Chapter 1

FUNDAMENTAL CONCEPTS OF THE THEORY OF TURBULENCE

A. RANDOM FIELDS

The modern theory of turbulence is a statistical theory. The statistical methods provide the most natural tool for the description of turbulent motion, as turbulence is in fact a consequence of instability of fluid flow in relation to the inevitable small fluctuations in the liquid (or the gas). Description of fluctuation phenomena which arise when sound or electromagnetic waves are propagated in a turbulent medium also requires statistical techniques. The mathematical aspect of this problem has been developed in considerable detail in recent years and comprehensive treatment of the mathematical approach will be found in the specialized work of Khinchin, Obukhov, Yaglom, and other authors (see, e.g., /1–9/). We will give here (without rigorous proof) some fundamental concepts and relations from the theory of random functions and fields.

§ 1. Random functions

Wind speed, air temperature, refractive index, and other similar physical quantities undergo irregular random fluctuations in a turbulent atmosphere. Figure 1 shows some typical recordings of various meteorological elements obtained with low-inertia equipment. Wind speed and temperature clearly undergo irregular fluctuations of different amplitudes and frequencies, which are randomly superimposed on one another. The fields of meteorological elements in a turbulent atmosphere are described in terms of random functions. The concept of a random function is a generalization of the more familiar concept of a random variable. For example, a discrete random variable may take on values from a certain set of numbers $\xi_1, \xi_2, \ldots$ (the sample space of ξ) with corresponding probabilities $p_1, p_2, \ldots$. We similarly regard $f(t)$ as a random function if there is a certain probability that it coincides with one of the functions of a give set $f_\alpha(t)$ (the sample space or the set of realizations), where α is a set of parameters taking on discrete or continuous values. To determine the probability characteristics of random functions in practice, we should be able to record as often as necessary the various realizations of these functions.

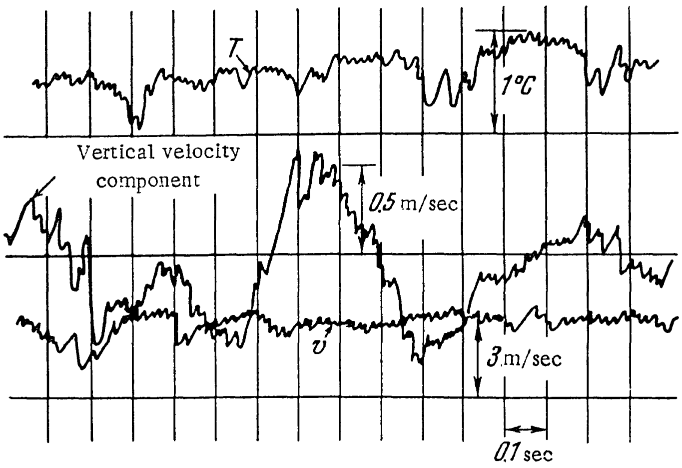

FIGURE 1. Simultaneous recordings of temperature T, wind velocity v, and vertical wind component in the ground air layer.

Consider two examples. 1) Let $f_\alpha(t) = \exp(-\alpha t^2)$, where α is a random variable uniformly distributed over the interval (0, 1). Each realization is a normal curve which markedly differs in its shape from the functions represented in Figure 1. Yet we are dealing with a random function in this case. 2) Let the sample space, i.e., the set of all possible realizations, comprise a single function, e.g., one of the functions shown in Figure 1. The outcome of every statistical test is thus always the same function, and we are forced to the conclusion that, despite the highly complex and "randomized" appearance, the function is fully determined and in no way random.

These examples were chosen to demonstrate that the apparent complexity of the curve or the recording is not always an indication of a random function. The realizations of the random function considered in example 1 above were so simple that the function was fully defined by specifying a single parameter. A much more common case, however, is that of a more diversified sample space which has to be described by a very large (often infinite) number of parameters. The probability to end up with a smooth function, like that in example 1, is generally much lower than the probability of obtaining a complex, irregular curve, like those of Figure 1. This explains why the intuitive picture of a random function is not unlike the curves in Figure 1.

The expansion coefficients of a random function in terms of some orthogonal system may be chosen as the parameters characterizing its sample space. This is not the usual choice, however, and we will now proceed to outline an alternative approach to describing the sample space.

Let $f(t)$ be some random function of time. Its values at any given time t_1 may differ in accordance with its random character. For each time t_1 we should thus specify the probability density $P_{t_1}(f_1)$ which determines the probability of an event $|f(t_1)-f_1| \leqslant \frac{1}{2}\,df_1$:

$$\mathcal{P}\left(|f(t_1)-f_1| \leqslant \frac{1}{2}\,df_1\right) = P_{t_1}(f_1)\,df_1{}^{*}\,. \qquad (1)$$

Given the probability density $P_{t_1}(f_1)$, we can find the statistical characteristics of the function $f(t)$ for the time t_1, such as the mean value (averaging is denoted by the brackets $\langle\ \rangle$)

$$\langle f(t_1)\rangle = \int\limits_{-\infty}^{\infty} P_{t_1}(f_1)\,f_1\,df_1 \qquad (2)$$

and the variance

$$\sigma^2(t_1) = \langle [f(t_1)-\langle f(t_1)\rangle]^2\rangle = \int\limits_{-\infty}^{\infty} P_{t_1}(f_1)\,[f_1-\langle f(t_1)\rangle]^2\,df_1. \qquad (3)$$

In general, the probability density $P_{t_1}(f_1)$ depends on the particular time t_1; this is also true, of course, for the values of $\langle f\rangle$, σ^2, etc., calculated using P_{t_1}. The variation of the probability density with time is a manifestation of a nonstationary process and may be associated, for example, with diurnal or yearly march of meteorological elements or some other factor.

The function $P_{t_1}(f_1)$ (which we call a one-point distribution function) can be used to determine the statistical characteristics at only one time. Given $P_{t_1}(f_1)$ we cannot establish, for instance, the probability that $f(t)$ changes by a certain prescribed amount during a time τ or that the derivative $f'(t)$ takes on a certain value. To solve these problems, we need the two-point distribution function $P_{t_1,t_2}(f_1,\ f_2)$ defined by the equality

$$\mathcal{P}\left(|f(t_1)-f_1| \leqslant \frac{1}{2}\,df_1;\ |f(t_2)-f_2| \leqslant \frac{1}{2}\,df_2\right) = P_{t_1,t_2}(f_1,\ f_2)\,df_1\,df_2. \qquad (4)$$

This distribution function gives the joint probability that two conditions are satisfied simultaneously: for $t=t_1$ the function $f(t)$ is close to f_1 and for $t=t_2$ it is close to f_2. The two-point distribution function carries much more information about the random process $f(t)$ than the function $P_{t_1}(f_1)$ does, but even the two-point distribution does not provide a comprehensive description of $f(t)$.

For a comprehensive description of a random process $f(t)$, we require that for any n we know the corresponding n-point distribution function

$$\mathcal{P}\left(|f(t_1)-f_1| \leqslant \frac{1}{2}\,df_1;\ldots;\ |f(t_n)-f_n| \leqslant \frac{1}{2}\,df_n\right) = P_{t_1,\ldots,t_n}(f_1,\ldots,f_n)\,df_1\ldots df_n, \qquad (5)$$

which depends on the arbitrary times $t_1,\ldots,t_n$.

The functions (5) enable us to determine the probability of any particular realization of the random process $f(t)$. In practice, however, it is fairly difficult to determine all the functions given by (5). This is feasible only for random processes of special kinds (e.g., the normal or gaussian process). Random functions are therefore often described using the simpler characteristics associated with $P_{t_1,t_2}(f_1,\ f_2)$.

* The notation $\mathcal{P}(|f(t_0)-f_0| \leqslant \frac{1}{2}\,df_0)$ is the probability that the inequality in parentheses is satisfied.

In what follows we will often encounter complex random functions $f(t) = \varphi(t) + i\psi(t)$. The probability distribution of a complex random function $f(t)$ is defined as the joint probability distribution of the pair of functions $\{\varphi(t), \psi(t)\}$. The one-point distribution function of $f(t)$ is thus defined by the equality

$$\mathcal{P}\left(|\varphi(t_1)-\varphi_1| \leqslant \tfrac{1}{2}d\varphi_1; |\psi(t_1)-\psi_1| \leqslant \tfrac{1}{2}d\psi_1\right) = P_{t_1}(\varphi_1, \psi_1)\,d\varphi_1\,d\psi_1.$$

Many-point probability distributions of complex random functions are obtained similarly.

If we are dealing with random functions of position, $f(x, y, z) = f(\boldsymbol{r})$ (they are called random fields), the probability distribution depends on the radius-vector $\boldsymbol{r}$. Thus, relations (1) and (4) are replaced by

$$\mathcal{P}\left(|f(\boldsymbol{r}_1)-f_1| \leqslant \tfrac{1}{2}df_1\right) = P_{\boldsymbol{r}_1}(f_1)\,df_1, \tag{6}$$

$$\mathcal{P}\left(|f(\boldsymbol{r}_1)-f_1| \leqslant \tfrac{1}{2}df_1; |f(\boldsymbol{r}_2)-f_2| \leqslant \tfrac{1}{2}df_2\right) = P_{\boldsymbol{r}_1,\boldsymbol{r}_2}(f_1, f_2)\,df_1\,df_2. \tag{7}$$

In some cases we will deal with functions of both position and time,* $f = f(\boldsymbol{r}, t)$; in general their probability distribution depends on both $\boldsymbol{r}$ and t. For example,

$$\mathcal{P}\left(|f(\boldsymbol{r}_1, t_1)-f_1| \leqslant \tfrac{1}{2}df_1; |f(\boldsymbol{r}_2, t_2)-f_2| \leqslant \tfrac{1}{2}df_2\right) = P_{\boldsymbol{r}_1, t_1; \boldsymbol{r}_2, t_2}(f_1, f_2)\,df_1\,df_2. \tag{8}$$

§ 2. Stationary random functions

A random function $f(t)$ is called stationary if its probability distribution functions (1.5) are invariant under time translation:

$$P_{t_1, t_2, \ldots, t_n}(f_1, \ldots, f_n) = P_{t_1+T, t_2+T, \ldots, t_n+T}(f_1, \ldots, f_n). \tag{1}$$

For $n = 1$ we have

$$P_{t_1}(f_1) = P_{t_1+T}(f_1).$$

Setting $T = -t_1$, we get

$$P_{t_1}(f_1) = P_0(f_1), \tag{2}$$

i.e., the one-point distribution function of a stationary process is independent of time. For $n = 2$, taking $T = -t_1$, we get

$$P_{t_1, t_2}(f_1, f_2) = P_{t_1+T, t_2+T}(f_1, f_2) = P_{0, t_2-t_1}(f_1, f_2), \tag{3}$$

i.e., the two-point distribution function of a stationary random process only depends on the distance $t_2 - t_1$ between the points t_1 and t_2.** It follows

* The word "random" is not quite adequate in some cases: the "random" velocity field in turbulence theory, say, satisfies certain exact relations, the equations of hydrodynamics. Perhaps a better term would be "statistically determined" functions.

** If (1) is satisfied only for $n = 1, 2$, $f(t)$ is called stationary in the wide sense (since with this definition we can always find functions for which (2) and (3) are satisfied but (1) is not observed for $n \geqslant 3$). Since we are dealing only with two-point characteristics of random processes, we will ignore this distinction.

from (2) that $\langle f(t)\rangle$, $\sigma^2(t)$, and other characteristics of a stationary process are independent of time:

$$\langle f(t)\rangle = \text{const}, \qquad \sigma^2(t) = \text{const}. \tag{4}$$

The most important characteristic of a random function is its correlation function, sometimes called the covariance:

$$B(t_1, t_2) = \langle [f(t_1) - \langle f(t_1)\rangle]\,[f^*(t_2) - \langle f^*(t_2)\rangle]\rangle \tag{5}$$

(asterisks denote complex conjugates). In (5), setting $t_1 = t_2$, we get

$$B(t, t) = \langle |f(t) - \langle f(t)\rangle|^2\rangle = \sigma^2(t). \tag{6}$$

For stationary processes $B(t_1 + T, t_2 + T) = B(t_1, t_2)$, so that B also depends only on $\tau = t_1 - t_2$:

$$B(\tau) = \langle [f(t+\tau) - \langle f\rangle]\cdot[f^*(t) - \langle f^*\rangle]\rangle, \tag{7}$$

$$B(0) = \langle |f(t) - \langle f\rangle|^2\rangle = \sigma^2. \tag{8}$$

The value of the correlation function at the origin is thus equal to the mean square of the fluctuations. If $f(t+\tau)$ and $f(t)$ are statistically independent (which is generally true for sufficiently large τ), the mean value of the product in the right-hand side of (7) is equal to the product of the means, each of which is zero. $B(\tau)$ thus characterizes the statistical dependence (or correlation) between the fluctuations of f at the times $t+\tau$ and t.

Consider some properties of the function $B(\tau)$. From (5) it follows that

$$B(t_2, t_1) = B^*(t_1, t_2), \tag{9}$$

and for stationary processes

$$B(-\tau) = B^*(\tau). \tag{10}$$

If $f(t)$ is a real function, i.e., $f^* = f$, we have

$$B(-\tau) = B(\tau \tag{11}$$

For a stationary random process we have the inequality

$$|B(\tau)| \leqslant B(0), \tag{12}$$

which shows that $B(\tau)$ has a maximum at $\tau = 0$.

Indeed, let $\langle f\rangle = 0$. Then for any complex λ we have the inequality

$$\langle [\lambda f(t_1) - f(t_2)]\,[\lambda^* f^*(t_1) - f^*(t_2)]\rangle \geqslant 0,$$

or

$$\lambda\lambda^*\sigma_1^2 - \lambda B(t_1, t_2) - \lambda^* B^*(t_1, t_2) + \sigma_2^2 \geqslant 0.$$

Let $\lambda = \alpha B^*(t_1, t_2)$, where $\alpha = \alpha^*$. The last inequality then takes the form

$$\alpha^2\sigma_1^2 |B(t_1, t_2)|^2 - 2\alpha |B(t_1, t_2)|^2 + \sigma_2^2 \geqslant 0.$$

As we know, a quadratic in α is nonnegative if its discriminant is nonpositive:

$$|B(t_1, t_2)|^4 - \sigma_1^2\sigma_2^2 |B(t_1, t_2)|^2 \leqslant 0.$$

Hence the inequality

$$|B(t_1, t_2)|^2 \leqslant \sigma_1^2\sigma_2^2,$$

which for stationary processes reduces to condition (12).

Let $f(t)$ be a stationary random process with zero mean. Consider the expression

$$A = \frac{1}{\sqrt{2T}} \int_{-T}^{T} \varphi(t) f(t)\, dt,$$

where $\varphi(t)$ is any complex function. Clearly $\langle |A|^2 \rangle \geqslant 0$, i.e.,

$$\left\langle \left| \frac{1}{\sqrt{2T}} \int_{-T}^{T} \varphi(t) f(t)\, dt \right|^2 \right\rangle = \frac{1}{2T} \int_{-T}^{T} dt_1 \int_{-T}^{T} dt_2 \varphi(t_1) \varphi^*(t_2) \langle f(t_1) f^*(t_2) \rangle =$$

$$= \frac{1}{2T} \int_{-T}^{T} \int_{-T}^{T} B(t_1 - t_2) \varphi(t_1) \varphi^*(t_2)\, dt_1\, dt_2 \geqslant 0. \qquad (13)$$

Inequality (13) is the condition of positive definiteness of the correlation function $B(\tau)$. This condition is of the utmost importance, as only functions $B(\tau)$ which are positive definite may be regarded as correlation functions. In (13) let

$$\varphi(t) = \frac{1}{\sqrt{2\pi}} e^{-i\omega t}.$$

Then (13) becomes the inequality

$$\langle |A|^2 \rangle = \frac{1}{4\pi T} \int_{-T}^{T} dt_1 \int_{-T}^{T} dt_2 e^{-i\omega (t_1 - t_2)} B(t_1 - t_2) \geqslant 0. \qquad (14)$$

We introduce the new variables of integration

$$\tau = t_1 - t_2 \quad \text{and} \quad t = \frac{1}{2}(t_1 + t_2)$$

and integrate over t; after simple manipulations we get

$$\langle |A|^2 \rangle = \frac{1}{2\pi} \left[\int_{0}^{2T} \left(1 - \frac{\tau}{2T}\right) e^{-i\omega\tau} B(\tau)\, d\tau + \int_{-2T}^{0} \left(1 + \frac{\tau}{2T}\right) e^{-i\omega\tau} B(\tau)\, d\tau \right] \geqslant 0. \qquad (15)$$

In (15), setting the limit $T \to \infty$, we obtain the condition

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega\tau} B(\tau)\, d\tau = W(\omega) \geqslant 0. \tag{16}$$

The function $W(\omega)$ is the Fourier transform of the correlation function $B(\tau)$. According to (16), it is real and nonnegative, since otherwise $B(\tau)$ is not a correlation function. Khinchin /6/ proved the inverse theorem: if $W(\omega) \geqslant 0$, the function

$$B(\tau) = \int_{-\infty}^{\infty} e^{+i\omega\tau} W(\omega)\, d\omega \tag{17}$$

is a correlation function of some stationary random process. (Formula (17) is the inverse of the Fourier transform (16).)

If $f(t)$ is a real function, we have $B(\tau) = B(-\tau)$. Then (16) and (17) may also be written in the form

$$W(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \cos\omega\tau\, B(\tau)\, d\tau, \tag{16'}$$

$$B(\tau) = \int_{-\infty}^{\infty} \cos\omega\tau\, W(\omega)\, d\omega. \tag{17'}$$

An important equality follows from (17) which elucidates the physical meaning of the function $W(\omega)$. Putting $\tau = 0$, in (17), we obtain

$$\sigma^2 = \langle [f(t)]^2 \rangle = \int_{-\infty}^{\infty} W(\omega)\, d\omega. \tag{18}$$

Let $f(t)$ represent an electric current fluctuation across a unit resistance. Then $f^2(t)$ is the instantaneous power. The mean power is

$$\langle f^2 \rangle = \sigma^2 = B(0).$$

From (18) we conclude that $W(\omega)$ is the power per unit frequency. In the literature concerned with radio physics, $W(\omega)$ is correspondingly called the noise power spectrum.

If $f(t)$ is the magnitude of the velocity vector in a fluid, $W(\omega)$ is proportional to the spectral energy density per unit mass. In turbulence theory this function is called the spectral density of the energy distribution. The considerable importance attached to the correlation function $B(\tau)$ in the theory of stationary processes and its applications is directly linked with the interpretation of its Fourier transform $W(\omega)$ as the power (or energy) spectrum of the process.

Like the correlation function $B(\tau)$ of a stationary random process $f(t)$, which can be represented as a Fourier integral (17), the random function $f(t)$ itself can be written as a Fourier — Stieltjes stochastic integral

$$f(t) = \int_{-\infty}^{\infty} e^{i\omega t} Z(d\omega) \tag{19}$$

(see, e.g., Yaglom /1/). The function Z is a random function of the frequency interval $\Delta\omega$. Since $\langle f(t)\rangle = 0$, $Z(\Delta\omega)$ must also satisfy the condition $\langle Z(\Delta\omega)\rangle = 0$. Using (19), we write for the correlation function $B(t_1 - t_2) = \langle f(t_1) f^*(t_2)\rangle$,

$$B(t_1 - t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\omega t_1 - \omega' t_2)} \langle Z(d\omega) Z^*(d\omega')\rangle. \tag{20}$$

Since for a stationary process $f(t)$ the left-hand side of (20) depends only on $t_1 - t_2$, the integrand should contain a factor $\delta(\omega - \omega')$, which ensures the appearance of the appropriate combination of t_1 and t_2. Hence,

$$\langle Z(d\omega) Z^*(d\omega')\rangle = \delta(\omega - \omega') F(d\omega, d\omega').$$

Using the last expression, we obtain

$$B(t_1 - t_2) = \int\int_{-\infty}^{\infty} e^{i\omega(t_1 - t_2)} F(d\omega, d\omega')\,\delta(\omega - \omega'). \tag{21}$$

Comparison with (17) shows that it is necessary to have the equality

$$F(d\omega, d\omega') = W(\omega)\, d\omega\, d\omega'.$$

In this case equation (20) agrees with formula (17).

We thus have the relation

$$\langle Z(d\omega) Z^*(d\omega')\rangle = \delta(\omega - \omega')\, W(\omega)\, d\omega\, d\omega'. \tag{22}$$

From (22) we conclude that the spectral amplitudes $Z(d\omega)$ at different frequencies are uncorrelated.

Consider the relationship between the spectral expansion of a stationary process and the ordinary Fourier integral (see, e.g., /165/). The function $f(t)$ may be expanded in a Fourier integral if the integral of $|f(t)|$ between infinite limits converges and if all the discontinuities of $f(t)$ are finite: $|f(t+0) - f(t-0)| < \infty$. A stationary random function clearly does not satisfy the first of these two requirements and therefore cannot be expanded directly into a Fourier integral. Consider the function

$$f_T(t) = \begin{cases} f(t) & \text{for } |t| < \frac{T}{2}, \\ 0 & \text{for } |t| > \frac{T}{2}, \end{cases}$$

which is zero outside the interval $(-\frac{T}{2}, \frac{T}{2})$, whereas inside this interval it coincides with the continuous real stationary random function $f(t)$ such that $\langle f(t)\rangle = 0$, $\langle f^2(t)\rangle < \infty$. The function $f_T(t)$, unlike $f(t)$, is directly representable as a Fourier integral:

$$\left.\begin{aligned} f_T(t) &= \int_{-\infty}^{\infty} e^{i\omega t}\varphi_T(\omega)\, d\omega, \\ \varphi_T(\omega) &= \frac{1}{2\pi}\int_{-T/2}^{T/2} e^{-i\omega t} f(t)\, dt. \end{aligned}\right\} \tag{23}$$

Consider the mean square of the modulus of $\varphi_T(\omega)$:

$$\langle|\varphi_T(\omega)|^2\rangle = \frac{1}{4\pi^2}\int_{-T/2}^{T/2}\int_{-T/2}^{T/2} e^{i\omega(t_2-t_1)}\langle f(t_1)f(t_2)\rangle\, dt_1\, dt_2 = \frac{1}{4\pi^2}\int_{-T/2}^{T/2}\int_{-T/2}^{T/2} e^{i\omega(t_2-t_1)}B(t_2-t_1)\, dt_1\, dt_2.$$

Replacing t_1 with a new variable τ defined by the equality $t_1 = t_2 + \tau$ and integrating over t_2, we find after simple manipulations

$$\langle|\varphi_T(\omega)|^2\rangle = \frac{T}{4\pi^2}\int_{-T}^{T}\cos\omega\tau\, B(\tau)\, d\tau - \frac{1}{2\pi^2}\int_0^T \tau\cos\omega\tau\, B(\tau)\, d\tau.$$

Multiplying by $2\pi/T$ and taking the limit $T\to\infty$, we obtain (always assuming that the limits in the right member of the equality exist)

$$\lim_{T\to\infty}\frac{2\pi\langle|\varphi_T(\omega)|^2\rangle}{T} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\cos\omega\tau\, B(\tau)\, d\tau + \frac{1}{\pi}\lim_{T\to\infty}\frac{1}{T}\int_0^T \tau\cos\omega\tau\, B(\tau)\, d\tau.$$

If

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \tau\,|B(\tau)|\, d\tau = 0 \tag{24}$$

(this condition is satisfied if, e.g., $|B(\tau)|$ decreases for $\tau\to\infty$ faster than τ^{-1}), the second term vanishes and we obtain the relation

$$W(\omega) = \lim_{T\to\infty}\frac{2\pi\langle|\varphi_T(\omega)|^2\rangle}{T}. \tag{25}$$

We see from this relation that $\langle|\varphi_T(\omega)|^2\rangle \sim T$, i.e., the function $\varphi_T(\omega)$ is of the order of $\sqrt{T}$, so that we may not take the limit $T\to\infty$ in (23). Therefore the ordinary spectral density $\varphi(\omega)$ cannot be determined for a stationary random function $f(t)$, and we have to deal only with the random spectral amplitude $Z(d\omega)$, which satisfies (22). Nevertheless, the expansion (19) is often replaced in the physical literature by an ordinary Fourier integral, requiring the random spectral densities to satisfy the relation

$$\langle\varphi(\omega_1)\varphi^*(\omega_2)\rangle = W(\omega_1)\delta(\omega_1-\omega_2), \tag{22'}$$

which leads to correct results. The meaning of condition (22') can be seen by examining expansion (23). If (23) is used to calculate $\langle\varphi_T(\omega_1)\varphi_T^*(\omega_2)\rangle$, then for $T\to\infty$ relation (22') is replaced by an expression which does not contain an exact expression for a δ-function but the function

$$\delta_T(\omega_1-\omega_2) = \frac{\sin\left[\frac{1}{2}(\omega_1-\omega_2)T\right]}{\pi(\omega_1-\omega_2)}. \tag{26}$$

For $T\to\infty$ we have $\delta_T(\omega)\to\delta(\omega)$ (since $\delta_T(0) = \frac{T}{2\pi}\to\infty$ and $\int_{-\infty}^{\infty}\delta_T(\omega)\, d\omega = 1$). For $\omega_1 = \omega_2$, relation (25) is again found. Thus, relation (22') should be interpreted as the limit of expansion (23) for $T\to\infty$.

