THE ASTROPHYSICAL JOURNAL, 188:657–665, 1974 March 15

ANISOTROPIC SPHERES IN GENERAL RELATIVITY*

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Received 1973 June 4; revised 1973 September 25

ABSTRACT

The possible importance of locally anisotropic equations of state for relativistic spheres is discussed by generalizing the equations of hydrostatic equilibrium to include these effects. The resulting change in maximum equilibrium mass M and surface redshift z is found analytically in the case of incompressibility (ρ = const.) and a highly idealized expression for the anisotropy. Bondi's analysis of isotropic spheres is generalized to include anisotropy, and the maximum surface redshift is investigated without reference to specific equations of state. A numerical model [with $p(r) = \frac{1}{3}\rho(r)$ and a special form of anisotropy] is then solved. In general, it is found that specific models lead to increases in z typically of the same order of magnitude as the fractional anisotropy.

Subject headings: equation of state — hydrodynamics — quasi-stellar sources or objects — relativity

I. INTRODUCTION

The past decade witnessed a formidable attack on three major problems in relativistic astrophysics—the properties of superdense matter; the details of gravitational collapse (Thorne 1971; Ruffini and Wheeler 1969); and the nature of quasars (Zel'dovich and Novikov 1972). Nevertheless, a number of fundamental problems associated with these subjects remain unsolved. In particular, one would like to know whether realistic evolutionary models lead to black holes, and whether quasars are local or must be cosmologically explained. Recent observations indicate that this latter question is far from resolved (Gunn 1971). If we venture to assume that strong gravitational fields and quasars may in some cases be closely related, then we come face to face with the difficult problem of the equation of state for superdense matter. The latter determines the maximum mass that can escape collapse to a black hole, as well as the maximum redshift from the object's surface. To date investigations of the physics of superdense matter have been limited to systems which are locally isotropic. The complexity of strong interactions—particularly in the areas of superfluidity and superconductivity—suggests that superdense matter may be anisotropic, at least in certain density ranges. The existence of local anisotropy has not yet been theoretically investigated. This is in part due to the complexity of the problem. It is therefore of considerable interest to determine the extent to which local anisotropy, *if it exists*, can alter the structure of massive objects.

As a preliminary step in this direction we consider spherically symmetric static distributions of matter which are *assumed* to be locally anisotropic. The equations of hydrostatic equilibrium for such systems are derived and investigated. We first consider an incompressible model with a highly idealized form of anisotropy which allows the structure equations to be integrated analytically. The resulting maximum mass and surface redshift (hereinafter abbreviated as SRS) are compared with the corresponding isotropic model. In order to eliminate model-dependent effects, we then study the questions of the maximum SRS by generalizing Bondi's analysis of isotropic spheres in general relativity (Bondi 1964) to include anisotropic stresses.

Throughout our discussion we use Einstein's theory of gravitation with cosmological constant $\Lambda = 0$, and relativistic units $c = 1$, $G = 1$. The metric is of signature -2; Greek indices range from 0 to 3.

II. ANISOTROPIC EQUATIONS OF STATE

That superdense matter is locally isotropic is an implicit assumption common to astrophysical studies of massive objects. This is a reasonable first approximation for matter whose dominant properties depend on chemical forces (Coulomb interactions, etc.). For most systems that depend on fully relativistic theories of gravitation, it is the strong interactions which dominate local physics. It is well known that these interactions are nearly as complex as general principles permit, being strongly velocity-, spin-, and parity-dependent. Consequently one may not conclude from *a priori* considerations alone whether or not strongly interacting matter is locally isotropic. Two basic questions arise in this regard: (1) Is strongly interacting matter described by a locally anisotropic equation

* Supported in part by National Science Foundation grant GP-34639X.

of state; and (2) if so, how large is the anisotropy? The answer to both questions requires detailed calculations in both the nonrelativistic ($\rho < 10^{15}$ g cm^{-3}) and the relativistic ($\rho > 10^{15}$ g cm^{-3}) density regimes.

At the present time there is reason to believe that anisotropy may occur in the $\rho < 10^{15}$ g cm^{-3} range. It is believed (Ruderman 1972) that the *p*-state contributions to neutron-neutron interactions may lead to a superfluid state with an anisotropic gap (the *s*-state yields an isotropic gap). Although such effects may be small, detailed calculations have yet to be performed in either density range.

Another possible source of local anisotropy would be a solid core. Recent calculations (Canuto 1973) indicate that a solid state may indeed occur for cold matter in the density range above 4×10^{14}–3×10^{15} g cm^{-3}. However, the exact nature of such a state is as yet unclear.

Although the above examples suggest strongly that some anisotropy may be present, it seems unlikely that the effect would be large. However, in view of the uncertainties and theoretical difficulties involved in treating anisotropic superdense matter in detail, we shall emphasize in the remainder of this work the maximum possible effects which it would be expected to have for spherically symmetric distributions of matter. The results which we obtain assume *maximum anisotropy* and therefore set upper limits. A more detailed analysis must involve specific model calculations in the representative density range which are based on microscopic interactions.

