RELATIVISTIC GRAVITATIONAL POTENTIAL AND ITS RELATION TO MASS-ENERGY

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Abstract. From the general theory of relativity a relation is deduced between the mass of a particle and the gravitational field at the position of the particle. For this purpose the fall of a particle of negligible mass in the gravitational field of a massive body is used. After establishing the relativistic potential and its relationship to the rest mass of the particle, we show, assuming conservation of mass-energy, that the difference between two potential-levels depends upon the value of the radial metric coefficient at the position of an observer. Further, it is proved that the relativistic potential is compatible with the general concept of the potential also from the standpoint of kinematics. In the third section it is shown that, although the mass-energy of a body is a function of the distance from it, this does not influence the relativistic potential of the body itself. From this conclusion it follows that the mass-energy of a particle in a gravitational field is anisotropic; isotropic is the mass only. Further, the possibility of an incidental feed-back between two masses is ruled out, and the law of the composition of the relativistic gravitational potentials is deduced. Finally, it is shown, by means of a simple model, that local inhomogeneities in the ideal fluid filling the Universe have negligible influence on the total potential in large regions.

1. Introduction

Despite the fact that the general theory of relativity has passed successfully through a number of tests, it is still possible to find problematic points in some applications of the theory.

One of the problems is whether the mass of a particle depends in any way on the gravitational field at the place of the particle [*15*]. The question whether the gravitational self-energy of a particle contributes to its mass and how this possible interaction occurs has often been discussed in the literature [*1*], [*2*].

According to Einstein, both gravitational and inertial mass depend upon gravitational energy. Dicke is doubtful about Einstein's conclusions. The problem of energy conservation, which arises in Dicke's theory, is solved by a possible violation of the equivalence principle. In his theory, gravitational self-energy contributes to the gravitational mass only, the inertial mass remains unchanged [*12*]. Recently, Williams *et al.* [*18*] made an important experiment proving again the principle of equivalence. The proof is based on the null-result for the Nordtvedt effect.

Another problem is whether it is possible to establish a potential in the general theory of relativity, with properties analogous to those of the classical potential considering both energy and kinematics. This is denied by most experts. The main problem is the definition of a reference-level for discussing potential energy. This is

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sometimes stated in the form, that it is impossible to localize the potential energy [13].

Consider the following paradox: A test particle of negligible mass is falling radially from a great distance to a massive black hole and is observed by a static observer above the Schwarzschild limit, which is possible by principle. If the radial coordinate of the observer is very close to that limit, the observed velocity of the particle will be near that of the light and the relativistic mass of the originally negligible particle will suddenly be comparable to the mass of the body generating the gravitational field. As a consequence there should be a violent dynamical interaction between the particle and massive body, which constitutes the paradox.

The aim of the present paper is not to give an exhaustive analysis of the problems, but to indicate a possible easy and logical way for the solution. This should be complete, self-consistent and in agreement with past experiments. Therefore in the following analysis we start from principles of the general theory of relativity, i.e. from the principle of equivalence [9], from invariability of the gravitational constant [10] and from the law of mass-energy conservation.

2. Mass and Relativistic Gravitational Potential

2.1. The Fall of a Particle of Negligible Mass in the Gravitational Field of a Massive Body

To deduce a relation between the mass and the gravitational potential, let us use the free fall of a particle in the gravitational field of a massive body with the mass incomparably greater than that of the falling particle itself. This assumption makes it possible to simplify the theory in the first phase of the analysis: one does not wish to complicate the theory in such a sense that the particle should influence in some way the mass of the massive body.

The particle begins to fall from rest ($\mathrm{d}r/\mathrm{d}t = 0$), from the flat spacetime. The mass of the particle is μ for $r \to \infty$ and $\mathrm{d}r/\mathrm{d}t = 0$. The space component of the particle's geodesic is a radial straight line relative to the massive body with mass $M \gg \mu$.

The symbol r stands for a Schwarzschild radial coordinate with origin in the center of the massive body, t is a time coordinate in the flat space-time.

In the geometrized system of units, the four-momentum of the particle is

$$\mathbf{p} = m_{(0)}\mathbf{u}, \tag{1}$$

where $\mathbf{u}$ is the four-velocity of the particle defined as

$$u^0 = (1 - v^2)^{-1/2} \tag{2}$$

$$u^i = v^i(1 - v^2)^{-1/2}, \qquad (i = 1, 2, 3) \tag{3}$$

where v^i are the velocity components in a local reference frame and $m_{(0)}$ the rest mass of the falling particle.