However, since direct application of the mathematically rigorous formula (19) does not lead to any additional difficulties compared to the conventional Fourier integral, we will in fact use relation (19).

Consider some examples of correlation functions and their spectral densities.

1. In applications a frequently used correlation function is

$$B(\tau)=a^2\exp\left(-\frac{|\tau|}{\tau_0}\right). \qquad (27)$$

The corresponding spectral density is readily computed:

$$W(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\omega\tau}a^2e^{-\frac{|\tau|}{\tau_0}}\,d\tau=\frac{a^2\tau_0}{\pi(1+\omega^2\tau_0^2)}. \qquad (28)$$

Clearly $W(\omega)>0$, and function (27) may indeed be a correlation function of a stationary random process.

2. It is easily verified that if

$$B(\tau)=a^2\exp\left(-\frac{\tau^2}{\tau_0^2}\right), \qquad (29)$$

we have

$$W(\omega)=\frac{a^2\tau_0}{2\sqrt{\pi}}\exp\left(-\frac{\omega^2\tau_0^2}{4}\right)>0. \qquad (30)$$

3. The spectral density

$$W(\omega)=A\frac{a^2\tau_0}{(1+\omega^2\tau_0^2)^{\nu+\frac{1}{2}}}>0,\ \text{where}\ A=\frac{\Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(\nu)}. \qquad (31)$$

is associated with the correlation function

$$B(\tau)=a^2\frac{1}{2^{\nu-1}\Gamma(\nu)}\cdot\left(\frac{\tau}{\tau_0}\right)^{\nu}K_\nu\left(\frac{\tau}{\tau_0}\right)\quad [B(0)=a^2], \qquad (32)$$

where $K_\nu(x)$ is Bessel's function of the second kind of an imaginary argument (Macdonald function). A correlation function of the form (32) was proposed by von Kármán as an approximation to the correlation functions arising in the theory of turbulence. The correlation functions of examples 1, 2, 3 and the corresponding spectral densities are plotted in Figures 2 and 3.

We will now establish a useful relation between the scales of the correlation functions and the spectral density function, which will be often used in the following. A characteristic scale of the correlation function $B(\tau)$ is the so-called integral scale τ_0,

$$\tau_0=\frac{1}{B(0)}\int_{-\infty}^{\infty}B(\tau)\,d\tau. \qquad (33)$$

(The area of a rectangle of height $B(0)$ and base τ_0 is equal to the area enclosed by the correlation function and the τ axis).

Similarly we introduce the integral scale of the spectral density $W(\omega)$, which is designated ω_0:

$$\omega_0 = \frac{1}{W(0)} \int_{-\infty}^{\infty} W(\omega)\, d\omega. \tag{34}$$

(This definition is effective only in the case where $W(\omega) \leqslant W(0)$.)

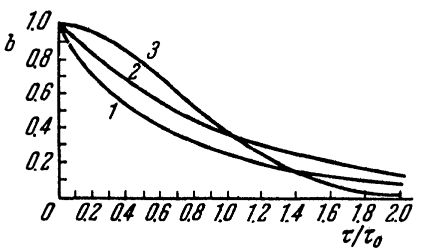

FIGURE 2. Examples of normalized correlation functions:
1) $b(\tau) = \exp\left(-\frac{|\tau|}{(\tau_0)}\right)$; 2) $b(\tau) = 2^{2/3}\left[\Gamma\left(\frac{1}{3}\right)\right]^{-1}\left(\frac{\tau}{\tau_0}\right)^{1/3} K_{1/3}\left(\frac{\tau}{\tau_0}\right)$;
3) $b(\tau) = \exp\left(-\frac{\tau^2}{\tau_0^2}\right)$.

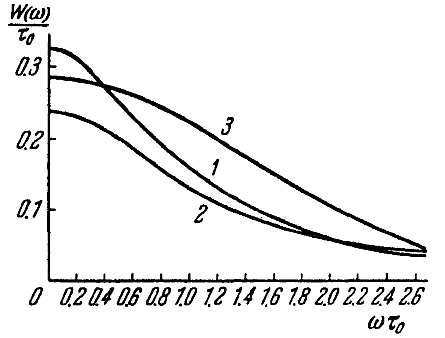

FIGURE 3. Spectral densities $W(\omega)$ corresponding to the correlation functions of Figure 2 (curves numbered as in Figure 2).

Using (16) and (17), we obtain

$$\tau_0 = \frac{1}{B(0)} 2\pi W(0), \qquad \omega_0 = \frac{1}{W(0)} B(0),$$

and hence the relation

$$\omega_0 \tau_0 = 2\pi, \tag{35}$$

between the "width" of the spectrum ω_0 and the correlation scale τ_0. Note that the τ_0 and ω_0 defined by (33) and (34) do not always exist (i.e., if the corresponding integral is divergent). However, if some alternative definition of τ_0 and ω_0 is devised for these improper cases (e.g., τ_0 can be defined as the point where $B(\tau_0) = 0.5\, B(0)$), a relation of the form (35) is again applicable, but now it acquires an additional coefficient of the order of unity.

In conclusion let us consider another important problem, associated with the actual construction of the statistical characteristics of a random process. In practice, we generally have access to relatively few realizations of the random process obtained under identical external conditions. The averaging over the ensemble is thus far from being very effective, and in deriving the statistical characteristics we are often restricted to time averaging over a single realization of the process.

Let $u(t)$ be a quantity whose mean value is to be obtained by time averaging; u may stand for the random function $f(t)$, or for its square, or for the product $f(t)\, f(t+\tau)$ with a fixed translation τ, etc. We regard $u(t)$ as a stationary random process with a known correlation function

$$B_u(t_1-t_2)=\langle [u(t_1)-\langle u\rangle]\,[u(t_2)-\langle u\rangle]\rangle$$

(we are dealing only with real functions $u(t)$).

The time average of the random function u over the period T is defined by

$$\bar{u}(t)\equiv\frac{1}{T}\int_0^T u(t-\tau)\,d\tau .$$

This average, like u itself, is a random variable. Successive determinations of $\bar{u}$ from different sections of the curve $u(t)$ will give somewhat different results. The variance between the time average $\bar{u}$ and the ensemble average $\langle u\rangle$ is given by

$$\sigma^2_{\bar{u}}\equiv\langle [\bar{u}(t)-\langle u\rangle]^2\rangle.$$

Inserting the expression for $\bar{u}$, we readily get

$$\sigma^2_{\bar{u}}=\frac{1}{T^2}\int_0^T\int_0^T B_u(t_1-t_2)\,dt_1dt_2=\frac{2}{T^2}\int_0^T dt\int_0^t B_u(\tau)\,d\tau .$$

We now prove that if

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T B_u(\tau)\,d\tau=0 \tag{36}$$

then

$$\lim_{T\to\infty}\sigma^2_{\bar{u}}=0 \tag{37}$$

(see, e.g., /2/). Indeed, by (36), for any $\frac{\delta}{2}>0$ there is T_0 such that

$$\left|\int_0^t B_u(\tau)\,d\tau\right|<\frac{\delta t}{2}\ \text{for}\ t>T_0 .$$

On the other hand, for any t we have the inequality

$$\left|\int_0^t B_u(\tau)\,d\tau\right|<B_u(0)\,t,$$

since $B_u(0)\geqslant B_u(t)$. Thus,

$$\left|\int_0^t B_u(\tau)\,d\tau\right|<\begin{cases}B_u(0)\,t & \text{for } t<T_0,\\ \frac{\delta}{2}\,t & \text{for } t>T_0.\end{cases}$$

Integrating this inequality over the interval $(0, T)$, where $T>T_0$, we obtain

$$\left|\int_0^T dt\int_0^t B_u(\tau)\,d\tau\right|\leqslant\frac{B_u(0)\,T_0^2}{2}+\frac{\delta}{4}(T^2-T_0^2).$$

If $T > T_0\sqrt{\frac{2B_u(0)}{\delta}}$, the first term or the right is at most equal to the second term, and their sum is at most $\delta T^2/2$; hence the inequality

$$\left|\int_0^T dt\int_0^t B_u(\tau)\,d\tau\right| \leqslant \frac{\delta}{2}T^2,$$

which immediately yields (37). Thus, if (36) is satisfied, the statistical averages can be replaced by time averages.

If the integral

$$\tau_u = \frac{1}{B_u(0)}\int_0^\infty B_u(\tau)\,d\tau < \infty$$

exists, condition (36) is always satisfied. The variance between the time average and the ensemble average is then easily estimated. Interchanging the integrations over t and τ in the expression for σ_u^2 and taking the integral over t, we obtain

$$\sigma_u^2 = \frac{2}{T}\int_0^T \left(1-\frac{\tau}{T}\right) B_u(\tau)\,d\tau.$$

If $T \gg \tau_u$, the second term in parentheses is negligible compared to unity, while the integral over the first term can be expressed in terms of τ_u. This leads to an approximate relation

$$\sigma_u^2 \approx \frac{2B_u(0)\,\tau_u}{T}, \qquad (38)$$

which can be conveniently used to choose the averaging period needed to achieve the desired accuracy. Relation (38) was first derived by Taylor in his development of the theory of eddy diffusion.

§ 3. Random functions with stationary increments

Real fluctuation processes are often described with sufficient accuracy by stationary random functions. These processes include, for instance, the voltage fluctuation across a resistor which is in thermodynamic equilibrium with the surrounding medium. There is another class of processes, however, which are not stationary. As an example we can mention the integral of a stationary process.

Discussion of nonstationary processes implies that we know how to exactly reproduce the conditions under which the process takes place. Indeed, the only way to determine the statistical characteristics of such a process is by averaging over an ensemble of realizations obtained under identical conditions.

Moreover, in dealing with nonstationary processes, it is necessary to have some sort of a natural initial time (zero point on the time scale) for each realization; otherwise, calculation of the statistical characteristics will entail time averaging and the process will become indistinguishable from a stationary one.

To avoid this difficulty, the correlation functions (2.5) are replaced in turbulence theory by so-called structure functions, first introduced by Kolomogorov /10, 11/. The underlying idea of this approach is the following. If $f(t)$ is a nonstationary random function, i.e., if $\langle f(t)\rangle$ is variable in time, $f(t)$ can be replaced by the difference

$$F_T(t) = f(t+T) - f(t)$$

for some fixed T. For not too large T, the low-frequency component of the function $f(t)$ does not affect the value of this difference, which thus may prove to be (perhaps approximately) a stationary function of time. If $F_T(t)$ is a random stationary function of t, $f(t)$ is called a random function with stationary first increments, or simply a random function with stationary increments.

As an example, consider a function of the form

$$f(t) = at + \xi(t), \tag{1}$$

where a is a real random variable and $\xi(t)$ is a stationary random process, such that $\langle a\xi(t)\rangle = 0$. Clearly, if $\langle \xi \rangle = 0$, we have

$$\langle f(t)\rangle = \langle a\rangle t. \tag{2}$$

It is also easy to show that

$$B(t, t+\tau) = \langle [f(t) - \langle f(t)\rangle] [f^*(t+\tau) - \langle f^*(t+\tau)\rangle]\rangle = \sigma_a^2 t(t+\tau) + B_\xi(\tau), \tag{3}$$

where

$$\sigma_a^2 = \langle a^2\rangle - \langle a\rangle^2 \text{ and } B_\xi(\tau) = \langle \xi(t)\xi^*(t+\tau)\rangle.$$

Therefore, the process $f(t)$ is not stationary, since its mean value and its correlation function both explicitly depend on the time t.

Now consider the first increment

$$F_T(t) = aT + \xi(t+T) - \xi(t). \tag{4}$$

Clearly,

$$\langle F_T(t)\rangle = \langle a\rangle T = \text{const}.$$

Calculation of $B_F(t, t+\tau)$ gives after simple manipulations

$$B_F(t, t+\tau) = \langle [F_T(t) - \langle F_T(t)\rangle] [F_T^*(t+\tau) - \langle F_T^*(t+\tau)\rangle]\rangle = \\ = \sigma_a^2 T^2 + 2B_\xi(\tau) - B_\xi(\tau - T) - B_\xi(\tau + T). \tag{5}$$

Both the mean value and the correlation function of the process $F_T(t)$ are independent of the time t (they depend, however, on the constant time translation T).

Consider a random function

$$f(t) = at^2 + bt + c + \xi(t),$$

where $\xi(t)$ is again a stationary process and a, b, c are random variables; the difference $F_T(t) = f(t+T) - f(t)$ in this case is no longer a stationary random function (the second increment $F_T(t+T_1) - F_T(t)$), however, is stationary). A random process $f(t)$ is thus a process with stationary first increments only if its mean is a linear function of time. However, for not too large time intervals any function may be approximately treated as linear (taking the first terms of its Taylor series), so that introduction of processes with stationary first increments will markedly extend the class of real random processes that can be treated within the framework of this theory.

Consider a real random process $f(t)$ with stationary first increments. Its mean $\langle f(t)\rangle$ in general is a function of t. Consider a new function

$$\xi(t) = f(t) - \langle f(t)\rangle,$$

for which $\langle \xi(t)\rangle = 0$. The random process $\xi(t)$ is also a process with stationary increments, since its correlation function may explicitly depend on t, although $\langle \xi(t)\rangle = 0$. Now take the difference

$$F_T(t) = \xi(t+T) - \xi(t);$$

this is a stationary random process, and its correlation function B_F should not depend explicitly on t. Computing $B_F(\tau)$, we get

$$B_F(\tau) = \langle F_T(t+\tau) F_T(t)\rangle = \langle [\xi(t+T) - \xi(t)]\,[\xi(t+\tau+T) - \xi(t+\tau)]\rangle. \qquad (6)$$

Using the algebraic identity

$$(a-b)(c-d) = \frac{1}{2}[(a-d)^2 + (b-c)^2 - (a-c)^2 - (b-d)^2],$$

we obtain

$$B_F(\tau) = \frac{1}{2}\{\langle[\xi(t+T) - \xi(t+\tau)]^2\rangle + \langle[\xi(t+\tau+T) - \xi(t)]^2\rangle - \\ - \langle[\xi(t+T) - \xi(t+\tau+T)]^2\rangle - \langle[\xi(t+\tau) - \xi(t)]^2\rangle\}. \qquad (7)$$

We now introduce the structure function $D_f(t_1, t_2)$ of the process $f(t)$, defined by

$$D_f(t_1, t_2) = \langle[\xi(t_1) - \xi(t_2)]^2\rangle, \qquad (8)$$

Equation (7) is then written in the form

$$B_F(\tau) = \frac{1}{2}D_f(t+T, t+\tau) + \frac{1}{2}D_f(t+\tau+T, t) - \\ - \frac{1}{2}D_f(t+\tau+T, t+T) - \frac{1}{2}D_f(t+\tau, t). \qquad (9)$$

Since by assumption $f(t)$ is a process with stationary increments, and $F_T(t)$ is therefore stationary, B_F should be independent of t and may depend on τ and T only. This will be so if $D_f(t_1, t_2)$ depends on the difference $t_1 - t_2$ only, i.e., if

$$\langle[\xi(t+\tau)-\xi(t)]^2\rangle = D_f(\tau). \tag{10}$$

Relation (9) takes the form

$$B_F(\tau) = \frac{1}{2} D_f(\tau+T) + \frac{1}{2} D_f(\tau-T) - D_f(\tau). \tag{11}$$

Substituting $\xi(t) = f(t) - \langle f(t)\rangle$ in (10) we obtain

$$D_f(\tau) = \langle\{[f(t+\tau)-f(t)] - \langle f(t+\tau)-f(t)\rangle\}^2\rangle. \tag{12}$$

Since the left member of (12) is a function of τ alone, $\langle f(t+\tau) - f(t)\rangle$ should also depend only on τ and be independent of t. Hence it follows that $\langle f(t)\rangle$ is a linear function of time,

$$\langle f(t)\rangle = a + bt \quad \text{and} \quad \langle f(t+\tau) - f(t)\rangle = b\tau. \tag{13}$$

Processes with stationary increments often have a constant mean $\langle f(t)\rangle$ (e.g., the locally isotropic random fields considered in the next section). In this case relation (12) takes the simpler form

$$D_f(\tau) = \langle[f(t+\tau)-f(t)]^2\rangle. \tag{14}$$

The structure function is a fundamental characteristic of a random process with stationary increments, and replaces the ideal of a correlation function. Roughly speaking, $D_f(\tau)$ characterizes the intensity of the fluctuations of $f(t)$ with periods less than or comparable with τ. Indeed, variations in $f(t)$ which are slow compared to τ do not affect the difference $f(t+\tau) - f(t)$ and therefore do not make a contribution to $D_f(\tau)$. Clearly $D_f(\tau)$ may also be constructed for ordinary stationary functions, which are a particular case of functions with stationary increments. If $f(t)$ is a stationary random function with zero mean, we have

$$D_f(\tau) = \langle[f(t+\tau)-f(t)]^2\rangle = \langle[f(t+\tau)]^2\rangle + \langle[f(t)]^2\rangle - 2\langle f(t+\tau)f(t)\rangle.$$

Since $f(t)$ is stationary, we have

$$\langle[f(t)]^2\rangle = \langle[f(t+\tau)]^2\rangle = B_f(0).$$

Thus for a stationary process

$$D_f(\tau) = 2[B_f(0) - B_f(\tau)]. \tag{15}$$

If $B_f(\infty) = 0$ (in practice this is almost always so), $D_f(\infty) = 2B(0)$. Using the above relation we can express the correlation function $B_f(\tau)$ in terms of the structure function $D_f(\tau)$:

$$B_f(\tau) = \frac{1}{2} D_f(\infty) - \frac{1}{2} D_f(\tau). \tag{16}$$

For stationary random processes the structure function $D_f(\tau)$ can be used on an equal footing with the correlation function; in some respects it is even more expedient. Indeed, when we approach a random process whose stationarity is not evident beforehand, a better policy is to construct its structure function, and not the correlation function. In practice, the construction of the structure function is always more reliable, since $D_f(\tau)$ is not affected by errors in the mean $\langle f(t)\rangle$. If the structure function is found to be constant for large τ, $B_f(\tau)$ can be readily computed using (16).

In relation (2.17) above the correlation function of a real stationary random process was represented as a Fourier integral. If $f(t)$ is stationary, so that both its correlation and structure function exist, we can use (15) to obtain the spectral expansion of the structure function $D(\tau)$ (the subscript f for D_f and B_f is henceforth omitted). Substituting (2.17') in (15), we find

$$D(\tau) = 2\int_{-\infty}^{\infty}[1-\cos\omega\tau]\,W(\omega)\,d\omega, \tag{17}$$

which relates the energy spectral density $W(\omega)$ to the structure function $D(\tau)$.

Consider the conditions that are imposed on the spectral density $W(\omega)$ in the following two cases: (a) the correlation function $B(\tau)$ expressed by integral (2.17') exists, and (b) the structure function $D(\tau)$ expressed by integral (17) exists. The integral (2.17) converges for $\tau = 0$ only if the function $W(\omega)$ increases near the origin more slowly than ω^{-1} and decreases at infinity faster than ω^{-1}, so that it is necessary that

$$\lim_{\omega\to 0}[\omega W(\omega)] = 0, \quad \lim_{\omega\to\infty}[\omega W(\omega)] = 0. \tag{18}$$

On the other hand, the convergence of the integral in (3.17) as $\omega \to 0$ is more easily ensured, since $1-\cos\omega\tau \sim \omega^2$ for $\omega \to 0$. For $\omega \to \infty$ the convergence condition of (3.17) is the same as before. Hence, (3.17) converges if

$$\lim_{\omega\to 0}[\omega^3 W(\omega)] = 0, \quad \lim_{\omega\to\infty}[\omega W(\omega)] = 0. \tag{19}$$

Conditions (19) are much less exacting than conditions (18). In this case, the function $W(\omega)$ may have a power singularity at the origin of the form $W(\omega) \sim \omega^{-\mu}$ with $\mu < 3$. The structure function $D(\tau)$ calculated using (17) will be quite meaningful, even though the correlation function $B(\tau)$ no longer exists. This point suggests that spectral expansion (17) obtained from the spectral expansion (2.17') for the correlation function holds true even when the original expansion (2.17') is not applicable. Yaglom /1/ showed that this is actually so.

Processes with stationary increments differ from stationary processes in two respects: their mean value may be a linear function of time and their spectrum may have a singularity at the origin.

The applicability of structure functions to processes with singularities at the origin (i.e., processes with infinite "energy" in the low-frequency region) explains their considerable practical significance, as processes of this kind are fairly common.

Comparing the methods of description of stationary random processes and of processes with stationary increments, we also note that descriptions using the spectral function $W(\omega)$ have certain advantages over the description in terms of either correlation or structure functions. When $W(\omega)$ is used, we need not even bother to distinguish between a stationary process and a process with stationary increments, as $W(\omega)$ exists in both cases and is always identifiable with the spectral density of energy. It is only in the last stage of calculations, when we are actually interested in $B(\tau)$ or $D(\tau)$, that we should apply (2.17) or (3.17), depending on whether $W(\omega)$ has an integrable singularity at the origin or not. The second advantage of the spectral approach is that $W(\omega)$ has a more straightforward physical meaning than either $B(\tau)$ or $D(\tau)$.

We will now derive relations expressing $W(\omega)$ in terms of $D(\tau)$. Differentiating (17) and seeing that $W(\omega)$ is an even function, we obtain a Fourier-type integral which can be inverted. Inversion gives

$$W(\omega)=\frac{1}{2\pi\omega}\int_0^\infty D'(\tau)\sin\omega\tau\,d\tau, \tag{20}$$

which may be used if the integral on the right converges.

For $\omega \neq 0$, the sufficient conditions for the convergence of this integral are the following:

(a) $\lim_{\tau\to\infty} D'(\tau)=0$;

(b) there exists $\alpha<1$ such that

$$\lim_{\tau\to 0}\tau^{1+\alpha}D'(\tau)=\text{const}<\infty. \tag{21}$$

Condition (b) can be replaced by a more strict though more convenient requirement

$$\lim_{\tau\to 0}\tau^2 D'(\tau)=0.$$

If we differentiate (17) twice, we also obtain a Fourier integral which can be inverted to give

$$W(\omega)=\frac{1}{2\pi\omega^2}\int_0^\infty D''(\tau)\cos\omega\tau\,d\tau. \tag{22}$$

For $\omega \neq 0$ relation (22) is valid if

(a) $\lim_{\tau\to\infty} D''(\tau)=0$;

(b) there exists $\alpha<1$ such that

$$\lim_{\tau\to 0}\tau^{\alpha}D''(\tau)=\text{const}<\infty \tag{23}$$

or a more strict condition

$$\lim_{\tau\to 0}\tau D''(\tau)=0.$$

In general conditions (21) and (23) are different, and both should be checked before embarking on a computation of $W(\omega)$ in order to choose the appropriate formula (see examples).