III. ANISOTROPIC SPHERES

We now consider a static equilibrium distribution of matter which is spherically symmetric, but whose stress tensor is in general locally anisotropic. Spherical symmetry implies that (in canonical coordinates) the stress-energy tensor $T_\nu{}^\mu$ is diagonal: $T_\nu{}^\mu = \text{diag}\,(\rho, -p_r, -p_\theta, -p_\varphi)$ and, moreover, $p_\theta = p_\varphi$, which we will denote by $p_\perp$. Without further specifications about the relations between p_r, $p_\perp$, and ρ, this form is completely general. The quantities p_r, $p_\perp$ can contain contributions from fluid pressures as well as other stresses. But for convenience, in the following we will always call p_r the radial "pressure" and $p_\perp$ the tangential "pressure" (no confusion with ordinary hydrostatic pressure should result). The space-time geometry and matter distribution are determined by the Einstein equations

$$G_\nu{}^\mu = \kappa T_\nu{}^\mu\,, \tag{2.1}$$

where $\kappa = 8\pi$. In Schwarzschild coordinates the metric can be written as

$$ds^2 = e^\nu dt^2 - e^\lambda dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\varphi^2 \tag{2.2}$$

with $T_\nu{}^\mu$ given by

$$T_\nu{}^\mu = \text{diag}\,(\rho, -p_r, -p_\perp, -p_\perp)\,. \tag{2.3}$$

The Einstein equations (2.1) then yield the set of equations

$$G_r{}^r = \kappa T_r{}^r \Leftrightarrow 8\pi p_r = e^{-\lambda}(\nu'/r + 1/r^2) - 1/r^2\,, \tag{2.4}$$

$$G_\theta{}^\theta = G_\varphi{}^\varphi = \kappa T_\theta{}^\theta$$
$$\Leftrightarrow 8\pi p_\perp = e^{-\lambda}[\tfrac{1}{2}\nu'' - \tfrac{1}{4}\lambda'\nu' + \tfrac{1}{4}(\nu')^2 + (\nu' - \lambda')/2r]\,, \tag{2.5}$$

$$G_t{}^t = \kappa T_t{}^t \Leftrightarrow 8\pi\rho = e^{-\lambda}(\lambda'/r - 1/r^2) + 1/r^2, \tag{2.6}$$

where a prime denotes differentiation with respect to r. We note that the only difference between the above equations and those describing isotropic spherically symmetric matter is that $p_r \neq p_\perp$. Examination of equations (2.4)–(2.6) shows that we have three structure equations in five unknowns (ν, λ, ρ, p_r, and $p_\perp$). Consequently it is necessary to specify in addition two equations of state, such as $p_r = p_r(\rho)$ and $p_\perp = p_\perp(\rho)$.[1] In case $p_\perp = p_r$ it is evident that the set (2.4)–(2.6) reproduces the equations for isotropic spherically symmetric matter (Tolman 1934).

We now proceed to construct the equations of hydrostatic equilibrium[2] (equations of motion). The procedure is essentially the same as for an isotropic fluid. Using equations (2.4) and (2.5), or, equivalently, the conservation equations $T^\mu{}_{\nu;\mu} = 0$, we obtain the radial equation for p_r:

$$\frac{dp_r}{dr} = -(\rho + p_r)\frac{\nu'}{2} + \frac{2}{r}(p_\perp - p_r)\,. \tag{2.7}$$

Equations (2.4) and (2.6) are then used to show that

$$\tfrac{1}{2}\nu' = \frac{m(r) + 4\pi r^3 p_r}{r(r - 2m)}\,, \tag{2.8}$$

where

$$m(r) = \int_0^r 4\pi r^2 \rho dr \tag{2.9}$$

is the mass inside a sphere of radius r as seen by a distant observer. Finally integration of (2.6) gives

$$e^{-\lambda} = 1 - 2m/r\,. \tag{2.10}$$

It will be noted that there is no equation governing the gradient of the tangential pressure $p_\perp$. This, however, is to be expected since these equations (subject to suitable boundary conditions to be discussed below) determine the radial variation in p_r and ρ. The equation of state $p_\perp = p_\perp(\rho)$ automatically gives $p_\perp(r)$. The equation of hydrostatic equilibrium for an anisotropic spherically symmetric fluid is therefore given by equations (2.7)–(2.8) plus the equations of state. The mass definition (2.9) is identical to the corresponding expression for an isotropic fluid.

[1] In general p_r and $p_\perp$ will depend on additional variables (such as entropy, magnetic fields, etc.), in which case further equations determining these will be needed. For collisionless systems (e.g., star clusters), ρ, p_r, $p_\perp$ are determined by the kinetic distribution function which depends on $g_{\mu\nu}$ itself.

[2] An alternate method involving variational principles can be used which is more conducive to discussions of stability. However, we shall not pursue this aspect of the problem here.

The second term in equation (2.7) is the only one in the structure equations which explicitly contains the tangential pressure. All other terms contain only ρ and p_r. Furthermore, in the Newtonian limit equation (2.7) reduces to the expression

$$\frac{dp_r}{dr} = -\frac{m\rho}{r^2} + \frac{2}{r}(p_\perp - p_r)\,. \tag{2.11}$$

The anisotropy term is thus of Newtonian origin,[3] at least in the special case of spherical symmetry.

A solution to equations (2.7)–(2.10) is possible only when boundary conditions have been imposed. As in the case of isotropy we require that the interior of any matter distribution be free of singularities, which imposes the condition $m(r) \to 0$ as $r \to 0$. Assuming that p_r is finite at $r = 0$, we have $\nu' \to 0$ as $r \to 0$. Therefore the gradient dp_r/dr will be finite at $r = 0$ only if $(p_\perp - p_r)$ vanishes at least as rapidly as r when $r \to 0$. We shall, in fact, require that

$$\lim_{r\to 0} \frac{p_\perp - p_r}{r} = 0\,. \tag{2.12}$$

Finally the radius R of a stellar model will be determined by the condition[4] $p_r(R) = 0$ [note that we do not necessarily require that $p_\perp(R)$ vanish; however, we will assume that $p_r(r)$, $p_\perp(r) \geq 0$ for all $r < R$]. An exterior vacuum Schwarzschild metric is always matchable to our interior solution across $r = R$ as long as $p_r(R) = 0$, even though $\rho(R)$ and $p_\perp(R)$ may be discontinuous.