In the flat space-time (indices of $\hat{\alpha}$-type) the particle, as well as the reference frame, are at rest – i.e., $\mathbf{u} = (1, 0, 0, 0)$, and according to (1)

$$p^{\hat{0}} = \mu u^{\hat{0}} = \mu. \tag{4}$$

At a point with a finite r coordinate and in a local reference frame, which is at rest (r is constant), is

$$p^0 = \mu u^0 = \mu(1 - v^2)^{-1/2} = m. \tag{5}$$

Local velocity of the particle in this reference frame is $v = \mathrm{d}l/\mathrm{d}\tau$; l is a proper radial coordinate and τ is proper time.

If we should stop the particle in such a way that its kinetic energy transformed into matter (which is possible by principle, but in the decisive majority of physical phenomena, which lead to the stoppage of a particle, the kinetic energy is simply taken away; see Appendix), the zero-component of four-momentum is

$$p^0_{(0)} = m_{(0)}u^0 = m_{(0)}. \tag{6}$$

The symbol $m_{(0)}$ denotes the rest mass of the particle after the stoppage at the point with the radial coordinate r.

Furthermore,

$$p^0 = m = m_{(0)} = p^0_{(0)}. \tag{7}$$

From the relations

$$p^0 = p^{\bar{\alpha}} \frac{\mathrm{d}x^0}{\mathrm{d}x^{\bar{\alpha}}} = p^{\bar{\delta}} \frac{\mathrm{d}x^0}{\mathrm{d}x^{\bar{\delta}}} = p^{\bar{\delta}} \frac{\mathrm{d}\tau}{\mathrm{d}T} \tag{8}$$

describing a transformation from the local Lorentz frame of the moving particle (T is a proper time in this frame) to the reference frame which is at rest (r is constant), and from (4) and (7) it follows that

$$m_{(0)} = m = \mu \frac{\mathrm{d}\tau}{\mathrm{d}T}. \tag{9}$$

At the point with coordinate r (finite) the curved space-time has a Schwarzschild metric. For the particle at rest ($\mathrm{d}r/\mathrm{d}t = \mathrm{d}\theta/\mathrm{d}t = \mathrm{d}\phi/\mathrm{d}t = 0$) its world line is described by the equation

$$\mathrm{d}s^2 = g_{00}\,\mathrm{d}t^2. \tag{10}$$

Further

$$\mathrm{d}s^2 = -\mathrm{d}\tau^2 \tag{11}$$

and

$$\mathrm{d}\tau = (-g_{00})^{1/2}\,\mathrm{d}t. \tag{12}$$

For a particle which continues to fall ($\mathrm{d}\theta/\mathrm{d}t = \mathrm{d}\phi/\mathrm{d}t = 0$), the geodesic is described by

$$\mathrm{d}s^2 = g_{00}\,\mathrm{d}t^2 + g_{rr}\,\mathrm{d}r^2, \tag{13}$$

where

$$g_{00} = -\left(1 - \frac{2M}{r}\right), \qquad g_{rr} = \left(1 - \frac{2M}{r}\right)^{-1}; \tag{14}$$

so that

$$ds^2 = -dT^2. \tag{15}$$

From (13), (14) and (15) it follows that

$$dT^2 = \left\{\left(1 - \frac{2M}{r}\right) - \left(1 - \frac{2M}{r}\right)^{-1}\left(\frac{dr}{dt}\right)^2\right\} dt^2. \tag{16}$$

The fall is described also by the equation

$$\frac{dr}{dt} = \left(1 - \frac{2M}{r}\right)\left(\frac{2M}{r}\right)^{1/2}, \tag{17}$$

in accordance with the equation of the geodesic in the space-time with the Schwarzschild metric [*5*].

In accordance with Equation (17) we have

$$dT = \left(1 - \frac{2M}{r}\right) dt, \tag{18}$$

which with (12) and (14) gives

$$\frac{d\tau}{dT} = \left(1 - \frac{2M}{r}\right)^{-1/2}; \tag{19}$$

thus, according to (9),

$$m_{(0)} = \mu\left(1 - \frac{2M}{r}\right)^{-1/2}, \tag{20}$$

which agrees with Møller [*15*].

