Q. Jl R. astr. Soc. (1972) 13, 40–62.

The Description of Foucault's Pendulum

W.B.Somerville

(University of London Observatory)

'Thus we may see,' quoth he, *'how the world wags.'**

SUMMARY

It is explained how the workings of Foucault's pendulum can be understood without recourse to a mathematical treatment. Computer-drawn pictures show the path of the bob over the ground, including the entertaining case where it swings across the equator. Incidental comments are made on the treatment given in some of the astronomical textbooks.

1. INTRODUCTION

Foucault's pendulum experiment provides a clear and convincing demonstration of the rotation of the Earth. However, it is convincing only if it is performed competently and with care; and it is clear only if the spectator has some understanding of how it is that the rotation of the Earth can affect the swing of a pendulum.

The observed result is that the plane of swing of a pendulum changes with time. In the Northern Hemisphere, the plane of swing moves in the same sense as the hands of a clock which is face up on the ground; in the Southern Hemisphere it moves in the opposite sense. The rate of rotation of the plane of swing is independent of the period of the pendulum and is such that the plane goes round once, through 360°, in a time†

$$T = 24\ \text{hr}/\sin\phi$$

where ϕ is the latitude. That is, the plane of swing would go round in that time if the pendulum could keep swinging for so long. At the pole, $T = 24$ hr. At the equator, T is infinite: that means that the plane of swing does not change at all.

*Shakespeare (1623).

†Actually, $T = (23^{\text{h}}\ 56^{\text{m}}{\cdot}03)/\sin\phi$, for it is the sidereal day rather than the mean solar day which is involved. The difference is scarcely significant.

PLATE I. Foucault's pendulum. (Crown Copyright, Science Museum, London)

[To face page 40

For practical reasons, certain conditions must be satisfied if the experiment is to succeed. The suspension must allow the pendulum to swing freely, without torque, in any plane. The bob must be heavy and the string long, to reduce the effects of air resistance and of random currents. The pendulum must be released from rest smoothly, to ensure that it does indeed swing in a plane. Plate I shows the Foucault pendulum at the Science Museum, South Kensington. The mass is 30 lb and the length 79 ft. It is released by burning the thread which restrains it. The period of oscillation of the pendulum is about 10 s; the plane of swing moves through 360° in 30·6 hr, or rather through 11° 45′ in 1 hr.

The movement of the plane of swing is a consequence of the Coriolis force produced by the rotation of the Earth. The Coriolis force, which is analogous to the centrifugal force, is a fictitious force introduced to describe motion relative to a rotating frame of reference (a rotating set of coordinate axes). Any moving object, viewed from a frame of reference fixed in the rotating Earth, appears to be acted on by this additional, non-physical force. The direction of the Coriolis force is at right angles to the direction of motion and also at right angles to the axis of the Earth's rotation. It depends only on the velocity of the motion and is independent of location; in particular, for motion near the surface of the Earth, it is independent of latitude. However, as with the Foucault pendulum, we are frequently concerned only with that part of the motion which is parallel to the ground. For horizontal motion, the horizontal component of the Coriolis force depends on the latitude; it is zero at the equator and a maximum at the poles. It is independent of the compass bearing of the motion.

The horizontal component of the Coriolis force is in the Northern Hemisphere usually to the right of the direction of motion and in the Southern Hemisphere to the left. This asymmetry comes from the relation of the two hemispheres to the axis of rotation. The Earth rotates in a right-handed (clockwise) sense about the south–north axis. The South Pole goes into the ground and the North Pole comes out of it.

Because of the Coriolis force, the bob of a pendulum in the Northern Hemisphere moves a little to the right in each swing. Over a period of time, the plane of swing is observed to have rotated, in the clockwise direction.

Effects of the Coriolis force are seen in other ways. Wind is a flow of air from a region of high barometric pressure to one of low pressure and it is observed that winds do not flow straight across the isobars. Around a centre of pressure, the winds spiral out or in. The prevailing winds in the Northern Hemisphere are from the north-east or the south-west, rather than from north or south. Other effects are that a river (in the Northern Hemisphere) erodes the right bank faster than the left and that an object dropped from a height tends to fall slightly to the east; the latter effect, arising from a vertical motion, is a maximum at the equator and zero at the pole. When a rocket is sent over a large distance, it is necessary to allow for the Coriolis force lest it be sadly misplaced.

It is popularly supposed that the sense of the vortex which develops when bath water runs out through the plug-hole is determined by the Coriolis force. However, in a definitive experiment Shapiro (1962; see also Andrade 1963) demonstrated that stray currents in the water generally have a dominant effect. To produce with consistency the predicted behaviour, an anti-clockwise spiral in the Northern Hemisphere, it is necessary to have an axially-symmetrical bath tub, to leave the water undisturbed for about 24 hr (which may appear a trifle unhygienic) and to remove the plug with caution. This effect is analogous to a cyclone, for the vortex is generated by the sideways deviation of the water flowing horizontally towards a centre of low pressure, or sink.

The theory of the Coriolis force is well understood and the application to Foucault's pendulum is trivial. It is so trivial that most books on dynamics do not discuss this application at all, or else they set it as an elementary problem. Where it is discussed, consideration usually is limited to an outline of the mathematics, leaving out the physics of the situation. Books which give a reasonable mathematical account include those by Ames & Murnaghan (1929), Becker (1954) and Ramsey (1951). It is not always easy to find the discussion: some books, including Ramsey's, do not have an index. For the dedicated, an outline of the theory is given in the Appendix to this paper.

Books on descriptive astronomy generally mention the motion of Foucault's pendulum as a proof of the rotation of the Earth and generally do not attempt to explain how the rotation of the Earth does produce the motion. This paper has arisen from an attempt to understand in simple, non-mathematical terms what it is that happens when the pendulum swings.

It should perhaps be mentioned that the 'proof' of the rotation of the Earth does not rest on Foucault's experiment. The merit of the experiment is that it demonstrates the rotation using an effect which is clearly visible. The rotation can be demonstrated in other ways. The simplest is to consider the daily motion of the stars. We know now that the stars are not all the same distance from us, held fixed in an invisible, rigid crystal spherical shell. They are individual bodies, at very different distances, and each has its peculiar motion. It is inconceivable that so many individual bodies could be whirling round us so rapidly, in perfect synchronization.

2. HOW THE PENDULUM MOVES

Consider a large pendulum with its point of support on the Earth's axis, vertically above the North Pole. The pendulum swings in a plane, across the pole. There is no sideways force on the bob and it continues to swing in the same plane in space, as the Earth rotates underneath. Viewed from above, the ground rotates anticlockwise under the pendulum. Viewed from the rotating Earth, the plane of swing of the pendulum rotates clockwise: the pendulum moves a little to the right in each swing. After 24 hr, the Earth has returned to its original position and so the plane of swing appears to have rotated round through 360°.

Consider a pendulum at the equator, swinging in the plane of the equator. There is no sideways force on the bob and so it continues to swing in the plane of the equator. The point of support is carried round by the Earth's rotation but it remains in the same plane. Viewed from the rotating Earth, the plane of swing does not change.

It is harder to picture what happens at intermediate latitudes. One way to overcome the problem of understanding the motion of the pendulum is to ignore the motion of the pendulum and concentrate on the motion of the ground. The best account is in the classic *Astronomy* by Russell, Dugan & Stewart (1945). It cannot be put more clearly than in their words:

> 'The northern edge of the floor of a room in the northern hemisphere is nearer the axis of the earth than is its southern edge, and therefore is carried more slowly eastward by the earth's rotation. Hence the floor must skew around continually, like a postage stamp gummed upon a whirling globe, anywhere except at the globe's equator. The pendulum is constrained by the force of gravity to follow the changes in the direction of the vertical, but is otherwise free. Its plane of vibration, therefore, will appear to deviate in the opposite direction from the real skewing motion of the ground, and at the same rate. In the northern hemisphere it apparently moves in the same direction as the hands of a watch; in the southern hemisphere, in the opposite direction.'

This is splendid but incomplete. It does not say anything of how the pendulum itself moves. It is not easy to visualize how the pendulum behaves as it swings over this little, rotating plot.