IV. INCOMPRESSIBLE MATTER

The qualitative effects of anisotropy may be seen by considering the idealized case of an incompressible fluid for which the energy density is independent of coordinates: $\rho(r) = \rho_0$. It will be recalled that this is the only case in which the isotropic structure equations are explicitly integrable. As a second equation of state, we will assume that

$$p_\perp - p = Cf(p, r)(p + \rho)r^n\,. \tag{3.1}$$

In equation (3.1) and hereinafter we omit the subscript r on the radial pressure. The constant C measures the relative strength of the anisotropy and may have either sign, while the constant $n > 1$. The anisotropy may vary with position in the star, and may depend nonlinearly on the radial pressure. These effects are contained in $f(p, r)$. For $n > 1$, equation (3.1) satisfies the boundary condition (2.12). In general, the functional form of equation (3.1) will be determined by the exact nature of the interactions between particles. If the curvature of space-time is comparable to the electron Compton wavelength (matter densities $\rho \simeq 10^{49}$ g cm^{-3}), or if it varies significantly over distances characterizing collective effects, then relativity may contribute to $p_\perp - p$ as well (Bowers and Zimmerman 1973).

In this section we are primarily interested in the type of effects which can result from anisotropy, and in their importance. We will therefore make two simplifying assumptions. First we require that the anisotropy vanish at the origin quadratically ($n = 2$). Next we assume that at least part of the anisotropy is gravitationally induced, and that it is nonlinear in the pressure. A simple form which exhibits these effects is

$$f(p, r) = \frac{\rho + 3p}{1 - 2m/r}\,. \tag{3.2}$$

It must be emphasized that this hypothetical model is chosen primarily because it allows us to explicitly integrate the structure equations rather than for any particular physical reasons. It will be noted that $p_\perp$ does not vanish at the surface ($p = 0$). However, as will be seen in § V, the results obtained here are qualitatively independent of the form of equation (3.1).

Substituting equations (3.1)–(3.2) into (2.7) and using (2.8), we obtain, after a slight rearrangement of terms,

$$\frac{dp}{dr} = -\left(\frac{4\pi}{3} - 2C\right)\frac{(\rho_0 + p)(\rho_0 + 3p)r}{1 - \frac{8}{3}\pi\rho_0 r^2}\,, \tag{3.3}$$

where we have used the relation $m(r) = 4\pi\rho_0 r^3/3$ which follows from the assumed coordinate independence of ρ_0 and (2.9). Integrating, we obtain the radial pressure

$$p = \rho_0\left[\frac{(1 - 2m/r)^Q - (1 - 2M/R)^Q}{3(1 - 2M/R)^Q - (1 - 2m/r)^Q}\right], \tag{3.4}$$

where the total mass $M = m(R)$, R is the radius of the system, and $Q = \frac{1}{2} - 3C/4\pi$. The central pressure is given by

$$p_c = \rho_0 \frac{1 - (1 - 2M/R)^Q}{3(1 - 2M/R)^Q - 1}\,. \tag{3.5}$$

$C = 0$ gives the expected result for an isotropic fluid. An equilibrium configuration exists for all values of M/R such that p_c is finite. The critical model results for that value of M/R such that p_c becomes infinite. From equation (3.5) this occurs when the denominator vanishes, or when

$$(2M/R)_{\rm crit} = 1 - (\tfrac{1}{3})^{2/1-\xi}\,, \tag{3.6}$$

where $\xi \equiv 3C/2$.

Since physically reasonable models must have $(M/R)_{\rm crit} > 0$, we therefore require $\xi \leq 1$ or $C \leq \frac{2}{3}$. From equation (3.3) this automatically guarantees that dp/dr will never become positive. The ratio $(2M/R)_{\rm crit}$ is plotted as function of ξ in figure 1. In the limit of weak anisotropy $|C| \ll 1$, equation (3.6) is

[3] This suggests that the "regenerative" pressure effect familiar in the isotropic case will not directly involve the tangential pressure $p_\perp$.

[4] As in isotropic models, the inclusion of an "atmosphere" will result in complications without affecting the generality of our conclusions as to maximum mass or redshift. If an atmosphere is included, we should require $p(r) = p(r) = P_{\rm ph}$ the "photospheric" pressure, where $r = R$.

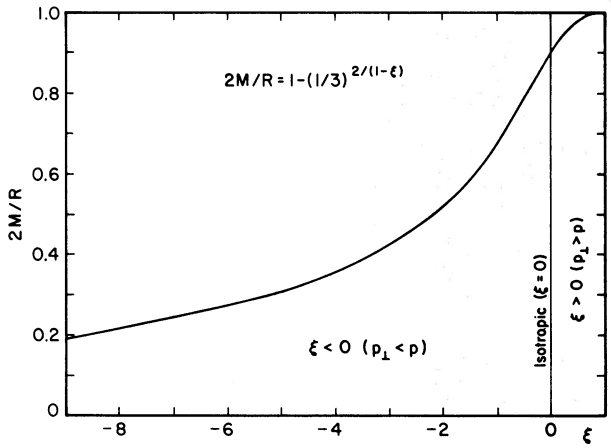

FIG. 1.—The ratio of the gravitational (Schwarzschild) radius $2M$ to stellar radius R for a spherically symmetric distribution of anisotropic matter with constant energy density $\rho = \rho_0$. The anisotropy is described by equations (3.1)–(3.2). For $\xi > 1$ the tangential pressure $p_\perp$ exceeds the radial pressure p; for $\xi < 1$, $p_\perp < p$. The isotropic case corresponds to $\xi = 0$. The assumed anisotropy can increase the maximum equilibrium mass by as much as 19% ($\xi = 1$).