2.2. The mass-energy of the falling particle and the establishing of the relativistic gravitational potential

The Hamiltonian for the motion of the particle along a geodesic in the time-independent (stationary) gravitational field is (see [*6*] and [*16*]),

$$H = E + V, \tag{21}$$

where E is the special relativistic mass of the particle (then m), as described in formula (5), and

$$V = E\cdot\left[\left(1 - \frac{2M}{r}\right)^{1/2} - 1\right] \tag{22}$$

is its potential energy. The Hamiltonian H is a constant of such a geodesic motion of the particle [17], which may be identified with the total energy of the particle. According to its physical significance it will be called the mass-energy and denoted m_e. Similarly E, which includes rest mass and kinetic energy of the particle, will be denoted m_{e_k} and finally the potential energy V is denoted m_{e_p}. Thus

$$m_e = m_{e_k} + m_{e_p}, \tag{23}$$

where

$$m_{e_k} = m \tag{24}$$

and

$$m_{e_p} = m \cdot \left[\left(1 - \frac{2M}{r}\right)^{1/2} - 1\right]. \tag{25}$$

Consider now a distant, static observer in the flat spacetime ($r \to \infty$) observing the falling particle. In the moment of the beginning of the fall (the particle is at the place of the observer) is

$$m_{e_k} = \mu \tag{26}$$

and

$$m_{e_p} = 0. \tag{27}$$

Mass-energy of the particle is therefore equal to the proper mass of the particle, hence μ. We come to the same conclusion for a general position of the particle in the field (r is finite) if we use relations (7), (20), (23), (24), and (25).

The function

$$\varphi = \left(1 - \frac{2M}{r}\right)^{1/2} - 1 \tag{28}$$

has, therefore, from the standpoint of the law of mass-energy conservation, the character of an effective relativistic potential. Let us for convenience call it 'the relativistic potential' only. One can arrive at the same conclusion another way.

Let us take the effective potential [11] for the radial motion only, and renormalize it, so that for non-relativistic conditions it is equal to the Newtonian potential. We obtain again the function φ as given by Equation (28).

The Newtonian potential is

$$\varphi_N = \lim_{2M/r \to 0} \varphi = -\frac{M}{r}. \tag{29}$$

Furthermore,

$$\lim_{r \to \infty} \varphi = \lim_{r \to \infty} \varphi_N = 0. \tag{30}$$

According to (20)

$$m_{(0)} = \mu(1 + \varphi)^{-1}. \tag{31}$$

In [13] Misner *et al.* write: "Moreover, 'local gravitational energy-momentum' has no weight. It does not curve space.... Unhappily, enormous time and effort were devoted in the past to trying to 'answer this question' before investigators realized the futility of the enterprise." On the other hand, among the quotations at the beginning of the same chapter, one finds Einstein's statement: "All forms of energy possess inertia." Misner's explanation that it is not possible to localize a potential energy, meets in every case certain logical difficulties. An observer (we assume a static one) is placed at the finite distance r from the source of the gravitational field. Why not relate the potential energy to the potential level on which the observer is placed (as is usual in practical engineering)? If the falling particle is just passing our observer, its potential energy simply equals zero, and the problem whether or not it is possible to localize a potential energy related to infinity, is for our observer actually solved. Moreover, with this new conception there is no violation of the theory of relativity. The praxis to always relate the potential energy to the distant zero-potential level without regard to the position of the observer is very usual in the classical celestial mechanics, but then the mass-equivalent of such an energy is without exception negligible.

Under those conditions, the mass-energy m_e (Hamiltonian H) of the particle is for a static observer constant, but for static observers in the different Schwarzschild distances r from the source of the gravitational field, it is different. For our static observer placed at the point B (its radial co-ordinate r is finite), where the falling particle is just passing, is

$$m_{e_k} = m_B, \tag{32}$$

where m_B is the special relativistic mass of the particle, further

$$m_{e_p} = 0 \tag{33}$$

and therefore, relating to (23),

$$m_e = m_B. \tag{34}$$

If the particle is situated at a point $A \not\equiv B$, the law of mass-energy conservation (23) (we assume the law is fulfilled for every static observer) implies that

$$m_e = m_B = m_A + m_A \, \Delta\varphi_{\langle A-B \rangle}, \tag{35}$$

where $\Delta\varphi_{\langle A-B \rangle}$ is the potential difference between the two points relative to the observer. But according to (31),

$$m_A = \mu(1 + \varphi_A)^{-1} \tag{36}$$

and

$$m_B = \mu(1 + \varphi_B)^{-1}. \tag{37}$$

Using (36) and (37), we obtain from (35) that

$$\Delta\varphi_{\langle A-B \rangle} = \frac{\varphi_A - \varphi_B}{1 + \varphi_B}. \tag{38}$$

$\Delta\varphi_{\langle A-B \rangle} = \varphi_A - \varphi_B$ only for $\varphi_B = 0$ or $\varphi_A = \varphi_B$.

