\gamma}. \tag{27}$$

For numerical purposes it is advantageous to express the coefficients of the total energy in terms of the radial component Ξ_α and the horizontal component H_α of the displacement eigenfunction (Y_α being a real spherical harmonic, see Appendix 2)

$$\delta r_\alpha=Re\left\{\left[\Xi_\alpha(r)e_r Y_\alpha+H_\alpha(r)\left(\left[\frac{\partial Y_\alpha}{\partial\theta}\right]e_\theta+\frac{1}{\sin\theta}\left[\frac{\partial Y_\alpha}{\partial\phi}\right]e_\phi\right)\right]e^{i\omega t}\right\}. \tag{28}$$

This representation is adopted, for instance, by Christensen-Dalsgaard (1981, 1982). As shown in Appendix 1 [Eq. (A.31)], Ξ_α and H_α directly relate to our basic fields δr and $\delta\mu$:

$\Xi_\alpha=\delta r_\alpha[s_0(r)]$ and $H_\alpha=-r\,\delta\mu_\alpha[s_0(r)]/[l_\alpha(l_\alpha+1)\mu_0]$

(l_α being the degree l of the multi-index α). Since Ξ_α and H_α are usually computed as functions of r, it is convenient to transform all integrals over s back into integrals over r, writing, for any function Q

$$\int_{s_c}^{s_s} ds\,\mu_0(s)\,Q(s)=4\pi\int_0^R dr\,r^2\varrho_0(r)\,Q(r). \tag{29}$$

The coefficients of the total thermal energy become

$$\left.\begin{aligned}V^{\text{th}}_{\alpha\beta\gamma}=Z_{\alpha\beta\gamma}\cdot\Bigg\{&-\frac{1}{6}\int_0^R dr\,r^2 p\left[\Gamma(\Gamma+1)+\left(\frac{\partial\Gamma}{\partial\ln\varrho}\right)_s\right]A_\alpha A_\beta A_\gamma\\&-\int_0^R dr\,p\,\Xi_\alpha\Xi_\beta\Xi'_\gamma+\frac{\Lambda_\gamma^2}{2}\int_0^R dr\,rp\Gamma A_\alpha A_\beta H_\gamma\\&+\int_0^R dr\,p\Gamma\,\Xi_\alpha\Xi_\beta A_\gamma+2\int_0^R dr\,rp\Gamma A_\alpha\Xi_\beta\Xi'_\gamma\Bigg\}.\end{aligned}\right\} \tag{30}$$

Here, A_α is the relative change of the specific volume of mode α

$$\left.\begin{aligned}A_\alpha(r)&=-\frac{\delta\varrho_\alpha}{\varrho_0}=-\frac{\delta\mu_\alpha}{\mu_0}+\frac{2\delta r_\alpha}{r_0}+\frac{\delta r'_\alpha}{r'_0}\\&=\frac{\Lambda_\alpha}{r}H_\alpha(r)+\frac{2\Xi_\alpha(r)}{r}+\Xi'_\alpha(r).\end{aligned}\right\} \tag{31}$$

$Z_{\alpha\beta\gamma}$ denotes the angular part of the integral (Appendix 2), $\Lambda_\alpha=l_\alpha(l_\alpha+1)$, $\Gamma=(\partial\ln p/\partial\ln\varrho)_s$, R is the radius of the star, and primes denote derivatives with respect to the space variable. Note that the coefficients (30) do not have the symmetry properties implied by (26) (their structure would be more complicated in a fully symmetric form). Of course, any antisymmetric part will disappear after insertion into (26).

The coefficients of the total gravitational energy are (technical details are relegated to appendix 4)

$$\left.\begin{aligned}
V^{\text{gr}}_{\alpha\beta\gamma}&=-\frac{G}{2}\sum_{j=1}^{5}(G^{(j)+}_{\alpha\beta\gamma}+G^{(j)-}_{\alpha\beta\gamma})\cdot Z_{\alpha\beta\gamma}\\
G^{(1)+}_{\alpha\beta\gamma}&=\frac{1}{3}l_\gamma(l_\gamma-1)(l_\gamma-2)\int_0^R dr\,F_{-l_\gamma+1)}(r)\int_0^r dr'\,F_{l_\gamma+1}(r)\,\Xi_\alpha(r')\,\Xi_\beta(r')\,\Xi_\gamma(r')\\
G^{(1)-}_{\alpha\beta\gamma}&=-\frac{1}{3}(l_\gamma+1)(l_\gamma+2)(l_\gamma+3)\int_0^R dr\,F_{l_\gamma+2}(r)\int_r^R dr'\,F_{-(l_\gamma+2)}(r')\,\Xi_\alpha(r')\,\Xi_\beta(r')\,\Xi_\gamma(r')\\
G^{(2)+}_{\alpha\beta\gamma}&=\Lambda_\gamma^2(l_\gamma+2)\int_0^R dr\,F_{-(l_\gamma+1)}(r)\Xi_\alpha(r)\Xi_\beta(r)\int_0^r dr'\,F_{l_\gamma+1}(r')\Xi_\gamma(r')\\
G^{(2)-}_{\alpha\beta\gamma}&=-\Lambda_\gamma^2(l_\gamma-1)\int_0^R dr\,F_{l_\gamma}(r)\Xi_\alpha(r)\Xi_\beta(r)\int_r^R dr'\,F_{-l_\gamma}(r')\Xi_\gamma(r')\\
G^{(3)+}_{\alpha\beta\gamma}&=-2\Lambda_\alpha^2 l_\alpha(l_\alpha-1)\int_0^R dr\,F_{-l_\alpha}(r)H_\alpha(r)\int_0^r dr'\,F_{l_\alpha}(r')\Xi_\beta(r')\Xi_\gamma(r')\\
G^{(3)-}_{\alpha\beta\gamma}&=-2\Lambda_\alpha^4\int_0^R dr\,F_{l_\alpha+1}(r)H_\alpha(r)\int_r^R dr'\,F_{-(l_\alpha+1)}(r')\Xi_\beta(r')\Xi_\gamma(r')\\
G^{(4)+}_{\alpha\beta\gamma}&=2\Lambda_\alpha^2 l_\alpha\Lambda_\beta^2\int_0^R dr\,F_{-l_\alpha}(r)H_\alpha(r)\int_0^r dr'\,F_{l_\alpha}(r')H_\beta(r')\Xi_\gamma(r')\\
G^{(4)-}_{\alpha\beta\gamma}&=-2\Lambda_\alpha^2(l_\alpha+1)\Lambda_\beta^2\int_0^R dr\,F_{l_\alpha+1}(r)H_\alpha(r)\int_r^R dr'\,F_{-(l_\alpha+1)}(r')H_\beta(r')\Xi_\gamma(r')\\
G^{(5)+}_{\alpha\beta\gamma}&=2\Lambda_\alpha^2\Lambda_\beta^2\int_0^R dr\,F_{-(l_\gamma+1)}(r)H_\alpha(r)\Xi_\beta(r)\int_0^r dr'\,F_{l_\gamma+1}(r')\Xi_\gamma(r')\\
G^{(5)-}_{\alpha\beta\gamma}&=2\Lambda_\alpha^2\Lambda_\gamma^2\int_0^R dr\,F_{l_\gamma}(r)H_\alpha(r)\Xi_\beta(r)\int_r^R dr'\,F_{-l_\gamma}(r')\Xi_\gamma(r')\\
F_l(r)&=\varrho_0(r)\,r^l.
\end{aligned}\right\} \tag{32}$$