A similar discussion can be given, recognizing that the rotation of the Earth can be represented by a vector $\boldsymbol{\omega}$ of magnitude $\omega = 360°/\text{day}$ and direction along the S–N axis (defined by the right-hand rule), and resolving it into horizontal and vertical components at P, $\omega \cos \phi$ and $\omega \sin \phi$. The local ground effectively rotates about the local vertical.

A good account of things in these terms is given in the Larousse encyclopedia (Rudaux & de Vaucouleurs 1959). However, this approach again considers the ground and ignores the pendulum.

An interesting contribution to clarity has recently been made by Sher (1969). He derives the skewing rate of the ground from spherical astronomy in terms of the rate of change of azimuth for a star on the horizon. Then—and this is the important part—he argues that if the bob were moving in a fixed direction in space, this direction would follow a star across the sky; but the bob is in fact constrained to move in the horizontal plane, so the motion is the same as the component of motion along the horizon—the change in azimuth—of a star which is at that moment on the horizon.

The description which I have evolved has the advantage of being pictorial.

Consider the diagrams, Figs 1 and 2. The pendulum, at the point P at latitude ϕ, is carried by the rotation of the Earth to the point P′. O is the centre of the Earth, C is the centre of the small circle of which PP′ is an arc and N is the North Pole. The tangents to the surface along the northerly meridians at P and P′ both meet the axis at K.

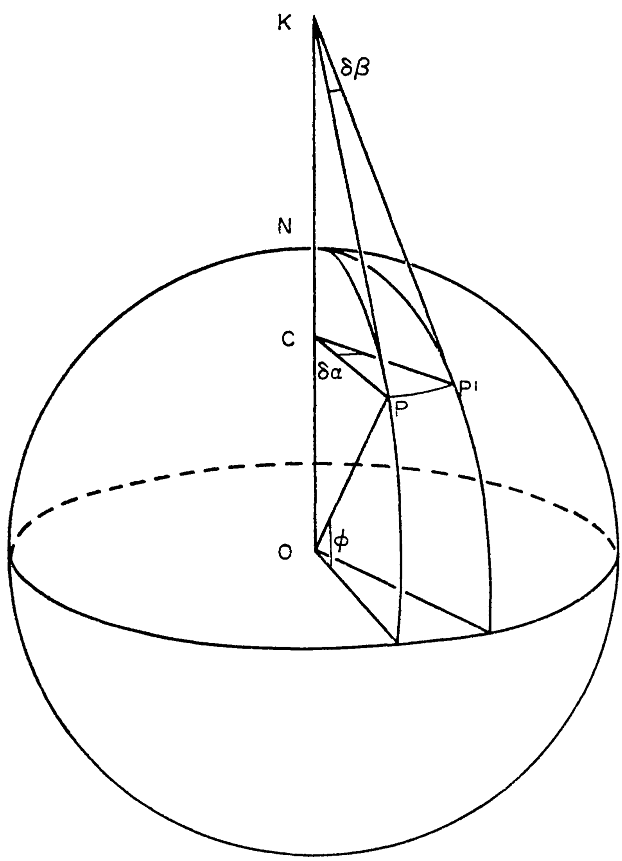

FIG. 1. The rotating Earth. The pendulum is carried from P to P′.

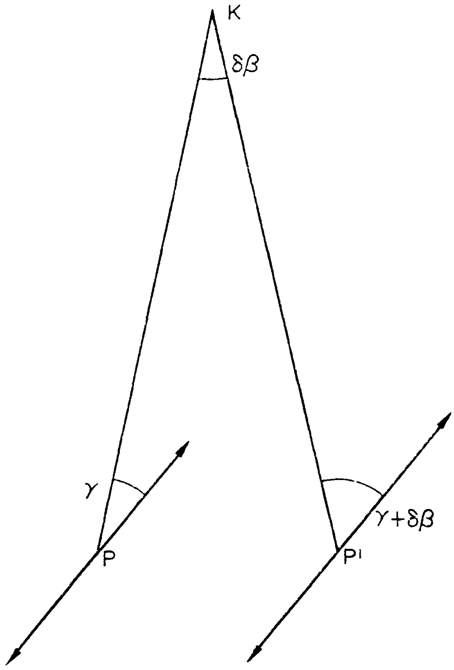

FIG. 2. The line of swing of the pendulum. This is a plane diagram; do not try to see a three-dimensional effect.

The effect arises from the difference between the small angles $\delta\alpha$ and $\delta\beta$. If the time interval between P and P′ is sufficiently small, then*

$$\delta\beta = \mathrm{PP'/KP}, \quad \delta\alpha = \mathrm{PP'/CP}.$$

But also
$$\sin\phi = \mathrm{CP/KP},$$
because OPK is a right angle, and so

$$\delta\beta = \delta\alpha \, . \sin\phi.$$

Now consider the pendulum. For sufficiently small $\delta\beta$ and for small angles of swing, we may regard the bob as moving in the plane PP′K, when it is at P and when it is at P′. If the pendulum is swinging one instant at P in the direction at an angle γ to the north, a few instants later at P′ it is swinging in the direction at $(\gamma+\delta\beta)$ to the north, for there is no sideways force on the bob (viewed from a non-rotating frame, outside the Earth). So, while the Earth rotates through $\delta\alpha$ from P to P′, the plane of swing of the pendulum is seen to move relative

*The exact relation is $\sin(\delta\beta/2) = \mathrm{PP'/2KP}$.
PP′ is the straight line between the points, not the arc.

to the meridian through $\delta\beta$. This angle is independent of the compass bearing γ of the swing.

Adding up for one rotation of the Earth all the elements $\delta\beta$, we see that in 24 hr the Earth rotates through a total angle $\alpha = 360°$ and the plane of swing of the pendulum appears to rotate through a total angle

$$\beta = 360° \,.\, \sin\phi\,.$$

It follows that the plane of swing appears to rotate through 360° in

$$T = 24\ \mathrm{hr}/\sin\phi\,.$$

This discussion applies to the extreme cases, as well as to intermediate latitudes. For a pendulum at the pole, the points P, P′ and K all coincide with N. In the limit as this happens, $\delta\beta = \delta\alpha$; so the period of the plane of swing is $T = 24$ hr. For a pendulum at the equator, K is at an infinite distance and the lines PK, P′K and NK are parallel. $\delta\beta$ is zero, whatever the value of $\delta\alpha$, so the plane of swing of the pendulum does not change. The discussion applies also to a pendulum in the Southern Hemisphere, if the point N is re-named S and the Earth is thought of as rotating from P′ to P.

3. WHAT THE TEXTBOOKS SAY

I have not attempted to make a general survey of all the books on astronomy or dynamics written since the time of Foucault (1851). Instead, I have examined a number of books which are currently available in libraries or in bookshops and accessible to our contemporary students. It is possible that I have missed some splendid accounts of the experiment, complete, accurate and clear, although experience with the books I have examined does not encourage that thought. However, I meet criticism with the defence that any account which I have not seen must be in a book which is not readily available.

It will be obvious that there are very many books which do not mention Foucault's pendulum at all, and not all of these are books on astronomy. Elementary astronomical textbooks sometimes mention the result, without any discussion. At a slightly more advanced level, the simple case of a pendulum at the North Pole is considered. On the other hand, detailed accounts of research about the Earth such as Jeffreys' book (1970) or Kuiper's (1954) are not interested in elementary demonstrations of the planet's motions. We are therefore concerned with introductory textbooks which have pretensions to a reasonable standard of science.

The most disquieting result of this survey is not that many books give accounts which are confusing and obscure, which is something that we must put up with, but that many say things which are misleading and actually wrong. This is serious, for they are attempting to explain in words a complicated phenomenon to someone who probably is very unsure about it. The commonest error is the remark that the pendulum continues to swing in the same plane in space, which in fact is true only in the polar case and for an equatorial pendulum swinging in the plane of the equator. For examples of this, with a variety of tenses, we have 'The earth [at Paris] had turned under the pendulum, which maintained a fixed direction in space' (Payne-Gaposchkin 1954), 'The pendulum . . . will maintain its plane of vibration in the direction in which it was originally started' (Mehlin 1959) and 'The direction of the plane of motion must remain the same in space' (Spencer Jones 1961). The notion of the *direction of a plane* is new to me. I wonder what it means. In a few descriptive words added to his mathematical treatment, Ramsey (1951) also uses it: 'The plane of oscillation of the pendulum remains fixed in direction in space'.