In accordance with (28)

$$1 + \varphi_B = \left(1 - \frac{2M}{r_B}\right)^{1/2}. \tag{39}$$

Because

$$\left(\frac{\mathrm{d}r}{\mathrm{d}l}\right)_B = (g_{rr_B})^{-1/2} = \left(1 - \frac{2M}{r_B}\right)^{1/2}, \tag{40}$$

we have, according to (38),

$$\Delta\varphi_{\langle A-B\rangle} = (\varphi_A - \varphi_B)(g_{rr_B})^{1/2}. \tag{41}$$

Thus, the difference between two relativistic potentials depends upon the local value of the relativistic potential or upon the value of the metric coefficient g_{rr} at the position of the observer.

It is important to note that

$$|\Delta\varphi_{\langle A-B\rangle}| \neq |\Delta\varphi_{\langle B-A\rangle}|, \tag{42}$$

$$\Delta\varphi_{\langle A-B\rangle} \neq \Delta\varphi_{\langle A-C\rangle} + \Delta\varphi_{\langle C-B\rangle} \tag{43}$$

and, the potential energy at a point A, relative to a point B,

$$m_{e_p\langle A-B\rangle} = m_A \cdot \Delta\varphi_{\langle A-B\rangle} = \mu \frac{\varphi_A - \varphi_B}{(1 + \varphi_A)(1 + \varphi_B)}. \tag{44}$$

One can establish such a $\Delta f_{\langle A-B\rangle}$, that (44) can be written

$$m_{e_p\langle A-B\rangle} = \mu\, \Delta f_{\langle A-B\rangle}, \tag{45}$$

i.e., in accordance with (44)

$$\Delta f_{\langle A-B\rangle} = \frac{\varphi_A - \varphi_B}{(1 + \varphi_A)(1 + \varphi_B)}. \tag{46}$$

For $\Delta f_{\langle A-B\rangle}$ is

$$\Delta f_{\langle A-B\rangle} = -\Delta f_{\langle B-A\rangle} \tag{47}$$

and

$$\Delta f_{\langle A-B\rangle} = \Delta f_{\langle A-C\rangle} + \Delta f_{\langle C-B\rangle}. \tag{48}$$

Despite the algebraic properties of Δf we shall use the potential difference $\Delta\varphi$ only, because it is related to the actual mass of the particle and thus in better agreement with the usual definition of the potential.

Furthermore, it is useful to show that the function φ has properties of the potential, not only from the standpoint of the law of mass-energy conservation, but also from that of kinematics. One has namely to prove that, for $v = 0$, a local acceleration a is equal to a local gradient of the potential difference (41); i.e., that

$$a = \frac{\mathrm{d}\psi}{\mathrm{d}l}, \tag{49}$$

where, according to (40)

$$\mathrm{d}l = (g_{rr})^{1/2} \cdot \mathrm{d}r \tag{50}$$

and

$$\mathrm{d}\psi = \lim_{A \to B} \Delta\varphi_{\langle A-B \rangle}, \tag{51}$$

which gives, in accordance with (41),

$$\mathrm{d}\psi = (g_{rr})^{1/2} \cdot \mathrm{d}\varphi. \tag{52}$$

Then,

$$\frac{\mathrm{d}\psi}{\mathrm{d}l} = \frac{\mathrm{d}\varphi}{\mathrm{d}r}. \tag{53}$$

Let us prove, therefore, that

$$a = \frac{\mathrm{d}\varphi}{\mathrm{d}r}. \tag{54}$$

We start from the Schwarzschild metric and solve the equation for a geodesic. (Under different conditions for the solution of the differential equations, which we obtain after substitution of non-zero connection coefficients into the general equation of the geodesic, and with the use of the equation describing the metric of the Schwarzschild field, we can arrive at the solutions explaining today already 'classic' (and, by observations, definitely confirmed) conclusions: namely, the angular advance of perihelion for Mercury and the gravitational deflection of a light in vicinity of the Sun. For a more detailed treatment see, e.g., [7].) If we solve those equations under conditions for a radial free fall (θ and ϕ are constant) from rest from a point with the co-ordinate $r = R$ ($v_R = 0$), the number of equations is reduced to three and the solution is

$$\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 = 2M \cdot \left(1 - \frac{2M}{R}\right)^{-1} \cdot \left(1 - \frac{2M}{r}\right)^2 \cdot \left(\frac{1}{r} - \frac{1}{R}\right). \tag{55}$$

(Compare with [5] for $c = 1$.) For a free fall from the flat spacetime ($R \to \infty$) we obtain the already used formula (17). It is pertinent to indicate that, by using formulas (12), (14), (17) and (50), we obtain for the locally measured velocity of the falling particle (under condition $R \to \infty$), the expression

$$v = \frac{\mathrm{d}l}{\mathrm{d}\tau} = \left(\frac{2M}{r}\right)^{1/2}. \tag{56}$$

By using (5) and (7) it is possible to arrive more easily in another way to the conclusion (20).