A stationary random process $f(t)$ can be represented as a stochastic Fourier—Stieltjes integral (2.19); similarly a process with stationary increments can also be represented as a spectral expansion. This expansion is best obtained by remembering that the derivative $\xi(t) = \frac{df(t)}{dt}$ of a process $f(t)$ with stationary increments is itself a stationary random process. Hence, $\xi(t)$ can be represented in the form (2.19):

$$\xi(t) = \frac{df(t)}{dt} = \xi_0 + \int_{-\infty}^{\infty} e^{i\omega t} Z_1(d\omega), \tag{24}$$

where ξ_0 is the mean value of $\xi(t)$ and $Z_1(d\omega)$ has the properties

$$\langle Z_1(d\omega)\rangle = 0,$$
$$\langle Z_1(d\omega)\, Z^*_1(d\omega')\rangle = \delta(\omega - \omega')\, W_1(\omega)\, d\omega\, d\omega'. \tag{25}$$

Integrating (24) over the interval $(0,\ t)$, we obtain

$$f(t) = f(0) + \xi_0 t + \int_{-\infty}^{\infty} [e^{i\omega t} - 1]\, \frac{Z_1(d\omega)}{i\omega}, \tag{26}$$

where $f(0)$ is some random variable.

Let $Z(d\omega) = \frac{1}{i\omega} Z_1(d\omega)$. Clearly,

$$\langle Z(d\omega)\rangle = 0 \text{ and } \langle Z(d\omega)\, Z^*(d\omega')\rangle = \frac{1}{\omega^2} W_1(\omega)\, \delta(\omega - \omega')\, d\omega\, d\omega'. \tag{27}$$

If the spectral density $W_1(\omega)$ of the derivative $\frac{df}{dt}$ is replaced by the spectral density

$$W(\omega) = \frac{1}{\omega^2} W_1(\omega)$$

of the original process $f(t)$, relations (26), (27) take the form

$$f(t) = f(0) + \xi_0 t + \int_{-\infty}^{\infty} [e^{i\omega t} - 1]\, Z(d\omega), \tag{28}$$

where

$$\langle Z(d\omega)\rangle = 0, \quad \langle Z(d\omega)\, Z^*(d\omega')\rangle = \delta(\omega - \omega')\, W(\omega)\, d\omega\, d\omega'. \tag{29}$$

Relations (28), (29) give the sought expansion of a process with stationary increments. In the particular case where $\langle f(t)\rangle$ is independent of t, relation (28) takes the form

$$f(t) = f(0) + \int_{-\infty}^{\infty} [e^{i\omega t} - 1]\, Z(d\omega). \tag{30}$$

It is easy to check that substitution of (28) in (12) gives relation (17), which expresses $D(\tau)$ in terms of $W(\omega)$ (remember that $f^*(t) = f(t)$).

Let us consider two examples.

1) Construct the structure function of the stationary random process from example 3 of the previous section. Using the equality $D(\tau) = 2B(0) - 2B(\tau)$, we obtain

$$D(\tau) = 2a^2\left[1 - \frac{2^{1-\nu}}{\Gamma(\nu)}\left(\frac{\tau}{\tau_0}\right)^{\nu} K_\nu\left(\frac{\tau}{\tau_0}\right)\right]. \tag{31}$$

For $\tau \ll \tau_0$ it is only necessary to retain the first two terms in the series expansion of $K_\nu(x)$. Simple manipulations then give

$$D(\tau) \approx 2a^2 \frac{\Gamma(1-\nu)}{\Gamma(1+\nu)}\left(\frac{\tau}{2\tau_0}\right)^{2\nu}, \tag{32}$$

i.e., $D(\tau) \sim \tau^{2\nu}$. For $\tau \gtrsim \tau_0$ the growth of $D(\tau)$ is slowed down and it approaches a constant value $2a^2$. The spectral density corresponding to (31) is the same as in the previous example:

$$W(\omega) = \frac{\Gamma\left(\nu + \frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(\nu)} \frac{a^2\tau_0}{(1+\omega^2\tau_0^2)^{\nu+\frac{1}{2}}}. \tag{33}$$

For $\omega \gg \frac{1}{\tau_0}$ we have

$$W(\omega) \approx \frac{\Gamma\left(\nu + \frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(\nu)} \frac{a^2}{\tau_0^{2\nu}|\omega|^{2\nu+1}}. \tag{34}$$

Putting

$$\frac{2a^2\Gamma(1-\nu)}{(2\tau_0)^{2\nu}\Gamma(1+\nu)} = C^2,$$

we can write (32) and (34) in the form

$$D(\tau) \approx C^2\tau^{2\nu},$$
$$W(\omega) \approx \frac{2^{2\nu-1}\Gamma(1+\nu)\Gamma\left(\frac{1}{2}+\nu\right)}{\sqrt{\pi}\,\Gamma(\nu)\Gamma(1-\nu)} C^2 |\omega|^{-(1+2\nu)}.$$

Using the known formulas

$$2^{2\nu-1}\Gamma(\nu)\Gamma\left(\nu+\frac{1}{2}\right) = \sqrt{\pi}\,\Gamma(2\nu),$$
$$\Gamma(\nu)\Gamma(1-\nu) = \frac{\pi}{\sin\pi\nu},$$

we rewrite the last expression in the form

$$W(\omega) \approx \frac{\nu}{\pi}\Gamma(2\nu)\sin(\pi\nu)\, C^2|\omega|^{-(1+2\nu)}.$$

Putting $2\nu = \mu$, we finally obtain the asymptotic formulas

$$D(\tau) \approx C^2\tau^{\mu}, \tag{35}$$

$$W(\omega) \approx \frac{1}{2\pi}\Gamma(\mu+1)\sin\frac{\pi\mu}{2}\, C^2|\omega|^{-(\mu+1)}. \tag{36}$$

Relations (35) and (36) are the asymptotic form of expressions (21) and (33), which approach constant values for $\tau \to \infty$ and $\omega \to 0$. Therefore, for large τ and small ω, relations (35) and (36) markedly differ from the starting relations (31) and (33).

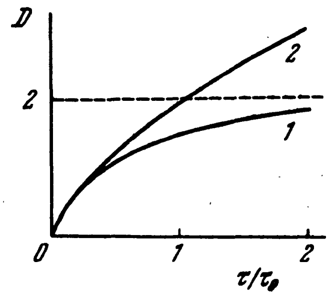

FIGURE 4. Examples of structure functions:

1) $D(\tau) = 2\left[1 - 2^{2/3}\times\right.$ $\left.\times\left[\Gamma\left(\frac{1}{3}\right)\right]^{-1}\left(\frac{\tau}{\tau_0}\right)^{1/3} K\left(\frac{\tau}{\tau_0}\right)\right]$;

2) $D(\tau) = 2^{1/3}\Gamma\left(\frac{2}{3}\right)\times$ $\times\left[\Gamma\left(\frac{4}{3}\right)\right]^{-1}\left(\frac{\tau}{\tau_0}\right)^{2/3}$.

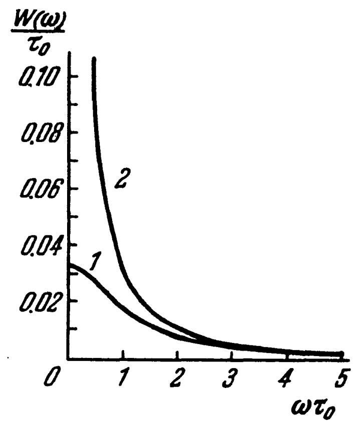

FIGURE 5. Spectral densities corresponding to the structure functions of Figure 4 (curves numbered as in Figure 4).

2) Consider the structure function

$$D(\tau) = C^2 |\tau|^{\mu} \qquad (0<\mu<2). \tag{37}$$

Expression (37) is assumed to apply for $0 < \tau < \infty$, and not only for $\tau \ll \tau_0$ as in the previous examples. We will now derive the spectral density $W(\omega)$ corresponding to (37). Differentiating (37), we get

$$D'(\tau) = \mu C^2 \tau^{\mu-1}.$$

If $\mu < 1$,

$$\lim_{\tau\to\infty} D'(\tau) = 0.$$

The condition

$$\lim_{\tau\to 0} \tau^2 D'(\tau) = 0$$

is satisfied for all $\mu > -1$, i.e., also for $\mu > 0$, as in our example. Inserting $D'(\tau)$ in (20) and integrating, we obtain

$$W(\omega) = \frac{C^2}{2\pi}\Gamma(\mu+1)\sin\frac{\pi\mu}{2}|\omega|^{-(\mu+1)}. \tag{38}$$

For $1<\mu<2$, the condition $\lim_{\tau\to\infty} D'(\tau)=0$ is no longer satisfied, but we have instead

$$\lim_{\tau\to\infty} D''(\tau)=0,\quad \lim_{\tau\to 0} \tau D''(\tau)=0$$

and $W(\omega)$ can be computed from (22). Integrating, we obtain the same relation (38), which is thus valid not only for $0<\mu<1$, but for all μ between 0 and 2.

The structure functions from examples 1 and 2 and their spectra are shown in Figures 4 and 5.

Comparison of the two examples shows that the structure function of the form given in (37), which is the asymptotic form of the structure function (31) of a stationary random process, may also be a structure function of a process with stationary increments. In the theory of turbulence we are generally dealing with the case where the asymptotic form of the structure functions is known for fairly small τ, while the behavior of these functions for large τ is not clear. It is then advisable to consider the random process as a process with stationary increments and whenever possible to generalize the asymptotic form of structure functions known for small τ to the entire range of τ values.

§ 4. Homogeneous and isotropic random fields

Let us now consider random functions of three variables (random fields). The concept of a random field is entirely analogous to the concept of a random process. Examples of random fields are provided by the wind velocity field in a turbulent atmosphere (a vector field, comprising the three random velocity components), the temperature field, the humidity field, and the dielectric constant field (all scalar fields). The mean value of a random field $f(\boldsymbol{r})$ and its correlation function can be defined as before:

$$B_f(\boldsymbol{r}_1, \boldsymbol{r}_2) = \langle [f(\boldsymbol{r}_1) - \langle f(\boldsymbol{r}_2)\rangle][f^*(\boldsymbol{r}_2) - \langle f^*(\boldsymbol{r}_2)\rangle]\rangle. \quad (1)$$

A generalization of the concept of a stationary random process is the concept of a homogeneous random field. A random field is said to be homogeneous if it has a constant mean and if its correlation function is unaffected by a simultaneous translation of $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ in the same direction by the same amount, i.e., if

$$\langle f(\boldsymbol{r})\rangle = \text{const},\quad B_f(\boldsymbol{r}_1, \boldsymbol{r}_2) = B_f(\boldsymbol{r}_1 + \boldsymbol{r}_0, \boldsymbol{r}_2 + \boldsymbol{r}_0). \quad (2)$$

Setting in the last relation $\boldsymbol{r}_0 = -\boldsymbol{r}_2$, we see that in a homogeneous field $B_f(\boldsymbol{r}_1, \boldsymbol{r}_2) = B_f(\boldsymbol{r}_1 - \boldsymbol{r}_2, 0)$, i.e., the correlation function of a homogeneous field depends only on the difference $\boldsymbol{r}_1 - \boldsymbol{r}_2$,

$$B_f(\boldsymbol{r}_1, \boldsymbol{r}_2) = B_f(\boldsymbol{r}_1 - \boldsymbol{r}_2).$$

A homogeneous random field is said to be isotropic if $B_f(\boldsymbol{r})$ depends only on $r = |\boldsymbol{r}|$, i.e., only on the distance between the observation points. For example, a field with a correlation function

$$B_f(\boldsymbol{r}_1-\boldsymbol{r}_2)=B_f[\alpha(x_1-x_2)+\beta(y_1-y_2)+\gamma(z_1-z_2)]$$

is homogeneous but not isotropic.

If a certain straight line is chosen in a homogeneous and isotropic random field, the value of the field along this line is represented by a random function of a single variable x. All the previous results for stationary random functions are fully applicable to this new function. In particular, the correlation function may be expressed as a Fourier integral:

$$B_f(x)=\int_{-\infty}^{\infty}\cos(\varkappa x)\,V(\varkappa)\,d\varkappa. \tag{3}$$

A more natural approach, however, calls for three-dimensional spectral expansions. A homogeneous random field may be represented as a three-dimensional stochastic Fourier–Stieltjes integral

$$f(\boldsymbol{r})=\iiint_{-\infty}^{\infty}e^{i\boldsymbol{\varkappa}\boldsymbol{r}}Z(d^3\varkappa), \tag{4}$$

where

$$d^3\varkappa=d\varkappa_1 d\varkappa_2 d\varkappa_3,$$

and the "amplitudes" $Z(d^3\varkappa)$ satisfy the relation

$$\langle Z(d^3\varkappa_1)\,Z^*(d^3\varkappa_2)\rangle=\delta(\boldsymbol{\varkappa}_1-\boldsymbol{\varkappa}_2)\,\Phi(\boldsymbol{\varkappa}_1)\,d^3\varkappa_1 d^3\varkappa_2, \tag{5}$$

where

$$\Phi(\boldsymbol{\varkappa})\geqslant 0,\quad \delta(\boldsymbol{\varkappa}_1-\boldsymbol{\varkappa}_2)=\delta(\varkappa_{1x}-\varkappa_{2x})\,\delta(\varkappa_{1y}-\varkappa_{2y})\,\delta(\varkappa_{1z}-\varkappa_{2z}).$$

Inserting expansion (4) in the relation

$$B_f(\boldsymbol{r}_1-\boldsymbol{r}_2)=\langle f(\boldsymbol{r}_1)\,f^*(\boldsymbol{r}_2)\rangle$$

(we assume that $\langle f(\boldsymbol{r})\rangle=0$) and using (5), we obtain

$$B_f(\boldsymbol{r}_1-\boldsymbol{r}_2)=\iiint_{-\infty}^{\infty}e^{i\boldsymbol{\varkappa}(\boldsymbol{r}_1-\boldsymbol{r}_2)}\Phi(\boldsymbol{\varkappa})\,d^3\varkappa\,. \tag{6}$$

Since for real random fields

$$B(\boldsymbol{r}_1-\boldsymbol{r}_2)=B(\boldsymbol{r}_2-\boldsymbol{r}_1),\quad \Phi(\boldsymbol{\varkappa})=\Phi(-\boldsymbol{\varkappa}),$$

equation (6) may be written in the form

$$B_f(\boldsymbol{r})=\iiint_{-\infty}^{\infty}\cos\boldsymbol{\varkappa}\boldsymbol{r}\,\Phi(\boldsymbol{\varkappa})\,d^3\varkappa. \tag{7}$$

The function $\Phi(\boldsymbol{\varkappa})$ may be expressed in terms of $B_f(\boldsymbol{r})$:

$$\Phi(\boldsymbol{\varkappa})=\frac{1}{(2\pi)^3}\iiint_{-\infty}^{\infty}\cos\boldsymbol{\varkappa}\boldsymbol{r}\,B_f(\boldsymbol{r})\,d^3r. \tag{8}$$

$B_f(\mathbf{r})$ and $\Phi(\boldsymbol{\varkappa})$ are thus Fourier transforms of one another.

If $f(\mathbf{r})$ is an isotropic random field, $B_f(\mathbf{r})$ is a function of r only. We may change over to spherical coordinates in (8) and integrate over the angular variables. The result gives

$$\Phi(\varkappa) = \frac{1}{2\pi^2\varkappa}\int_0^\infty rB_f(r)\sin\varkappa r\,dr, \tag{9}$$

where $\varkappa = |\boldsymbol{\varkappa}|$.

In an isotropic random field the spectral density $\Phi(\boldsymbol{\varkappa})$ is thus a function of one variable, the magnitude of the vector $\boldsymbol{\varkappa}$. We can now simplify expression (7) for isotropic fields. Introducing spherical coordinates in the $\boldsymbol{\varkappa}$ space and integrating over the angular variables, we find

$$B_f(r) = \frac{4\pi}{r}\int_0^\infty \varkappa\Phi(\varkappa)\sin\varkappa r\,d\varkappa. \tag{10}$$

Let us now derive a useful relation between the three-dimensional spectral density $\Phi(\varkappa)$ of an isotropic random field and its one-dimensional spectral density $V(\varkappa)$. Inverting (3) and remembering that $B_f(r)$ is an even function, we get

$$V(\varkappa) = \frac{1}{\pi}\int_0^\infty B_f(r)\cos\varkappa r\,dr. \tag{11}$$

Differentiating this expression gives

$$\frac{dV(\varkappa)}{d\varkappa} = -\frac{1}{\pi}\int_0^\infty B_f(r)\sin(\varkappa r)\,r\,dr. \tag{12}$$

Comparing (12) with (9), we see that

$$\Phi(\varkappa) = -\frac{1}{2\pi\varkappa}\frac{dV(\varkappa)}{d\varkappa}. \tag{13}$$

Formula (13) gives the three-dimensional spectral density of an isotropic random field if the one-dimensional spectral density $V(\varkappa)$ is known.

Consider a few examples of three-dimensional correlation functions and their spectra.

1)
$$B_f(r) = a^2 e^{-\left|\frac{r}{r_0}\right|}. \tag{14}$$

Using the results of example 1 on p. 10 and seeing that expansion (3) is completely analogous to expansion (2.17') for a stationary random process, we write directly for the one-dimensional spectral density $V(\varkappa)$

$$V(\varkappa) = \frac{a^2 r_0}{\pi(1+\varkappa^2 r_0^2)}. \tag{15}$$

We can now apply (13) to find $\Phi(\varkappa)$:

$$\Phi(\varkappa) = \frac{a^2 r_0^3}{\pi^2(1+\varkappa^2 r_0^2)^2}. \tag{16}$$

2) Similarly for the correlation function

$$B_f(r) = a^2 e^{-\left(\frac{r}{r_0}\right)^2} \tag{17}$$

we get

$$V(\varkappa) = \frac{a^2 r_0}{2\sqrt{\pi}} e^{-\frac{\varkappa^2 r_0^2}{4}}, \quad \Phi(\varkappa) = \frac{a^2 r_0^3}{8\pi\sqrt{\pi}} e^{-\frac{\varkappa^2 r_0^2}{4}}. \tag{18}$$

3) Finally, for the correlation function

$$B_f(r) = \frac{a^2}{2^{\nu-1}\Gamma(\nu)} \left(\frac{r}{r_0}\right)^{\nu} K_\nu\left(\frac{r}{r_0}\right) \tag{19}$$

we have

$$V(\varkappa) = \frac{\Gamma\left(\nu + \frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(\nu)} \frac{a^2 r_0}{(1 + \varkappa^2 r_0^2)^{\nu + \frac{1}{2}}},$$

$$\Phi(\varkappa) = \frac{\Gamma\left(\nu + \frac{3}{2}\right)}{\pi\sqrt{\pi}\,\Gamma(\nu)} \frac{a^2 r_0^3}{(1 + \varkappa^2 r_0^2)^{\nu + \frac{3}{2}}}. \tag{20}$$

§ 5. Locally homogeneous and isotropic random fields

Homogeneous and isotropic random fields provide only a crude approximation to real meteorological fields. For example, the statistical characteristics of atmospheric turbulence are generally functions of altitude. Therefore, as with nonstationary random processes, the three-dimensional structure of meteorological and other fields is best treated by the method of structure functions.

The difference of the field values $f(\boldsymbol{r})$ between two points $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ is mainly influenced by those inhomogeneities of the field f whose dimensions do not exceed the distance $|\boldsymbol{r}_1 - \boldsymbol{r}_2|$. If this distance is not excessively large, the largest inhomogeneities do not affect the difference $f(\boldsymbol{r}_1) - f(\boldsymbol{r}_2)$ and the structure function (which will be considered only for real random fields)

$$D(\boldsymbol{r}_1, \boldsymbol{r}_2) = \langle\{[f(\boldsymbol{r}_1) - \langle f(\boldsymbol{r}_1)\rangle] - [f(\boldsymbol{r}_2) - \langle f(\boldsymbol{r}_2)\rangle]\}^2\rangle \tag{1}$$

may depend on $\boldsymbol{r}_1 - \boldsymbol{r}_2$ only. On the other hand, the correlation function $B(\boldsymbol{r}_1, \boldsymbol{r}_2)$ is sensitive to inhomogeneities of all scales and it may depend on each of the two arguments separately,* and not just on the difference $\boldsymbol{r}_1 - \boldsymbol{r}_2$. In this case we introduce the concept of local homogeneity /10/.

A random field $f(\boldsymbol{r})$ is said to be locally homogeneous if the distribution functions of the random variable $f(\boldsymbol{r}_1) - f(\boldsymbol{r}_2)$ are invariant under translation of the pair of points $\boldsymbol{r}_1$, $\boldsymbol{r}_2$.

* For example, the correlation functions of atmospheric meteorological fields depend not only on the distance between the two observation points but also on the average height above the ground, i.e., in fact on the coordinates of each of the two observation points.

The mean value of the difference $\langle f(\boldsymbol{r}_1) - f(\boldsymbol{r}_2)\rangle$ and the structure function (1) of a locally homogeneous random field thus depend only on $\boldsymbol{r}_1 - \boldsymbol{r}_2$:

$$D(\boldsymbol{r}_1 + \boldsymbol{r}, \boldsymbol{r}_1) = D(\boldsymbol{r}).$$

A locally homogeneous random field is said to be locally isotropic if the distribution functions of the difference $f(\boldsymbol{r}_1) - f(\boldsymbol{r}_2)$ are invariant under rotation and reflection of the vector $\boldsymbol{r}_1 - \boldsymbol{r}_2$. The structure function of a locally isotropic random field depends only on $|\boldsymbol{r}_1 - \boldsymbol{r}_2|$:

$$D(\boldsymbol{r}) = D(r). \tag{2}$$

A locally homogeneous random field may be represented in a spectral form analogous to the expansion of a process with random increments, (3.28):

$$f(\boldsymbol{r}) = f(0) + \boldsymbol{a}\boldsymbol{r} + \iiint\limits_{-\infty}^{\infty} [e^{i\varkappa r} - 1]\, Z(d^3\varkappa), \tag{3}$$

where $f(0)$ is a random variable and $\boldsymbol{a}$ is a random vector. If the field f is locally isotropic, no preferred directions are observed. The vector $\boldsymbol{a}$ is isotropically distributed and $\langle a\rangle = 0$. In most cases the vector $\boldsymbol{a}$ itself may be taken equal to zero; in this case the spectral expansion of a locally isotropic field takes the form

$$f(\boldsymbol{r}) = f(0) + \iiint\limits_{-\infty}^{\infty} [e^{i\varkappa r} - 1]\, Z(d^3\varkappa). \tag{4}$$

The function $Z(d^3\varkappa)$ satisfies the same relations as before:

$$\left.\begin{aligned} \langle Z(d^3\varkappa)\rangle &= 0, \\ \langle Z(d^3\varkappa)\, Z^*(d^3\varkappa')\rangle &= \delta(\boldsymbol{\varkappa} - \boldsymbol{\varkappa}')\, \Phi(\boldsymbol{\varkappa})\, d^3\varkappa\, d^3\varkappa', \end{aligned}\right\} \tag{5}$$

where $\Phi \geqslant 0$ is the spectral density of the field.