This mis-statement probably comes from a simple misunderstanding. It is true that there is no sideways force and that the only forces acting on the bob of the pendulum are gravity and the tension in the string. But the direction of gravity changes. This makes the plane in space of the swing change also. Only at the pole does the direction of the vertical remain fixed in space and there the plane of swing does remain the same.

Another difficulty arises with the account given by Struve, Lynds & Pillans (1959). They describe the motion correctly and very clearly, for the two extreme cases, but then they attempt to calculate the effect at a particular intermediate latitude. Their mathematical derivation is so simple as to be nearly incomprehensible; on close scrutiny, it turns out to be formally correct but sadly misleading. Consider a pendulum swinging in the plane of the meridian and suppose that the eastward velocity of the bob is the same as that of the ground under the point of support*. At the northern end of the swing, the eastward velocity of the bob is larger than that of the ground there, at the southern end it is smaller. The plane of swing therefore rotates, clockwise over the ground, and the rate of rotation can be derived by elementary geometry. This discussion is correct, but it must surely leave many readers with the impression that the rate of rotation depends on the compass bearing of the swing. If the bob goes less far to the north, surely the effect must be smaller? If the pendulum swings from east to west, surely the effect must be zero? In fact, as we have seen, the effect is independent of the compass bearing γ.

*It may not be, for it depends on how the pendulum was launched. This is not important: it is differences in the eastward velocity which matter.

Payne-Gaposchkin's book contains in a footnote the remark that 'The detailed theory of the Foucault pendulum should take account of the curvature of the earth's orbit, which changes the orientation of a parallel of longitude by about one degree a day.' I suspect that this is intended simply to refer to the difference between the sidereal day and the mean solar day. However, it is worded in a way that can set the mind spinning, with thoughts of a second Coriolis force from the angular velocity about the Sun. This force varies with time of day as well as depending on latitude. Its maximum contribution is $15°/365 = 0°{\cdot}04$ per hour to the rotation of the plane of swing. It is therefore quite negligible.

Despite these criticisms, I consider this still to be one of the best introductory textbooks. A second edition has recently appeared (Payne-Gaposchkin & Haramundanis 1970). The only relevant changes of substance are in the discussion of other effects of the Coriolis force, including 'The Coriolis force . . . causes . . . the vertical deviation of falling bodies from east to west', which seems to have it the wrong way round, and 'It may also cause the deflection of short-range projectiles fired in northerly or southerly directions, but its effect on high altitude projectiles is reversed because of the variation of the projectile's distance from the centre of the earth', which is not very clear. The allusion probably is to the interesting effect of the instantaneous *elevation* of the motion on the horizontal component of the force. For simplicity, consider motion in the meridian plane, at latitude ϕ north. For motion parallel to the Earth's axis, at elevation ϕ from the north horizon, the Coriolis force is zero and there is no deflection. For smaller northerly elevations, the Coriolis force is to the east. For larger northerly elevations, including motion vertically upwards, and for all southerly elevations, the Coriolis force is to the west. Thus, for northerly elevations between ϕ and 90°, the horizontal velocity is to the north and the deviation is to the west—to the left, reversing the usual rule. This does not depend on the altitude of the motion but only on the elevation, which, in any case, is continuously changing. The effect on the total trajectory can be rather complicated.

In what can hardly be called a current textbook, Young (1889) gives an interesting description of the motion at intermediate latitudes, which is both clear and correct (but, I think, inferior to the account given here). Consider Fig. 1. In one day, the line PK traces out a conical surface. If it is cut along a radius PK and spread out flat, this surface forms a sector of a circle, with central angle $\beta = 360° \sin \phi$. Gravity is always perpendicular to this surface and in it there is no sideways force on the bob. In one day, the pendulum goes round the arc defined by the sector. In this surface, the swing remains in the same direction but the direction of the north meridian changes by β.

I have seen a similar account in a modern American book, which I found in the second-hand department of H.K.Lewis' shop. Alas! Not only did I churlishly fail to buy the book, foolishly I did not even note the author's name. I have never seen the book since. Despite a careful search of the astronomical libraries and bookshops of London, I have not found it. I take consolation from the thought that this must surely mean it is not common in this country.*

4. THE PATH OF THE SWING

In our discussion of the motion of the pendulum, we have considered it as moving in a plane but deviating slightly to the side so that the plane appears to rotate with time. That is correct, but a little imprecise. It is interesting to look more closely at the path which the bob follows over the ground.

Mathematically, the motion projected onto the ground can be described as a rotating ellipse. The equations can be solved exactly and it is easy to draw a graph of the track. It is even easier if a computer is available which has graphical output. Indeed, this part of the work was started as a way of using the graph plotter newly fitted to the IBM 1130 computer at the Observatory in Mill Hill. As might have been predicted, the computer work became the major part of the study, wagging the rest.

Fig. 3 shows the path of the bob over the ground. The 'Ratio' is the quantity R defined in the Appendix; it is such that the number of complete double swings, to-and-fro, in the period 24 hr/sin ϕ is $(1+R^2)^{\frac{1}{2}}$. The value $R = 20$ has been chosen for pictorial convenience. For an actual pendulum, the value is much larger—for the Science Museum pendulum, it is almost 9000—and the sideways motion in one swing is correspondingly much less. R need not be an integer.

The centre of the diagram is vertically below the point of support. The pendulum is pulled to one side and released from rest, at the top of the diagram. As the bob falls towards the centre, it deviates to the right and does not pass under the point of support. The first swing, from rest to rest, is shown in Fig. 3 (a); the X's mark equal intervals of time. Each swing is like the previous one and the pendulum moves to the right each time. In Fig. 3 (b), after 20 double swings, the pendulum is almost back where it started. When $(1+R^2)^{\frac{1}{2}}$ is integer, it returns exactly to the starting point.

*The book referred to here is *Astronomy* by A.Krause (Oliver & Boyd 1961, p. 61). It is not of American, but of German origin, which may have been what confused me.

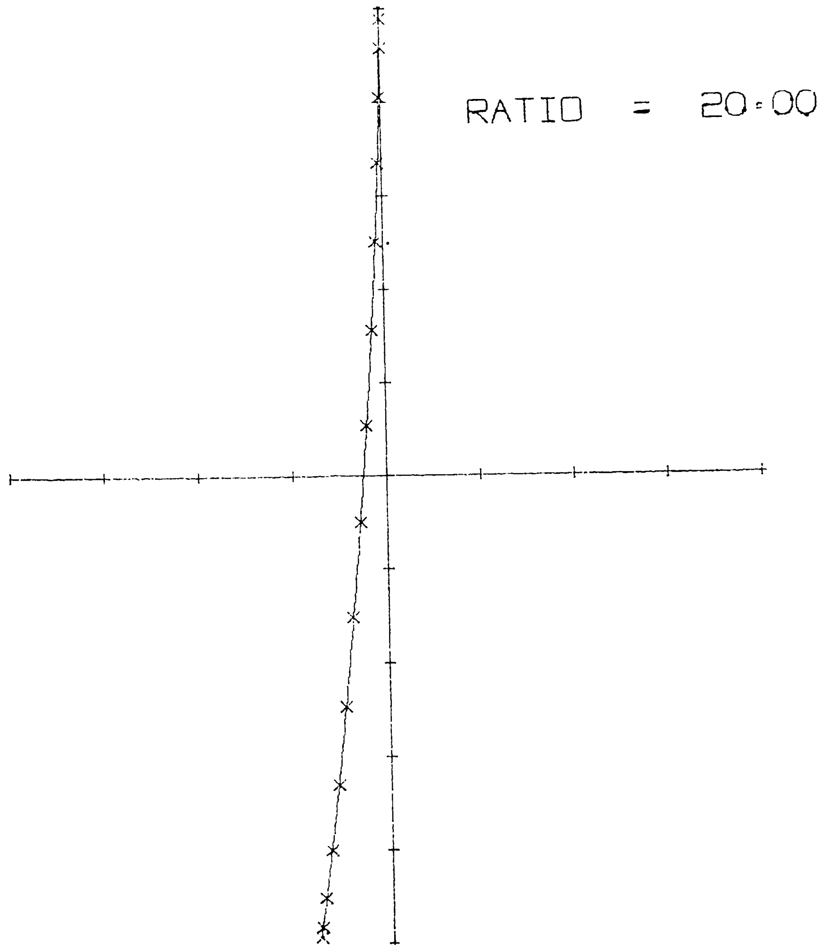

FIG. 3 (a)

Fig. 3 (a) and (b) shows the motion of the bob over the ground, as viewed from the rotating Earth. Fig. 3 (c) shows, for a pendulum at the pole, the motion in a plane fixed in space—the motion which remains the same while the Earth rotates underneath. The inclined axes show how far the Earth has turned, in the time for the one swing shown; this is the same swing which is viewed from the rotating Earth in Fig. 3 (a). The path in space is half an ellipse. The pendulum was released from rest at a point on the rotating Earth: it was moving with the Earth and has an angular momentum about the point of support.