Taking the derivative of the expression (55) with respect to t and after resubstitution of the same expression under conditions $r = R$ for $v_R = 0$, we obtain:

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = -\frac{M}{r^2}\cdot\left(1 - \frac{2M}{r}\right). \tag{57}$$

The local acceleration a can be expressed as

$$a = -\frac{\mathrm{d}^2 l}{\mathrm{d}\tau^2}, \tag{58}$$

and by (12) and (50) again

$$\frac{\mathrm{d}^2 l}{\mathrm{d}\tau^2} = \frac{(g_{rr})^{1/2}\cdot \mathrm{d}^2 r}{-g_{00}\cdot \mathrm{d}t^2}. \tag{59}$$

Using (14), from (58) it follows that

$$a = \left(1 - \frac{2M}{r}\right)^{-3/2}\cdot\frac{\mathrm{d}^2 r}{\mathrm{d}t^2}. \tag{60}$$

Referring to (57), we finally find that

$$a = \frac{M}{r^2}\cdot\left(1 - \frac{2M}{r}\right)^{-1/2}. \tag{61}$$

According to (54), we ask if

$$\frac{\mathrm{d}\varphi}{\mathrm{d}r} = \frac{M}{r^2}\cdot\left(1 - \frac{2M}{r}\right)^{-1/2}. \tag{62}$$

This is actually fulfilled by function φ given by relation (28). Referring again to the Equations (29) and (30) for the non-relativistic and boundary conditions, it is possible to claim definitively that function φ is the relativistic potential.

Here one may ask: Is the last statement justified, when, in the deduction of the Schwarzschild metric, one identifies the constant M with the mass of a source of the gravitational field with use of

$$-g_{00} = 1 + 2\chi, \tag{63}$$

where χ is the Newtonian potential? The answer is that relation (63) is deduced for weak fields only ([*4*] and [*8*]), i.e., under the boundary relativistic conditions. But such conditions are possible to use (and really are used) for determination of the constants without limitation of the validity of the general solution. Consequently, the constant M represents the mass under general relativistic conditions just because, with relativistic boundary conditions, the function φ becomes equal to the Newtonian potential as described by (29).

2.3. Some general conclusions

The mass of the particle is related to the local value of the relativistic gravitational potential, as described by (31). The expression 'mass' means the mass when the particle according to (7) is momentarily at rest; otherwise, it means the relativistic mass relative to a static observer. The interaction between the particle and the massive body takes place on the purely spacelike hypersurface ($dt = 0$; $ds^2 > 0$), i.e. immediately. The particle interacts with the time-constant (stationary) gravitational field of the massive body. The particle itself does not influence the mass of the massive body, due to the assumption made at the beginning.

In accordance with (31), the mass of the particle at rest depends upon the local value of the relativistic gravitational potential φ, which is a scalar. This means that the inertia is isotropic. This conclusion is in agreement with [3].

Despite the high improbability of a spontaneous stoppage-process, in which kinetic energy of the falling particle would be transformed into matter, we can take every particle with rest mass $m_{(0)}$, which is at rest on the potential level $\varphi(r)$, as if it would be a result of such a stoppage-process; and, with use of formula (31), we can calculate its proper mass μ pertaining to the particle in the flat (Minkowski) space-time.

If a particle falls from the flat space-time (from rest for $r \to \infty$), then Equation (4) is always fulfilled in the local Lorentz frame connected with the particle. It means that the relativistic potential φ for a particle which is moving along a geodesic, according to Equations (7) and (31), is equal to zero, because the rest mass of the particle remains equal to its proper mass μ. (More simply and perhaps even more correctly: In the local Lorentz frame of the moving particle, the influence of the relativistic potential φ is eliminated.)

If the particle falls from a point with finite radial co-ordinate r towards the massive body, then in the local Lorentz frame connected with the particle, the potential difference $\Delta\varphi$ is equal to zero (is eliminated). It is so because the rest mass is equal to the mass the particle had at the point with the radial co-ordinate r.

3. Finite Bodies and Other Applications

3.1. The problem of gravitational self-energy and mass of a body

A body with a mass M is formed by a thin spherical shell with Schwarzschild radius r. The relativistic potential on such a shell is determined by formula (28). But it means that the mass of every particle pertaining to this formation will increase in accordance with the relation (31). Therefore, for an observer situated on the surface of such a shell, the mass of the whole body will increase in the same way. It would appear that this can lead to the wrong opinion, that the mass of the formation determines itself by the pertinent relativistic potential in the feed-back way.