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Here, the notation is as in (30), and again an asymmetric representation of the coefficients has been adopted.

4.2. Total kinetic energy

Technical aspects of the calculation of the coefficients are discussed in Appendix 1. Here, we merely quote the result. The factors $Z_{\alpha\beta\gamma}$ and $\hat{Z}_{\alpha\beta\gamma}$ result from the angular part of the integral (Appendix 2). As for the other energy components, the structure of the kinetic energy coefficients $K_{\alpha\beta\gamma}$ is presented in an asymmetric form. The antisymmetric part in the indices (β, γ) will again disappear when the coefficients are inserted into (26).

$$\left.\begin{aligned} K_{\alpha\beta\gamma} = &-\frac{1}{2}\Lambda_\alpha^2 Z_{\alpha\beta\gamma} T^{(2)}_{\gamma\beta\alpha} + \frac{1}{2}\Lambda_\alpha^2 \hat{Z}_{\alpha\beta\gamma} T^{(1)}_{\alpha\beta\gamma} - \hat{Z}_{\beta\alpha\gamma}(T^{(4)}_{\beta\alpha\gamma} + T^{(7)}_{\alpha\beta\gamma}) \\ &- \Lambda_\alpha^2 Z_{\alpha\beta\gamma}(T^{(4)}_{\alpha\beta\gamma} + T^{(8)}_{\alpha\beta\gamma}) + \hat{Z}_{\alpha\beta\gamma} T^{(2)}_{\alpha\beta\gamma} + Z_{\alpha\beta\gamma}\,(2\Lambda_\beta^2 T^{(4)}_{\beta\alpha\gamma} \\ &+ 2\Lambda_\alpha^2 T^{(4)}_{\alpha\beta\gamma} + \Lambda_\beta^2 T^{(6)}_{\alpha\beta\gamma} + \Lambda_\alpha^2 T^{(5)}_{\alpha\beta\gamma} + 6 T^{(2)}_{\alpha\beta\gamma} + 2 T^{(9)}_{\alpha\beta\gamma} \\ &+ 2 T^{(9)}_{\beta\alpha\gamma} + 2 T^{(3)}_{\alpha\beta\gamma}) \\ T^{(1)}_{\alpha\beta\gamma} &= \int_0^R dr\, F_1 \cdot H_\alpha H_\beta H_\gamma; \qquad T^{(2)}_{\alpha\beta\gamma} = \int_0^R dr\, F_1 \Xi_\alpha \Xi_\beta H_\gamma \\ T^{(3)}_{\alpha\beta\gamma} &= \int_0^R dr\, F_3 \Xi'_\alpha \Xi'_\beta H_\gamma; \qquad T^{(4)}_{\alpha\beta\gamma} = \int_0^R dr\, F_1 H_\alpha \Xi_\beta H_\gamma \\ T^{(5)}_{\alpha\beta\gamma} &= \int_0^R dr\, F_2 H_\alpha \Xi'_\beta H_\gamma; \qquad T^{(6)}_{\alpha\beta\gamma} = \int_0^R dr\, F_2 \Xi'_\alpha H_\beta H_\gamma \\ T^{(7)}_{\alpha\beta\gamma} &= \int_0^R dr\, F_2 \Xi_\alpha H'_\beta H_\gamma; \qquad T^{(8)}_{\alpha\beta\gamma} = \int_0^R dr\, F_2 H_\alpha \Xi_\beta H'_\gamma \\ T^{(9)}_{\alpha\beta\gamma} &= \int_0^R dr\, F_2 \Xi'_\alpha \Xi_\beta H_\gamma; \qquad F_n(r) = \varrho_0(r)\, r^n . \end{aligned}\right\} \quad (33)$$

4.3. Hamiltonian function

The Hamiltonian corresponding to the Lagrangian (26) is obtained by the Legendre transformation

$$H(\boldsymbol{p},\boldsymbol{q}) = \varepsilon^2 \boldsymbol{p}\cdot\dot{\boldsymbol{q}} - L(\boldsymbol{q},\dot{\boldsymbol{q}}). \quad (34)$$

The momenta conjugate to $\boldsymbol{q}$ are

$$\boldsymbol{p} = \frac{1}{\varepsilon^2}\left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\right). \quad (35)$$