Substituting (3) in (1), we obtain the expansion

$$D(\boldsymbol{r}) = 2\iiint\limits_{-\infty}^{\infty} [1 - \cos \boldsymbol{\varkappa}\boldsymbol{r}]\, \Phi(\boldsymbol{\varkappa})\, d^3\varkappa. \tag{6}$$

For a locally isotropic field, it follows from (4) that $\langle f(\boldsymbol{r})\rangle = \text{const}$ and relation (1) takes the simpler form

$$D(\boldsymbol{r}) = \langle [f(\boldsymbol{r} + \boldsymbol{r}') - f(\boldsymbol{r}')]^2\rangle. \tag{7}$$

In this case the spectral density $\Phi(\boldsymbol{\varkappa})$ depends only on the magnitude of the vector $\boldsymbol{\varkappa}$:

$$\Phi(\boldsymbol{\varkappa}) = \Phi(\varkappa).$$

In (6) we introduce spherical coordinates in $\boldsymbol{\varkappa}$-space and integrate over the angular variables, obtaining

$$D(r) = 8\pi \int\limits_0^{\infty} \left(1 - \frac{\sin \varkappa r}{\varkappa r}\right) \Phi(\varkappa)\, \varkappa^2\, d\varkappa. \tag{8}$$

To obtain the inverse of (6), we take its gradient and invert the resulting Fourier integral, which gives

$$\Phi(\boldsymbol{\varkappa}) = \frac{\boldsymbol{\varkappa}}{16\pi^3\varkappa^2}\iiint\limits_{-\infty}^{\infty} \sin \boldsymbol{\varkappa r} \,\mathrm{grad}\, D(\boldsymbol{r})\, d^3r. \tag{9}$$

If the field $f(\boldsymbol{r})$ is locally isotropic then $D(\boldsymbol{r}) = D(r)$, and relation (9) can be simplified. In this case

$$\boldsymbol{\varkappa}\nabla D(\boldsymbol{r}) = \frac{\boldsymbol{\varkappa r}}{r} D'(r).$$

Introducing spherical coordinates and integrating over the angular variables, we obtain

$$\Phi(\varkappa) = \frac{1}{4\pi^2\varkappa^3}\int\limits_0^{\infty} D'(r)\,[\sin \varkappa r - \varkappa r \cos \varkappa r]\, dr. \tag{10}$$

The integral in (10) converges (for $\varkappa \neq 0$) if

a) $\lim\limits_{r\to\infty} rD'(r) = 0$,

b) there exists $\alpha < 1$ such that

$$\lim_{r\to 0} r^{3+\alpha} D'(r) = \mathrm{const} < \infty \tag{11}$$

or a more strict condition

$$\lim_{r\to 0} r^4 D'(r) = 0.$$

Operating with a Laplacian on (6), we obtain a different inversion formula:

$$\Phi(\boldsymbol{\varkappa}) = \frac{1}{16\pi^3\varkappa^2}\iiint\limits_{-\infty}^{\infty} \cos \boldsymbol{\varkappa r}\, \Delta D(\boldsymbol{r})\, d^3r,$$

which for locally isotropic fields takes the form

$$\Phi(\varkappa) = \frac{1}{4\pi^2\varkappa^2}\int\limits_0^{\infty} \frac{\sin \varkappa r}{\varkappa r} \frac{d}{dr}[r^2 D'(r)]\, dr. \tag{12}$$

The integral in (12) converges (for $\varkappa \neq 0$) if

a) $\lim\limits_{r\to\infty} r^{-1}\frac{d}{dr}[r^2 D'(r)] = 0$,

b) there exists $\alpha < 1$ such that

$$\lim_{r\to 0} r^{\alpha} \frac{d}{dr}[r^2 D'(r)] = \mathrm{const} < \infty \tag{13}$$

or a more strict condition

$$\lim_{r\to 0} r \frac{d}{dr}[r^2 D'(r)] = 0.$$

As in the case of processes with stationary increments, conditions (11) and (13) are fundamentally different, and relations (10) and (12) should therefore be applied only after the validity of these conditions has been checked.

For a locally isotropic field we may also introduce a one-dimensional spectral expansion, as in (4.3):

$$D(r) = 2\int_{-\infty}^{\infty} [1 - \cos \varkappa r]\, V(\varkappa)\, d\varkappa. \tag{14}$$

The inversion formula for (14) is analogous to the corresponding formula (3.20) for processes with stationary increments:

$$V(\varkappa) = \frac{1}{2\pi\varkappa}\int_{0}^{\infty} \sin \varkappa r D'(r)\, dr. \tag{15}$$

Relation (15) is valid if the following conditions are satisfied:

a) $\lim\limits_{r\to\infty} D'(r) = 0$,

b) there exists $\alpha < 1$ such that

$$\lim_{r\to 0} r^{1+\alpha} D'(r) = \text{const} < \infty$$

or the more strict condition

$$\lim_{r\to 0} r^2 D'(r) = 0. \tag{16}$$

Differentiating (15) and comparing the result with (10), we obtain a relation between $\Phi(\varkappa)$ and $V(\varkappa)$, analogous to (4.13):

$$\Phi(\varkappa) = -\frac{1}{2\pi\varkappa}\frac{dV(\varkappa)}{d\varkappa}. \tag{17}$$

This expression can also be drived from (12). The relationship between the one-dimensional and the three-dimensional spectral density of a locally isotropic random field is thus the same as for an isotropic field.

The integral in the spectral expansion (6) converges if

a) there exists $\alpha < 1$ such that

$$\lim_{\varkappa\to 0} \varkappa^{4+\alpha}\Phi(\varkappa) = 0; \tag{18a}$$

b) there exists $\beta > 1$ such that

$$\lim_{\varkappa\to\infty} \varkappa^{2+\beta}\Phi(\varkappa) = 0. \tag{18b}$$

The first of these two requirements indicates that the function $\Phi(\varkappa)$ may have a singularity at the origin of the form $\varkappa^{-\mu}$ with $\mu < 5$.

The techniques for describing locally homogeneous random fields is thus applicable to the description of random fields which have infinite "energy" in the large scale region.

Besides the expansion of locally isotropic fields and their structure functions in three-dimensional Fourier integrals, we will often use two-dimensional expansions in a plane x = const:

$$f(x, y, z) = f(x, 0, 0) + \iint_{-\infty}^{\infty} [e^{i(\varkappa_2 y + \varkappa_3 z)} - 1]\, u(d\varkappa_2, d\varkappa_3, x). \tag{19}$$

Here $f(x, 0, 0)$ is a random function and $u(d\varkappa_2, d\varkappa_3, x)$ satisfies the relation

$$\langle u(d\varkappa_2, d\varkappa_3, x)\, u^*(d\varkappa_2', d\varkappa_3', x')\rangle = \\ = \delta(\varkappa_2 - \varkappa_2')\,\delta(\varkappa_3 - \varkappa_3')\,F(\varkappa_2, \varkappa_3, x - x')\,d\varkappa_2\,d\varkappa_3\,d\varkappa_2'\,d\varkappa_3', \tag{20}$$

where

$$F(\varkappa_2, \varkappa_3, x) = F(\varkappa_2, \varkappa_3, -x).$$

Consider the difference between $f(x, y, z)$ at two points in the plane $x =$ = const. Using expansion (19), we obtain

$$f(x, y, z) - f(x, y', z') = \iint_{-\infty}^{\infty} [e^{i(\varkappa_2 y + \varkappa_3 z)} - e^{i(\varkappa_2 y' + \varkappa_3 z')}]\, u(d\varkappa_2, d\varkappa_3, x).$$

We calculate the correlation function of two such differences in the planes x and x':

$$\langle [f(x, y, z) - f(x, y', z')]\,[f^*(x', y, z) - f^*(x', y', z')]\rangle = \\ = \iiiint_{-\infty}^{\infty} [e^{i(\varkappa_2 y + \varkappa_3 z)} - e^{i(\varkappa_2 y' + \varkappa_3 z')}]\,[e^{-i(\varkappa_2' y + \varkappa_3' z)} - e^{-i(\varkappa_2' y' + \varkappa_3' z')}] \times \\ \times \langle u(d\varkappa_2, d\varkappa_3, x)\, u^*(d\varkappa_2', d\varkappa_3', x')\rangle.$$

Using (20) we find

$$\langle [f(x, y, z) - f(x, y', z')]\,[f^*(x', y, z) - f^*(x', y', z')]\rangle = \\ = 2\iint_{-\infty}^{\infty} \{1 - \cos[\varkappa_2(y - y') + \varkappa_3(z - z')]\}\, F(\varkappa_2, \varkappa_3, x - x')\, d\varkappa_2\, d\varkappa_3. \tag{21}$$

In a locally isotropic field the correlation between the differences

$$f(x, y, z) - f(x, y', z') \quad \text{and} \quad f^*(x', y, z) - f^*(x', y', z')$$

is obviously caused only by inhomogeneities which are larger than the distance $|x - x'|$, i.e., $l \geqslant |x - x'|$. Since the scale l corresponds to a wavenumber $\varkappa \sim \frac{2\pi}{l}$, the correlation between these differences is determined only by that part of the spectrum where the wavenumbers satisfy the inequality $\varkappa|x - x'| \leqslant 1$. Hence, the function $F(\varkappa_2, \varkappa_3, x - x')$, which is the spectral density of

$$\langle [f(x, y, z) - f(x, y', z')]\ [f^*(x', y, z) - f^*(x', y', z')]\rangle,$$

rapidly diminishes for $\varkappa|x - x'| > 1$. Using the algebraic identity

$$(a - b)(c - d) = \frac{1}{2}[(a - d)^2 + (b - c)^2 - (a - c)^2 - (b - d)^2],$$

we can express the left-hand side of equation (21) in terms of the structure function of the field f, which gives

$$D_f(x - x', y - y', z - z') - D_f(x - x', 0, 0) = \\ = 2\iint_{-\infty}^{\infty} \{1 - \cos[\varkappa_2(y - y') + \varkappa_3(z - z')]\}\, F(\varkappa_2, \varkappa_3, x - x')\, d\varkappa_2\, d\varkappa_3. \tag{22}$$

Setting $\xi = x - x' = 0,\ y - y' = \eta,\ z - z' = \zeta$, we obtain

$$D_f(0, \eta, \zeta) = 2\iint\limits_{-\infty}^{\infty} [1 - \cos(\varkappa_2\eta + \varkappa_3\zeta)]\, F(\varkappa_2, \varkappa_3, 0)\, d\varkappa_2\, d\varkappa_3, \tag{23}$$

i.e., the function $F(\varkappa_2, \varkappa_3, 0)$ is the two-dimensional spectral density of $D(0, \eta, \zeta)$. If the field is locally isotropic in the plane x = const, the function $F(\varkappa_2, \varkappa_3, |x|)$ depends only on $\sqrt{\varkappa_2^2 + \varkappa_3^2}$ and we have

$$D_f(\rho) = 4\pi\int\limits_0^{\infty} [1 - J_0(\varkappa\rho)]\, F(\varkappa, 0)\varkappa d\varkappa. \tag{24}$$

Here

$$\rho^2 = \eta^2 + \zeta^2 \text{ and } F(\varkappa_2, \varkappa_3, 0) = F(\sqrt{\varkappa_2^2 + \varkappa_3^2}, 0).$$

We now derive the relationship between $\Phi(\varkappa)$ and $F(\varkappa_2, \varkappa_3, x)$. Inserting the spectral expansion (6) in the left-hand side of (22), we obtain

$$2\iiint\limits_{-\infty}^{\infty} [\cos\varkappa_1\xi - \cos(\varkappa_1\xi + \varkappa_2\eta + \varkappa_3\zeta)]\, \Phi(\boldsymbol{\varkappa})\, d^3\varkappa =$$

$$= 2\iint\limits_{-\infty}^{\infty} [1 - \cos(\varkappa_2\eta + \varkappa_3\zeta)]\, F(\varkappa_2, \varkappa_3, \xi)\, d\varkappa_2\, d\varkappa_3. \tag{25}$$

Using the equality $\cos\varkappa_1\xi - \cos(\varkappa_1\xi + \varkappa_2\eta + \varkappa_3\zeta) = \cos\varkappa_1\xi - \cos\varkappa_1\xi\ \cos\ (\varkappa_2\eta + \varkappa_3\zeta) + \sin\varkappa_1\xi\ \sin\ (\varkappa_2\eta + \varkappa_3\zeta)$ and noting that the integral of the product

$$\sin\varkappa_1\xi \sin(\varkappa_2\eta + \varkappa_3\zeta)\ \Phi(\boldsymbol{\varkappa})$$

is zero as this function is odd with respect to $\boldsymbol{\varkappa}$, we obtain

$$\iint\limits_{-\infty}^{\infty} [1 - \cos(\varkappa_2\eta + \varkappa_3\zeta)]\, d\varkappa_2\, d\varkappa_3 \int\limits_{-\infty}^{\infty} \cos(\varkappa_1\xi)\, \Phi(\boldsymbol{\varkappa})\, d\varkappa_1 =$$

$$= \iint\limits_{-\infty}^{\infty} [1 - \cos(\varkappa_2\eta + \varkappa_3\zeta)]\, F(\varkappa_2, \varkappa_3, \xi)\, d\varkappa_2\, d\varkappa_3, \tag{26}$$

from which it follows that

$$F(\varkappa_2, \varkappa_3, \xi) = \int\limits_{-\infty}^{\infty} \cos(\varkappa_1\xi)\, \Phi(\boldsymbol{\varkappa})\, d\varkappa_1. \tag{27}$$

Inverting this Fourier integral, we get

$$\Phi(\boldsymbol{\varkappa}) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} \cos(\varkappa_1\xi)\, F(\varkappa_2, \varkappa_3, \xi)\, d\xi. \tag{28}$$

Using the last formula we can express the spectral expansion of $D(\boldsymbol{r})$ in terms of $F(\varkappa_2, \varkappa_3, \xi)$. Inserting (28) in (6) and using the relation

$$\delta(\xi) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} \cos\varkappa\xi\, d\varkappa,$$

we obtain after simple calculations

$$D(x, y, z) = 2\iint_{-\infty}^{\infty} F(\varkappa_2, \varkappa_3, 0)\, d\varkappa_2\, d\varkappa_3 - 2\iint_{-\infty}^{\infty} F(\varkappa_2, \varkappa_3, x)\cos(\varkappa_2 y + \varkappa_3 z)\, d\varkappa_2\, d\varkappa_3. \qquad (29)$$

For a locally isotropic field,

$$F(\varkappa_2, \varkappa_3, x) = F\left(\sqrt{\varkappa_2^2 + \varkappa_3^2}, x\right).$$

Introducing new integration variables

$$\varkappa_2 = \varkappa\cos\varphi, \quad \varkappa_3 = \varkappa\sin\varphi$$

and putting

$$y = \rho\cos\alpha, \quad z = \rho\sin\alpha, \text{ where } \rho^2 = y^2 + z^2,$$

we integrate using the well-known formula

$$\int_{-\pi}^{\pi} \cos[\varkappa\rho\cos(\varphi - \alpha)]\, d\varphi = 2\pi J_0(\varkappa\rho)$$

and obtain

$$D(r) = 4\pi\int_0^{\infty} F(\varkappa, 0)\varkappa\, d\varkappa - 4\pi\int_0^{\infty} F(\varkappa, x) J_0(\varkappa\rho)\varkappa\, d\varkappa, \qquad (30)$$

where $r^2 = x^2 + \rho^2$. In the particular case $x = 0$ relation (30) reduces to (24).

The correlation function $B(r)$ of a homogeneous field $f(\boldsymbol{r})$, if it exists, can be expressed in terms of $F(\varkappa_2, \varkappa_3, x)$ by the following relation, which is proved without much difficulty:

$$B(x, y, z) = \iint_{-\infty}^{\infty} \cos(\varkappa_2 y + \varkappa_3 z) F(\varkappa_2, \varkappa_3, x)\, d\varkappa_2\, d\varkappa_3. \qquad (31)$$

If the field $f(\boldsymbol{r})$ is isotropic in the plane $x = \text{const}$, we get

$$B(r) = 2\pi\int_0^{\infty} F(\varkappa, x) J_0(\varkappa\rho)\varkappa\, d\varkappa. \qquad (32)$$

In the following we will often use the expression for the correlation function in a plane perpendicular to the x axis, which is obtained from (32) taking $x = 0$:

$$B(\rho) = 2\pi\int_0^{\infty} F(\varkappa, 0) J_0(\varkappa\rho)\varkappa\, d\varkappa. \qquad (33)$$

Consider a few examples.

1) The structure function of a homogeneous and isotropic random field treated in example 3 of the previous section can be expressed in terms of

its correlation function by the standard relation

$$D(r) = 2B(0) - 2B(r).$$

Substituting

$$B(r) = \frac{a^2}{2^{\nu-1}\Gamma(\nu)}\left(\frac{r}{r_0}\right)^{\nu} K_{\nu}\left(\frac{r}{r_0}\right),$$

we obtain

$$D(r) = 2a^2\left[1 - \frac{1}{2^{\nu-1}\Gamma(\nu)}\left(\frac{r}{r_0}\right)^{\nu} K_{\nu}\left(\frac{r}{r_0}\right)\right]. \qquad (34)$$

The spectral density corresponding to (34) is

$$\Phi(\varkappa) = \frac{\Gamma\left(\nu + \frac{3}{2}\right)}{\pi\sqrt{\pi}\,\Gamma(\nu)} \frac{a^2 r_0^3}{(1 + \varkappa^2 r_0^2)^{\nu + \frac{3}{2}}}. \qquad (35)$$

For $r \ll r_0$,

$$D(r) \approx 2a^2 \frac{\Gamma(1-\nu)}{\Gamma(1+\nu)}\left(\frac{r}{2r_0}\right)^{2\nu}, \text{ i.e., } D(r) \sim r^{2\nu}.$$

2) Consider the structure function

$$D(r) = C^2 r^{\mu} \quad (0 < \mu < 2). \qquad (36)$$

The one-dimensional spectral density corresponding to this function (see (3.38)) is

$$V(\varkappa) = \frac{\Gamma(\mu + 1)}{2\pi} \sin\left(\frac{\pi\mu}{2}\right) C^2 \varkappa^{-(\mu+1)}. \qquad (37)$$

Using (17), we find the three-dimensional spectral density $\Phi(\varkappa)$:

$$\Phi(\varkappa) = \frac{\Gamma(\mu + 2)}{4\pi^2} \sin\left(\frac{\pi\mu}{2}\right) C^2 \varkappa^{-(\mu+3)}. \qquad (38)$$

Let us also compute the two-dimensional spectral density $F(\varkappa, x)$ corresponding to the structure function $C^2 r^{\mu}$. Inserting expression (38) in the right-hand side of (27) and integrating, we obtain

$$F(\varkappa, x) = \frac{C^2}{\pi^2} \sin\frac{\pi\mu}{2} \cdot 2^{\frac{\mu}{2} - 1}\,\Gamma\left(1 + \frac{\mu}{2}\right) \frac{(\varkappa x)^{1+\frac{\mu}{2}} K_{1+\frac{\mu}{2}}(\varkappa x)}{\varkappa^{\mu+2}}. \qquad (39)$$

Since for $z \gg 1$

$$K_{\nu}(z) \approx \sqrt{\frac{\pi}{2z}}\, e^{-z},$$

the function (39) for $\varkappa x \gg 1$ rapidly approaches zero, which corresponds to the previously mentioned general property of $F(\varkappa, x)$.

For

$$\nu = \frac{\mu}{2} \text{ and } \frac{a^2}{r_0^{2\nu}} = \frac{2^{2\nu-1}\Gamma(1+\nu)}{\Gamma(1-\nu)} C^2$$

the structure function of the previous example coincides with the structure function $C^2 r^\mu$ for $r \ll r_0$. The spectra of these functions coincide for all $\varkappa r_0 \gg 1$ (see Figures 4, 5).

§ 6. Space-time random fields

Actual meteorological fields are random functions both in space and in time. We will briefly consider their description, as it has much in common with the previous treatment.

If a real field $f(\boldsymbol{r}, t)$ is stationary in time and homogeneous in space, it can be described by means of a space-time correlation function

$$B(\boldsymbol{r}, \tau) = \langle f(\boldsymbol{r} + \boldsymbol{r}_1, t + \tau) f(\boldsymbol{r}_1, t) \rangle \tag{1}$$

(it is assumed that $\langle f \rangle = 0$). $B(\boldsymbol{r}, \tau)$ is even with respect to $\boldsymbol{r}$ and τ; it furthermore satisfies the relation

$$|B(\boldsymbol{r}, \tau)| \leqslant B(0, 0)$$

and is positive definite. $B(\boldsymbol{r}, \tau)$ may be represented as a Fourier integral

$$B(\boldsymbol{r}, \tau) = \iiiint_{-\infty}^{\infty} \cos(\boldsymbol{\varkappa r} + \omega\tau)\, u(\boldsymbol{\varkappa}, \omega)\, d^3\varkappa\, d\omega, \tag{2}$$

where $u(\boldsymbol{\varkappa}, \omega) \geqslant 0$ is a four-dimensional (space-time) spectral density. Setting $\tau = 0$ in (2), we obtain

$$B(\boldsymbol{r}, 0) = B(\boldsymbol{r}) = \iiint_{-\infty}^{\infty} \cos \boldsymbol{\varkappa r} \left[\int_{-\infty}^{\infty} u(\boldsymbol{\varkappa}, \omega)\, d\omega\right] d^3\varkappa. \tag{3}$$

Comparing (3) with (4.7), we find

$$\Phi(\boldsymbol{\varkappa}) = \int_{-\infty}^{\infty} u(\boldsymbol{\varkappa}, \omega)\, d\omega. \tag{4}$$

Similarly, setting $\boldsymbol{r} = 0$ in (2), we find

$$B(0, \tau) = B(\tau) = \int_{-\infty}^{\infty} \cos \omega\tau \left[\iiint_{-\infty}^{\infty} u(\boldsymbol{\varkappa}, \omega)\, d^3\varkappa\right] d\tau, \tag{5}$$

hence the relation

$$W(\omega) = \iiint_{-\infty}^{\infty} u(\boldsymbol{\varkappa}, \omega)\, d^3\varkappa. \tag{6}$$

A stationary and homogeneous random field $f(\boldsymbol{r}, t)$ can be represented as a stochastic Fourier—Stieltjes integral

$$f(\boldsymbol{r}, t)=\iiiint_{-\infty}^{\infty} e^{i(\varkappa r+\omega t)} Z(d^3\varkappa, d\omega), \tag{7}$$

where the function Z satisfies the conditions

$$\begin{gathered}\langle Z(d^3\varkappa, d\omega)\rangle=0, \\ \langle Z(d^3\varkappa, d\omega) Z^*(d^3\varkappa', d\omega')\rangle=\delta(\boldsymbol{\varkappa}-\boldsymbol{\varkappa}')\, \delta(\omega-\omega')\, u(\boldsymbol{\varkappa}, \omega)\, d^3\varkappa\, d^3\varkappa'\, d\omega\, d\omega'.\end{gathered} \tag{8}$$

The substitution of (7) in (1) gives (2).

As an important example, consider the case when all the time changes in $f(\boldsymbol{r}, t)$ are associated with a simple translation of the spatial field distribution with a constant velocity $\boldsymbol{v}$; this translation is not accompanied by any mixing (a "frozen" field). Then

$$f(\boldsymbol{r}, t+t')=f(\boldsymbol{r}-\boldsymbol{v}t', t).$$

Inserting this expression in (1) we obtain

$$\begin{gathered}B(\boldsymbol{r}, \tau)=\langle f(\boldsymbol{r}+\boldsymbol{r}_1, t+\tau) f(\boldsymbol{r}_1, t)\rangle= \\ =\langle f(\boldsymbol{r}+\boldsymbol{r}_1-\boldsymbol{v}t-\boldsymbol{v}\tau, 0)\ f(\boldsymbol{r}_1-\boldsymbol{v}t, 0)\rangle=B(\boldsymbol{r}-\boldsymbol{v}\tau),\end{gathered}$$

i.e., for a "frozen" field moving with a constant velocity $\boldsymbol{v}$ we have

$$B(\boldsymbol{r}, \tau)=B(\boldsymbol{r}-\boldsymbol{v}\tau). \tag{9}$$

Expression (9) is readily seen to be equivalent to the relation

$$u(\boldsymbol{\varkappa}, \omega)=\delta(\omega+\boldsymbol{\varkappa v})\ \Phi(\boldsymbol{\varkappa}) \tag{10}$$

between spectral densities. Indeed, if we insert (10) in (2), we get (9).

Let the field $f(\boldsymbol{r}, t)$ be statistically isotropic. Then $\Phi(\boldsymbol{\varkappa})=\Phi(\varkappa)$. Introducing spherical coordinates with the polar axis pointing along the vector $\boldsymbol{v}$, we substitute (10) in (6):

$$W(\omega)=\int_0^{\infty} \varkappa^2 d\varkappa \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta\, d\theta\, \delta(\varkappa v\cos\theta+\omega)\, \Phi(\varkappa). \tag{11}$$

Integrating over φ and introducing a new variable $\cos\theta=\xi$, we write (11) in the form

$$W(\omega)=2\pi\int_0^{\infty} \varkappa^2 \Phi(\varkappa)\, d\varkappa \int_{-1}^{1} \delta(\varkappa v x+\omega)\, dx. \tag{12}$$

Seeing that

$$\delta(\varkappa v x+\omega)=\frac{1}{\varkappa v}\, \delta\left(x+\frac{\omega}{\varkappa v}\right),$$

we obtain

$$\int_{-1}^{1} \delta(\varkappa v x + \omega)\, dx = \frac{1}{\varkappa v} \int_{-1}^{1} \delta\left(x + \frac{\omega}{\varkappa v}\right) dx = \begin{cases} \frac{1}{\varkappa v} & \text{for } \left|\frac{\omega}{\varkappa v}\right| < 1, \\ 0 & \text{for } \left|\frac{\omega}{\varkappa v}\right| > 1. \end{cases}$$

Substituting this expression in (12), we obtain

$$W(\omega) = \frac{2\pi}{v} \int_{|\omega|/v}^{\infty} \Phi(\varkappa)\, \varkappa\, d\varkappa. \tag{13}$$

Differentiating (13), we obtain the relation

$$\Phi(\varkappa) = -\frac{v^2}{2\pi\varkappa} W'(\varkappa v). \tag{14}$$

Expressions (13) and (14) relate the space and the time (frequency) spectra of an isotropic "frozen" random field. The conditions under which real fields in a turbulent medium can be considered frozen are treated in Part B of this chapter.