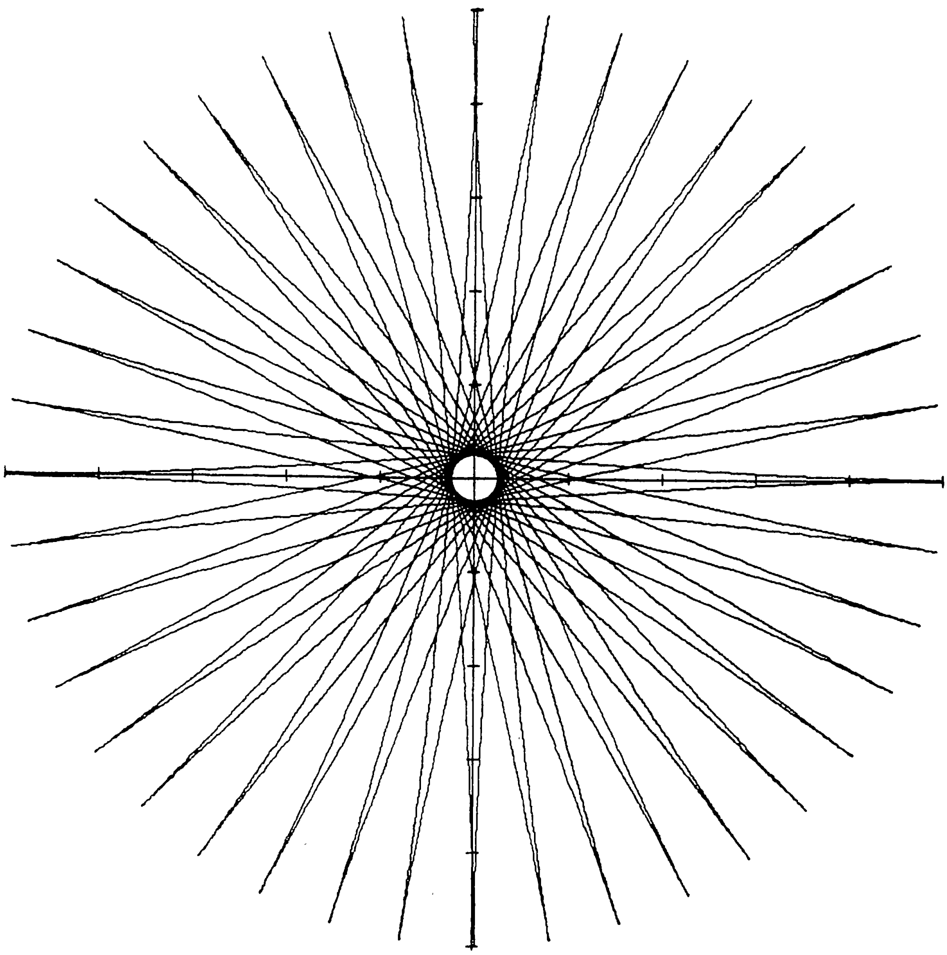

FIG. 3 (b)

It is possible to imagine the pendulum launched in some other way. If, for example, it were started by a horizontal impulse from a position under the point of support—easy to imagine but not easy to achieve—it would have zero angular momentum about the point of support. The result is shown in Fig. 4. The bob starts moving down the page and deviates to the right. At the extremity of its swing, it is moving with a velocity which exactly balances the Earth's rotation: its angular momentum remains zero. After half the period (Fig. 4), it is back under the point of support. For the same R, the plane of swing rotates at the same rate as in the other case, but the detailed path is different. For the polar case, the motion in a plane fixed in space now really is a straight line.

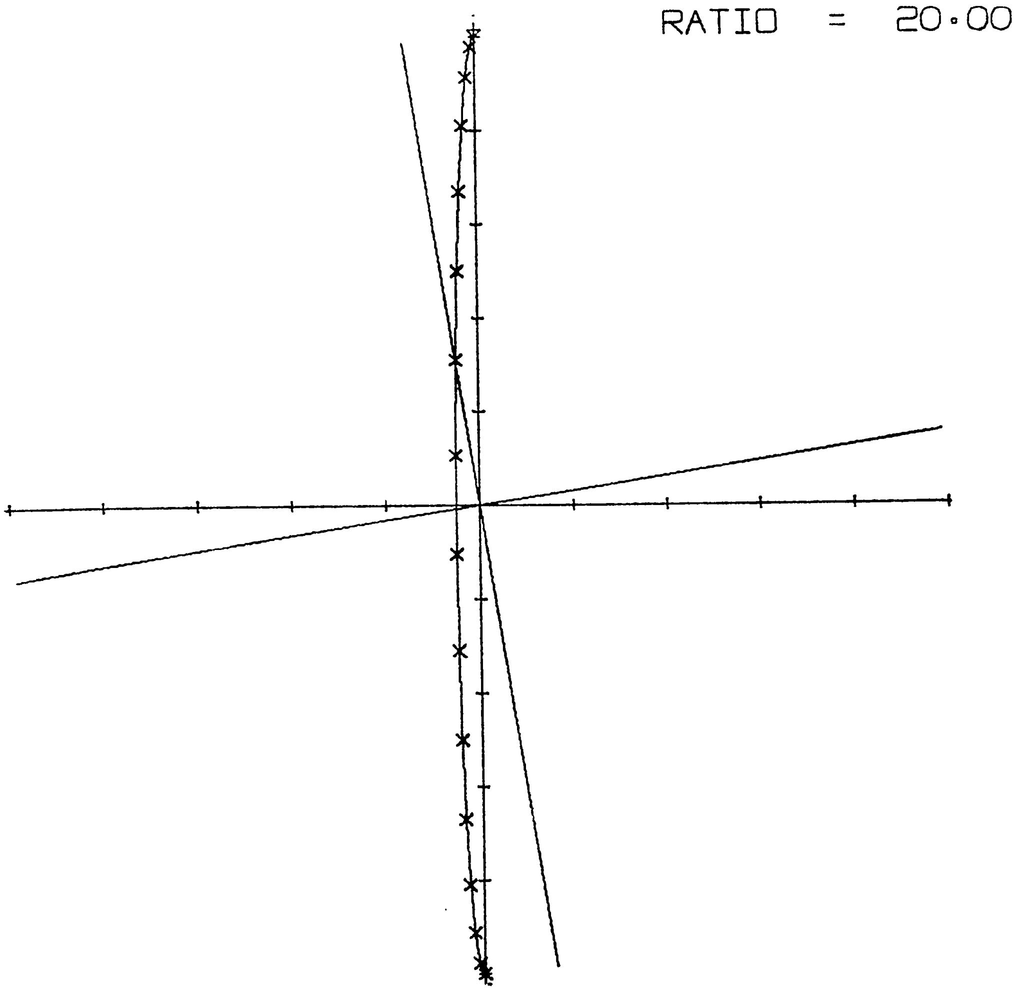

FIG. 3 (c)

FIG. 3 (a), (b) and (c). The swing of a pendulum released in the usual way.