Here, the insertion of a factor ε^{-2} arranges that $\boldsymbol{p}$ is of same order as $\dot{\boldsymbol{q}}$. For the Hamiltonian function, $\dot{\boldsymbol{q}}$ must be known as a function of $\boldsymbol{p}$ and $\boldsymbol{q}$. The first order correction to $\dot{\boldsymbol{q}}$ is sufficient in order to get the Hamiltonian function to third order. Fortunately, this first order correction cancels out in the final result, and need therefore not be known explicitly. Setting

$$\dot{\boldsymbol{q}} = \boldsymbol{p} + \varepsilon \dot{\boldsymbol{q}}^{(1)}, \quad (36)$$

we obtain in fact [using (35)]

$$\begin{aligned} H(\boldsymbol{p},\boldsymbol{q}) &= \varepsilon^2 \boldsymbol{p}^2 + \varepsilon^3 \boldsymbol{p}\dot{\boldsymbol{q}}^{(1)} - L(\boldsymbol{q},\boldsymbol{p}) - \varepsilon\dot{\boldsymbol{q}}^{(1)} \cdot \left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\right)_{\dot{\boldsymbol{q}}=\boldsymbol{p}} + O(\varepsilon^4) \\ &= \varepsilon^2 \boldsymbol{p}^2 - L(\boldsymbol{q},\boldsymbol{p}) + O(\varepsilon^4), \end{aligned}$$

which is independent of $\boldsymbol{q}^{(1)}$; in the notation of (26),

$$\begin{aligned} H(\boldsymbol{p},\boldsymbol{q}) = \varepsilon^2 &\left\{\frac{1}{2}\sum_{j=1}^{F} p_j^2 + \frac{1}{2}\sum_{j=1}^{F} \omega_j^2 q_j^2\right\} + \\ &+ \varepsilon^3 \left\{-\sum_{j,k,l=1}^{F} K_{jkl}\, q_j p_k p_l + \sum_{j,k,l=1}^{F} V_{jkl}\, q_j q_k q_l\right\}. \end{aligned} \quad (37)$$

5. Conclusion and outlook

In this paper we have developed a mode-coupling formalism capturing an important class of non-linear, non-radial adiabatic stellar motions, namely genuine oscillations about a dynamically stable equilibrium state. Just as in the case of radial oscillations (Papers I and II), this formalism is generated through a variational principle. Such an approach has the following main advantages over the conventional method, in which an expansion of the stellar fields in the eigenfunctions is directly substituted into the full hydrodynamic equations. (i) As in the case of purely radial motions, the variational setting automatically unfolds the Hamiltonian structure of the resulting equations of motion. Besides the manifest algorithmic relevance of this property (conservation of oscillation energy serves as a simple test for the numerical integration), the mere realization of the Hamiltonian nature of the oscillations conveys a non-trivial piece of theoretical information on the character of the admitted motions. If we assume that the underlying Hamiltonian is "generic" (i.e. possesses no special properties that would be destroyed if it were slightly disturbed), then we are sure that in addition to regular (periodic and multi-periodic) pulsations the non-radial (bounded) motions of the star also display chaotic oscillations, provided that the oscillation amplitude, or equivalently, the oscillation energy, is high enough (see Berry, 1978). (To find out, how large the critical energy is beyond which these oscillations arise, we need of course to investigate the Hamiltonian closer, either analytically or numerically.) (ii) Also as in the case of purely radial motions, an appropriate choice of the independent and dependent variables has enabled us to build conservation of mass and entropy directly into the variational formulation. (iii) To account for specifically non-radial properties, the formalism has been so designed as strictly to preserve the zero circulation (13) along any closed loop lying on a constant-entropy shell. In this way, the formalism suppresses all kinds of stationary motions. (iv) Likewise, the particular selection of the two dependent fields, the radial displacement field δr and the stratification field $\delta\mu$, enables us to avoid introducing *toroidal modes* in the representation of the velocity field without throwing away non-linear *toroidal velocity field* components. Unlike the treatment by previous authors (Dziembowski, 1982; Buchler and Regev, 1983), which discards the appearance of toroidal velocity fields at the outset, our formalism has the advantage of taking care of (non-linear) toroidal velocity components if included in the initial conditions (and if compatible with the zero-circulation (13)); or it may generate toroidal velocity components in the course of the non-linear evolution.

In our non-radial treatment, we have opted for specific entropy s as the independent space variable. Entropy thus replaces the mass variable $m(r)$ of the radial variational formulation. This asymmetry with the radial problem was dictated by an apparent algebraic simplification of the formalism rather than by an intrinsic necessity. The very fact that all final formulae for the coupling coefficients can be rewritten in terms of conventional

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polar coordinates (r, θ, ϕ) instead of (s, θ, ϕ) indicates however that the choice of the independent variables is largely a matter of taste.

It may be objected that the variational approach presented here suffers a serious drawback, not shared by the conventional direct method, when one wants to extend it to realistic non-adiabatic oscillations. In fact, in the presence of dissipation, the variational principle (7) ceases to describe the motion. However, as stressed elsewhere (Perdang, 1984a, b), non-adiabatic effects may be dealt with, provided that the characteristic time over which dissipation is operating is sufficiently longer than the dynamical time scale; appropriate techniques are the standard asymptotic 2-time method (or an improved version thereof) or the averaging procedure. If this condition is satisfied, then the lowest-order short-time equations precisely coincide with the adiabatic equations. If the adiabatic solutions are regular, an asymptotically rigorous scheme exists to compute the long-term amplitude variations. If the adiabatic solutions are chaotic, then the recipe proposed in Perdang (1984a, b) provides an approximate solution to this problem. We wish to emphasize in passing that the proponents of the conventional approach (Dziembowski, 1982; Buchler and Regev, 1983) have so far discarded short-time chaotic solutions altogether.

The next move is a systematic investigation of the character of the non-linear, non-radial oscillations. We believe in fact that prior to any application of this formalism to specific stellar models several points need to be clarified.

(1) How does the surface pattern evolve, as the star is undergoing non-radial chaotic oscillations? How does the variability of this pattern compare with the evolution of a regular non-linear oscillation surface pattern? Can one trace simple morphological features capable of revealing a chaotic behaviour (e.g. a time-run of correlations between different surface points etc.), which would be turned into a practical test of differentiating non-radial chaos from regular motions.?