approximately

$$(2M/R)_{\rm crit} \simeq \tfrac{8}{9}[1 + \tfrac{1}{8}\xi + \theta(\xi^2)] \,. \qquad (3.7)$$

In the opposite limit (strong anisotropy) $\xi = 1$, and we reach the Schwarzschild limit:

$$(2M/R)_{\rm crit} = 1 \,. \qquad (3.8)$$

Equation (3.7) gives the expected result for $\xi = 0$ (isotropic fluid) that $R = 9r_g/8$, where $r_g = 2M$ is the gravitational radius.

For a given ρ_0 and C the critical mass is

$$M_{\rm crit} = \left(\frac{3}{32\pi\rho_0}\right)^{1/2}[1 - (\tfrac{1}{3})^{2/(1-\xi)}]^{3/2} \,. \qquad (3.9)$$

Equation (3.9) and figure 1 show that the critical mass is less than the isotropic value when $C < 0$. When $C > 0$, the critical mass exceeds the isotropic limit. Specifically, for a given value of ρ_0, the ratio of the anisotropic critical mass M_a to the critical isotropic mass M_i ($\xi = 0$) can approach arbitrarily close to

$$M_a(\xi = 1)/M_i \simeq 1.19 \,, \qquad (3.10)$$

where $M_a(\xi = 1)$ corresponds to a configuration uniformly filling up to its own Schwarzschild radius. This represents a maximum of $\leqslant 19$ percent increase in the stable mass. The results of relativistic model calculations for slowly rotating fully relativistic isotropic stars indicate that the increase in maximum equilibrium mass is likely to be less than 20 percent (Thorne 1971; Hartle and Thorne 1968). The maximum mass shift due to anisotropy (at least in this idealized model) is therefore seen to be comparable to the rotational correction, provided that we allow arbitrarily large anisotropy.

In addition to altering the maximum stellar mass, anisotropy will in general affect the redshift z at the surface (SRS), given by

$$z = (1 - 2M/R)^{-1/2} - 1 \,. \qquad (3.11)$$

The SRS for critical configurations is, from equation (3.6),

$$z_{\rm crit} = 3^{1/1-\xi} - 1 \,. \qquad (3.12)$$

For $\xi = 0$ we recover the well-known isotropic result $z_{\rm crit} = 2$. The introduction of anisotropy (with $C > 0$) removes the upper limit, since as $\xi \to 1$ the critical stellar surface can be arbitrarily close to the horizon, and $z_{\rm crit}$ arbitrarily large. Thus, at least within the context of incompressible models, anisotropy is capable of explaining quasar redshifts larger than 2. For example, the quasar 4C 25.5 has an observed $z = 2.358$. To account for this value with equation (3.12) we need only $C = 0.063$.

Although the above results suggest that anisotropy may play a role in relativistic stellar structure and the theory of quasars, this model is too hypothetical to allow us to draw firm conclusions. We therefore turn to a different kind of analysis which is independent of the form of the anisotropy or equation of state.

V. BONDI ANALYSIS FOR ANISOTROPIC MATTER

An elegant method has been employed by Bondi (1964) to obtain a model-independent upper limit for the SRS of a locally isotropic distribution of matter within the framework of general relativity. This method may be readily generalized to the case of locally anisotropic matter. We discuss the maximum SRS in terms of this approach, and integrate numerically the structure equations for an isothermal model with $p = \frac{1}{3}\rho$ and some special form of $p_\perp$.

a) Bondi Analysis for Anisotropic Matter

To examine the constraints which are imposed on the maximum SRS by the equations of hydrostatic equilibrium, we now, following Bondi, make the following change in variables:

$$u = m(r)/r \,, \qquad (4.1)$$

$$v = 4\pi r^2 p \,, \qquad (4.2)$$

$$x = (p_\perp - p)/p \,, \qquad (4.3)$$

where x, the *fractional anisotropy*, is in general a function of u, v, and r. Equations (2.7)–(2.9) then reduce to the form

$$\frac{1}{r}\frac{dr}{du} = \frac{(1 - 2u)(dv/du - \alpha)}{H(u, v, x)} \,, \qquad (4.4)$$

$$4\pi r^2 \rho = u\,\frac{(dv/du - \beta)}{(dv/du - \alpha)} \,. \qquad (4.5)$$

The functions α, β, and H are defined by

$$H = 2v(1 - 2u)(1 + x) - (u + v)^2 \,, \qquad (4.6)$$

$$\alpha = -(u + v)/(1 - 2u) \,, \qquad (4.7)$$

$$\beta = -\frac{v}{u}\frac{(5 - 2u - v)}{(1 - 2u)} - \frac{2vx}{u} \,. \qquad (4.8)$$

Since r is defined implicitly in terms of u and v via equation (4.4), in the following we will always treat $x = x[u, v, r(u, v)]$ as a function of u and v only. Note that α is independent of x, and that equations (4.4)–(4.8) reduce to Bondi's expressions for $x = 0$ (isotropy). The expressions above may be used to divide the (u, v)-plane into several regions of interest (fig. 2). The construction of each curve has been discussed by Bondi for the special case $x = 0$ (Bondi 1964). The introduction of anisotropy results only in quantitative shifts of these curves. We outline below the results which we have obtained.