1972QJRAS..13...40S

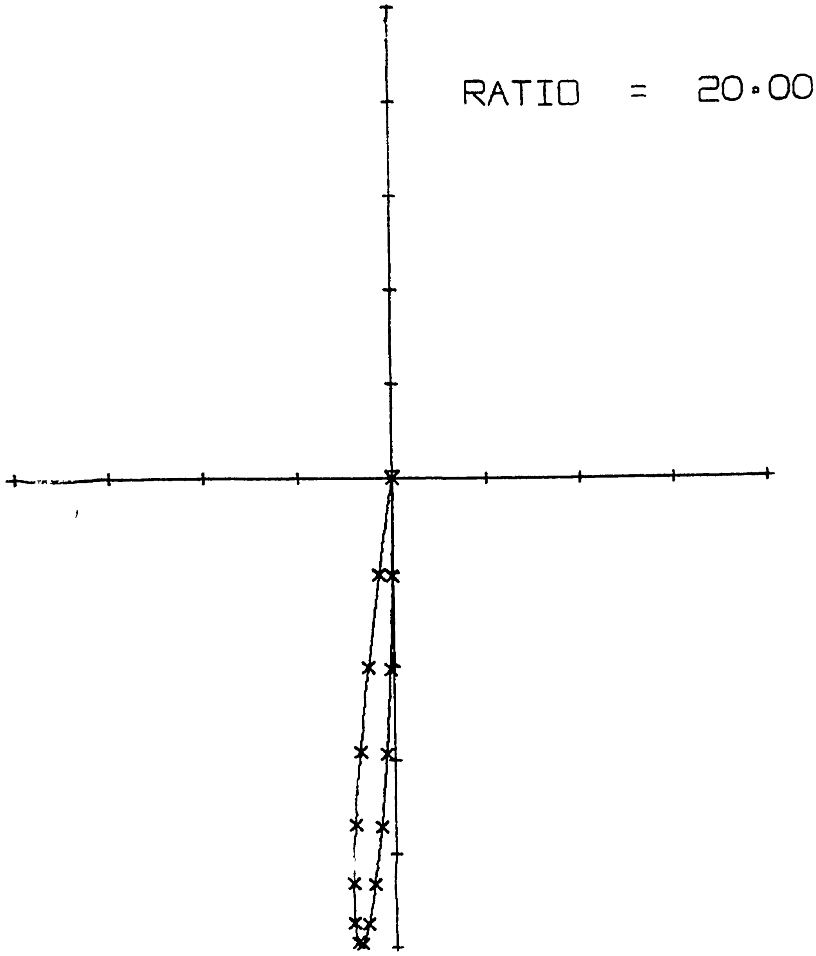

FIG. 4. The swing of a pendulum started with an impulse.

Fig. 5 shows some of the other patterns which are produced, for various numbers of swings with different R from the two initial situations. The larger R is, the smaller is the sideways motion in one swing. As $R \to \infty$ the swing becomes a straight line. For a pendulum released in the conventional manner, the distance of closest approach to the centre is

$$d = a/(1+R^2)^{\frac{1}{2}}$$

where a is the semi-amplitude along the ground. For the Science Museum pendulum, $a = 3$ ft and $d \simeq 0{\cdot}1$ mm which is small but could perhaps be detected optically. At $R = 0$, the pendulum stays fixed, held rigidly at its starting point.

4.1 *The equatorial pendulum*

For a pendulum swinging along the equator, the Coriolis force is zero and the swing remains along the equator. If a pendulum at the equator is swinging in some other direction, on each swing it ventures a little north and south, where the Coriolis force is not quite zero. It

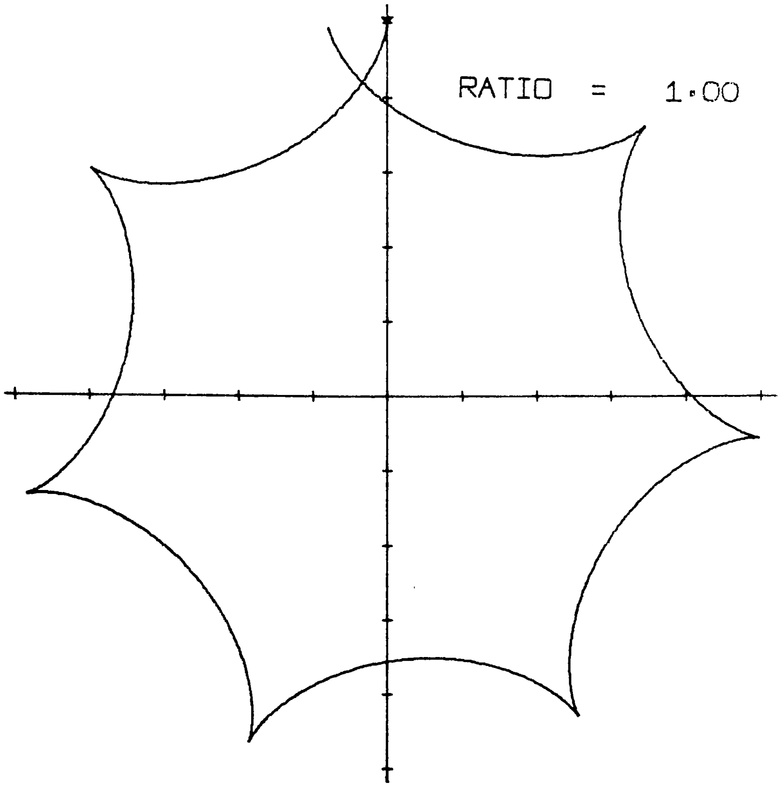

FIG. 5 (a)

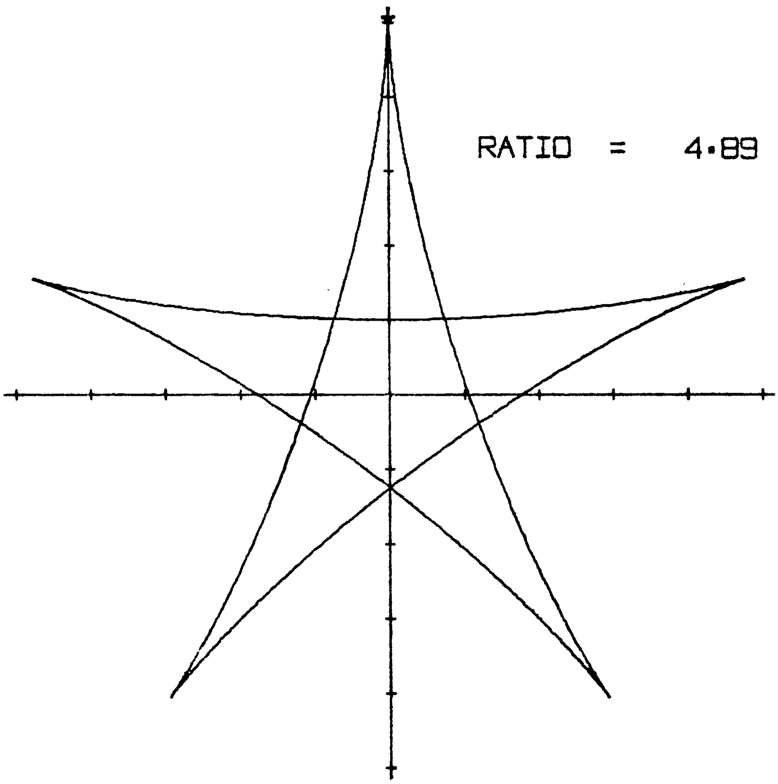

FIG. 5 (b)