(2) How do the critical amplitude levels, at which non-linear effects become noticeable and at which chaotic oscillations set in, vary with the number of coupled non-radial modes; more specifically, what is the influence of the degrees l and the azimuthal quantum numbers m of the coupled modes on the onset of chaos? This question, besides its theoretical relevance, is of direct interest in testing the suggestion that the 160 min solar oscillation is a non-linear many-mode coupling effect (Perdang and Blacher, 1984b; Däppen and Perdang, 1984). Regarding the latter point, a preliminary numerical experiment has already been performed using the formalism of the present paper (Däppen, 1984). It demonstrates – as expected theoretically – the actual occurrence of non-radial chaos and a reduction to $\lesssim 1.5\%$ of the critical relative surface-amplitude level as a result of the coupling of radial modes ($n = 22, 23, 24$) with a few non-radial modes ($l = 1, n = 22, 23$; $l = 2, n = 21, 22$).

(3) How does the coupling between (dynamically stable) *gravity modes* and *acoustic modes* manifest itself? Are chaotic oscillations of reasonable surface amplitudes allowed that are the result of the interaction between modes of these two classes and not of the interaction between modes of each class only?

Finally, we should like to point out that the variational formalism developed in this paper lends itself to a systematic extension covering simple stationary flow fields such as rotation or constant vorticity.

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Appendix 1: velocity field and kinetic energy

1. Justification of the Clebsch-type form of the velocity field

We briefly show that the radial displacement $\delta r(s, \theta, \phi; t)$ and the stratification disturbance $\delta\mu(s, \theta, \phi; t)$ uniquely determine the free functions $a(\boldsymbol{r}, t)$ and $b(\boldsymbol{r}, t)$ of the velocity field written in the form (14). The function $b(\boldsymbol{r}, t)$ is determined by the local form of the entropy conservation condition,

$$\dot{s} + \boldsymbol{u} \cdot \nabla s = 0, \tag{A.1}$$

(in the following, the dot denotes partial derivative with respect to time). Substituting Eq. (14) into (A.1), we obtain the functional form of $b(\boldsymbol{r}, t)$, which is seen to be single-valued. We then have

$$\boldsymbol{u}(\boldsymbol{r}, t) = -[\dot{s}/(\nabla s)^2]\,\nabla s + \nabla^* a \tag{A.2}$$

where $\nabla^* a$ stands for the projection of ∇a onto the surface of constant entropy

$$\nabla^* a = \nabla a - [\nabla a \cdot \nabla s/(\nabla s)^2]\,\nabla s\,. \tag{A.3}$$

This leaves us with a single unknown field $a(\boldsymbol{r}, t)$ which is required to be single-valued (see Sect. 3). To determine $a(\boldsymbol{r}, t)$, we substitute Eq. (A.2) into the local mass conservation equation

$$\dot{\varrho} + \mathrm{div}(\varrho \boldsymbol{u}) = 0\,. \tag{A.4}$$

We then obtain a linear partial differential equation for the unknown field $a(r, t)$

$$\mathrm{div}(\varrho\,\nabla^* a) = -\dot{\varrho} + \mathrm{div}[\dot{s}\,\nabla s/(\nabla s)^2]\,. \tag{A.5}$$

Lynden-Bell and Katz (1981) have shown that this equation has a unique solution $a(\boldsymbol{r}, t)$ (under broader conditions than required in our case), provided that $\varrho(\boldsymbol{r}, t)$ and $s(\boldsymbol{r}, t)$ are given. But the latter fields are directly obtained from the radial position $r(s, \theta, \phi; t)$ and the stratification field $\mu(s, \theta, \phi; t)$. Hence, the knowledge of the free scalar fields r and ϱ (or δr and $\delta\mu$) secures a unique velocity field in the Clebsch-type representation (14).

2. The field $a(\boldsymbol{r}, t)$ as an explicit function of δr and $\delta\mu$

We shall write the ε-expansion in the following form, a subscript 0 referring to the equilibrium state

$$\left.\begin{aligned} \varrho &= \varrho_0 + \varepsilon\varrho_1 + \varepsilon^2\varrho_2 + \ldots \\ s &= s_0 + \varepsilon s_1 + \varepsilon^2 s_2 + \ldots \\ a &= \varepsilon a_1 + \varepsilon^2 a_2 + \ldots \\ b &= \varepsilon b_1 + \varepsilon^2 b_2 + \ldots \quad . \end{aligned}\right\} \tag{A.6}$$

Here, two remarks are in order. First, unless otherwise stated, an Eulerian picture is used, i.e. s, ϱ, a, b are regarded as functions of $\boldsymbol{r}$ and t. Second, a and b have no stationary components (see Sect. 3), and therefore their expansions begin with a first-order term. We also introduce the "tangential" operator [the notation is borrowed from Dziembowski (1982)]

$$\nabla_H = \left(0, \frac{\partial}{\partial\theta}, \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right). \tag{A.7}$$

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To obtain an explicit solution of Eq. (A.5), we expand the latter equation in powers of ε, using (A.6). As shown in Sect. 3 of this appendix, only a_1 will be needed in the coupling coefficients that describe the first non-linear correction to the equations of motion. Identifying first order coefficients, we have

$$\operatorname{div}\left(\varrho_0 \frac{1}{r} \nabla_H a_1\right) = -\dot{\varrho}_1 + \operatorname{div}\,[\dot{s}_1 e_r / s_0'(r)]\,. \tag{A.8}$$

The left-hand side of this equation becomes, noting that $\varrho_0(\boldsymbol{r}) = \varrho_0(|\boldsymbol{r}|)$,

$$\operatorname{div}\left(\varrho_0 \frac{1}{r} \nabla_H a_1\right) = \varrho_0 \hat{\Delta} a_1\,. \tag{A.9}$$

Here, $\hat{\Delta}$ denotes the angular part of the Laplace operator, obeying

$$\hat{\Delta} Y_{lm} = -l(l+1)\, Y_{lm}/r^2\,. \tag{A.10}$$

The right-hand side of (A.8) is more complicated. First, ϱ_1 has to be expressed as a function of μ, using Eq. (16) to first order of ε. Next, $\operatorname{div}(\varrho_0 e_r \dot{s}_1/s_0)$ is evaluated again through Eq. (16). We finally obtain

$$\hat{\Delta} a_1 = -\frac{1}{\mu_0}\left(\frac{\partial \delta\mu}{\partial t}\right)_s\,. \tag{A.11}$$

In sectors of constant angular momentum, this equation is solved explicitly. The Woltjer-type Eq. (21) and an expansion of a_1 in spherical harmonics

$$a_1 = \sum_{l,m} a_{lm}^{(1)}(r,t)\; Y_{lm}(\theta,\phi)\,, \tag{A.12}$$

together with property (A.10), lead to

$$a_{lm}^{(1)}(r,t) = \frac{-r^2}{l(l+1)} \sum_n \dot{q}_{nlm}(t)\, \delta\mu_{nlm}\,[s_0(r)]/\mu_0\,. \tag{A.13}$$

For $l=0$, this expression is undetermined. It transpires, however, from the Clebsch representation (A.2) that without loss we can set $a_{00}^{(1)} \equiv 0$.