In analyzing equations (4.4)–(4.8), the following assumptions will be made:

i) The quantities p, $p_\perp$, and ρ are finite at the origin, and nonnegative everywhere;

ii) The region $2m/r > 1$ (inside the horizon) will not be considered, since no static equilibrium configuration can exist inside its own horizon.

iii) The model radius R is given by the solution of $p(R) = 0$.

The curves A = constant are the integral curves of $dv/du = \alpha$ and correspond to mass shells of infinite density (eqs. [4.4]–[4.5]). *Their position in the* (u, v)-*plane is unaffected by anisotropy*. The curves $H(u, v, x) = 0$ divide a given model into a core ($H > 0$), and an envelope ($H < 0$). The position of this curve is determined by the amount and nature of the anisotropy. For $x > 0$ ($p_\perp > p$) it is shifted away from the v-axis and intersects curves $dv/du = \alpha$ corresponding to larger values of the constant A. For $x < 0$, the opposite trend results (see fig. 2). Note that the boundary condition $x = 0$ at $u = v = 0$ guarantees that $H = 0$ always passes through the origin, independent of the form of x. Every equilibrium configuration will for a given equation of state and x be represented by a particular structure curve in the (u, v)-plane. The general properties of such a curve are: (1) $v = 0$ when $u = 0$, and $dv/du = 3p_c/\rho_c$ at the origin; (2) the curve lies in the quadrant $u \geq 0$, $v \geq 0$; (3) the surface condition $p = 0$ requires that the curve return to $v = 0$ except for infinite-radius models (the further constraint $\rho = 0$ at the surface requires that the curve meet the u-axis tangentially); (4) the structure curve moves toward nondecreasing values of A in the core ($H > 0$), crosses $H = 0$ tangentially to an A = constant curve, and in the envelope ($H < 0$) moves toward nonincreasing values of A; (5) the structure curve is constrained to move between the two curves $dv/du = \alpha$ (infinite density) and $dv/du = \beta$ (zero density).

The requirements (1), (2), and (5) arise from assumption (i), while requirement (3) is a statement of the surface boundary condition $p(R) = 0$. The requirement (4) is a consequence of our choosing the parameter r to increase outwardly ($dr > 0$). Figure 2 shows two possible structure curves for realistic systems with the surface boundary conditions $p = \rho = 0$ and $p = 0$ but $\rho \neq 0$.

As in the isotropic case, the maximum value of u_s at the surface is obtained for a model whose structure curves rises as rapidly as possible in the core and descends as gradually as possible in the envelope. This corresponds to a model with nonzero pressure but zero energy-density in the core, and a thin infinite-density shell for an envelope. The resulting curve in the (u, v)-plane moves from the origin to the intersection of $H = 0$ and some $dv/du = \alpha$ curve A_2, and then descends along A_2 until the u-axis is reached. In the case $x = 0$ the maximum value of u_s thus obtained is ~ 0.485 ($\Leftrightarrow A = 9 \Leftrightarrow z \simeq 4.77$) (Bondi 1964). Examination of figure 2 shows that anisotropy with $x > 0$ ($x < 0$) results in an increase (decrease) in the maximum SRS.

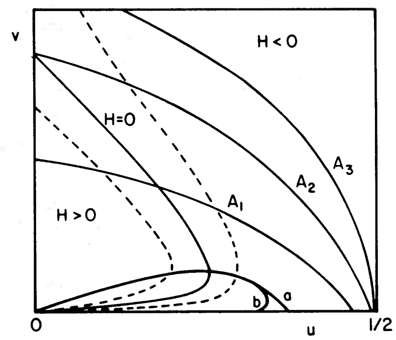

FIG. 2.—Bondi diagram of the (u, v)-plane. The curve $H = 0$ separates the core ($H > 0$) and envelope ($H < 0$) regions. The position of the curve depends on the fractional anisotropy x, shifting toward (away from) the v-axis as $|x|$ increases, where $x < 0$ ($x > 0$). The curves A = constant (independent of x) correspond to thin mass shells of infinite density, and approach the vertical asymptote $u = \frac{1}{2}$ as $A \to \infty$. For a given equation of state and x, the structure of a finite radius "star" is given by a curve in the (u, v)-plane which begins at $u = v = 0$ and ends on the u-axis, crossing $H = 0$ tangent to one of the A = constant curves. Two representative structure curves for realistic matter are shown (*heavy lines*) which correspond to the surface boundary conditions (*a*) $p(R) = 0$; (*b*) $p(R) = \rho(R) = 0$. Notice that these curves fall much more rapidly than the thin mass-shell curves A = constant.

Since the maximum SRS is achieved by moving along A = constant curves in the envelope which are independent of anisotropy, we shall concentrate on the core region in the remainder of our discussion.

b) *Maximum Redshift for* $x \neq 0$

For the isotropic case $x = 0$, the limiting SRS is $z_{\max}{}^0 = 4.77$ ($\Leftrightarrow u_{\max}{}^0 = 0.485$), whereas for any physically realizable types of matter a much smaller value is in general expected. We now wish to consider limitations on $z_{\max}$ which are independent of the exact form of the equation of state or the anisotropy [subject only to assumption (i) above]. In order to maximize z, the structure curve must reach a curve $dv/du = \alpha$ with the largest possible constant A when crossing $H = 0$. The coordinates (u_0, v_0) of the point of intersection of $H = 0$ with an A = constant curve is given by the solution of the equations

$$2v(1 - 2u)(1 + x) = (u + v)^2 \qquad (4.9)$$

and

$$v = [A(1 - 2u)]^{1/2} - 1 + u\,. \qquad (4.10)$$

Thus a given value of A is attained at $H = 0$ only if the fractional anisotropy takes a value at (u_0, v_0) equal to

$$x(u_0, v_0) = \frac{(A + 3 - 4u_0 - 4)[A(1 - 2u_0)]^{1/2}}{2\{[A(1 - 2u_0)]^{1/2} - 1 + u_0\}} . \quad (4.11)$$