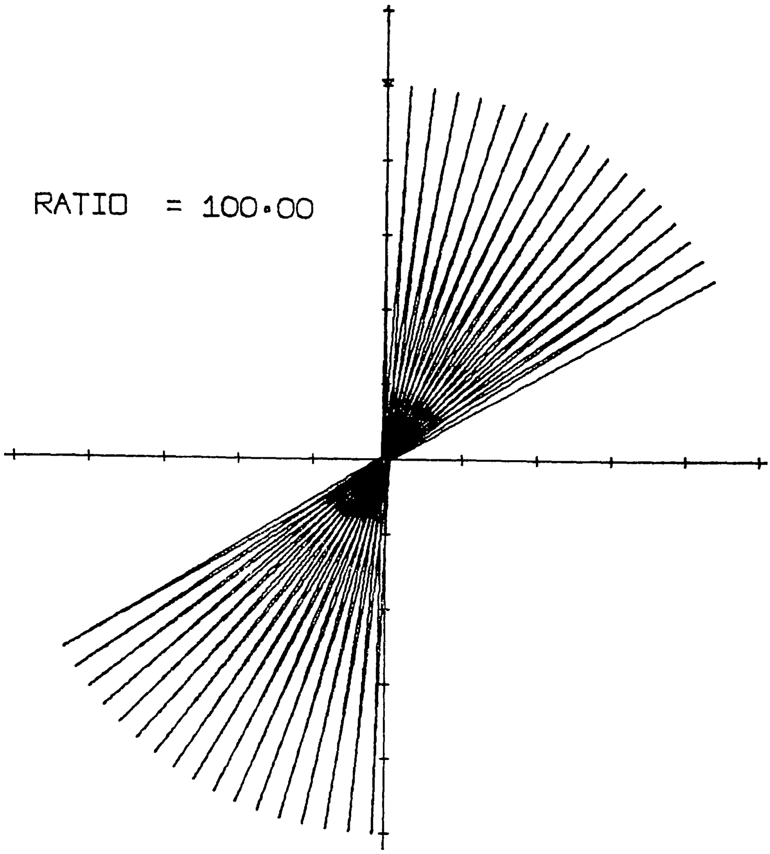

Fig. 5 (c)

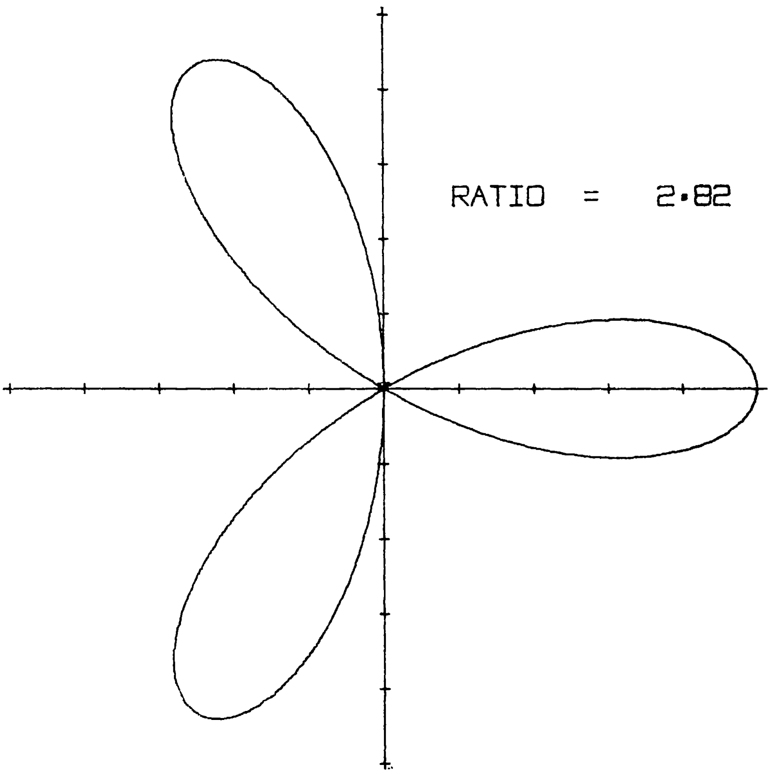

Fig. 5 (d)

therefore moves not quite in a straight line; in practice, the effect is undetectable. With the point of support on the equator, there is no rotation of the plane of swing and after one double swing the pendulum returns to its starting point.

Mathematically, this is a harder problem than the theory of a pendulum at high latitudes. The strength of the Coriolis force depends on the distance from the equator: for small distances, it can be taken as proportional to the north–south coordinate Y. At high latitudes, the force is essentially independent of position in the X–Y plane and it is easy to solve the equations of motion analytically. In the equatorial case, they have had to be solved numerically. Attempts at solution revealed deficiencies, first in my numerical methods and then in IBM's compiler program. When these were overcome, after months of effort, it seemed grimly appropriate that the path traced out had the shape of a tear drop.

It is tempting to suppose that the Coriolis force on the bob is proportional to the latitude, to the right in the Northern Hemisphere and to the left in the Southern. That is what happens for anything moving always horizontally, such as a river; the horizontal is a curved surface, the geoid. But the bob of a pendulum does not move horizontally. It swings in an arc, curved in the opposite sense to the

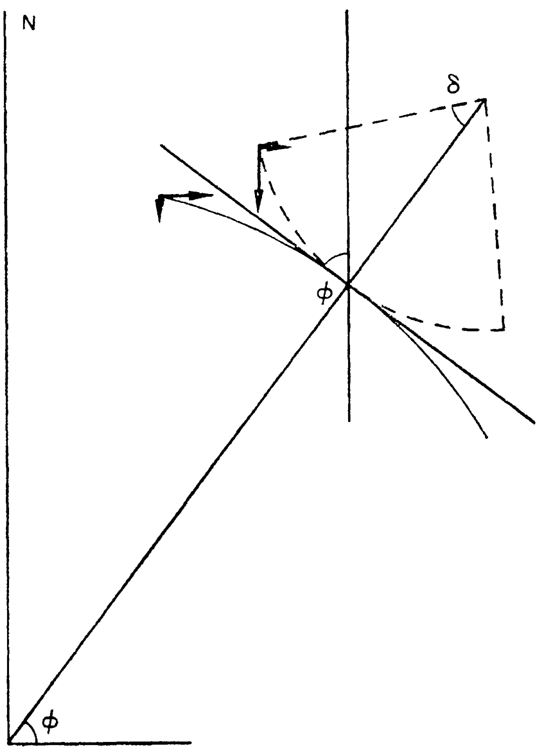

FIG. 6 (a)

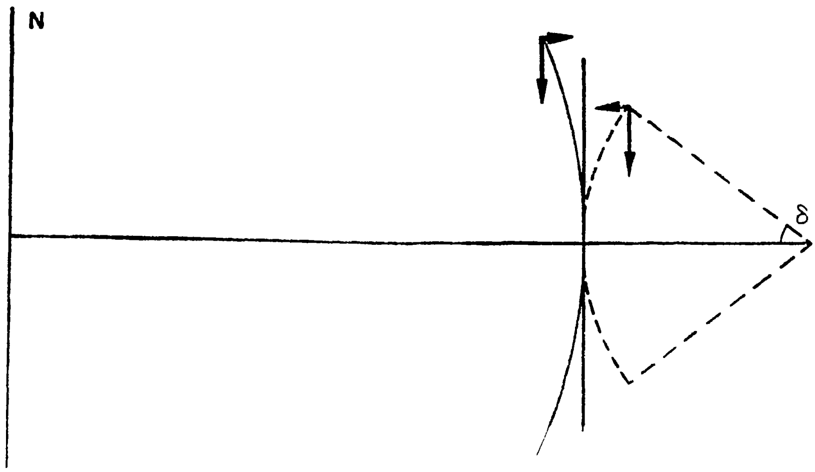

Fig. 6 (b)

Fig. 6 (a). At latitude ϕ, if $\delta < \phi$, the components of velocity perpendicular to the axis are in the same direction, for the pendulum and for horizontal motion. 6 (b). At the equator, the components of velocity perpendicular to the axis are always in opposite directions.

surface of the Earth. At high latitudes this does not matter, if the angle of swing is small; the effect of a large elevation on the motion has been mentioned in Section 3. At the equator, the elevation completely determines the motion, no matter how small the angle of swing.

The situation is illustrated in Fig. 6, where for simplicity we consider motion along the meridian. At a high latitude ϕ, it is a reasonable approximation to consider the horizontal motion of the bob as being in a plane, so long as the angle of swing δ is smaller than ϕ, which in practice it is. It is the component of velocity perpendicular to the axis which produces the Coriolis force and this is in the same sense, away from the axis or towards the axis, for the bob, for motion in a plane and for motion in the curved horizontal surface. The error introduced by this approximation is of second order. If $\delta > \phi$, at one point in the swing the bob is moving parallel to the Earth's axis and there the Coriolis force is zero. In part of the swing the component of velocity perpendicular to the axis is in the opposite sense to that for horizontal motion; the horizontal part of the Coriolis force is to the left. If the bob were swung right round overhead in a circle at constant velocity, the eastward and westward forces would cancel out. The net Coriolis force on a Big Wheel is zero; a couple is present, however.