3. *Expansion of the total kinetic energy of the star*

In terms of the scalar fields a and b, the kinetic energy becomes

$$K = \frac{1}{2}\int d^3r\, \varrho u^2 = \frac{1}{2}\int d^3r\, \frac{\varrho \dot{s}^2}{(\nabla s)^2} + \frac{1}{2}\int d^3r\, \varrho(\nabla^* a)^2 \equiv K_s + K_a \tag{A.14}$$

(all integrals are over the moving volume $\mathcal{V}(t)$ of the star). To obtain this expression we have taken account of the orthogonality of ∇s and $\nabla^* a$.

Since $(\nabla s)^2 = (s')^2 + O(\varepsilon^2)$ (s', partial derivative of $s(r,\theta,\phi;t)$ with respect to r) we have

$$K_s = \frac{1}{2}\int d^3r\, \varrho \left(\frac{\dot{s}}{s'}\right)^2 + O(\varepsilon^4)\,. \tag{A.15}$$

$\dot{s}/s'$ is readily expressed in terms of our δr-field:

$$\frac{\dot{s}}{s'} = -\left(\frac{\partial r}{\partial t}\right)_s = -\varepsilon\left(\frac{\partial \delta r}{\partial t}\right)_{s,\theta,\phi}\,. \tag{A.16}$$

(A.15) and (A.16) yield the third-order contribution to K_s

$$K_s^{(3)} = \frac{1}{2}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \int_{s_c}^{s_s} ds\, \delta\mu(s,\theta,\phi;t) \left(\frac{\partial \delta r}{\partial t}\right)^2_{s,\theta,\phi}\,. \tag{A.17}$$

Using (A.3, A.7), one can write K_a in the form

$$K_a = \varepsilon^2 \frac{1}{2}\int d^3r\, \frac{\varrho_0}{r^2}(\nabla_H a_1)^2 + \varepsilon^3 \left\{\frac{1}{2}\int d^3r\, \frac{\varrho_1}{r^2}(\nabla_H a_1)^2 + \int d^3r\, \frac{\varrho_0}{r^2} \nabla_H a_1 \cdot \nabla_H a_2 - \int d^3r\, \frac{\varrho_0}{r} (\nabla_H a_1 \cdot \boldsymbol{\sigma}_H)(\nabla a_1 \cdot e_r)\right\} \tag{A.18}$$

Here, we have introduced the dimensionless vector $\boldsymbol{\sigma}_H$ of the deviation fo ∇s from its equilibrium value ∇s_0. We define this vector implicitly by

$$\nabla s = s'(e_r + \varepsilon \boldsymbol{\sigma}_H) + O(\varepsilon^2)\,. \tag{A.19}$$

The most difficult component of K_a is the third integral in (A.18) involving a_1 and a_2 together. One could obtain it by a second iteration of equation (A.5), a quite tedious task. Fortunately, we do not need to know a_2 explicitly. Instead, Gauss's divergence theorem can be used to reduce the integral $\int d^3r\, \varrho_0 \nabla_H a_1 \cdot \nabla_H a_2$ to an expression involving only a_1 and ϱ_2.
To do so, we start out with observing that

$$\int d^3r\, \operatorname{div}(a_1 \varrho \boldsymbol{u}) = 0\,. \tag{A.20}$$

(Note that there is no normal flux component through the moving boundary.) Pulling out a_1 from the divergence we obtain, taking account of the continuity equation (A.4)

$$0 = \int d^3r\, a_1 \dot{\varrho} + \int d^3r\, \frac{\dot{s}\varrho}{(\nabla s)^2} \nabla a_1 \cdot \nabla s - \int d^3r\, \varrho \nabla a_1 \cdot \nabla^* a\,. \tag{A.21}$$

If we expand this equation in a power series of ε, then the integral $\int d^3r\, \varrho_0 \nabla_H a_1 \cdot \nabla_H a_2$ appears in the third order coefficient (being zero); therefore the latter generates a new expression for this integral. To this end we expand $\nabla^* a$ in a power series of ε

$$\nabla^* a = \varepsilon \frac{1}{r} \nabla_H a_1 + \varepsilon^2 \left\{\frac{1}{r}[\nabla_H a_2 - (\nabla_H a_1 \cdot \boldsymbol{\sigma}_H) e_r] - (\nabla a_1 \cdot e_r)\, \boldsymbol{\sigma}_H\right\}. \tag{A.22}$$

Equation (A.21) then becomes

$$\begin{aligned} 0 = \varepsilon^2 &\left\{\int d^3r\, \frac{\varrho_0}{r^2}(\nabla_H a_1)^2 + \int d^3r\, \varrho_0 \left(\frac{\partial \delta r}{\partial t}\right)_s (\nabla a_1 \cdot e_r) - \int d^3r\, a_1 \dot{\varrho}_1\right\} \\ + \varepsilon^3 &\left\{\int d^3r\, \frac{\varrho_0}{r}(\nabla a_1 \cdot \nabla_H a_2) \right. \\ &+ \int d^3r\, \frac{\varrho_1}{r^2}(\nabla_H a_1)^2 - 2\int d^3r\, \frac{\varrho_0}{r}(\nabla a_1 \cdot e_r)(\nabla_H a_1 \cdot \boldsymbol{\sigma}_H) \\ &+ \int d^3r\, \varrho_1 \left(\frac{\partial \delta r}{\partial t}\right)_s (\nabla a_1 \cdot e_r) \\ &\left. + \int d^3r\, \frac{\varrho_0}{r}\left(\frac{\partial \delta r}{\partial t}\right)_s (\nabla_H a_1 \cdot \boldsymbol{\sigma}_H) - \int d^3r\, a_1 \dot{\varrho}_2\right\}. \end{aligned} \tag{A.23}$$