If the field $f(\boldsymbol{r}, t)$ is locally homogeneous in $\boldsymbol{r}$ and has the property of stationary increments, it is described by the structure function

$$D(\boldsymbol{r}, \tau) = \langle\{[f(\boldsymbol{r} + \boldsymbol{r}', t + \tau) - f(\boldsymbol{r}', t)] - \langle f(\boldsymbol{r} + \boldsymbol{r}', t + \tau) - f(\boldsymbol{r}', t)\rangle\}^2\rangle. \tag{15}$$

$D(\boldsymbol{r}, \tau)$ may be represented in the form

$$D(\boldsymbol{r}, \tau) = 2 \iiiint_{-\infty}^{\infty} [1 - \cos(\boldsymbol{\varkappa r} + \omega\tau)]\, u(\boldsymbol{\varkappa}, \omega)\, d^3\varkappa\, d\omega. \tag{16}$$

Relations (4) and (6) are also valid in this case. The field $f(\boldsymbol{r}, t)$ can be represented as a stochastic Fourier–Stieltjes integral

$$f(\boldsymbol{r}, t) = f(0, 0) + \boldsymbol{a r} + bt + \iiiint_{-\infty}^{\infty} [e^{i(\boldsymbol{\varkappa r} + \omega t)} - 1]\, Z(d^3\varkappa, d\omega), \tag{17}$$

with $Z(d^3\varkappa, d\omega)$ satisfying relations (8).

For a locally isotropic field with stationary increments $\boldsymbol{a} = 0$ (see §5).

For a "frozen" locally isotropic random field, relation (9) is replaced by

$$D(\boldsymbol{r}, \tau) = D(\boldsymbol{r} - \boldsymbol{v}\tau), \tag{18}$$

whereas the relations (10), (13), and (14) between spectra are not changed.*

§ 7. Locally homogeneous fields with smoothly varying mean characteristics

We have dealt so far with locally homogeneous random fields whose structure function $D(\boldsymbol{r}_1, \boldsymbol{r}_2)$ satisfied the condition $D(\boldsymbol{r}_1, \boldsymbol{r}_2) = D(\boldsymbol{r}_1 - \boldsymbol{r}_2)$ within

* An interesting example of a random field whose space and time characteristics are related by certain constraints which follow from the wave equation is considered in /12/.

a certain region G with dimensions of the order L_0. In the theory of turbulence, the structure functions of random fields are of a universal character for $|r_1 - r_2| \ll L_0$, i.e., their dependence on $r_1 - r_2$ is the same irrespective of the actual position of the points r_1, r_2 in the turbulent region, provided that they are sufficiently close to one another. On passing to another region G' in the turbulent medium, which is also of the size L_0 and is at a distance L_0 from G, the dependence of the structure function on $r_1 - r_2$ does not change, although the general intensity of the fluctuations, i.e., the numerical coefficient in $D(r_1, r_2)$, may change. We thus arrive at a relation of the form

$$D(r_1, r_2) = C^2\left(\frac{r_1 + r_2}{2}\right) D_0(r_1 - r_2). \quad (1)$$

Here the function $C^2(R)$ describes the smooth variation in the intensity of fluctuations on passing from one part of the turbulent region to another; it changes appreciably only when R is incremented by an amount on the order of L_0. The function $D_0(r_1 - r_2)$ describes the local structure of the random field and is the same for all the regions $G, G', \ldots$ The function $D_0(r)$ is meaningful only for $r \ll L_0$. However, if it is extended in a purely formal way to larger values of the argument, we can write the spectral expansion

$$D_0(r) = 2\iiint_{-\infty}^{\infty} [1 - \cos \varkappa r]\, \Phi_0(\varkappa)\, d^3\varkappa. \quad (2)$$

Substituting (2) in (1), we obtain

$$D(r_1, r_2) = 2C^2\left(\frac{r_1 + r_2}{2}\right) \iiint_{-\infty}^{\infty} [1 - \cos \varkappa (r_1 - r_2)]\, \Phi_0(\varkappa)\, d^3\varkappa. \quad (3)$$

This relation can be written as

$$D(r_1, r_2) = 2\iiint_{-\infty}^{\infty} [1 - \cos \varkappa (r_1 - r_2)]\, \Phi\left(\varkappa, \frac{r_1 + r_2}{2}\right) d^3\varkappa, \quad (4)$$

where

$$\Phi\left(\varkappa, \frac{r_1 + r_2}{2}\right) = C^2\left(\frac{r_1 + r_2}{2}\right) \Phi_0(\varkappa). \quad (5)$$

The relative spectral distribution of fluctuations in f is thus the same for all the regions $G, G', \ldots$, where it is described by the function $\Phi_0(\varkappa)$; different regions only differ in the general intensity of the fluctuations, which is expressed by the function

$$C^2\left(\frac{r_1 + r_2}{2}\right).$$

As before, we can introduce a two-dimensional spectral density

$$F_f\left(\varkappa_2, \varkappa_3, x - x', \frac{r + r'}{2}\right),$$

related to the functions

$$D_f(\boldsymbol{r}, \boldsymbol{r}') \text{ and } \Phi_f\left(\boldsymbol{\varkappa}, \frac{\boldsymbol{r}+\boldsymbol{r}'}{2}\right)$$

by expressions analogous to (5.22)—(5.28):

$$F_f\left(\varkappa_2, \varkappa_3, x-x', \frac{\boldsymbol{r}+\boldsymbol{r}'}{2}\right) = \int\limits_{-\infty}^{\infty} \cos\left[\varkappa_1(x-x')\right] \Phi_f\left(\varkappa_1, \varkappa_2, \varkappa_3, \frac{\boldsymbol{r}+\boldsymbol{r}'}{2}\right) d\varkappa. \tag{6}$$

(In (4)—(6) we disregard the difference in the values of $C_f^2\left(\frac{\boldsymbol{r}_1+\boldsymbol{r}_2}{2}\right)$ at two close points the distance between which is $|\boldsymbol{r}_1-\boldsymbol{r}_2| \ll L_0$, since this function changes appreciably only when its argument is changed by an amount of the order of L_0).

Let us briefly consider the spectral expansion of the random field $f(\boldsymbol{r})$. For simplicity, we investigate an example where the correlation function of f exists and is of the form

$$B_f(\boldsymbol{r}, \boldsymbol{r}') = \sigma_f^2\left(\frac{\boldsymbol{r}+\boldsymbol{r}'}{2}\right) b_f(|\boldsymbol{r}-\boldsymbol{r}'|).$$

(Similar problems are treated by Silverman /166/.)

Let $f(\boldsymbol{r})$ be expressed by a stochastic Fourier—Stieltjes integral

$$f(\boldsymbol{r}) = \iiint\limits_{-\infty}^{\infty} e^{i\boldsymbol{\varkappa}\boldsymbol{r}} Z(d^3\varkappa) \tag{7}$$

(we assume that $\langle f\rangle = 0$).

Consider the expression $\langle f(\boldsymbol{r}) f(\boldsymbol{r}')\rangle = B_f(\boldsymbol{r}, \boldsymbol{r}')$:

$$\langle f(\boldsymbol{r}) f^*(\boldsymbol{r}')\rangle = \int\limits_{-\infty}^{\infty}\!\!\cdots\!\!\int e^{i(\boldsymbol{\varkappa}\boldsymbol{r}-\boldsymbol{\varkappa}'\boldsymbol{r}')} \langle Z(d^3\varkappa) Z^*(d^3\varkappa')\rangle. \tag{8}$$

We change to new coordinates

$$\boldsymbol{\rho} = \boldsymbol{r}-\boldsymbol{r}', \quad \boldsymbol{R} = \frac{1}{2}(\boldsymbol{r}+\boldsymbol{r}').$$

Then

$$\boldsymbol{\varkappa}\boldsymbol{r} - \boldsymbol{\varkappa}'\boldsymbol{r}' = (\boldsymbol{\varkappa}-\boldsymbol{\varkappa}')\boldsymbol{R} + \frac{1}{2}(\boldsymbol{\varkappa}+\boldsymbol{\varkappa}')\boldsymbol{\rho}$$

and

$$\langle f(\boldsymbol{r}) f^*(\boldsymbol{r}')\rangle = \int\limits_{-\infty}^{\infty}\!\!\cdots\!\!\int e^{i(\boldsymbol{\varkappa}-\boldsymbol{\varkappa}')\boldsymbol{R}} e^{i\frac{\boldsymbol{\varkappa}+\boldsymbol{\varkappa}'}{2}\boldsymbol{\rho}} \langle Z(d^3\varkappa) Z^*(d^3\varkappa')\rangle. \tag{9}$$

To agree with our assumption

$$\langle f(\boldsymbol{r}) f^*(\boldsymbol{r}')\rangle = \sigma_f^2(\boldsymbol{R}) b_f(\boldsymbol{\rho}).$$

Hence

$$\langle Z(d^3\varkappa) Z^*(d^3\varkappa')\rangle = M(\boldsymbol{\varkappa}-\boldsymbol{\varkappa}') \Phi_f^{(0)}\left(\frac{\boldsymbol{\varkappa}+\boldsymbol{\varkappa}'}{2}\right) d^3\varkappa\, d^3\varkappa', \tag{10}$$

where clearly $M(0) \geqslant 0$ and $\Phi_f^{(0)}(\varkappa) \geqslant 0$. Inserting this expression in (9) and changing variables, we obtain

$$\sigma_f^2(\boldsymbol{R}) = \iiint\limits_{-\infty}^{\infty} M(\varkappa)\, e^{i\varkappa R}\, d^3\varkappa, \tag{11}$$

$$b_f(\rho) = \iiint\limits_{-\infty}^{\infty} \Phi_f^{(0)}(\varkappa)\, e^{i\varkappa\rho}\, d^3\varkappa. \tag{12}$$

If σ_f^2 = const, $M(\varkappa - \varkappa') = \sigma_f^2\delta(\varkappa - \varkappa')$ and formula (10) reduces to (5.5). It follows from (10) that for small σ_f^2 the spectral components $Z(d^3\varkappa)$ and $Z^*(d^3\varkappa')$ that are close together are correlated: if $|\varkappa - \varkappa'| \lesssim L_0^{-1}$, then $\langle Z(d^3\varkappa)\, Z^*(d^3\varkappa')\rangle \neq 0$. (We see from (11) that the function $M(\varkappa)$ is markedly different from zero in an interval of the order of L_0^{-1}, since $\sigma_f(\boldsymbol{R})$ markedly changes over an interval of the order of L_0.)

§ 8. Vector random fields

In the following we will often encounter vector random fields, such as the velocity field in a turbulent flow, the electromagnetic field propagating in a turbulent medium, etc.

The description of the statistical structure of a vector random field has much in common with the description of a scalar field, One could start by defining correlation and structure functions for each of the vector field components. These correlation functions, however, will not reflect the relationship between the different field components, and it is necessary to investigate also the cross-correlations between the different components. We are thus led to consider the correlation tensor

$$B_{ik}(\boldsymbol{r}) = \langle [v_i(\boldsymbol{r} + \boldsymbol{r}_1) - \langle v_i(\boldsymbol{r} + \boldsymbol{r}_1)\rangle]\, [v_k(\boldsymbol{r}_1) - \langle v_k(\boldsymbol{r}_1)\rangle]\rangle. \tag{1}$$

In what follows we take $\langle v_k\rangle = 0$, i.e., we consider only deviations of the field from its mean value. Locally homogeneous fields will be described using the structure tensor

$$D_{ik}(\boldsymbol{r}) = \langle [v_i(\boldsymbol{r} + \boldsymbol{r}_1) - v_i(\boldsymbol{r}_1)]\, [v_k(\boldsymbol{r} + \boldsymbol{r}_1) - v_k(\boldsymbol{r}_1)]\rangle. \tag{2}$$

For a statistically homogeneous field B_{ik} and D_{ik} are related by

$$D_{ik}(\boldsymbol{r}) = 2B_{ik}(0) - 2B_{ik}(\boldsymbol{r}). \tag{3}$$

Let us consider a statistically isotropic (locally isotropic) vector field.

In an isotropic scalar field $\langle f(\boldsymbol{r}_1)\, f(\boldsymbol{r}_2)\rangle$ is independent of the orientation of the vector $\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$. In vector fields, however, isotropy is a more involved concept. Consider, for example, the correlation tensor component $B_{11}(\boldsymbol{r}) = \langle v_1(\boldsymbol{r}_1)\, v_1(\boldsymbol{r}_2)\rangle$. Here $v_1(\boldsymbol{r}_k)$ is the projection of the vector $\boldsymbol{v}(\boldsymbol{r}_k)$ on the x_1 axis. Let the points $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ lie on the x_1 axis. Then B_{11} is the correlation of the longitudinal (relative to the vector $\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$) components of the vector $\boldsymbol{v}$. Now suppose that the vector $\boldsymbol{r}$ is rotated around the point $\boldsymbol{r}_1$

through $\pi/2$, so that the points $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ lie on the axis x_2. In this case $v_1(\boldsymbol{r}_1)$ and $v_1(\boldsymbol{r}_2)$ are as before the projections of $\boldsymbol{v}$ on x_1. The vector $\boldsymbol{r}$, however, is perpendicular to this axis, so that $B_{11}(\boldsymbol{r})$ now is a correlation of the transverse (relative to $\boldsymbol{r}$) components of the vector $\boldsymbol{v}$. The values of B_{11} in these two cases clearly need not be the same, since the position of the investigated component of the vector $\boldsymbol{v}$ relative to the vector $\boldsymbol{r}$ has changed. In a statistically isotropic vector field each tensor component $B_{ik}(\boldsymbol{r})$ thus depends on the orientation of the vector $\boldsymbol{r}$.

Now at the point $\boldsymbol{r}_1$ we introduce a unit vector $\boldsymbol{l}$ pointing in an arbitrary direction and at the point $\boldsymbol{r}_2$ a unit vector $\boldsymbol{m}$, also arbitrarily oriented. The orientation of the vector $\boldsymbol{r}$ is characterized by a unit vector $\boldsymbol{n}$ directed along $\boldsymbol{r}$, i.e., $\boldsymbol{n}=\frac{\boldsymbol{r}}{r}$, where $r=|\boldsymbol{r}|$. Consider the correlation of the projections $\boldsymbol{lv}(\boldsymbol{r}_1)$ and $\boldsymbol{mv}(\boldsymbol{r}_2)$ of the field $\boldsymbol{v}$ at these two points in the direction of $\boldsymbol{l}$ and $\boldsymbol{m}$, respectively:

$$B=\langle(\boldsymbol{lv}(\boldsymbol{r}_1))\cdot(\boldsymbol{mv}(\boldsymbol{r}_2))\rangle=\langle l_k v_k(\boldsymbol{r}_1)\,m_j v_j(\boldsymbol{r}_2)\rangle=l_k m_j\langle v_k(\boldsymbol{r}_1)\,v_j(\boldsymbol{r}_2)\rangle=l_k m_j B_{kj}(\boldsymbol{r}).$$

We now rotate all the three vectors $\boldsymbol{n},\boldsymbol{l},\boldsymbol{m}$ as a rigid triad through an arbitrary angle, keeping r constant. The mutual orientation of the vectors $\boldsymbol{n},\boldsymbol{l},\boldsymbol{m}$ clearly has not changed, so that the scalar products $\boldsymbol{nl},\boldsymbol{nm},\boldsymbol{lm}$ are conserved. Clearly, if the field $\boldsymbol{v}(\boldsymbol{r})$ is statistically isotropic, B is not affected by this rotation. The definition of a statistically isotropic vector field thus amounts to the requirement that the expression

$$B=l_k m_j B_{kj}(\boldsymbol{r})$$

should be invariant under simultaneous rotation of the three vectors $\boldsymbol{n},\ \boldsymbol{l},\boldsymbol{m}$.

Since the only quantities conserved in this rotation are $r,\boldsymbol{nl},\boldsymbol{mn},\boldsymbol{lm}$, B is a function of these quantities only:

$$l_k m_j B_{kj}(\boldsymbol{r})=B(r,\ \boldsymbol{ln},\ \boldsymbol{mn},\ \boldsymbol{lm}).$$

The left-hand side of this equality is linear both in the components of the vector $\boldsymbol{l}$ and those of the vector $\boldsymbol{m}$. Therefore on the right-hand side of this equation $\boldsymbol{lm}$ must enter only linearly, and $\boldsymbol{ln}.\boldsymbol{mn}$ only as a product. B should thus have the form

$$B(r,\ \boldsymbol{ln},\ \boldsymbol{mn},\ \boldsymbol{lm})=P(r)(\boldsymbol{lm})+Q(r)(\boldsymbol{ln})(\boldsymbol{mn})=$$
$$=P(r)\,l_j m_j+Q(r)\,l_k n_k m_j n_j=l_k m_j\,[P(r)\,\delta_{kj}+Q(r)\,n_k n_j].$$

Comparing this expression with the relation $B=l_k m_j B_{kj}(\boldsymbol{r})$ and seeing that l_k,m_j are quite arbitrary, we obtain

$$B_{kj}(\boldsymbol{r})=P(r)\,\delta_{kj}+Q(r)\,n_k n_j, \tag{4}$$

which is a general expression for an isotropic tensor of second rank. A similar argument applies to the structure tensor of the field $\boldsymbol{v}$ (in this case we do not consider the values of the field $\boldsymbol{v}$ at the two points $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ but the corresponding differences). For D_{jk} we thus obtain

$$D_{jk}(\boldsymbol{r})=R(r)\,\delta_{jk}+S(r)\,n_j n_k. \tag{5}$$

Another more formal derivation of (4) and (5) is based on the following consideration: the expression for $B_{ik}(\boldsymbol{r})$ of an isotropic field $\boldsymbol{v}$ may contain only the unit vector along $\boldsymbol{r}$ and the unit isotropic tensor δ_{jk}; it may not contain any other vector or tensor, as this would imply some preferred direction in space, in contradiction to the assumption of isotropy. The only conceivable expression for a tensor of second rank which contains nothing but n_j and δ_{jk} is clearly (4). The functions $P(r)$, $Q(r)$, $R(r)$, $S(r)$ entering (4) and (5) are scalar functions of the scalar argument $r = |\boldsymbol{r}|$.

Thus, while in general a second-rank tensor has nine independent components and a symmetric tensor six components, the isotropy condition reduces the number of independent components to two.

Let the axis $z = x_3$ of the coordinate system point along the vector $\boldsymbol{r}$. The components of $\boldsymbol{n}$ are then [0, 0, 1]. Taking both the subscripts j, k in (4) and (5) be 1 or 2 simultaneously, we obtain

$$B_{11} = B_{22} = P, \quad D_{11} = D_{22} = R.$$

For $j = k = 3$ we have

$$B_{33} = P + Q, \quad D_{33} = R + S.$$

The transverse components B_{11}, B_{22} are equal. They are generally designated by the symbols

$$B_{11} = B_{22} = B_{tt}, \quad D_{11} = D_{22} = D_{tt}.$$

The longitudinal component B_{33}, D_{33} is generally denoted B_{rr}, D_{rr}. Expressing the functions P, R, Q, S in terms of B_{rr}, B_{tt}, D_{rr}, D_{tt} and substituting in (4), (5), we obtain

$$B_{ik}(\boldsymbol{r}) = B_{tt}(r)\,\delta_{ik} + [B_{rr}(r) - B_{tt}(r)]\,n_i n_k, \tag{4'}$$
$$D_{ik}(\boldsymbol{r}) = D_{tt}(r)\,\delta_{ik} + [D_{rr}(r) - D_{tt}(r)]\,n_i n_k. \tag{5'}$$

We know from the theory of vector fields that any sufficiently smooth vector field can be decomposed into a potential and a solenoidal part. A similar decomposition is valid for random fields too. First consider a solenoidal vector field. A suitable example is provided by the velocity field in an incompressible fluid or by the magnetic induction field. A solenoidal field satisfies the equation

$$\operatorname{div} \boldsymbol{v}(\boldsymbol{r}) = \frac{\partial v_i(\boldsymbol{r})}{\partial x_i} = 0. \tag{6}$$

(In the following the convention of summation over repeated tensor indices is adopted, unless otherwise specified. Thus, $\operatorname{div} \boldsymbol{v} = \sum_{i=1}^{3} \frac{\partial v_i}{\partial x_i}$, and in tensor notation we write simply $\frac{\partial v_i}{\partial x_i}$. The subscripts of B_{rr}, B_{tt}, D_{rr}, D_{tt} are not tensor indices and no summation is implied.)

Differentiating the expression

$$B_{ik}(\boldsymbol{r}_1 - \boldsymbol{r}_2) = \langle v_i(\boldsymbol{r}_1)\, v_k(\boldsymbol{r}_2)\rangle$$

(remember that $\langle v\rangle = 0$) with respect to the coordinates $(r_1)_j$ we sum over the indices i, j. Using condition (6) of a solenoidal field, we obtain

$$\frac{\partial B_{ik}(r_1 - r_2)}{\partial x_{1i}} = \left\langle \frac{\partial v_i(r_1)}{\partial x_{1i}} v_k(r_2) \right\rangle = 0.$$

Differentiation of $B_{ik}(r_1 - r_2)$ with respect to the coordinates of r_1 is clearly equivalent to differentiation with respect to $r = r_1 - r_2$. Therefore the correlation tensor of a solenoidal vector field satisfies the condition

$$\frac{\partial B_{ik}(r)}{\partial x_i} = 0, \quad \frac{\partial B_{ik}(r)}{\partial x_k} = 0. \tag{7}$$

A similar condition is obtained for the structure tensor of a locally homogeneous random field. It is best derived by differentiating (3) and inserting (7):

$$\frac{\partial D_{ik}(r)}{\partial x_i} = 0, \quad \frac{\partial D_{ik}(r)}{\partial x_k} = 0. \tag{8}$$

Condition (8) has been derived for the structure tensor of a statistically homogeneous field, but it is also valid for a locally homogeneous field, when the correlation tensor does not exist.

Using conditions (7) and (8), we can establish a relationship between the longitudinal and the transverse components of the tensors B_{ik}, D_{ik} for statistically isotropic (locally isotropic) fields. Differentiating (4') and (5') and using the relations

$$\begin{gathered}
\frac{\partial r}{\partial x_i} = \frac{\partial \sqrt{x_1^2 + x_2^2 + x_3^2}}{\partial x_i} = \frac{x_i}{r} = n_i, \\
\frac{\partial n_k}{\partial x_i} = \frac{\partial}{\partial x_i}\left(\frac{x_k}{r}\right) = \frac{\delta_{ik}}{r} - \frac{x_k}{r^2} n_i = \frac{\delta_{ik} - n_i n_k}{r}, \\
\frac{\partial n_i}{\partial x_i} = \frac{\delta_{ii} - n_i n_i}{r} = \frac{3-1}{r} = \frac{2}{r}, \\
n_i \frac{\partial n_k}{\partial x_i} = n_i \frac{\delta_{ik} - n_i n_k}{r} = \frac{n_k - n_k}{r} = 0,
\end{gathered}$$

we obtain

$$\frac{\partial B_{ik}}{\partial x_i} = B'_{tt} n_k + [B'_{rr} - B'_{tt}]\, n_k + [B_{rr} - B_{tt}] \frac{2}{r} n_k.$$

Hence,

$$B'_{rr} + \frac{2}{r} B_{rr} = \frac{1}{r^2} \frac{d}{dr}(r^2 B_{rr}) = \frac{2}{r} B_{tt}$$

or

$$B_{tt}(r) = \frac{1}{2r} \frac{d}{dr}(r^2 B_{rr}). \tag{9}$$

A similar relation is obtained between the structure tensor components

$$D_{tt}(r) = \frac{1}{2r} \frac{d}{dr}(r^2 D_{rr}). \tag{10}$$

Relations (9), (10), first derived by von Kármán, enable us to express the transverse component of the correlation (structure) tensor of a solenoidal and statistically homogeneous and isotropic (locally homogeneous and isotropic) vector field in terms of its longitudinal component and the number of independent components is thus reduced to one.