At the equator, $\phi = 0$, and so $\delta > \phi$ for all δ. No matter how small δ is, the velocity perpendicular to the axis is opposite to that for a horizontal motion. The Coriolis force is, therefore, always to the left in the Northern Hemisphere and to the right in the Southern, and its strength does not depend on the latitude ϕ north or south but on the angle δ of the pendulum.

Results of calculations for the equatorial pendulum are shown in Fig. 7. The X-axis represents the equator; in previous diagrams, the orientation of the axes was irrelevant. The sideways Coriolis force is proportional to the Y distance, to the left in the Northern Hemisphere. The 'Ratio' has a different significance here; it is defined in the Appendix. To show the motion more clearly, in Fig. 7 (a) and (b) the X scale has been expanded by a factor 10.

In Fig. 7 (a), the pendulum is released from rest due north of the point of support, which is on the equator. It deviates to the left of its path in the Northern Hemisphere. As it crosses the equator, moving horizontally, the Coriolis force is zero; it is moving in an ellipse as an ordinary simple pendulum. In the Southern Hemisphere, the Coriolis force is to the right and adds on to the tendency to elliptical motion. The pendulum swings round to the west and back to its starting point.

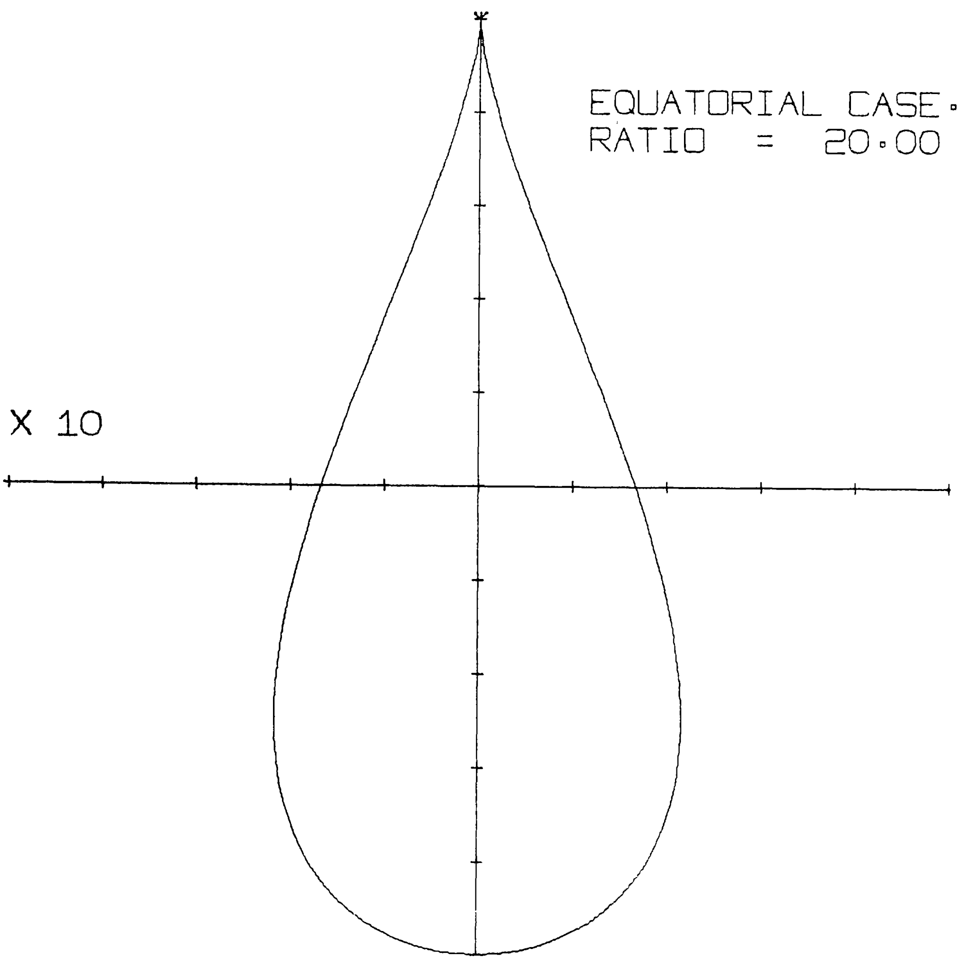

FIG. 7 (a)

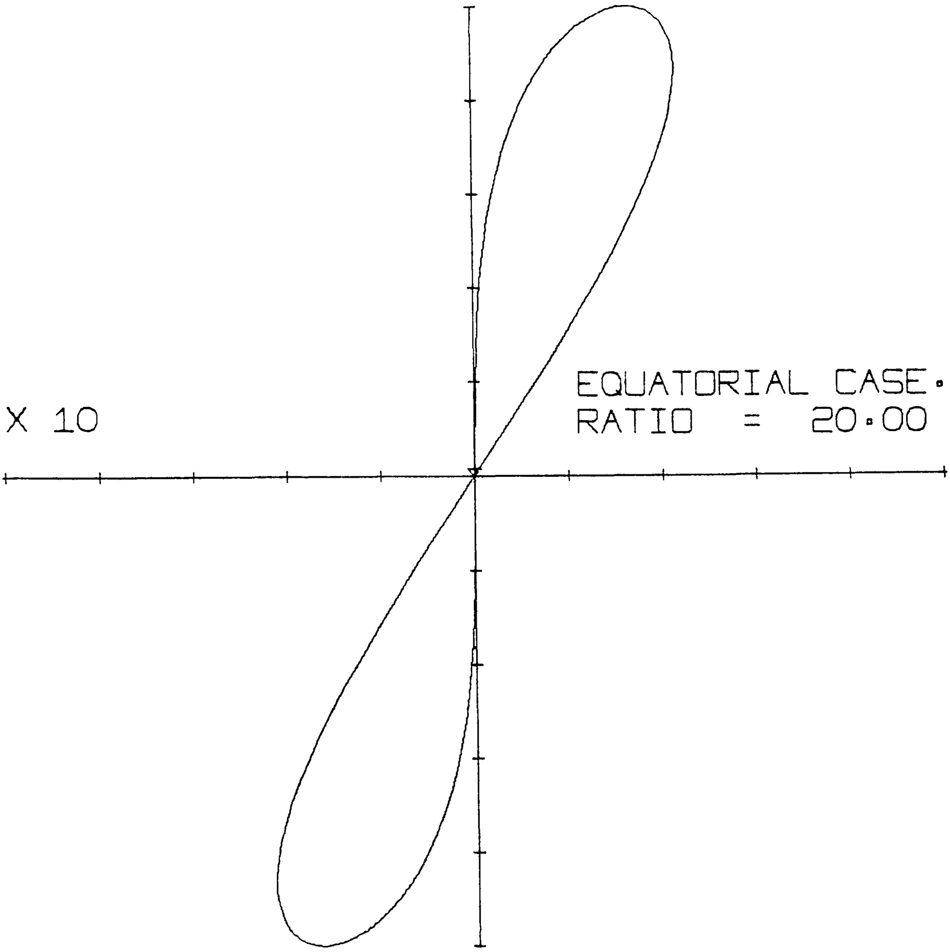

FIG. 7 (a) and (b). The swing of an equatorial pendulum.

Fig. 7 (b) shows the pendulum started with an impulse from the centre. It moves due south and deviates to the right of its path. It passes always under the point of support. It deviates to the left of its path in the Northern Hemisphere and is moving again due south with the impulsive velocity when it returns to its starting point.

REFERENCES

Ames, J.S. & Murnaghan, F.D., 1929. *Theoretical Mechanics*, p. 225, Ginn and Company, Boston.

Andrade, E.N. da C., 1963. *Nature*, **197**, 480.

Becker, R.A., 1954. *Introduction to Theoretical Mechanics*, p. 255, McGraw-Hill, New York.

Foucault, L., 1851. *Comptes Rendus*, **32**, 135.