Since $\nabla a_1 \cdot \nabla a_2 = \nabla_H a_1 \cdot \nabla_H a_2$, the integral $\int d^3r\, \varrho_0 \nabla_H a_1 \cdot \nabla_H a_2$ can now be read off from the third-order coefficient. Therefore

$$\begin{aligned} K_a = \frac{\varepsilon^2}{2}\int d^3r\, \frac{\varrho_0}{r^2}(\nabla_H a_1)^2 + \varepsilon^3 &\left\{-\frac{1}{2}\int d^3r\, \frac{\varrho_1}{r^2}(\nabla_H a_1)^2 \right. \\ &+ \int d^3r\, \frac{\varrho_0}{r}(\nabla_H a_1 \cdot \nabla_H)(\nabla a_1 \cdot e_r) - \int d^3r\, \varrho_1 \left(\frac{\partial \delta r}{\partial t}\right)_s (\nabla a_1 \cdot e_r) \\ &\left. - \int d^3r\, \frac{\varrho_0}{r}\left(\frac{\partial \delta r}{\partial t}\right)_s (\nabla_H a_1 \cdot \boldsymbol{\sigma}_H) + \int d^3r\, a_1 \cdot \dot{\varrho}_2\right\}. \end{aligned} \tag{A.24}$$

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All components of (A.24) are straightforward except the last integral, which involves $\dot{\varrho}_2$. One could compute $\dot{\varrho}_2$ from the defining Eq. (16), but this would again be a tedious manipulation. A faster computation of the integral $\int d^3r\,\dot{\varrho}_2 a_1$ consists in using the Woltjer-type expansion for a_1, so that the time derivatives can be performed in the coefficients outside the integral. More precisely, with the expansion

$$a_1 = \sum \dot{q}_\alpha(t)\, a_\alpha^{(1)}(r,\theta,\phi) \tag{A.25}$$

we obtain (the contribution from the derivative of the moving boundary being of order ε^4)

$$\int d^3r a_1 \dot{\varrho}_2 = \sum_\alpha \dot{q}_\alpha(t) \int d^3r\,\dot{\varrho}_2 a_\alpha^{(1)} = \sum_\alpha \dot{q}_\alpha(t)\, \frac{d}{dt} \int d^3r\,\varrho_2 a_\alpha^{(1)} + O(\varepsilon^4) \tag{A.26}$$

The function ϱ_2 in turn is now computed from the expansion of (16). The result is $[\delta r' \equiv (\partial\delta r/\partial s)_{\theta,\phi,t}$ etc.]

$$\frac{\varrho_2}{\varrho_0} = -2\frac{\delta\mu}{\mu_0}\frac{\delta r}{r_0} - \frac{\delta\mu}{\mu_0}\frac{\delta r'}{r_0'} + 3\left(\frac{\delta r}{r_0}\right)^2 + 2\frac{\delta r}{r_0}\cdot\frac{\delta r'}{r_0'} + \left(\frac{\delta r'}{r_0'}\right)^2 \tag{A.27}$$

Therefore, K_α can be expressed in terms of δr and $\delta\mu$. The calculation of the coefficients of the total kinetic energy (33) is now straightforward. The matrix elements $Z_{\alpha\beta\gamma}$ and $\hat{Z}_{\alpha\beta\gamma}$ appearing in (33) are defined in Appendix 2.

4. Expression of the fields δr and $\delta\mu$ in the linear approximation in terms of the usual radial and horizontal component of the velocity eigenfunctions

To establish the connection between the components δr_α and $\delta\mu_\alpha$ of the eigenfunctions of mode α on the one hand, and Ξ_α and H_α [Eq. (28)] on the other hand, we write the velocity field $\boldsymbol{u}$ to first order (see A.16, A.23)

$$\boldsymbol{u} = \left(\frac{\partial\delta r}{\partial t}\right)_s \boldsymbol{e}_r + \frac{1}{r}\nabla_H a_1\,. \tag{A.28}$$

Inserting the Woltjer-type expression (21) and expanding δr and a_1 in spherical harmonics, we obtain

$$\boldsymbol{u} = \sum_\alpha \dot{q}_\alpha(t) \left\{\delta r_\alpha[s_0(r)]\, Y_\alpha \boldsymbol{e}_r + \frac{1}{r} a_\alpha^{(1)} \nabla_H Y_\alpha\right\}. \tag{A.29}$$

This expression has to be compared with the velocity as obtained from representation (28)

$$\boldsymbol{u} = \sum_\alpha \dot{q}_\alpha(t)\; [\Xi_\alpha(r)\, Y_\alpha \boldsymbol{e}_r + H_\alpha(r)\, \nabla_H Y_\alpha] \tag{A.30}$$

Insertion of (A.13) into (A.14) yields

$$\Xi_\alpha(r) = \delta r_\alpha[s_0(r)]\,,$$
$$H_\alpha(r) = -\frac{r}{l_\alpha(l_\alpha+1)}\,\frac{\delta\mu_\alpha[s_0(r)]}{\mu_0}\,. \tag{A.31}$$

Appendix 2: angular integrals and selection rules

1. Spherical harmonics. Notations

In the context of non-linear adiabatic mode coupling, the expansion in complex eigenfunctions, requiring complex amplitudes $q(t)$, unnecessarily increases the numerical effort in solving the amplitude equations. Therefore, we have here selected *real* eigenfunctions and, in particular, following Morse and Feshbach (1953), we have adopted a set of *real* spherical harmonics defined as follows

$$Y_{lm}^{(r)} = \left\{\begin{array}{l} P_l^{|m|}(\cos\theta)\,\cos(m\phi)\;(0 \leqq m \leqq l) \\ P_l^{|m|}(\cos\theta)\,\sin(m\phi)\;(-l \leqq m < 0)\,. \end{array}\right\} \tag{A.32}$$

With this definition we have

$$\int_0^{2\pi} d\phi \int_0^{\pi} d\theta\,\sin\theta\,[Y_{lm}^{(r)}(\theta,\phi)]^2 = \frac{4\pi}{\varepsilon_l\cdot(2l+1)}\cdot\frac{(l+|m|)!}{(l-|m|)!}, \tag{A.33}$$

where $\varepsilon_0 = 1$, and $\varepsilon_l = 2$, $l = 1, 2, 3 \ldots$.