Consider a combination of a solenoidal and statistically homogeneous and isotropic (or locally isotropic) vector field and a homogeneous and isotropic (or locally isotropic) scalar field $A(\boldsymbol{r})$. Consider the correlation function

$$B_k(\boldsymbol{r}_1-\boldsymbol{r}_2)=\langle v_k(\boldsymbol{r}_1)\,A(\boldsymbol{r}_2)\rangle. \tag{11}$$

Since in isotropic fields all directions in space are equivalent, the vector $B_k(\boldsymbol{r})$ may be taken along the vector argument $\boldsymbol{r}$ of the correlation function. Thus,

$$B_k(\boldsymbol{r})=B(r)\,n_k,$$

where $B(r)$ is a scalar function of the scalar argument r. Differentiating equation (11) and taking into account that the field is solenoidal, i.e., using equation (6), we obtain

$$\frac{\partial B_k(\boldsymbol{r}_1-\boldsymbol{r}_2)}{\partial x_{k1}}=\frac{\partial B_k(\boldsymbol{r})}{\partial x_k}=0.$$

Hence

$$\frac{\partial B_k(\boldsymbol{r})}{\partial x_k}=B'(r)\,n_kn_k+B(r)\frac{\partial n_k}{\partial x_k}=B'(r)+\frac{2}{r}B(r)=\frac{1}{r^2}\frac{d}{dr}\left[r^2B(r)\right]=0.$$

Integrating the above expression gives $r^2B(r)=\text{const}$, therefore

$$B_k(\boldsymbol{r})=\frac{\text{const}}{r^2}n_k.$$

Clearly, $B_k(\boldsymbol{r})$ should remain finite for $\boldsymbol{r}=0$. We thus conclude that the constant in the expression for B_k must be zero, and we have

$$B_k(\boldsymbol{r}_1-\boldsymbol{r}_2)=\langle v_k(\boldsymbol{r}_1)\,A(\boldsymbol{r}_2)\rangle=0. \tag{12}$$

In this case, a statistically isotropic scalar field and a solenoidal vector field have a zero cross correlation coefficient. A similar equality can be established for locally isotropic fields:

$$D_k(\boldsymbol{r}_1-\boldsymbol{r}_2)=\langle[v_k(\boldsymbol{r}_1)-v_k(\boldsymbol{r}_2)]\,[A(\boldsymbol{r}_1)-A(\boldsymbol{r}_2)]\rangle=0. \tag{13}$$

Let us now consider a statistically homogeneous and isotropic potential vector field. A suitable example is provided by the gradient of a statistically homogeneous and isotropic scalar field. Let

$$u_k(\boldsymbol{r})=\frac{\partial A(\boldsymbol{r})}{\partial x_k},$$

where $A(\boldsymbol{r})$ is a homogeneous and isotropic scalar field with the correlation function

$$\langle A(\boldsymbol{r}_1)\,A(\boldsymbol{r}_2)\rangle=R(|\boldsymbol{r}_1-\boldsymbol{r}_2|)\quad(\langle A\rangle=0).$$

Differentiation with respect to the coordinates of r_1 and r_2 yields

$$\left\langle \frac{\partial A(r_1)}{\partial x_{i1}} \frac{\partial A(r_2)}{\partial x_{k2}} \right\rangle = \langle u_i(r_1) u_k(r_2) \rangle = \frac{\partial^2 R(|r_1 - r_2|)}{\partial x_{i1} \partial x_{k2}} = -\frac{\partial^2 R(r)}{\partial x_i \partial x_k},$$

i.e.,

$$B_{ik}(r) = -\frac{\partial^2 R(r)}{\partial x_i \partial x_k}.$$

Carrying out the differentiation, we obtain

$$B_{ik}(r) = -\left(R'' - \frac{R'}{r}\right) n_i n_k - \frac{R'}{r} \delta_{ik}.$$

On the other hand, the tensor B_{ik} should have the form (4'); comparing the two expressions, we find

$$B_{tt} = -\frac{R'}{r}, \quad B_{rr} = -R''.$$

Eliminating R between these expressions, we obtain a relationship between the transverse and the longitudinal components of the correlation tensor of a homogeneous and isotropic potential vector field:

$$B_{rr}(r) = \frac{d\,[rB_{tt}(r)]}{dr}. \qquad (14)$$

Similarly

$$D_{rr}(r) = [rD_{tt}(r)]'.$$

Relation (14) was first derived by Obukhov /2/.

Relations (9) and (14) show how a statistically homogeneous and isotropic vector field with correlation functions falling off to zero at infinity can be decomposed into a potential and a solenoidal part. Let B_{rr}, B_{tt} be the longitudinal and the transverse correlation functions of the initial field. We write

$$\left.\begin{aligned} B_{rr}(r) &= B_{rr}^{(p)}(r) + B_{rr}^{(s)}(r), \\ B_{tt}(r) &= B_{tt}^{(p)}(r) + B_{tt}^{(s)}(r), \end{aligned}\right\} \qquad (15)$$

where the superscripts p and s denote the potential and the solenoidal components. According to (9) and (14), we have

$$B_{tt}^{(s)}(r) = \frac{1}{2r}\frac{d}{dr}[r^2 B_{rr}^{(s)}(r)],$$

$$B_{rr}^{(p)}(r) = \frac{d}{dr}[rB_{tt}^{(p)}(r)].$$

Together with relations (15), these equations are sufficient in order to find from known $B_{rr}(r)$ and $B_{tt}(r)$ the four functions $B_{tt}^{(s)}$, $B_{rr}^{(s)}$, $B_{tt}^{(p)}$, $B_{rr}^{(p)}$. Under the conditions that $B_{tt}(r)$, $B_{rr}(r) \xrightarrow[r\to\infty]{} 0$, the set of equations has a unique solution /2/. The potential and the solenoidal field components are uncorrelated with each other, as there exists a general theorem that states that statistically homogeneous and isotropic solenoidal and potential fields are not correlated. Indeed, if $v_k(r)$ is a solenoidal field and $u_k(r) = \frac{\partial A(r)}{\partial x_k}$ is a potential field, we have using (12)

$$\langle v_k(\boldsymbol{r}_1)\, u_j(\boldsymbol{r}_2)\rangle = \left\langle v_k(\boldsymbol{r}_1)\frac{\partial A(\boldsymbol{r}_2)}{\partial x_{2j}}\right\rangle = \frac{\partial}{\partial x_{2j}}\langle v_k(\boldsymbol{r}_1)\, A(\boldsymbol{r}_2)\rangle = 0,$$

i.e.,

$$\langle v_k(\boldsymbol{r}_1)\, u_j(\boldsymbol{r}_2)\rangle = 0. \qquad (16)$$

Consider the spectral expansions of correlation and structure functions for homogeneous and isotropic (locally homogeneous and isotropic) vector fields. These functions are given by

$$B_{ik}(\boldsymbol{r}) = \iiint\limits_{-\infty}^{\infty} \cos \boldsymbol{\varkappa}\boldsymbol{r}\, \Phi_{ik}(\boldsymbol{\varkappa})\, d^3\varkappa, \qquad (17)$$

$$D_{ik}(\boldsymbol{r}) = 2\iiint\limits_{-\infty}^{\infty} [1 - \cos \boldsymbol{\varkappa}\boldsymbol{r}]\, \Phi_{ik}(\boldsymbol{\varkappa})\, d^3\varkappa. \qquad (18)$$

Since for isotropic fields all directions in wavenumber space are equivalent, Φ_{ik} has the form

$$\Phi_{ik}(\boldsymbol{\varkappa}) = F(\varkappa)\,\delta_{ik} + G(\varkappa)\,\varkappa_i\varkappa_k, \qquad (19)$$

where F and G are scalar functions of the scalar argument $\varkappa$. Let us find the form of the spectral tensor for a solenoidal field. Inserting (17) in (7) gives the equation

$$\frac{\partial B_{ik}(\boldsymbol{r})}{\partial x_k} = -\iiint \sin(\boldsymbol{\varkappa}\boldsymbol{r})\varkappa_k\Phi_{ik}(\boldsymbol{\varkappa})\, d^3\varkappa = 0,$$

which shows that

$$\varkappa_k\Phi_{ik}(\boldsymbol{\varkappa}) = 0. \qquad (20)$$

A similar equation is obtained after inserting (18) in (8). Substituting (19) in (20) we get

$$\varkappa_i F + \varkappa_i\varkappa^2 G = 0.$$

Thus G can be expressed in terms of F, and the expression for Φ_{ik} takes the form

$$\Phi_{ik}(\boldsymbol{\varkappa}) = \left(\delta_{ik} - \frac{\varkappa_i\varkappa_k}{\varkappa^2}\right) F(\varkappa). \qquad (21)$$

The tensor $\delta_{ik} - \frac{\varkappa_i\varkappa_k}{\varkappa^2}$ may be represented as a projection operator in the direction at right angles to the vector $\boldsymbol{\varkappa}$.

The correlation tensor of a potential vector field satisfies the equation

$$\frac{\partial B_{ik}(\boldsymbol{r})}{\partial x_j} - \frac{\partial B_{ij}(\boldsymbol{r})}{\partial x_k} = 0, \qquad (22)$$

which is readily derived from the condition for a potential random field (curl $\boldsymbol{u}=0$), just as equation (7) is derived from (6). Inserting (17) in (22), we get

$$\varkappa_j \Phi_{ik}(\boldsymbol{\varkappa}) - \varkappa_k \Phi_{ij}(\boldsymbol{\varkappa}) = 0. \tag{23}$$

Using expression (19), we reduce (23) to the equality $F=0$. The spectral tensor of a homogeneous and isotropic potential vector field thus has the form

$$\Phi_{ik}(\boldsymbol{\varkappa}) = \varkappa_i \varkappa_k G(\varkappa). \tag{24}$$

Consider the spectral expansions of the longitudinal and the transverse structure functions. Contracting (5') over i, k, we obtain

$$D_{ii} = 2D_{tt} + D_{rr}.$$

Contraction of (18) gives

$$D_{ii}(\boldsymbol{r}) = 2D_{tt}(r) + D_{rr}(r) = 2\iiint_{-\infty}^{\infty} [1 - \cos \boldsymbol{\varkappa r}]\, \Phi_{ii}(\varkappa)\, d^3\varkappa.$$

The trace of the spectral density tensor $\Phi_{ii}(\varkappa)$ is only a function of the magnitude of the wavenumber $\varkappa$. Therefore we may introduce spherical coordinates in the integrand and carry out explicitly the integration over the angular variables. This gives

$$D_{ii}(r) = 2D_{tt}(r) + D_{rr}(r) = 8\pi \int_0^{\infty} \left(1 - \frac{\sin \varkappa r}{\varkappa r}\right) \Phi_{ii}(\varkappa)\, \varkappa^2\, d\varkappa. \tag{25}$$

Further calculations are carried out separately for solenoidal and potential fields.

1. Solenoidal fields. In this case D_{tt} and D_{rr} are related by (10). Hence,

$$D_{ii}(r) = D_{rr} + \frac{1}{r}\frac{d}{dr}[r^2 D_{rr}] = \frac{1}{r^2}\frac{d}{dr}[r^3 D_{rr}(r)].$$

Integrating this differential equation and taking into account that $D_{rr}(0)=0$, we get

$$D_{rr}(r) = \frac{1}{r^3}\int_0^r \rho^2 D_{ii}(\rho)\, d\rho. \tag{26}$$

Inserting the spectral expansion (25) in the right-hand side of (26), changing the order of integration of ρ and $\varkappa$ and integrating over ρ, we obtain

$$D_{rr}(r) = 8\pi \int_0^{\infty} \left[\frac{1}{3} + \frac{\cos \varkappa r}{\varkappa^2 r^2} - \frac{\sin \varkappa r}{\varkappa^3 r^3}\right] \Phi_{ii}(\varkappa)\, \varkappa^2\, d\varkappa =$$
$$= 16\pi \int_0^{\infty} \left[\frac{1}{3} + \frac{\cos \varkappa r}{\varkappa^2 r^2} - \frac{\sin \varkappa r}{\varkappa^3 r^3}\right] F(\varkappa)\, \varkappa^2\, d\varkappa, \tag{27}$$

where in the last equality we used the relation $\Phi_{ii} = 2F$, which follows from (21). The spectral expansion of $D_{tt}(r)$ can be obtained by taking half the difference of (25) and (27):

$$D_{tt}(r) = 4\pi \int_0^\infty \left[\frac{2}{3} - \frac{\sin \varkappa r}{\varkappa r} - \frac{\cos \varkappa r}{\varkappa^2 r^2} + \frac{\sin \varkappa r}{\varkappa^3 r^3}\right] \Phi_{ii}(\varkappa)\varkappa^2 d\varkappa =$$

$$= 8\pi \int_0^\infty \left[\frac{2}{3} - \frac{\sin \varkappa r}{\varkappa r} - \frac{\cos \varkappa r}{\varkappa^2 r^2} + \frac{\sin \varkappa r}{\varkappa^3 r^3}\right] F(\varkappa)\varkappa^2 d\varkappa. \qquad (28)$$

Note that for $\varkappa r \to 0$ the expression in brackets in (27) is approximately $\varkappa^2 r^2/30$, and the corresponding expression in (28) is approximately $4\varkappa^2 r^2/30$. Therefore, the integrals in (27) and (28) converge at the origin even if the function $F(\varkappa)$ has a singularity of the type $\varkappa^{-\alpha}$ with $\alpha < 5$.

2. Potential fields. Similar calculations lead to the results

$$D_{tt}^{(p)}(r) = \frac{1}{r^3}\int_0^r \rho^2 D_{ii}(\rho)\, d\rho, \qquad (29)$$

$$D_{tt}^{(p)}(r) = 8\pi \int_0^\infty \left[\frac{1}{3} + \frac{\cos \varkappa r}{\varkappa^2 r^2} - \frac{\sin \varkappa r}{\varkappa^3 r^3}\right] \Phi_{ii}(\varkappa)\varkappa^2 d\varkappa, \qquad (30)$$

$$D_{rr}^{(p)}(r) = 8\pi \int_0^\infty \left[\frac{1}{3} - \frac{\sin \varkappa r}{\varkappa r} - \frac{2\cos \varkappa r}{\varkappa^2 r^2} + \frac{2\sin \varkappa r}{\varkappa^3 r^3}\right] \Phi_{ii}(\varkappa)\varkappa^2 d\varkappa. \qquad (31)$$

For $\varkappa r \to 0$ the expression in brackets in the last relation approaches zero as $\varkappa^2 r^2/10$.

B. THE MICROSTRUCTURE OF TURBULENT FLOW

As we know, the flow is turbulent for large Reynolds numbers $\mathrm{Re} = \frac{vl}{\nu}$ (v is the characteristic velocity, l the characteristic scale of flow, ν the kinematic viscosity of the fluid). Unlike laminar flow, where the velocity is a determinate function, turbulent flow is characterized by a velocity which is a random function of position and time. Consequently, the outcomes of different tests carried out under identical external conditions present us with different realizations of the turbulent velocity field.

Turbulent fluid flow is described by the general equations of hydrodynamics, which comprise the Navier–Stokes equations and the equations of continuity. Turbulent motion is treated with good accuracy in terms of the incompressible fluid model (this is permissible when the characteristic velocities are small compared to the velocity of sound c and the ratio of the characteristic scale over which the velocity changes markedly to the time needed for this change to occur is also small compared to c).

In some elementary cases, these equations can be applied to solve the problem of stability of motion and find the critical Reynolds number (see, e.g., /164/).

The onset of turbulence is the result of the breaking up of eddies on an ever descending scale which produces a redistribution of kinetic energy:

it is transferred from large-scale components of motion to progressively smaller ones. Kolmogorov advanced a hypothesis that the small-scale structure of motion has the important property of being locally homogeneous and isotropic. This hypothesis introduced substantial simplification into the description of small-scale turbulence and led to great advances in the theory.

§ 9. Structure and spectral functions of the velocity field in turbulent flow

As we have already noted, turbulent motion can be regarded with good accuracy as incompressible flow for the characteristic velocities and periods of motion observed in the real atmosphere. Furthermore, according to Kolmogorov's hypothesis the small-scale velocity field can be regarded as locally homogeneous and isotropic.

The incompressibility condition

$$\operatorname{div} \boldsymbol{v} = 0 \tag{1}$$

indicates that the velocity field is a solenoidal random field. It may therefore be described using the mathematical techniques of the preceding section. The structure functions of the velocity field are expressed by relations (5'), (8), (10), (18), (21), (27), and (28) of the preceding section.

Let us consider in more detail the relation of the spectral density of the velocity field to the kinetic energy of turbulence. Suppose at this stage that the turbulence is homogeneous and isotropic, so that correlation functions exist, as well as structure functions. An expression analogous to (8.25) for the trace of the velocity correlation tensor takes the form

$$B_{ii}(\boldsymbol{r}) = 4\pi \int_0^\infty \frac{\sin \varkappa r}{\varkappa r} 2F(\varkappa)\varkappa^2 \, d\varkappa.$$

Putting $\boldsymbol{r} = 0$, we obtain

$$B_{ii}(0) = 8\pi \int_0^\infty F(\varkappa)\varkappa^2 \, d\varkappa.$$

But for $\boldsymbol{r} = 0$ we have the equality

$$B_{ii}(0) = \langle v_1^2 \rangle + \langle v_2^2 \rangle + \langle v_3^2 \rangle = \langle \boldsymbol{v}^2 \rangle.$$

Let T be the average kinetic energy per unit mass of the fluid (in a coordinate system moving with the average flow velocity). We then have the expansion

$$T = \int_0^\infty 4\pi\varkappa^2 F(\varkappa) \, d\varkappa.$$

In the theory of turbulence it is generally convenient to deal with the spectral density of the average kinetic energy per unit mass, defined by

$$T = \int_0^\infty E(\varkappa)\,d\varkappa. \tag{2}$$

Expressing F in terms of E, we find

$$F(\varkappa) = \frac{E(\varkappa)}{4\pi\varkappa^2}.$$

The spectral tensor of the velocity field is related to $E(\varkappa)$ by

$$\Phi_{ik}(\varkappa) = \frac{1}{4\pi\varkappa^2}\left(\delta_{ik} - \frac{\varkappa_i\varkappa_k}{\varkappa^2}\right)E(\varkappa), \tag{3}$$

and the longitudinal and the transverse structure functions are expressed in terms of $E(\varkappa)$ by the relations

$$D_{rr}(r) = 4\int_0^\infty \left[\frac{1}{3} + \frac{\cos\varkappa r}{\varkappa^2 r^2} - \frac{\sin\varkappa r}{\varkappa^3 r^3}\right]E(\varkappa)\,d\varkappa, \tag{4}$$

$$D_{tt}(r) = 2\int_0^\infty \left[\frac{2}{3} - \frac{\sin\varkappa r}{\varkappa r} - \frac{\cos\varkappa r}{\varkappa^2 r^2} + \frac{\sin\varkappa r}{\varkappa^3 r^3}\right]E(\varkappa)\,d\varkappa. \tag{5}$$

§ 10. Energy dissipation in a turbulent flow

An important characteristic of turbulent motion is the amount of energy converted into heat by viscous forces per unit time per unit mass of the fluid. This quantity is given by (see /13/)

$$\varepsilon = \frac{\nu}{2}\left[\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right]^2 = \nu\left[\left(\frac{\partial v_i}{\partial x_k}\right)^2 + \frac{\partial v_i}{\partial x_k}\frac{\partial v_k}{\partial x_i}\right].$$

We will be mainly concerned with the average value of the energy dissipation rate $\langle\varepsilon\rangle$, which is designated in what follows by the symbol ε, omitting the averaging brackets. Averaging the expression for ε, we obtain

$$\varepsilon = \nu\left[\left\langle\left(\frac{\partial v_i}{\partial x_k}\right)^2\right\rangle + \left\langle\frac{\partial v_i}{\partial x_k}\frac{\partial v_k}{\partial x_i}\right\rangle\right]. \tag{1}$$

The terms on the right-hand side of this equality can be expressed using the derivatives of the structure functions D_{ik} at the origin. To this end we write $\left\langle\left(\frac{\partial v_i}{\partial x_k}\right)^2\right\rangle$ in the form

$$\left\langle\left(\frac{\partial v_i}{\partial x_k}\right)^2\right\rangle = \lim_{|r_1 - r_2|\to 0}\left\langle\frac{\partial v_i(r_1)}{\partial x_{k1}}\frac{\partial v_i(r_2)}{\partial x_{k2}}\right\rangle. \tag{2}$$

The right-hand side of (2) can be transformed, using the assumption of homogeneity:

$$\left\langle\frac{\partial v_i(r_1)}{\partial x_{k1}}\frac{\partial v_i(r_2)}{\partial x_{k2}}\right\rangle = \frac{\partial}{\partial x_{k1}}\frac{\partial}{\partial x_{k2}}\langle v_i(\boldsymbol{r}_1)\,v_i(\boldsymbol{r}_2)\rangle =$$

$$= \frac{\partial^2 B_{ii}(r_1 - r_2)}{\partial x_{k1}\partial x_{k2}} = -\frac{\partial^2 B_{ii}(r_1 - r_2)}{\partial x_{k1}\partial x_{k1}} = -\frac{\partial^2 B_{ii}(r)}{\partial x_k^2}. \tag{3}$$

Seeing that $\frac{\partial^2}{\partial x_k^2} = \Delta$ is the Laplacian and using the equality $\Delta B_{ii} = -\frac{1}{2}\Delta D_{ii}$, which follows from (8.3), we obtain

$$\left\langle\left(\frac{\partial v_i(r)}{\partial x_k}\right)^2\right\rangle = \frac{1}{2}\left[\Delta D_{ii}(r)\right]_{r=0} = \frac{1}{2}\Delta D_{ii}(0). \tag{4}$$

The second term in (1) is similarly transformed:

$$\left\langle\frac{\partial v_i}{\partial x_k}\frac{\partial v_k}{\partial x_i}\right\rangle = \lim_{|r_1-r_2|\to 0}\left\langle\frac{\partial v_i(r_1)}{\partial x_{k_1}}\frac{\partial v_k(r_2)}{\partial x_{i2}}\right\rangle = \lim_{|r_1-r_2|\to 0}\frac{\partial}{\partial x_{k1}}\frac{\partial}{\partial x_{i2}}B_{ik}(r_1 - r_2) = 0 \tag{4'}$$

since $\frac{\partial B_{ik}}{\partial x_i} = 0$. We therefore finally obtain

$$\varepsilon = \frac{\nu}{2}\Delta D_{ii}(0). \tag{5}$$

The function $D_{ii}(r)$ has the following properties: $D_{ii}(0) = 0$ and $D'_{ii}(0) = 0$ (the first condition follows from the definition of the structure function and the latter from its being an even function). The Taylor series expansion of $D_{ii}(r)$ at small r therefore starts with the quadratic term:

$$D_{ii}(r) = ar^2 + \ldots \tag{6}$$

Computing $\Delta D_{ii} = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dD_{ii}}{dr}\right)$, we obtain $\Delta D_{ii} = 6a + \ldots$ and consequently $\Delta D_{ii}(0) = 6a$. Inserting this result in (5), we obtain $\varepsilon = 3\nu a$, so that $a = \frac{\varepsilon}{3\nu}$ and

$$D_{ii}(r) = \frac{\varepsilon}{3\nu}r^2 + \ldots \tag{7}$$

We now use relation (8.26), which expresses D_{rr} in terms of $D_{ii}(r)$. Inserting expansion (7) in (8.26) and integrating, we obtain

$$D_{rr}(r) = \frac{1}{15}\frac{\varepsilon}{\nu}r^2 + \ldots \tag{8}$$

Using expression (8.10), which relates D_{tt} to D_{rr}, we also obtain the expansion

$$D_{tt}(r) = \frac{2}{15}\frac{\varepsilon}{\nu}r^2 + \ldots \tag{9}$$

The structure functions of the velocity field for small r are thus determined by the rate of dissipation of the kinetic energy of turbulence ε and by the viscosity ν.