Goldstein, H., 1950. *Classical Mechanics*, p. 135, Addison-Wesley, Cambridge, Massachusetts.

Jeffreys, H., 1970. *The Earth*, 5th edition, University Press, Cambridge.

Jones, H.S., 1961. *General Astronomy*, 4th edition, p. 9, Arnold, London.

Kuiper, G.P., (ed.), 1954. *The Earth as a Planet*, University Press, Chicago.

Mehlin, T.G., 1959. *Astronomy*, p. 280, Wiley, New York.
Payne-Gaposchkin, C., 1954. *Introduction to Astronomy*, p. 33, Prentice-Hall, Englewood Cliffs, New Jersey.
Payne-Gaposchkin, C. & Haramundanis, K., 1970. *Introduction to Astronomy*, 2nd edition, p. 48, Prentice-Hall, Englewood Cliffs, New Jersey.
Ramsey, A.S., 1951. *Dynamics*, Part II, p. 100, University Press, Cambridge.
Rudeaux, L. & de Vaucouleurs, G., 1959. *Larousse Encyclopedia of Astronomy*, p. 25, Batchworth Press, London.
Russell, H.N., Dugan, R.S. & Stewart, J.Q., 1945. *Astronomy*, Vol. I, p. 115, Ginn and Company, Boston.
Shakespeare, W., 1623. *As You Like It* II, vii, 23, Heming and Condell, London.
Shapiro, A.H., 1962. *Nature*, **196**, 1080.
Sher, D., 1969. *J. R. astr. Soc, Canada*, **63**, 227.
Struve, O., Lynds, B. & Pillans, H., 1959. *Elementary Astronomy*, p. 48, Oxford University Press, New York.
Young, C.A., 1889. *General Astronomy*, p. 95, Ginn and Company, Boston.

Appendix

After the usual manner of such things, we may consider our theoretical pendulum to be a point mass suspended by a string which is light and inextensible, without any friction at the support or any air resistance. Motion relative to a frame of reference rotating with uniform angular velocity $\boldsymbol{\omega}$ is described by the equation (see, for example, Goldstein 1950)

$$m\ddot{\mathbf{r}} = \mathbf{F} - 2m\boldsymbol{\omega}\times\dot{\mathbf{r}} - m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})\,. \qquad (1)$$

Here, $\mathbf{r}$ is defined relative to the rotating axes—axes fixed in the Earth. $\mathbf{F}$ contains all the real forces acting on the particle, in this case the gravitational force and the tension $\mathbf{T}$ in the string.

The Coriolis force $-2m\boldsymbol{\omega}\times\dot{\mathbf{r}}$ is perpendicular to $\boldsymbol{\omega}$ and to $\dot{\mathbf{r}}$, defined by the right-hand rule for vector products, and is independent of location.

The final term in equation (1) is the centrifugal force. The position vector $\mathbf{r}$ of the bob from the centre of the Earth changes by a negligible amount in the swing of a pendulum, so this term is effectively constant. In accord with the usual definition of the vertical as the direction of a plumb-line, we may conveniently absorb the centrifugal force into the gravitational force to obtain an effective force $m\mathbf{g}$. Then

$$m\ddot{\mathbf{r}} = \mathbf{T} + m\mathbf{g} - 2m\boldsymbol{\omega}\times\dot{\mathbf{r}}\,. \qquad (2)$$

Consider, at latitude ϕ, a coordinate frame with x to the east, y to the north and z vertically upwards, with origin where the z axis meets the axis of the Earth, a distance Z from the point of support of the pendulum. The Cartesian components of our vectors being

$$\mathbf{T} = (-Tx/l,\ -Ty/l,\ -T(z-Z)/l)\,, \qquad (3)$$

$$\mathbf{g} = (0, 0,\ -g)\,, \qquad (4)$$

$$\boldsymbol{\omega} = (0, \omega \cos \phi, \omega \sin \phi)\,, \tag{5}$$

where

$$l = [x^2+y^2+(z-Z)^2]^{\frac{1}{2}} \tag{6}$$

is the length of the pendulum, we have

$$m\ddot{x} = -Tx/l - 2m\omega(\dot{z}\cos\phi - \dot{y}\sin\phi)\,, \tag{7}$$

$$m\ddot{y} = -Ty/l - 2m\omega\dot{x}\sin\phi\,, \tag{8}$$

$$m\ddot{z} = -T(z-Z)/l - mg + 2m\omega\,\dot{x}\cos\phi\,. \tag{9}$$

In the usual approximation for the simple pendulum that the angle of swing is small, x and y remain small in comparison to $(z-Z) \simeq -l$ and $\dot{z}$ small in comparison to $\dot{x}$ and $\dot{y}$. Equation (9) then gives

$$T = mg \tag{10}$$

and equations (7) and (8)

$$\ddot{x} = -gx/l + 2\omega\dot{y}\sin\phi\,, \tag{11}$$

$$\ddot{y} = -gy/l - 2\omega\dot{x}\sin\phi\,. \tag{12}$$

These equations are unaltered if the axes are rotated through any angle θ about the vertical: so x now need not be to the east.

For a horizontal velocity $v = (\dot{x}^2+\dot{y}^2)^{\frac{1}{2}}$, the magnitude of the Coriolis force is $2\omega v \sin\phi$, independent of the compass bearing of v. It is perpendicular to the direction of v, to the right in the Northern Hemisphere. Our theory applies equally to the Southern Hemisphere, if ϕ there is given negative values.

Introducing

$$R = \left(\frac{g}{l}\right)^{\frac{1}{2}} \frac{1}{\omega \sin\phi} \tag{13}$$

and measuring time t in units $1/\omega \sin \phi$, we have

$$\ddot{x} - 2\dot{y} + R^2 x = 0\,, \tag{14}$$

$$\ddot{y} + 2\dot{x} + R^2 y = 0\,. \tag{15}$$

The general solution is

$$(x+iy) = (\mathrm{A}\sin\mu t + \mathrm{B}\cos\mu t)\,\mathrm{e}^{it}, \tag{16}$$

where

$$\mu = (1+R^2)^{\frac{1}{2}} \tag{17}$$

and the complex constants A and B are determined by the initial conditions. For a pendulum released from rest at $(x,y) = (0,a)$,

$$\left.\begin{aligned} x &= -(a/\mu)\sin\mu t\cos t + a\cos\mu t\sin t \\ y &= (a/\mu)\sin\mu t\sin t + a\cos\mu t\cos t \end{aligned}\right\}. \tag{18}$$

For a pendulum launched impulsively from (0,0),

$$\left.\begin{aligned} x &= -a\sin\mu t\sin t\,, \\ y &= -a\sin\mu t\cos t \end{aligned}\right\}. \tag{19}$$

Equations (18) represent an ellipse and (19) a straight line, rotating with period $1/\omega \sin \phi = 1$. In the Southern Hemisphere, where ϕ is negative, the path is traced in the reverse direction.

For the equatorial case, we must return to equations (7), (8) and (9), with $\phi = 0$, and retain terms to a higher order in small quantities. Thus, x, y, $\dot{x}$, $\dot{y}$, $\ddot{x}$ and $\ddot{y}$ being of first order and $(l+z-Z)$, $\dot{z}$ and $\ddot{z}$ of second order (Ames & Murnaghan 1929),

$$T = mg - 2m\omega\dot{x}\,, \tag{20}$$

and

$$\dot{z} = (x\dot{x}+y\dot{y})/l \tag{21}$$

and the equations of motion are

$$\ddot{x} = -gx/l - 2\omega y\dot{y}/l\,, \tag{22}$$

$$\ddot{y} = -gy/l + 2\omega y\dot{x}/l\,. \tag{23}$$

Now defining

$$R_1 = (gl)^{\frac{1}{2}}/\omega \tag{24}$$

and measuring t in units l/ω, we have

$$\ddot{x} + 2y\dot{y} + R_1^2 x = 0\,, \tag{25}$$

$$\ddot{y} - 2y\dot{x} + R_1^2 y = 0\,. \tag{26}$$

These equations have been solved numerically. Comparing them with equations (14) and (15), it is seen that there is an additional factor y in the second term and also that the sign has changed—the Coriolis force is in the opposite direction.