In order to discuss selection rules and to evaluate the angular part of the coupling coefficients, the usual set of normalized complex spherical harmonics proves more useful

$$Y_{lm}^{(c)} = \sqrt{\frac{(2l+1)\cdot(l-m)!}{4\pi\cdot(l+m)!}}\; P_l^m(\cos\theta)\, e^{im\phi}\,. \tag{A.34}$$

To avoid confusion, we use in this appendix the superscripts *r* and *c* for the real and complex set of spherical harmonics; in the remainder of the paper, Y always stands for a real spherical harmonic.

2. Angular integrals

The angular parts, $Z_{\alpha\beta\gamma}$ and $\hat{Z}_{\alpha\beta\gamma}$, of the coefficients (30, 32, 33) are defined by

$$Z_{\alpha\beta\gamma} = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\,\sin\theta\; Y_\alpha^{(r)} Y_\beta^{(r)} Y_\gamma^{(r)}$$
$$\hat{Z}_{\alpha\beta\gamma} = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\,\sin\theta\; Y_\alpha^{(r)}\, \nabla_H Y_\beta^{(r)} \cdot \nabla_H Y_\gamma^{(r)}\,. \tag{A.35}$$

As shown by Dziembowski (1982), an integration by parts reduces $\hat{Z}_{\alpha\beta\gamma}$ to $Z_{\alpha\beta\gamma}$

$$\hat{Z}_{\alpha\beta\gamma} = \frac{1}{2}\,[l_\beta(l_\beta+1) + l_\gamma(l_\gamma+1) - l_\alpha(l_\alpha+1)]\; Z_{\alpha\beta\gamma}\,. \tag{A.36}$$

It suffices, therefore, to discuss the selection rules for $Z_{\alpha\beta\gamma}$ only.

3. Selection rules

The selection rules are easiest expressed in terms of the complex spherical harmonics. The *complex* integral

$$Z_{\alpha\beta\gamma}^{(c)} = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\,\sin\theta\; \overline{Y_\alpha^{(c)}}\, Y_\beta^{(c)}\, Y_\gamma^{(c)} \tag{A.37}$$

is evaluated by a decomposition of $Y_\beta Y_\gamma$ in irreducible representations of the rotation group [see Blatt and Weisskopf (1952), Appendix A, Eq. (5.11)]

$$Y_{lm}^{(c)}(\theta,\phi)\, Y_{l'm'}^{(c)}(\theta,\phi) \tag{A.38}$$
$$= \sum_{LM} \sqrt{\frac{(2l+1)(2l'+1)}{4\pi(2L+1)}} \cdot C_{ll'}(L,0;0,0)\, C_{ll'}(L,M;m,m')\, Y_{LM}^{(c)}(\theta,\phi)$$

where $C_{ll'}(L,M;m,m')$ are the Clebsch-Gordan coefficients. Details about these coefficients are given in Landau and Lifshitz

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Table 1. Examples of allowed ($Z \neq 0$) and forbidden ($Z=0$) couplings of three modes. The azimuthal quantum numbers m are assumed to allow the coupling

	$Z_{ll'l''} \neq 0$	$Z_{ll'l''}=0$
One-mode couplings	000	111[b]
	222	333[b]
	444	555[b]
	...	...
Two-mode couplings	011	100[b]
	022	200[a]
	211	221[b]
	...	...
Three-mode couplings	123	012[ab]
	134	013[a]
	235	024[a]
	...	034[ab]
		124[b]
		234[b]
		...

[a] Triangle condition violated
[b] Parity condition violated

(1977). We here merely need the basic theorem: $C_{ll'}(L,M;m,m')$ is non-zero if and only if the following three conditions are simultaneously satisfied (a) $m+m'=M$, (b) $L\in\{|l-l'|, |l-l'|+1,\dots, l+l'\}$ (triangle condition), and (c) $l+l'+L=$ even (parity condition).

Conditions (b) and (c) remain directly valid for the *real* angular integral (A.35). Condition (a), however, loses its beauty (except in special cases such as $m=m'=m''=0$). Its counterpart is found by transforming the real spherical harmonics of (A.35) into complex spherical harmonics (A.34); with obvious notations

$$Z^{(r)}_{lml'm'l''m''}=w_1 Z^{(c)}_{lml'm'l''m''}+w_2 Z^{(c)}_{l,-ml'm'l''m''}+\dots + w_8 Z^{(c)}_{l,-ml',-m'l'',-m''} \tag{A.39}$$

where $w_1,\dots,w_8$ are functions of the normalization constants (A.33). One realizes how the simple property (a) has been spoiled: if $m+m'\neq m''$, $Z_{lml'm'l''m''}$ can still vanish (some terms on the right hand side might cancel), and if $m+m'=m''$, $Z_{lml'm'l''m''}$ could be non-zero, if one of the combinations $\pm m \pm m' \pm m''$ is not equal to zero.

Table 1 shows a few examples of "allowed" and "forbidden" coupling coefficients among modes of low degrees l.

Appendix 3: stars with non-monotonic entropy profile

Our basic assumption of a monotonically increasing specific entropy is not satisfied in stars with convection zones (where specific entropy is practically constant), or in stars with a marked space-dependent chemical compositon (where specific entropy can even *decrease* outwards). The first difficulty is not brushed away by remarking that in *real* convection zones specific entropy still remains slightly increasing outwards. Though in principle the formalism could be used, the nearly vanishing entropy gradient would be a source of numerical troubles.