Let us now establish a relation between the spectral energy density $E(\varkappa)$ introduced in §9 and the energy dissipation rate ε. Differentiating (7) twice and then setting $r = 0$ (so that all the terms after the first term in (7) vanish), we obtain $D''_{ii}(0) = \frac{2}{3}\frac{\varepsilon}{\nu}$.

We now use relation (8.25). Expanding $1 - \frac{\sin \varkappa r}{\varkappa r}$ in a series and setting $\Phi_{ii} = \frac{E(\varkappa)}{2\pi\varkappa^2}$, we obtain

$$D_{ii}(r) = 4\int_0^\infty\left(\frac{\varkappa^2 r^2}{6} + \ldots\right)E(\varkappa)\,d\varkappa.$$

Differentiating this expression twice and then setting $r = 0$, we find

$$D''_{ii}(0) = \frac{4}{3}\int_0^\infty \varkappa^2 E(\varkappa)\, d\varkappa = \frac{2}{3}\frac{\varepsilon}{\nu},$$

from which we obtain the important relation

$$\varepsilon = 2\nu \int_0^\infty \varkappa^2 E(\varkappa)\, d\varkappa, \tag{10}$$

which expresses ε in terms of $E(\varkappa)$. The fact that $D_{rr}(r) \sim r^2$ for small r is equivalent to the convergence of the integral (10). Since this integral contains a rapidly growing factor $\varkappa^2$, it will converge only if the function $E(\varkappa)$ falls off sufficiently rapidly (faster than $\varkappa^{-3}$) for large $\varkappa$. A more detailed discussion of the meaning of relation (10) will be given in §12.

§ 11. Kolmogorov's equation

All the above results were derived from considerations of isotropic and incompressible turbulence and from the expression for ε. The Navier — Stokes equation

$$\frac{\partial v_i}{\partial t} = -v_k \frac{\partial v_i}{\partial x_k} - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \Delta v_i, \tag{1}$$

where p is the pressure and ρ the density, was not used. We can try to apply this equation in order to obtain an additional equation for the structure function $D_{rr}(r)$. This can be done multiplying (1) by $v_l(\boldsymbol{r})$ and then averaging the resulting equation. The terms $\left\langle v_l(\boldsymbol{r}') \frac{\partial v_i(\boldsymbol{r})}{\partial t}\right\rangle$ and $\langle v_l(\boldsymbol{r}')\, \Delta v_i(\boldsymbol{r})\rangle$ can be transformed to $\frac{\partial D_{il}(r, t)}{\partial t}$ and $\Delta D_{il}(r, t)$. However, the nonlinear term of this equation, $v_k \frac{\partial v_i}{\partial x_k}$ will contain an expression of the form $\langle v_l(\boldsymbol{r}')\, v_k(\boldsymbol{r})\, v_i(\boldsymbol{r})\rangle$ after multiplication by $v_l(\boldsymbol{r}')$.

We introduce a tensor

$$D_{ikl}(\boldsymbol{r}) = \langle [v_i(\boldsymbol{r}+\boldsymbol{r}') - v_i(\boldsymbol{r}')]\, [v_k(\boldsymbol{r}+\boldsymbol{r}') - v_k(\boldsymbol{r}')]\, [v_l(\boldsymbol{r}+\boldsymbol{r}') - v_l(\boldsymbol{r}')]\rangle. \tag{2}$$

Using the equation of incompressibility and the isotropy condition we have previously expressed the tensor $D_{ik}(\boldsymbol{r})$ in terms of the single scalar function $D_{rr}(\boldsymbol{r})$; the tensor (2) can also be expressed in terms of the corresponding longitudinal function

$$D_{rrr}(r) = \langle \{\boldsymbol{n}\, [\boldsymbol{v}(\boldsymbol{r}+\boldsymbol{r}') - \boldsymbol{v}(\boldsymbol{r}')]\}^3 \rangle, \tag{3}$$

where $\boldsymbol{n} = \frac{\boldsymbol{r}}{r}$.

In this case, multiplying (1) by $v_l(\boldsymbol{r}')$ and averaging, we can express all the quantities entering the averaged equation in terms of $D_{rr}(r)$ and $D_{rrr}(r)$.

After fairly laborious manipulations, we end up with the equation

$$-\frac{2}{3}\varepsilon-\frac{1}{2}\frac{\partial D_{rr}}{\partial t}=\frac{1}{6r^4}\frac{\partial(r^4D_{rrr})}{\partial r}-\frac{\nu}{r^4}\frac{\partial}{\partial r}\left(r^4\frac{\partial D_{rr}}{\partial r}\right). \tag{4}$$

Equation (4) describes the decay of turbulence. Actually, equation (1) contains no external forces which may act as a source of energy. On the other hand, (1) contains a viscous term $\nu\Delta v_i$ which causes energy dissipation. The solution of (1) should therefore decay in time. This also applies to the possible solutions of (4).

The statement of the problem, however, can be modified. If we introduce external energy sources with power ε per unit fluid mass, it will compensate the viscous dissipation of energy so that steady-state conditions are established. In this case the term $\frac{\partial D_{rr}}{\partial t}$ in (4) will vanish and ε should be considered as an external parameter, equal to the power expended by the external sources in compensating energy dissipation. The equation for steady-state turbulence thus takes the form

$$-\frac{2}{3}\varepsilon=\frac{1}{r^4}\left[\frac{1}{6}\frac{d(r^4D_{rrr})}{dr}-\nu\frac{d}{dr}\left(r^4\frac{dD_{rr}}{dr}\right)\right]. \tag{5}$$

Multiplying (5) by r^4, integrating over r, and remembering that structure functions vanish at $r=0$, we obtain

$$D_{rrr}(r)=-\frac{4}{5}\varepsilon r+6\nu\frac{dD_{rr}}{dr}. \tag{6}$$

Equation (6) relating the functions D_{rrr} and D_{rr} was obtained by Kolmogorov.

Since equation (6) contains, besides $D_{rr}(r)$, a new unknown function $D_{rrr}(r)$, it cannot be integrated. If we try to derive an additional equation for D_{rrr}, we encounter a similar difficulty: the equation for D_{rrr} will contain new unknown functions, fourth-order moments of velocity differences. Any attempt to obtain a closed system of equations for the structure functions thus meets with fundamental difficulties. For this reason all further results of the theory of turbulence require introduction of additional hypotheses originating from various physical considerations.

§ 12. Structure of small-scale turbulence at very high Reynolds numbers

We have previously derived the relations

$$T=\int_0^\infty E(\varkappa)\,d\varkappa, \tag{1}$$

$$\varepsilon=2\nu\int_0^\infty \varkappa^2E(\varkappa)\,d\varkappa, \tag{2}$$

expressing the kinetic energy T and the rate of energy dissipation ε (both these quantities refer to a unit fluid mass) in terms of the spectral energy density $E(\varkappa)$. The experimental data show that $E(\varkappa)$ has a maximum in the region of small $\varkappa$ (large scales) and falls off fairly rapidly as $\varkappa$ increases. Near a certain maximum number $\varkappa = \varkappa_m$, the function $E(\varkappa)$ begins to decrease even more rapidly toward zero. A typical curve of $E(\varkappa)$ is shown in Figure 6. The principal value of the integral expressing T in terms of $E(\varkappa)$ is obtained by integrating over the region of small $\varkappa$ (in Figure 6 the corresponding section I of the $\varkappa$ axis is marked by a thick line). This range of wavenumbers is called the energy range. The parameter $\varkappa_m$ characterizing the wavenumbers which correspond to the smallest inhomogeneities in the velocity field depends on the Reynolds number Re of the flow, increasing as Re increases.

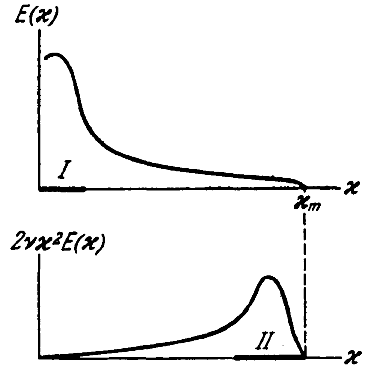

FIGURE 6. A typical curve of the spectral energy density in turbulent flow $E(\varkappa)$ (top) and of the spectral density of the energy dissipation rate $2\nu\varkappa^2 E(\varkappa)$ (bottom):

Section I on the wavenumber axis corresponds to the energy range of the turbulence spectrum, section II corresponds to the viscous range (dissipation range), and the intermediate section is the inertial range of wavenumbers.

Let us now consider the function $2\nu\varkappa^2 E(\varkappa)$, whose integral defines the dissipation rate ε. The factor $\varkappa^2$ vanishes at $\varkappa = 0$ and is small for small values of $\varkappa$, so that the product $\varkappa^2 E(\varkappa)$ is small in the energy range reaching a maximum for large wavenumbers. According to experimental data for $\varkappa \ll \varkappa_m$ the function $E(\varkappa)$ falls off more rapidly than the function $\varkappa^2$ increases, so that the maximum of $\varkappa^2 E(\varkappa)$ is situated near the point $\varkappa_m$, beyond which $E(\varkappa)$ rapidly vanishes. A typical curve of the function $2\nu\varkappa^2 E(\varkappa)$ is also shown in Figure 6. The region near $\varkappa = \varkappa_m$, where the integral gives the main contribution to ε, is called the dissipation region, or the viscous region (in Figure 6 this is section II marked by a thick line on the $\varkappa$ axis). The reason that energy dissipation is mainly confined to the region of high wavenumbers is because ε is proportional to the square of the velocity gradient (see (10.1), whose maximum is associated with the smallest inhomogeneities of the velocity field. Since the scale of an inhomogeneity l and the corresponding wavenumber $\varkappa$ are related by $\varkappa l \sim 2\pi$ (see (2.35)), we see that ε is associated with the largest wavenumbers in the turbulence spectrum.

If the Reynolds number is sufficiently high, the dissipation range around $\varkappa_m$ is separated from the energy range by a certain range of wavenumbers known as the inertial range. The inertial region becomes larger as Re increases (the ratio of the largest to the smallest inhomogeneities is of the order of $Re^{3/4}$ /13/).

Consider steady-state turbulence. In this case, as we have noted in the preceding section, external sources are required to inject energy into the turbulent flow so as to offset the dissipation of energy in the viscous range of wavenumbers; the power of the energy sources should be ε per unit mass of the fluid. These sources generate in the range where the bulk of the turbulent flow energy is concentrated.

The main energy losses, however, occur in the dissipation region, which is separated from the energy region by the inertial range (see Figure 6). The entire power ε is thus transmitted without any appreciable losses through the inertial range to the viscous dissipation region. The energy transfer through the spectrum from small to large wavenumbers, i.e., from large-scale inhomogeneities (eddies) to small-scale ones, can be visualized as a process of eddy fragmentation. If the Reynolds number of the initial flow is high, it becomes unstable and the size of the resulting eddies is of the order of the initial scale of the flow L_0. The Reynolds number characterizing the motion of these eddies is smaller than the Reynolds number of the initial flow, but it is still sufficiently high to make these eddies unstable and cause further fragmentation into smaller eddies. During this fragmentation the energy of a large decaying eddy is transferred to smaller eddies, i.e., a flow of energy is established from small to large wavenumbers.

Each fragmentation event reduces the Reynolds number of the product eddies. The fragmentation goes on until eddies with a subcritical Reynolds number are formed. These eddies are stable and show no tendency to decay into still smaller eddies. The number of successive fragmentations is clearly greater the larger the Reynolds number Re of the initial flow. The smallest size of the eddies produced at the end of this chain of fragmentations is therefore smaller the greater Re. In other words, the maximum wavenumber $\varkappa_m$ characterizing the size of the smallest eddies increases with increasing Re. A finite inertial range is observed when the viscous range is separated from the energy range. This is possible when $\mathrm{Re} \gg \mathrm{Re}_{cr}$ (Re_{cr} is the critical Reynolds number). In practice, an inertial range is observed for $\mathrm{Re} > 10^6 - 10^7$.

We see that the turbulence structure in the inertial wavenumber range is entirely determined by the energy ε transferred through the spectrum, and the structure in the viscous range is determined by ε and ν. The latter result is confirmed by expressions (10.8) and (10.9) for D_{rr} and D_{tt}. An additional argument in favor of this conclusion is that Kolmogorov's equation contains only ε and ν.

The inertial and the viscous range of the turbulence spectrum taken together constitute the so-called equilibrium range. We introduce a characteristic scale L_0 which is equal in its order of magnitude to the largest velocity inhomogeneities from the energy interval. L_0, called the outer scale of turbulence, gives the distance over which the mean flow velocity changes appreciably. The scales r from the equilibrium range are small compared to L_0, i.e., $r \ll L_0$.

Kolmogorov (1941) advanced a hypothesis according to which the structure of turbulence in the equilibrium range at very high Reynolds numbers is determined only by the parameters ε and ν, whereas in the inertial subrange of the equilibrium range it depends only on ε and is independent of ν. These hypotheses enable us to establish the form of the function $D_{rr}(r)$ in the inertial subrange.

According to the first hypothesis, for all $r \ll L_0$ the function $D_{rr}(r)$ is of the form

$$D_{rr}(r) = D_{rr}(\varepsilon, \nu, r). \tag{3}$$

There is only one combination of ε and ν which has the dimensions of length*

$$l_0 = \sqrt[4]{\frac{\nu^3}{\varepsilon}}, \tag{4}$$

* The dimension of ε is $\mathrm{cm}^2 \cdot \mathrm{sec}^{-2}$ in CGS units and watt/kg in SI units. The dimension of ν is $\mathrm{cm}^2 \cdot \mathrm{sec}^{-1}$.

and one which has the dimensions of velocity

$$v_0 = \sqrt[4]{\varepsilon\nu}. \tag{5}$$

The parameter l_0 is called the inner scale of turbulence.

Since the viscosity ν enters the definition of l_0 and v_0, these two parameters clearly characterize the scale and the velocity of the smallest inhomogeneities (eddies) in turbulent flow. This is also evident from the fact that the factor $l_0 v_0/\nu$ (the Reynolds number for eddies of scale l_0 and characteristic velocity v_0) is clearly equal to unity.

The only dimensionless combination that can be formed from the parameters ε, ν, and r is the ratio r/l_0. According to the well-known Π-theorem of dimensional analysis (see /14/), it thus follows that D_{rr}, whose dimension is that of velocity squared, may be represented in the form

$$D_{rr}(r) = v_0^2 f\left(\frac{r}{l_0}\right) \quad \text{for} \quad r \ll L_0, \tag{6}$$

where $f(x)$ is some nondimensional function of a nondimensional argument. Kolmogorov's second hypothesis enables us to establish the asymptotic form of the function $f(x)$ for $x \gg 1$, i.e., for $r \gg l_0$. In this case, the viscosity ν should drop out from (6). Writing (6) in the form

$$D_{rr}(r) = \varepsilon^{1/2}\nu^{1/2} f\left(\frac{\varepsilon^{1/4} r}{\nu^{3/4}}\right),$$

we see that this expression is independent of ν only if $f(x) = C^2 x^{2/3}$. We thus get

$$D_{rr}(r) = C^2 \varepsilon^{2/3} r^{2/3} \quad \text{for} \quad l_0 \ll r \ll L_0, \tag{7}$$

which is the "2/3 law" of Kolmogorov and Obukhov.* For $r \ll l_0$ we have, according to the preceding analysis, $D_{rr}(r) = \frac{1}{15}\frac{\varepsilon}{\nu} r^2$. The two curves represented by these asymptotic expressions intersect at $r = l_1 = (15\,C^2)^{3/4} l_0$. The exact form of the function $D_{rr}(r)$ near the point $r = l_1$, however, cannot be established from these two hypotheses, and yet the detailed behavior of $D_{rr}(r)$ near this point largely influences the spectral function $E(\varkappa)$ near $\varkappa = \varkappa_m$.

Using relation (8.10), $D_{tt} = \frac{1}{2r}\frac{d}{dr}(r^2 D_{rr})$, we find for D_{tt} in the inertial subrange

$$D_{tt}(r) = \frac{4}{3} C^2 \varepsilon^{2/3} r^{2/3} \quad \text{for} \quad l_0 \ll r \ll L_0.$$

Inserting (7) in Kolmogorov's equation (11.6), we obtain the relation

$$D_{rrr}(r) = -\frac{4}{5}\varepsilon r + 4C^2 \nu \varepsilon^{2/3} r^{-1/3} = -\frac{4}{5}\varepsilon r\left[1 - 5C^2\left(\frac{l_0}{r}\right)^{4/3}\right]. \tag{8}$$

* This law was derived by a different method by A. M. Obukhov in 1941 (see /15, 16/). Later, similar results were obtained by L. Onsager, C. von Weizsäcker, and W. Heisenberg /167—169/.

Since in the region of interest $\frac{l_0}{r} \ll 1$, the second term in square brackets can be omitted and we obtain

$$D_{rrr}(r) = -\frac{4}{5}\varepsilon r \quad \text{for} \quad l_0 \ll r \ll L_0. \tag{9}$$

From (9) it follows that the third moment of the longitudinal velocity difference Δv_r is negative. From this and from the fact that the mean longitudinal velocity difference is zero, we conclude that most of the time Δv_r has a small positive value (i.e., the particles diverge slowly) and only occasionally large negative values of Δv_r are recorded (i.e., relatively infrequently, but stronger convergent motions).

An important characteristic of the probability distribution of the longitudinal velocity difference is the skewness S of this distribution, defined by

$$S = \frac{D_{rrr}(r)}{[D_{rr}(r)]^{3/2}}. \tag{10}$$

Substituting (7) and (9), we get

$$S = -\frac{4}{5C^3}. \tag{11}$$

In the inertial subrange the skewness S is thus constant and the coefficient C^2 in (7) is expressed in terms of this constant value by

$$C^2 = \left(\frac{4}{5|S|}\right)^{2/3}. \tag{12}$$

Let us now establish the form of the function $E(\varkappa)$, equivalent to expression (7) for $D_{rr}(r)$. The best approach is to calculate the trace of the tensor D_{ik}.

It follows from (8.18) that the function $D_{ii}(\boldsymbol{r})$ has an integral representation of the same form as the structure function of a scalar field (see (5.6)):

$$D_{ii}(r) = 2\iiint\limits_{-\infty}^{\infty}[1-\cos\boldsymbol{\varkappa r}]\,\frac{E(\varkappa)}{2\pi\varkappa^2}\,d^3\varkappa,$$

where $E(\varkappa)/2\pi\varkappa^2$ takes the part of the function $\Phi(\varkappa)$. $D_{ii}(r)$ may also be expanded in a one-dimensional integral

$$D_{ii}(r) = 2\int\limits_{-\infty}^{\infty}[1-\cos\varkappa r]\,V(\varkappa)\,d\varkappa,$$

where according to (5.15)

$$V(\varkappa) = \frac{1}{2\pi\varkappa}\int\limits_0^{\infty}\sin\varkappa r\,D'_{ii}(r)\,dr$$

and $\Phi(\varkappa) = \frac{E(\varkappa)}{2\pi\varkappa^2}$ is expressed in terms of $V(\varkappa)$ using (5.17):

$$\Phi(\varkappa) = -\frac{1}{2\pi\varkappa}\frac{dV(\varkappa)}{d\varkappa}.$$

From the above relations, we obtain

$$E(\varkappa) = -\frac{\varkappa}{2\pi}\frac{d}{d\varkappa}\left[\frac{1}{\varkappa}\int_0^\infty \sin\varkappa r\, D'_{ii}(r)\, dr\right], \tag{13}$$

from which $E(\varkappa)$ can be found if $D_{ii}(r)$ is known.

The function $D_{ii}(r) = 2D_{tt} + D_{rr}$ in the inertial range is given by

$$D_{ii}(r) = \frac{11}{3} C^2 \varepsilon^{2/3} r^{2/3}, \qquad l_0 \ll r \ll L_0. \tag{14}$$

First suppose that $D_{ii}(r)$ is expressed by (14) for all r ($0 \leqslant r \leqslant \infty$) and not only for the range of r indicated above. At a later stage we will indicate how the function $E(\varkappa)$ should be modified so as to make $D_{ii} \sim r^2$ for $r \ll l_0$. Inserting (14) in (13) and integrating, we obtain after simple manipulations

$$E(\varkappa) = \frac{55}{27\pi}\Gamma\left(\frac{2}{3}\right)\cos\frac{\pi}{6}\; C^2\varepsilon^{2/3}\varkappa^{-5/3} = 0{,}76 C^2 \varepsilon^{2/3}\varkappa^{-5/3}. \tag{15}$$

Note that the three-dimensional spectral density $F(\varkappa)$ corresponding to the "2/3 law" is given by

$$F(\varkappa) = \frac{E(\varkappa)}{4\pi\varkappa^2} = 0{,}061 C^2 \varepsilon^{2/3}\varkappa^{-11/3}. \tag{16}$$

If we use (15) to compute the energy dissipation

$$\varepsilon = 2\nu \int_0^\infty \varkappa^2 E(\varkappa)\, d\varkappa,$$

the result is infinite, since the integral diverges for large $\varkappa$. This divergence is a result of the extension of (14) to the entire range $r > 0$, whereas in point of fact it is only applicable for $r \gg l_0$. To avoid this divergence, one should remember that for $\varkappa \sim \varkappa_m = \frac{2\pi}{l_0}$ the function $E(\varkappa)$ is no longer described by (15). For $\varkappa \gtrsim \varkappa_m$, $E(\varkappa)$ decreases faster than $\varkappa^{-3}$ and this ensures the convergence of the integral expressing ε in terms of $E(\varkappa)$. However, Kolmogorov's simple hypotheses are insufficient to establish the form of the function $E(\varkappa)$ in the dissipation range and therefore various authors were forced to introduce additional hypotheses.

As an example, we will consider Obukhov and Yaglom's treatment /17/, which is distinguished by its relative simplicity. As we have noted, the skewness S of the probability distribution for the longitudinal velocity difference is constant in the inertial range. Evidently S is also constant for small r (in the viscous range). Indeed, $D_{rr} \sim r^2$ for $r \ll l_0$, which points to a linear dependence of the velocity difference on r. If this is so, $D_{rrr}(r)$ should be proportional to r^3 and the ratio $D_{rrr}(r)/[D_{rr}(r)]^{3/2}$ should be independent of r. Obukhov and Yaglom therefore postulated that S is constant for all $r \ll L_0$, and not only in the inertial range. It is thus assumed that

$$S = \frac{D_{rrr}(r)}{[D_{rr}(r)]^{3/2}} = \text{const} \quad \text{for} \quad r \ll L_0.$$

This relation permits expressing $D_{rrr}(r)$ in terms of $D_{rr}(r)$:

$$D_{rrr}(r) = S\,[D_{rr}(r)]^{3/2}.$$

Inserting the last expression in Kolmogorov's equation, we obtain an ordinary differential equation for $D_{rr}(r)$:

$$6\nu\frac{dD_{rr}}{dr} - S\,[D_{rr}]^{3/2} + \frac{4}{5}\varepsilon r = 0.$$

Together with the initial condition $D_{rr}(0) = 0$, this equation defines a single function $D_{rr}(r)$ which was computed in /17/ by numerical integration. In /18/, however, it was shown that when the result is applied to compute the function $E(\varkappa)$ (using relation (13), say), $E(\varkappa)$ turns out to be negative for some $\varkappa$. This clearly establishes the inadequacy of the hypothesis of constant S for all r in the inertial and viscous ranges.

Alternative assumptions were introduced by other authors (e.g., /20/) in order to obtain a closed system of equations for $E(\varkappa)$ or $D_{rr}(r)$. This approach, however, invariably involves certain fundamental difficulties.