If the mass of the body for an observer on its surface is

$$\mathscr{M} = M(1 + \varphi)^{-1}, \tag{64}$$

then the supposition

$$\varphi = \left(1 - \frac{2\mathscr{M}}{r}\right)^{1/2} - 1, \tag{65}$$

is incorrect: instead of $\mathscr{M}$ in this relation we have to use the proper mass M of the body. This follows from the fact that in the derivation of the Schwarzschild metric, we have in the formulae (14) for g_{00} and g_{rr} a general constant instead of M. Only from boundary conditions, when $r \to \infty$, can one conclude that this constant is equal to M, i.e. the mass characterizing the body at great distances [*4*]. The relativistic potential is then determined exclusively by the variable r and the constant M. But this does not mean that everywhere outside the body its mass is equal to M, which is frequently assumed in the literature [*14*]. The mass as such is really given by Equation (64). For a distant observer ($r \to \infty$), the increase of mass is compensated by the negative gravitational self-energy, in such a way as explained in the text directly preceding Equation (28).

In order to fulfill the equivalence principle it remains to show that, for a nearby observer, $\mathscr{M}$ is not only the inertial mass, but also the gravitational mass. At a point B, with the Schwarzschild coordinate r, the length-unit is contracted in the radial direction (relative to a distant observer). Consequently, for a static observer at B, the radial distance r will be observed as

$$L = r\left(\frac{\mathrm{d}l}{\mathrm{d}r}\right)_B, \tag{66}$$

which, in accordance with (39) and (40), means that

$$L = r(1 + \varphi_B)^{-1}. \tag{67}$$

The distance L we call the local radial coordinate. In contradistinction to l (proper radial coordinate), the L is not an integral over $\mathrm{d}l$ from zero to r.

For the observer at point B a relativistic potential φ_B is determined by the mass $\mathscr{M}$ and the local radial coordinate L. Formula (28) then yields

$$\varphi_B = \left(1 - \frac{2\mathscr{M}}{L}\right)^{1/2} - 1. \tag{68}$$

But, according to (64) and (67), we arrive at the relation

$$\varphi_B = \left(1 - \frac{2M}{1 + \varphi_B}\frac{1 + \varphi_B}{r}\right)^{1/2} - 1. \tag{69}$$

We thus again obtain the feed-back-free relation (28). Simultaneously, the equivalence principle is fulfilled.

This solution at the same time explains a paradox concerning the mass of a particle falling into the vicinity of the event horizon ($r \to 2M$). According to (28) and (31), the mass of the particle, even though of negligible value at a great distance r, would increase in the vicinity of the Schwarzschild limit to such values that it would become comparable to the mass of the black hole, to which the Schwarzschild limit pertains. But, for every static observer, the special relativistic mass of the particle falling from infinity is equal to the proper mass μ multiplied with $(1 + \varphi)^{-1}$ and the locally observed mass (mass-energy) $\mathscr{M}$ of the black hole is equal to the same multiple of its proper mass M. Every static observer can thus claim that

$$m:\mathscr{M} = \mu:M. \tag{70}$$

Finally it is necessary to specify the concept of mass, which for simplicity was used several times in this section instead of the mass-energy. We shall understand henceforth under the expression 'mass of the formation' a value given by formula (64), when the observer is situated at the surface of the shell. The mass is then invariant relative to r. If the observer is situated above the surface of the shell, we shall call the $\mathscr{M}$, which is not invariant relative to r, the locally observed mass-energy of the formation. For $r \to \infty$ the mass-energy of the formation is identical to its proper mass M. The situation for a particle is analogous, but correct terminology was strictly kept in the relevant sections.

3.2. Isotropy of the Mass and Anisotropy of the Mass-Energy

It is easy to see that the conclusion on the isotropy of mass holds in a Schwarzschild gravitational field only locally; or stated more precisely: it holds only for the locally observed mass-energy at the point where the particle is situated (therefore, for the mass). This is in accordance with the measurements [*3*].

But generally the mass-energy of the particle is anisotropic. For a distant observer ($r \to \infty$) the mass-energy, considering the change of a length unit in the radial direction, will be equal to μ, and in the transverse direction it will be equal to $m_{(0)}$ in accordance with (31), where φ is the relativistic potential at the point of the particle. According to the definition of the Schwarzschild radial co-ordinate, the geometry in the transverse direction for every observer is locally Euclidean (the curvature of the equipotential surface may be neglected). We determine here the mass-energy of the particle from the shape of the equipotential surfaces of its own gravitational field. These equipotential surfaces have locally (and on the background of the gravitational field of the central body) the shape of concentric spherical shells.

For an observer, situated at a finite r coordinate, the mass-energy of a particle in infinity is μ in the transverse direction and $m_{(0)}$ in the radial direction according to (31); φ is the relativistic gravitational potential of the central body at the place of the observer.