Any model that reaches this value of A must encounter a maximum anisotropy $x \geq x(u_0, v_0)$. For $A \gg 1$, equation (4.11) tells us that $x(u_0, v_0) \sim A^{1/2}$. An immediate consequence is that $z = \infty$ ($\Leftrightarrow A = \infty$) is attainable only for models with infinite anisotropy: $u_s \rightarrow \frac{1}{2}$ *only if* $x \rightarrow \infty$, *independent of the equation of state or form of anisotropy.*

Conversely, suppose from local considerations we can set an upper limit to the allowed values of x—say, $x \leq x_0$. Then the maximum A attainable for models with the allowed values of x is, from equation (4.11),

$$A_{\max}{}^{1/2} \leq (2 + x_0)(1 - 2u_0)^{1/2} + [(x_0{}^2 + 6x_0 + 1) - 2u_0(x_0 + 1)(x_0 + 4)]^{1/2} . \quad (4.12)$$

$A_{\max}$ takes on its largest value (for fixed x_0) when $u_0 = 0$, in which case

$$\lim_{u_0 = 0} A_{\max}{}^{1/2} \leq 2 + x_0 + (1 + 6x_0 + x_0{}^2)^{1/2} ; \quad (4.13)$$

in this case the curve in the core rises vertically along the v-axis until it reaches $H = 0$. This is the anisotropic generalization of Bondi's thin-mass-shell model, a limiting case. The resulting SRS for some representative values of x_0 are given in table 1. It must be emphasized, however, that for more realistic models, the increase in SRS for a given x_0 is significantly lower than these limits, as will be demonstrated in the numerical example.

c) *Simple Models of Anisotropy*

We have shown above that the maximum A-value (and therefore the maximum SRS) attained by a specific model is essentially determined by the shape of the curve $H = 0$. It is therefore of some interest to illustrate the $H = 0$ curves for some simple models of the anisotropy. Two representative models that satisfy the boundary condition $x = 0$ at $u = v = 0$ are

Case I:

$$x(u, v) = \tfrac{1}{2}av , \quad (4.14)$$

Case II:

$$x(u, v) = \frac{1}{2}\frac{av + bu}{(1 - 2u)} , \quad (4.15)$$

where $a, b \geq 0$ are constants. Case II closely resembles (3.1)–(3.2) except that now $p_\perp = (1 + x)p$ vanishes at the surface ($p = 0$) automatically.

Case I.—The $H = 0$ curves for $a = 0$, 0.1, 0.5, 1.0, and $a \gg 1$ are shown in figure 3. The intersection of $H = 0$ and the v-axis is given by $v = 2/(1 - a)$, for $a \leq 1$. Therefore, arbitrarily large values of A ($\Leftrightarrow z \rightarrow \infty$) are in principle attainable if $a = 1$ through massless core models (at the expense of encountering infinite anisotropy, of course). For $a > 1$, however, $H = 0$ does *not* intersect the positive v-axis at all, and all models must have a massive core.

Case II.—The $H = 0$ curves for this two-parameter family are shown in figure 4 for $a = 0$, 0.5, and 1.0 and for $b = 0$, 1, 2, 3. In this case the parameter a shifts the $H = 0$ curves vertically (up for increasing a), while b shifts the curve horizontally (increasing u with increasing b). The intersection of $H = 0$ and the v-axis is again given by $v = 2/(1 - a)$. For $a = 1$, the $H = 0$ curves approach vertical asymptotes defined by $u = 2/(6 - b)$. By assumption (ii) above, we limit discussion to models lying within $u \leq \frac{1}{2}$. To ensure

TABLE 1

MAXIMUM A, u_s, AND SURFACE REDSHIFT z FOR VARIOUS VALUES OF THE FRACTIONAL ANISOTROPY x

x	$(A_{\max})^{1/2}$	$u_{\max}$	$z_{\max}$
0 (isotropic)	3	0.485	4.77
0.1	3.37	0.488	5.73
0.5	4.56	0.494	8.01
∞	∞	0.500	∞

NOTE.—The model corresponds to "massless" pressure in the core, and a thin shell envelope with infinite density. Only for infinite anisotropy ($x = \infty$) is the horizon reached (z infinite).

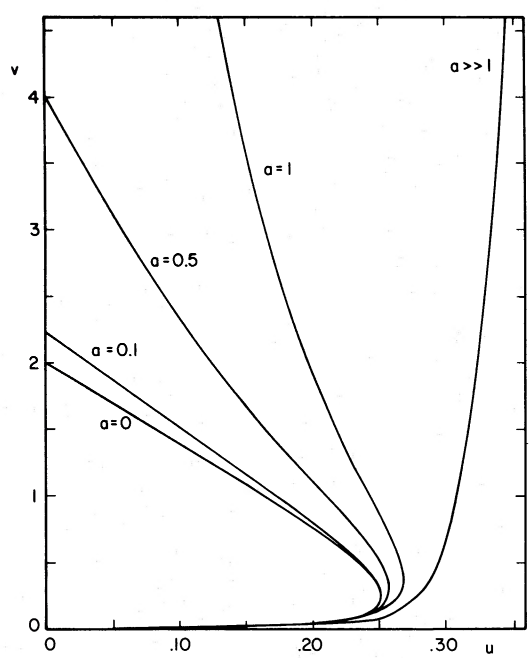

FIG. 3.—The $H = 0$ curves separating core and envelope regions for the fractional anisotropy (4.14) and $a = 0$, 0.1, 0.5, 1.0, and $a \gg 1$. The last two cases do not intersect $u = 0$ for $v > 0$.