A solution which overcomes both of these difficulties is to split the star into a finite number of co-moving zones, in each of which specific entropy s is (strictly) monotonically increasing or decreasing or constant. The integrals of total kinetic and potential energy (3, 7, 8, 9) must be replaced by sums of integrals over the individual zones. Furthermore, in a zone where specific entropy is constant, it has to be replaced by a suitable co-moving function.

For our formalism, the three required properties of s are: (a) s is co-moving, (b) there must be a one-to-one correspondence between (s,θ,ϕ) and (r,θ,ϕ) at any time t, and (c) the circulation (13) is conserved for loops on surfaces of constant s.

In the zones where s is strictly monotonically increasing or decreasing, it has all these three properties and can therfore be used without modification. In a zone of constant specific entropy, there is no one-to-one correspondence between (s,θ,ϕ) and (r,θ,ϕ). However, in such zones it suffices to use any co-moving (monotonically increasing or decreasing) quantity λ [the "load" of Lynden-Bell and Katz (1981] instead of s. For example, the mass within arbitrarily chosen co-moving nested shells, scaled by a constant factor to yield the dimension of entropy, trivially fulfills requirements (a) and (b). But (c) is also satisfied: in a zone of constant specific entropy, circulation is conserved for *all* loops, and therefore *a fortiori* for those lying on surfaces of constant load λ.

Notice that the coupling coefficients are eventually re-expressed in the original space variables (r,θ,ϕ) (Eqs. 30, 32, 33), so that the sum of integrals over the individual zones (replacing the single integral) and the "load" (in zones of constant s) only appear in the intermediate theoretical steps of the discussion.

Appendix 4: expansion of the total gravitational energy of the star

To compute the coefficients of the total gravitational energy, we expand $|\boldsymbol{r}-\boldsymbol{r}'|^{-1}$ in spherical harmonics [see e.g. Morse and Feshbach (1953), Eq. (10.3.37)]

$$\frac{1}{|\boldsymbol{r}_1-\boldsymbol{r}_2|}=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\varepsilon_l\cdot\frac{(l-|m|)!}{(l+|m|)!}\cdot\left(\frac{r_<^l}{r_>^{l+1}}\right)\cdot Y^{(r)}_{lm}(\theta,\phi)\,Y^{(r)}_{lm}(\theta',\phi'). \tag{A.40}$$

Here, $Y^{(r)}_{lm}$ and ε_l are defined as in Appendix 2.1., and $r_{\gtrless}=\{{\min \atop \max}\}(|\boldsymbol{r}_1|, |\boldsymbol{r}_2|)$. In terms of our basic fields we have

$$|\boldsymbol{r}|=r_0(s)+\delta r(s,\theta,\phi;t) \tag{A.41}$$

We need the third order expansion in δr of the function of r_1 and r_2 $\left(\frac{r_<^l}{r_>^{l+1}}\right)$.

Let us first define the auxiliary functions

$$A_l^{00}(x,y)=x^l\cdot y^{-l-1}, \tag{A.42}$$

and their partial derivatives

$$A_l^{jk}(x,y)=\frac{\partial^{(j+k)}A_l^{00}}{\partial x^j\cdot\partial y^k}. \tag{A.43}$$

Furthermore we introduce the "ordered" functions

$$[A_l^{jk}](x,y)=\begin{Bmatrix}A_l^{jk}(x,y) & (x\leqq y)\\ A_l^{kj}(y,x) & (y<x).\end{Bmatrix} \tag{A.44}$$

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With these definitions and the shorthand notation $[A_l^{jk}]$ for $[A_l^{jk}](r_0(s_1), r_0(s_2))$ and δr_i for $\delta r(s_i, \theta, \phi)$ $(i=1,2)$ we obtain

$$\left.\begin{aligned}\left(\frac{r_<^l}{r_>^{l+1}}\right) &= [A_l^{00}] + [A_l^{10}]\,\delta r_1 + [A_l^{01}]\,\delta r_2 + \frac{1}{2}\Big\{[A_l^{20}](\delta r_1)^2 \\ &\quad + 2[A_l^{11}]\,\delta r_1 \delta r_2 + [A_l^{02}]\,(\delta r_2)^2\Big\} \\ &\quad + \frac{1}{6}\Big\{[A_l^{30}](\delta r_1)^3 + 3[A_l^{21}]\,(\delta r_1)^2 \delta r_2 \\ &\quad + 3[A_l^{12}]\,\delta r_1 (\delta r_2)^2 + [A_l^{03}]\,(\delta r_2)^3\Big\} + O[(\delta r_i)^4]\end{aligned}\right\} \quad \text{(A.45)}$$

Next, we perform the two angular integrations in the total gravitational energy (7, 8). The third order contribution involves double angular integrals over a product of 5 spherical harmonics. These integrals belong to the following two types ($\int d\Omega$ stands for $\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta$)

$$I_1 = \sum_{lm} \int d\Omega \int d\Omega'\, Y_\alpha(\Omega)\, Y_\beta(\Omega)\, Y_\gamma(\Omega)\, Y_{lm}(\Omega)\, Y_{lm}(\Omega'), \quad \text{(A.46)}$$

$$I_2 = \sum_{lm} \int d\Omega \int d\Omega'\, Y_\alpha(\Omega)\, Y_\beta(\Omega)\, Y_\gamma(\Omega')\, Y_{lm}(\Omega)\, Y_{lm}(\Omega').$$

In I_1, all terms with $l>0$ vanish, because of the $d\Omega'$ integration. Since $Y_{00}=1$, we finally obtain

$$I_1 = 4\pi \int d\Omega\, Y_\alpha(\Omega)\, Y_\beta(\Omega)\, Y_\gamma(\Omega) = 4\pi Z_{\alpha\beta\gamma} \quad \text{(A.47)}$$

In I_2, the $d\Omega'$ integration similarly cancels all terms with $(l, m) \neq \gamma$. We get

$$I_2 = N_\gamma Z_{\alpha\beta\gamma} \quad \text{(A.48)}$$

with N_γ being the normalization factor of the real spherical harmonics (A.33). From these expressions we can compute the third order coefficients (32) of the total gravitational energy.

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