The mass-energy of a particle situated at a point A in an arbitrary direction of a unit vector $\mathbf{n}$ (in the co-ordinate system of the particle) will, for an observer situated at a point B, be given by

$$m_e = m_{(0)}\mathbf{n}\cdot\left(\frac{1 + \varphi_A}{1 + \varphi_B}, 1\right). \tag{71}$$

If **w** is a unit vector in the coordinate system of the observer, the same relation will be

$$m_e = m_{(0)}(w_r + w_t)\left[\left(w_r \frac{1 + \varphi_B}{1 + \varphi_A}\right)^2 + w_t^2\right]^{-1/2}. \tag{72}$$

The first component of the vector is the radial one and the second one is the transverse one.

If the observer is situated on another radial line than that of the particle, then

$$\mathbf{w} = \mathbf{s}\cdot\begin{pmatrix} \cos\beta, & \sin\beta \\ -\sin\beta, & \cos\beta \end{pmatrix}, \tag{73}$$

where β is an angle counted from the radial line of the observer to that of the particle, and **s** is an arbitrary unit vector at point B.

3.3. A GRAVITATIONAL INTERACTION BETWEEN TWO BODIES WITH COMPARABLE MASSES

If one analyses a situation where instead of masses μ and M ($M \gg \mu$) there are two masses with comparable values μ_1 and μ_2 (relative to flat space-time), the whole analysis is complicated by the fact that the mass of the attracting body is influencing the mass of the attracted body. Both bodies play two parts at the same time, both as attracting and attracted bodies, with all the consequences for their masses.

The question is whether the interaction will appear in the form of a feed-back between the masses, and if so, whether that coupling has a convergent or a divergent character.

We stated above that the relativistic potential at a specific point of space is invariant relative to the position of the observer. Because the radial mass-energy of both bodies is constant for a distant observer (we assume that velocities in other directions than that of the connecting line between the bodies are equal to zero), despite the increase of masses of bodies, their relativistic potentials, at least in the radial direction (connection line direction), are determined, analogously to the transition from Equations (68) to (69), by their proper masses μ_1 and μ_2 only. Therefore, even for a local observer there will be no feed-back between the masses. It can happen that when the bodies are approaching, as seen by distant observer, for a local observer the bodies are incidentally moving apart.

If the two bodies are particles of comparable masses on an equipotential surface of a body with much larger proper mass M than the particle masses $m_{(0)1}$ and $m_{(0)2}$, then the particles will gravitationally influence one another on the equipotential surface (on the background of the potential of the more massive body) in such a manner as is determined by their masses $m_{(0)}$ (not their proper masses μ) and their Schwarzschild distance. Here we have neglected the curvature of the equipotential surface in a certain small region.

If it is possible to assume that in a sufficiently large region the potential of the massive body is constant in the radial direction, the interaction between the particles has naturally a locally isotropic character: in such an interaction, the 'proper masses' of the particles are locally $m_{(0)}$ and their distance is determined by a set of measurement rods, which, despite the anisotropy for a distant observer, is locally isotropic.

3.4. The composition of the relativistic gravitational potentials

At a point where the relativistic potential of the body is equal to φ_1, let us assume that in a sufficiently large region of space the gravitational field is everywhere homogeneous (φ_1 is approximately constant). In this region we have a particle with mass $m_{(0)} \ll M$ (M is the proper mass of the body). At a certain point S in this region the relativistic potential of the particle on the background potential of the body is equal to φ_2.

On the background-field in this region it is possible to derive the formula for the Schwarzschild metric, in the vicinity of the particle, in such a manner that for a sufficiently great local r' (from the particle) the relativistic potential is equal to the potential of the background, i.e. the local relativistic potential of the particle is equal to zero.

A test particle with proper mass $\mu_{00} \ll m_{(0)}$ is placed into the gravitational field of the body, where the relativistic potential is equal to φ_1 (to a point R, far away from our particle). Then the mass of the test particle is

$$\mu_0 = \frac{\mu_{00}}{1 + \varphi_1}. \tag{74}$$

Hereafter the test particle is placed into the point S and its mass becomes

$$\mu = \frac{\mu_0}{1 + \varphi_2}. \tag{75}$$

If we establish the representative relativistic potential φ such that

$$\mu = \frac{\mu_{00}}{1 + \varphi}, \tag{76}$$

then according to (74) and (75)

$$\varphi = (1 + \varphi_1)(1 + \varphi_2) - 1. \tag{77}$$

It is also possible to state that the local relativistic potential on the background of the gravitational field of the body is equal to the potential difference related to the Minkowski (flat) space-time, because the mass as well as the relativistic potential related to the flat space-time are the same for all static observers. Then

$$\varphi_2 = \Delta\varphi_{\langle S-R \rangle}, \tag{78}$$

which again leads to formula (77).