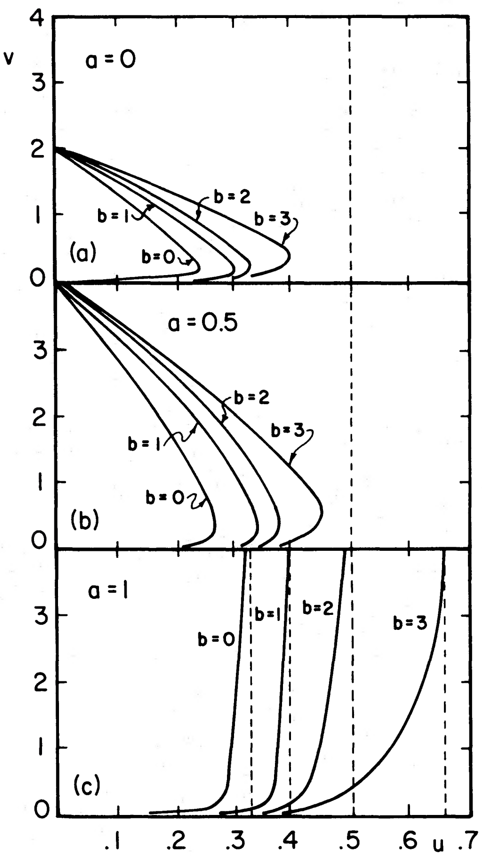

FIG. 4.—The curve $H = 0$ which separates core and envelope regions for the fractional anisotropy (eq. [4.15]). The parameter $b = 0, 1, 2$, and 3 in each figure. The curves in (*c*) approach the dashed lines $u =$ const. asymptotically. The curve $a = 1$, $b = 3$ does not satisfy condition (ii) in § V. For $a = 1$ all curves can lead to model with arbitrarily large surface redshift.

this, it is convenient to restrict the parameters a and b to: $a \leq 1$ and $b \leq 2$. It is again obvious from equation (4.15) that any model reaching $u = \frac{1}{2}$ must have $x \rightarrow \infty$.

d) *A Numerical Example*

We now present the results of the numerical integration of equations (4.4)–(4.5) for the equation of state $p = \frac{1}{3}\rho$ with the anisotropy as discussed in Case II above. We consider the case $a = 1$ and $b = 0$. To facilitate comparison with the corresponding isotropic Bondi models, we integrate only to the boundary of the core ($H = 0$), then match it with a thin-mass-shell ($A =$ const.) envelope (Bondi's model A3). Other more realistic envelope models could of course be used. But since we are primarily interested in the increase in maximum SRS, this case suffices to illustrate the point. Also since the $p = \frac{1}{3}\rho$ curve lies rather low in the (u, v)-plane (fig. 5) in the isotropic case, we expect that the maximum x encountered in our model will not be outrageously large (as will be verified by numerical integration).

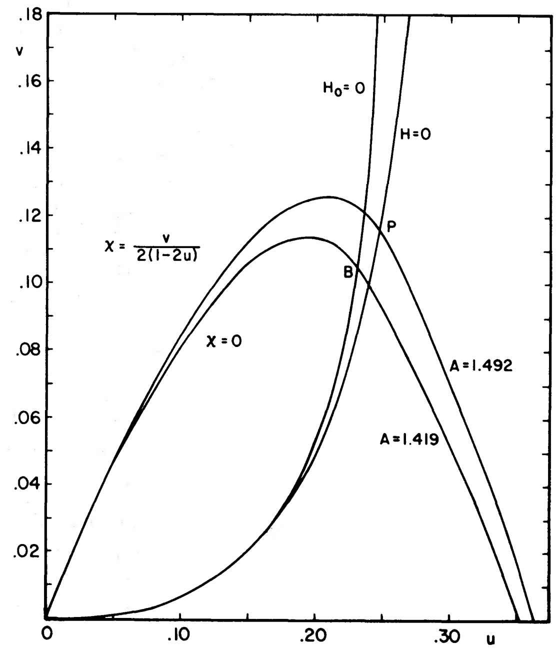

FIG. 5.—Structure curves for two equilibrium stellar models of § V: (*a*) a core with $p = \frac{1}{3}\rho$, fractional anisotropy (4.15), and a thin-mass-shell envelope given by (4.11) with $A = 1.492$; (*b*) the same but with $x = 0$. The isotropic curve lies below the anisotropic curve for all values of u. The curve $H_0 = H(u, v, 0) = 0$ and $H = H(u, v, x) = 0$ are also shown. The points P and B mark the separation of anisotropic and isotropic models into core and envelope. The structure curves follow $A =$ const. curves (thin mass shells) from the points P and B to the u-axis.

The results of integrating equations (4.4) and (4.5) as described above are shown in figure 5. We also plot the corresponding $x = 0$ model for comparison (Bondi 1964). The curve ($x \neq 0$) intersects $H = 0$ at $(u_0, v_0) = (0.2463, 0.1163)$, which corresponds to $A \simeq 1.492$. Employing a thin-mass-shell envelope with this value of A leads to $u_s \simeq 0.365$ and a SRS

$$z_{\text{aniso}} = (1 - 2u_s)^{-1/2} - 1 \simeq 0.923\,. \tag{4.16}$$

In the isotropic limit ($x = 0$) Bondi has shown that $u_s \simeq 0.352$, which gives

$$z_{\text{iso}} \simeq 0.838\,. \tag{4.17}$$

A comparison of expressions (4.16) and (4.17) shows that the SRS is increased by ~ 10 percent. The fractional anisotropy necessary to produce this is $x \simeq 0.114$ at the point (u_0, v_0). The incorporation of a more nearly realistic envelope will lead to a lower u_s, just as in the isotropic model. In any case, we see in this model that the fractional increase in SRS and the fractional anisotropy are comparable. This indicates that *in realistic situations, the increase in SRS will probably be of the same magnitude as the amount of anisotropy incorporated, independent of the details of* *the model.* (Compare the thin-mass-shell model limits of table 1.)

VI. STABILITY

The previous sections have resulted in upper limits on the static properties of massive objects which include local anisotropy. The stability of such objects is also of importance. Although we shall not solve this problem here, several comments are in order. First we should distinguish two aspects of the problem: (1) stability with respect to macroscopic motions of the system; (2) microscopic stability.

The first is just the requirement that the second variation in total energy (thermodynamic + "gravitational") at constant entropy be positive. This leads to an equation which sets a lower limit on the adiabatic index for a macroscopically stable system (Chiu 1968). A solution of that equation requires a knowledge of the anisotropy. Since no realistic expressions for $p_\perp - p$ exist, we shall not pursue this aspect of the problem further.

Perhaps a more relevant question is that of microscopic stability. This is usually implicitly included via the equations of state. Specifically one assumes that the chemical potential considered as a function of density is nondecreasing. It is the onset of microscopic instability which leads for example to a phase transition from a normal to a superfluid state and, in the case of neutron-neutron interactions (§ II), to a possible anisotropic contribution to the system. It is also possible that anisotropic effects may exist only in a finite density range. Returning to the example of a superfluid neutron phase, one might expect dissociation of the *n*-*n* pairs with increasing density at a point where the Fermi energy of the particles exceeds the pairing energy. A more detailed discussion of these questions requires a specification of the anisotropy.

VII. CONCLUSIONS

The equations of hydrostatic equilibrium for a locally anisotropic, static, and spherically symmetric distribution of matter have been set up and solved for two particular cases: (1) an incompressible model $\rho(r) = \rho_0$; (2) an isothermal model with $p(r) = \frac{1}{3}\rho(r)$. In the first example a highly idealized expression for the anisotropy was used, which has the advantage of leading to simple analytic solutions. The second model incorporates a more nearly plausible expression for the anisotropy and was evaluated numerically. On the basis of these investigations we draw the following conclusions for anisotropy in spherically symmetric matter:

1) The stress-energy tensor may be put in the form

$$T_\nu{}^\mu = \mathrm{diag}\,(\rho, -p, -p_\perp, -p_\perp)\,.$$

2) If the fractional anisotropy $x = (p_\perp - p)/p$ is positive, the maximum equilibrium mass and surface redshift exceed their corresponding isotropic ($x = 0$) values; if $x < 0$, then $M_{\max}$ and $z_{\max}$ are less than their isotropic values.

3) Although $z_{\max} = 4.77$ for isotropic matter, anisotropy with $x > 0$ permits (in principle) an arbitrarily large $z_{\max}$.

4) An infinite surface redshift is possible for equilibrium anisotropic matter, but only if the fractional anisotropy is infinite.

These results have been shown to be quite general, subject only to the conditions that p and ρ be everywhere nonnegative, finite at the origin, and $p_\perp - p = 0$ at the origin.

The physics of relativistic stars and of quasars has to date been based on the assumption that matter is locally isotropic. Our study indicates that anisotropy—if present in the density range expected for relativistic stars (densities up to at least 10^{15} g cm^{-3})—*may* have a nonnegligible influence on such parameters as the maximum equilibrium mass and surface redshift. A complete theoretical understanding of neutron stars and the mass limits against gravitational collapse to black holes, as well as the possible connection between quasars and relativistic compact objects, requires a knowledge of the possible anisotropic properties of dense matter. In particular, we need to determine the density ranges for which anisotropy is possible. In situations of astrophysical interest we might expect to find anisotropy in relativistic systems which are in rotation, or which contain strong magnetic fields. In these cases, however, space-time will no longer be spherically symmetric, and the equations of hydrostatic equilibrium must be modified. Although the spherically symmetric case is less difficult to solve, it may be necessary to include anisotropy in rotating stellar models for consistency. The reason for this is that in fully relativistic slowly rotating models the mass is increased by no more than $\sim$20 percent (Thorne 1971). However, we have seen (at least in terms of the incompressible model) that large anisotropy could in principle result in a comparable mass change with the direction dependent on the sign of x, the fractional anisotropy. However, further speculation on the magnitude of these effects is academic until we have a better understanding of anisotropic equations of state in dense matter.[5]

Finally we point out that at least one of the difficulties involved in associating quasars with local objects—the apparent inability to obtain surface redshifts greater than about $z_{\max} \simeq 2$ (for realistic systems)—might be eliminated by employing anisotropic stresses, although in realistic models this seems in general to involve inconceivably large anisotropies.

We would like to thank Dr. J. Campbell for help with some of the numerical work.

[5] One plausible physical constraint on the magnitude of the anisotropy might be a "tangential causality condition" $(v_s)_{\rm tang}{}^2 \equiv (\partial p_\perp/\partial \rho)_s \leq c^2$, where s is the entropy.

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