If the number of bodies were larger, the representative relativistic potential would be

$$\varphi = \prod_{i=1}^{n} (1 + \varphi_i) - 1. \tag{79}$$

In the Newtonian approximation ($|\varphi_i| \ll 1$)

$$\varphi \approx \sum_{i=1}^{n} \varphi_i \tag{80}$$

and

$$\mu \approx \mu_{00}\left(1 - \sum_{i=1}^{n} \varphi_i\right). \tag{81}$$

4. The Effect of Local Inhomogeneities in the Universe on the General Gravitational Potential

Up to now we analysed situations which may be characterized as local perturbations in the Universe. It is advisable to show that such local perturbations of the homogeneity of the fluid (substratum) filling the Universe does not have any global effect on the potential of the Universe as a whole. In accordance with the observations, these perturbations (on the level of galaxies and groups of galaxies) are situated uniformly and therefore a certain type of homogeneity exists.

In the frame of a model it is possible to find, for every locality with higher mass density than that of the ideal fluid (a positive perturbation), another locality at the same distance, where a lower density (a negative perturbation) exists.

For simplicity let us assume that in a certain region of the Universe there exists the same potential everywhere. On a level, where it is necessary to consider inhomogeneities of the substratum, this presumption is quite acceptable. The homogeneous fluid is therefore equipotential in the considered region.

However, in the fluid, a partial negation of its total continuity (a vacuum 'bubble') is found. This arose because matter from the region of the negation were transferred to its close vicinity (at the distance Δ).

We can consider the negation as a body with negative mass $m^- = -\mu < 0$ relative to the substratum. The fluid from the region of the negation forms a body with the mass $m^+ = \mu > 0$ relative to the substratum.

An analogous formula for the relativistic potential of the negation would be, in accordance with (38) for $\varphi_A = 0$

$$\varphi^- = \frac{-\varphi^+}{1 + \varphi^+}, \tag{82}$$

where φ^+ is a relativistic potential of the 'positive' body which is situated at the same distance r as the negation.

The representative relativistic potential at the distance r ($r \gg \Delta$) from the model-perturbation is in accordance with (77)

$$\varphi = (1 + \varphi^+)(1 + \varphi^-) - 1 \tag{83}$$

and according to (82) is $\varphi = 0$.

But Equation (82) is derived from an analogy only, and its validity need not be general. Therefore, it is better to use a Newtonian approximation; due to the low density our case becomes relativistic only when the dimensions of the region are comparable with the dimensions of the Universe. But such great inhomogeneities do not exist in the Universe. In the Newtonian approximation we have

$$\varphi_N^+ = -\frac{m^+}{r} = -\frac{\mu}{r} \tag{84}$$

and

$$\varphi_N^- = -\frac{m^-}{r} = \frac{\mu}{r}, \tag{85}$$

and according to (80) the conclusion is the same, $\varphi = 0$.

This means that in the substratum the potential remains constant, except in a small region in the vicinity of the perturbation.

Appendix

It might be objected that it is impossible to stop a particle falling in the gravitational field of the central massive body and to conserve, at the same time, a four-momentum of the particle as mentioned above in part 2.1. One possible solution would yield two particles with geodesics centrally symmetrical relative to the central body. The stoppage process should be mediated by two identical photons emitted from the central body so that they would move on the centrally symmetrical null-geodesics directly against the falling test particles. After a head-on encounter between the photons and the particles the energy of the photons, as measured by the static observers, should be radiated in a centrally symmetric manner relative to each stopped particle. Note that the energy of the photon is higher in the reference system of the falling particle than in that of an observer who is static relative to the central body. After this process the total energy of the photons emitted by our two particles will be equal to the sum of the energy of two originally emitted help photons. During the process, their total momentum is always zero. Therefore the four-momentums are conserved both in the system of the two particles and in that of the help photons.

If the transformation of the potential energy into matter and vice versa would take place continuously, the following mechanism would in principle be possible: From a rigid circle with its center in the central body two opposite radial wires are suspended. This rigid supporting system has a negligible mass. Two lifts are moving (climbing) along the wires on the centrally symmetric world-lines relative to the central body. Each lift has a mechanism which can transform its own mass into energy for ascension, or its potential gravitational energy into mass. The velocity of the lift (a 'particle' in the subsequent paper) is theoretically infinitesimally small, but in practice always non-relativistic. We call this transformation of matter into potential energy of the particle (and vice versa) the adiabatical transformation.

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