An interesting attempt to determine the function $E(\varkappa)$ in the dissipation range is due to Novikov /21/. He derived a relation

$$E(\varkappa) = C_1\varepsilon^{2/3}\varkappa^{-5/3}(\varkappa l_0)^{2k-\frac{4}{3}}\exp(-al_0^2\varkappa^2), \tag{17}$$

where k is some number from the interval $0.5 < k < 2$ and

$$a \approx 2\,(k^2 - k + 1)^{1/2}.$$

Relation (17) is valid for $\varkappa l_0 \gg 1$. If, however, we demand that for $\varkappa l_0 < 1$ it reduces to (15), which is applicable in the inertial subrange, we find that k should be equal to $2/3$. In this case (17) takes the form

$$E(\varkappa) = \frac{a^{2/3}}{\Gamma\left(\frac{2}{3}\right)}\varepsilon^{2/3}\varkappa^{-5/3}\exp(-al_0^2\varkappa^2). \tag{18}$$

The constant C_1 was chosen so that (18) satisfy the normalization condition

$$2\nu\int_0^\infty \varkappa^2 E(\varkappa)d\varkappa = \varepsilon.$$

Equation (18) satisfies the condition $E(\varkappa) > 0$. Moreover, in a model with a spectral density given by (18) all integrals of the form $\int_0^\infty \varkappa^{2n}E(\varkappa)\,d\varkappa$ converge, and the mean squares of all the derivatives of the velocity field are finite.

Using (9.4) we can compute the longitudinal structure function $D_{rr}(r)$, corresponding to the spectral density (18). Integration can be carried out by expanding the function

$$\frac{1}{3} + \frac{\cos\varkappa r}{\varkappa^2 r^2} - \frac{\sin\varkappa r}{\varkappa^3 r^3}$$

into a power series. The result gives

$$D_{rr}(r) = 2a\,(l_0\varepsilon)^{2/3}\left[{}_1F_1\left(-\frac{1}{3},\ \frac{5}{2},\ -\frac{r^2}{4al_0^2}\right) - 1\right], \tag{19}$$

where

$$ {}_1F_1(\alpha, \gamma, z) = 1 + \frac{\alpha}{\gamma}\frac{z}{1!} + \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}\frac{z^2}{2!} + \ldots = \frac{\Gamma(\gamma)}{\Gamma(\alpha)}\sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha)}{\Gamma(n+\gamma)}\frac{z^n}{n!} $$

is a confluent hypergeometric function (Kummer function). Using this expansion, we obtain from (19) for $r \ll l_0$

$$ D_{rr}(r) = \frac{1}{15}\frac{\varepsilon}{\nu}r^2 + \ldots $$

For $r \gg l_0$ we can use the asymptotic expansion

$$ {}_1F_1(\alpha, \gamma, z) = \frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha)}(-z)^{-\alpha}G(\alpha, \alpha-\gamma+1, -z) + \\ + \frac{\Gamma(\gamma)}{\Gamma(\alpha)}e^z z^{\alpha-\gamma}G(\gamma-\alpha, 1-\alpha, z), \qquad (20) $$

where

$$ G(\alpha, \beta, z) = 1 + \frac{\alpha\beta}{1!z} + \frac{\alpha(\alpha+1)\beta(\beta+1)}{2!z^2} + \ldots $$

Since in our case $z = -\frac{r^2}{4al_0^2} < 0$, we can drop the exponentially small second term of the asymptotic expansion. Retaining only the first term in the expression for G (i.e., putting $G \approx 1$) and ignoring unity compared to $\left(\frac{r}{l_0}\right)^{2/3}$, we obtain the asymptotic expression

$$ D_{rr}(r) \approx \frac{\sqrt[3]{2}\,\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{17}{6}\right)} a^{2/3}\varepsilon^{2/3}r^{2/3}. \qquad (21) $$

Comparing this expression with (7), we obtain the following relation between the constants C^2 and a:

$$ C^2 = \frac{\sqrt[3]{2}\,\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{17}{6}\right)} a^{2/3} \approx 0.972a^{2/3}. \qquad (22) $$

In this case expression (19) satisfies the necessary requirements for $r \ll l_0$ and $r \gg l_0$ and enables us to establish the form of $D_{rr}(r)$ for the intermediate values of r as well.

The substantial experimental material on hand in fact indicates that $E(\varkappa)$ is proportional to $\varepsilon^{2/3}\varkappa^{-5/3}$ in the inertial range. A recent experimental study of the spectral density $E(\varkappa)$ in the viscous range (see Part C) was published in /22/. The results of this study indicate that for $\varkappa \gtrsim l_0^{-1}$ the function $E(\varkappa)$ decreases with increasing $\varkappa$ more slowly than $\exp(-al_0^2\varkappa^2)$. The spectra in /22/ are adequately approximated by the expression

$$ E(\varkappa) = \frac{3a^{4/3}}{20\Gamma\left(\frac{5}{3}\right)}\varepsilon^{2/3}\varkappa^{-5/3}e^{-\sqrt{al_0\varkappa}}, \qquad (23) $$

where the numerical factor is chosen so that

$$\varepsilon = 2\nu \int_0^\infty \varkappa^2 E(\varkappa)\, d\varkappa.$$

The numerical value of the factor $3\alpha^{4/3}/20\Gamma\left(\frac{5}{3}\right)$ obtained in /22/ is 1.35± 0.06, which corresponds to $\alpha = 4.78 \pm 0.17$.

Expression (23), like (18), decreases fairly rapidly, so that integrals of the form $\int_0^\infty \varkappa^{2n} E(\varkappa)\, d\varkappa$ converge for arbitrarily large n. Unlike (18), however, its experimental justification is meager at present.

Using (23) and (9.25) to compute $D_{rr}(r)$ we find that for $r \ll l_0$

$$D_{rr} = \frac{1}{15} \frac{\varepsilon}{\nu} r^2 + \ldots,$$

and for $r \gg l_0$, $D_{rr} = C^2 \varepsilon^{2/3} r^{2/3}$, where C^2 and α are related by the expression (it is derived by comparing (23) and (15))

$$\alpha^{4/3} = \frac{550\sqrt{3}}{54\pi}\left[\Gamma\left(\frac{5}{3}\right)\right]^2 C^2 = 4.58 C^2. \tag{24}$$

(For $\alpha = 4.78 \pm 0.17$ we have $C^2 = 1.77 \pm 0.08$.)

In what follows we will use expression (18) as a convenient approximation to $E(\varkappa)$.

§ 13. The microstructure of the temperature field in turbulent flow

The velocity v is not the only random function of position and time in turbulent flow. Temperature, humidity, pressure, dielectric constant, and other characteristics of the air undergo fluctuations due to turbulence.

We will now examine the microstructure of the temperature field in a random turbulent medium. The origin of temperature fluctuations in such a medium is clear from the physical interpretation of the phenomenon. Suppose that the mean temperature in a turbulent medium is a function of position (e.g., the height above the ground). Since turbulent motion causes mixing, some of the fluid of temperature T may be transported to a point of the medium originally occupied by a fluid of a different temperature T'. Temperature fluctuations thus may develop at any point in the medium. The microstructure of the temperature field was treated by Obukhov /23/, Yaglom /24/, Corrsin /25/, and others.

Here we consider this problem following the treatment of /23, 24/. The temperature distribution in a moving medium is described by the equation of heat conduction

$$\frac{dT}{dt} = \frac{\partial T}{\partial t} + v_k \frac{\partial T}{\partial x_k} = \chi \Delta T,$$

where $\frac{d}{dt}$ is the total derivative with respect to time, χ is the thermal diffusivity of the medium, T is the deviation of the temperature from its mean $\langle T\rangle$, which is regarded as being constant, and v_k are the velocity fluctuations. Using the incompressibility of the fluid $\frac{\partial v_k}{\partial x_k} = 0$ and taking $\chi = \text{const}$, we write this equation in the form

$$\frac{\partial T}{\partial t} + \frac{\partial}{\partial x_k}\left(T v_k - \chi \frac{\partial T}{\partial x_k}\right) = 0. \tag{1}$$

Let T' be the temperature at some other point with the coordinates x_i'. Multiplying (1) by T', we absorb the temperature T' inside the derivative $\frac{\partial}{\partial x_k}$, as it is independent of x_k:

$$T' \frac{\partial T}{\partial t} + \frac{\partial}{\partial x_k}\left(v_k T T' - \chi \frac{\partial T T'}{\partial x_k}\right) = 0. \tag{2}$$

We now write the same equation interchanging the points x_k and x_k'. T and T' are also interchanged, and v_k and $\frac{\partial}{\partial x_k}$ are replaced by v_k' and $\frac{\partial}{\partial x'_k}$:

$$T \frac{\partial T'}{\partial t} + \frac{\partial}{\partial x_k'}\left(v_k' T' T - \chi \frac{\partial T T'}{\partial x_k'}\right) = 0. \tag{3}$$

Adding (2) and (3) and seeing that

$$T' \frac{\partial T}{\partial t} + T \frac{\partial T'}{\partial t} = \frac{\partial T T'}{\partial t},$$

we find

$$\frac{\partial T T'}{\partial t} + \frac{\partial}{\partial x_k}\left(v_k T T' - \chi \frac{\partial T T'}{\partial x_k}\right) + \frac{\partial}{\partial x_k'}\left(v_k' T' T - \chi \frac{\partial T T'}{\partial x_k'}\right) = 0. \tag{4}$$

Let us take the average of equation (4). Assuming a statistically homogeneous temperature field, we obtain for the correlation function

$$B_T(\boldsymbol{r} - \boldsymbol{r}', t) = \langle T T'\rangle \tag{5}$$

and

$$\frac{\partial B_T(\boldsymbol{r} - \boldsymbol{r}', t)}{\partial t} + \frac{\partial}{\partial x_k}\langle v_k T T'\rangle + \frac{\partial}{\partial x_k'}\langle v_k' T' T\rangle - \chi \Delta B_T - \chi \Delta' B_T = 0, \tag{6}$$

where $\Delta' = \frac{\partial^2}{\partial x_k'^2}$ is the Laplacian at the point $\boldsymbol{r}'$. We will now express equation (6) in terms of the structure function

$$D_T(\boldsymbol{r} - \boldsymbol{r}', t) = \langle (T - T')^2\rangle. \tag{6'}$$

Expanding the right-hand side of (6'), we obtain

$$D_T(\boldsymbol{r} - \boldsymbol{r}', t) = \langle T^2(\boldsymbol{r}, t)\rangle + \langle T^2(\boldsymbol{r}, t)\rangle - 2B_T(\boldsymbol{r} - \boldsymbol{r}', t). \tag{7}$$

Since by assumption the temperature field is statistically homogeneous, we have $\langle T^2\rangle = \langle T'^2\rangle$ and

$$D_T(\boldsymbol{r} - \boldsymbol{r}', t) = 2\langle T^2(\boldsymbol{r}, t)\rangle - 2B_T(\boldsymbol{r} - \boldsymbol{r}', t). \tag{8}$$

Differentiation of (8) with respect to time gives

$$\frac{\partial D_T(\boldsymbol{r}-\boldsymbol{r}',\ t)}{\partial t}=2\frac{\partial\langle T^2(\boldsymbol{r},\ t)\rangle}{\partial t}-2\frac{\partial B_T(\boldsymbol{r}-\boldsymbol{r}',\ t)}{\partial t}. \tag{9}$$

By analogy with the energy dissipation rate ε we introduce the parameter

$$N=-\frac{\partial}{\partial t}\left(\frac{\langle T^2\rangle}{2}\right). \tag{10}$$

From (9) we then have

$$\frac{\partial B_T(\boldsymbol{r}-\boldsymbol{r}',\ t)}{\partial t}=-2N-\frac{1}{2}\frac{\partial D_T(\boldsymbol{r}-\boldsymbol{r}',\ t)}{\partial t}. \tag{11}$$

Taking the Laplacian of both sides of (8) and noting that because of homogeneity $\langle T^2\rangle$ is independent of $\boldsymbol{r}$, we obtain

$$\Delta D_T(\boldsymbol{r}-\boldsymbol{r}',\ t)=-2\Delta B_T(\boldsymbol{r}-\boldsymbol{r}',\ t). \tag{12}$$

Now, since B_T depends only on $\boldsymbol{r}-\boldsymbol{r}'$, $\Delta' B_T=\Delta B_T$ and (12) may be written in the form

$$\Delta B_T+\Delta' B_T=-\Delta D_T(\boldsymbol{r}-\boldsymbol{r}',\ t), \tag{13}$$

where on the right-hand side we may take $\Delta=\frac{\partial^2}{\partial(x_k-x_k')^2}$.

We have expressed the first and the last two terms of (6) in terms of the function $D_T(\boldsymbol{r},\ t)$. To express the quantities $\langle v_k T T'\rangle$ and $\langle v_k T' T\rangle$ in terms of the difference characteristics of the fields $\boldsymbol{v}$ and T, we introduce the function

$$D_{kTT}(\boldsymbol{r}-\boldsymbol{r}',\ t)=\langle[v_k(\boldsymbol{r},\ t)-v_k(\boldsymbol{r}',\ t)][T(\boldsymbol{r},\ t)-T(\boldsymbol{r}',\ t)]^2\rangle. \tag{14}$$

Expanding the product on the right-hand side of (14), we obtain

$$D_{kTT}=\langle v_k T^2\rangle-2\langle v_k T T'\rangle+\langle v_k T'^2\rangle-\langle v_k' T^2\rangle+2\langle v_k' T' T\rangle-\langle v_k' T'^2\rangle. \tag{15}$$

Since the fields v_k and T are statistically homogeneous, $\langle v_k T^2\rangle$ is independent of position so that $\langle v_k T^2\rangle=\langle v_k' T'^2\rangle$. Therefore, the first and the last term in (15) cancel.

Now consider the term $\langle v_k T'^2\rangle$. This is the correlation between the velocity at the point $\boldsymbol{r}$ and the square of the temperature at the point $\boldsymbol{r}'$. According to the general theorem proven in §8, the solenoidal velocity field (incompressible flow) is uncorrelated with any scalar field, in particular with T^2, provided the fields v_k and T are statistically homogeneous and isotropic. Consequently,

$$\langle v_k T'^2\rangle=\langle v_k' T^2\rangle=0. \tag{16}$$

Using this equality, we may write (15) in the form

$$D_{kTT}=2\langle v_k' T' T\rangle-2\langle v_k T T'\rangle. \tag{17}$$

Note that $\langle v_k T T' \rangle$ depends only on $\boldsymbol{\rho} = \boldsymbol{r} - \boldsymbol{r}' = \{\xi_1, \xi_2, \xi_3\}$ and therefore

$$\frac{\partial}{\partial x_k} \langle v_k T T' \rangle = \frac{\partial}{\partial \xi_k} \langle v_k T T' \rangle,$$
$$\frac{\partial}{\partial x_k'} \langle v_k' T' T \rangle = -\frac{\partial}{\partial \xi_k} \langle v_k' T' T \rangle.$$

Differentiating (17) with respect to ξ_k and using these relations, we find

$$-\frac{1}{2} \frac{\partial D_{kTT}(\rho, t)}{\partial \xi_k} = \frac{\partial}{\partial x_k'} \langle v_k' T' T \rangle + \frac{\partial}{\partial x_k} \langle v_k T T' \rangle. \tag{18}$$

Inserting (11), (13), and (18) in equation (6), we obtain

$$-2N - \frac{1}{2} \frac{\partial D_T(\boldsymbol{\rho}, t)}{\partial t} - \frac{1}{2} \frac{\partial D_{kTT}(\boldsymbol{\rho}, t)}{\partial \xi_k} + \chi \Delta D_T(\boldsymbol{\rho}, t) = 0. \tag{19}$$

The quantity $D_{kTT} = \langle [v_k - v_k'] [T - T']^2 \rangle$ is a vector. Since we are dealing with a statistically isotropic model, this vector may only be directed along the vector $\boldsymbol{n} = \boldsymbol{\rho}/\rho$, i.e.,

$$D_{kTT} = n_k D_{rTT}(\rho), \tag{20}$$

where $D_{rTT}(\rho)$ is a scalar function of the scalar argument ρ, and it is interpreted as the mean value of the product formed from the square of the temperature difference at the points $\boldsymbol{r}$ and $\boldsymbol{r} + \boldsymbol{\rho}$ and the difference of the longitudinal velocity components at the same points. Differentiation of (20) yields

$$\frac{\partial D_{kTT}}{\partial \xi_k} = \frac{2}{\rho} D_{rTT} + D_{rTT}' = \frac{1}{\rho^2} \frac{d}{d\rho} [\rho^2 D_{rTT}(\rho)].$$

Inserting this expression in (19) and taking into account that

$$\Delta D_T = \frac{1}{\rho^2} \frac{d}{d\rho} \left[\rho^2 \frac{dD_T}{d\rho} \right],$$

we obtain the equation

$$-2N - \frac{1}{2} \frac{\partial D_T}{\partial t} - \frac{1}{2\rho^2} \frac{d}{d\rho} [\rho^2 D_{rTT}] + \frac{\chi}{\rho^2} \frac{d}{d\rho} \left[\rho^2 \frac{dD_T}{d\rho} \right] = 0. \tag{21}$$

Equation (21) is analogous to equation (11.4) for the velocity field. Just as the dissipation rate ε in (11.4) is related to the mean square of the velocity field, the parameter N can be expressed in terms of the mean square of the temperature gradient. To derive this relation, we multiply equation (1) by T, taken at the same point $\boldsymbol{r}$ and average:

$$\left\langle T \frac{\partial T}{\partial t} \right\rangle + \left\langle T \frac{\partial T v_k}{\partial x_k} \right\rangle - \chi \left\langle T \frac{\partial}{\partial x_k} \frac{\partial T}{\partial x_k} \right\rangle = 0. \tag{22}$$

Using the incompressibility equation, we write the second term in (22) in the form

$$\left\langle T \frac{\partial T v_k}{\partial x_k} \right\rangle = \left\langle T v_k \frac{\partial T}{\partial x_k} \right\rangle = \left\langle v_k \frac{\partial}{\partial x_k} \left(\frac{T^2}{2} \right) \right\rangle = \frac{\partial}{\partial x_k} \left\langle \frac{T^2 v_k}{2} \right\rangle.$$

The last term in (22) can be written as

$$-\chi\left\langle T\frac{\partial}{\partial x_k}\frac{\partial T}{\partial x_k}\right\rangle=-\frac{\partial}{\partial x_k}\left[\chi\left\langle T\frac{\partial T}{\partial x_k}\right\rangle\right]+\chi\left\langle\left(\frac{\partial T}{\partial x_k}\right)^2\right\rangle.$$

Using these expressions, we write (22) in the form

$$\frac{\partial}{\partial t}\frac{\langle T^2\rangle}{2}+\chi\left\langle\left(\frac{\partial T}{\partial x_k}\right)^2\right\rangle+\frac{\partial}{\partial x_k}\left[\frac{1}{2}\langle v_kT^2\rangle-\chi\left\langle T\frac{\partial T}{\partial x_k}\right\rangle\right]=0. \qquad (23)$$

Since, as already mentioned, the velocity field is uncorrelated with the scalar field T^2, the first term in square brackets vanishes. The second term may be written as $\frac{1}{2}\frac{\partial\langle T^2\rangle}{\partial x_k}$, and therefore it also vanishes because the temperature field is homogeneous ($\langle T^2\rangle=\text{const}$). In this case, we have the equation

$$\frac{\partial}{\partial t}\frac{\langle T^2\rangle}{2}+\chi\left\langle\left(\frac{\partial T}{\partial x_k}\right)^2\right\rangle=0 \qquad (24)$$

or

$$N=\chi\left\langle\left(\frac{\partial T}{\partial x_k}\right)^2\right\rangle. \qquad (25)$$

We now return to equation (21). It describes the damping of the temperatue fluctuations with time. Since the initial equation (1) did not contain any heat sources, the process of heat conduction should eventually result in a complete equalization of temperature in the medium.

As with equation (11.4), we can somewhat modify the statement of the problem. Consider external heat sources of intensity $N=\chi\langle(\operatorname{grad} T)^2\rangle$, which compensate for the temperature equalizing tendency. This will ensure some sort of a statistical steady-state distribution with time-independent D_T. N should be treated as some external parameter which characterizes the intensity of the fluctuations. Equation (21) thus takes the form

$$-2N-\frac{1}{2\rho^2}\frac{d}{d\rho}(\rho^2D_{rTT})+\frac{\chi}{\rho^2}\frac{d}{d\rho}\left(\rho^2\frac{dD_T}{d\rho}\right)=0. \qquad (26)$$

Multiplying (26) by ρ^2 and integrating over ρ from 0 to r, we obtain

$$-2N\frac{r^3}{3}-\frac{1}{2}r^2D_{rTT}(r)+\chi r^2\frac{dD_T(r)}{dr}=0,$$

and this leads us to an equation obtained by Yaglom /24/

$$D_{rTT}=-\frac{4}{3}Nr+2\chi D_T'(r), \qquad (27)$$

which is analogous to Kolmogorov's equation for the velocity field.

Consider distances r which are small compared with the inner scale of turbulence l_0. Over these distances $T-T'\sim r$ and $D_T(r)\sim r^2$, so that the right-hand side of (27) is proportional to r. Also $D_{rTT}\sim r^3$. Therefore, for $r\ll l_0$ we can ignore D_{rTT} and $D_T'(r)=\frac{2}{3\chi}Nr$. Integrating this equation we obtain

$$D_T(r)=\frac{1}{3}\frac{N}{\chi}r^2+\ldots,\quad r\ll l_0. \qquad (28)$$

This gives another relation

$$D_T''(0) = \frac{2}{3}\frac{N}{\chi}. \tag{29}$$

Expression (28) is analogous to the expression $D_{rr}(r) = \frac{1}{15}\frac{\varepsilon}{\nu}r^2 + \ldots$ for the velocity field. However, equation (27), like Kolmogorov's equation, is insufficient for determining the function $D_T(r)$, as it includes a new unknown function D_{rTT}. The determination of $D_T(r)$ requires introduction of additional hypotheses, as in the case of the velocity field.

One of these hypotheses used by Obukhov /23/ maintains that for $r \ll L_0$, $D_T(r)$ is a function of N, χ, r, and ε (the latter parameter is the only factor characterizing the structure of small-scale turbulence). For $l_0 \ll r \ll L_0$, on the other hand, $D_T(r)$ should be independent of χ. The only combination of N, χ, and ε with the dimension of temperature is

$$T_0 = \sqrt[4]{\frac{\chi N^2}{\varepsilon}}, \tag{30}$$

and the only combination with the dimension of length is

$$l_0 = \sqrt[4]{\frac{\chi^3}{\varepsilon}}. \tag{31}$$

Since (30) and (31) contain the thermal diffusivity χ, these parameters characterize the smallest inhomogeneities of the temperature field. The function $D_T(r) = D_T(N, \chi, \varepsilon, r)$, according to the Π-theorem, should have the form

$$D_T(r) = T_0^2 F\left(\frac{r}{l_1}\right) = N\sqrt{\frac{\chi}{\varepsilon}}\, F\left(\frac{r\varepsilon^{1/4}}{\chi^{3/4}}\right). \tag{32}$$

For $r \gg l_1$, χ should drop out of (39). From this it follows that $F(x) = a^2 x^{2/3}$ for $x \gg 1$, where a is a numerical constant of the order of unity. In this case

$$D_T(r) = a^2 \frac{N}{\varepsilon^{1/3}} r^{2/3} = C_T^2 r^{2/3}, \text{ where } C_T^2 = \frac{a^2 N}{\varepsilon^{1/3}} \text{ and } l_1 \ll r \ll L_0. \tag{33}$$

Therefore, in the inertial subrange $D_T(r)$ is also proportional to $r^{2/3}$. Since in the inertial subrange $D_T(r)$ is expressed by (33), the last term in equation (27) is small compared to the other terms. In this case,

$$D_{rTT}(r) = -\frac{4}{3}Nr, \quad l_0 \ll r \ll L_0. \tag{34}$$

Consider the parameter

$$S' = \frac{D_{rTT}}{\sqrt{D_{rr}} D_T}. \tag{35}$$

Using (33), (34) and the relation $D_{rr} = C^2 \varepsilon^{2/3} r^{2/3}$, we find

$$S' = -\frac{4}{3Ca^2} = \text{const}. \tag{36}$$

Yaglom /24/ adopted the condition $S' = \text{const}$ as an additional hypothesis which he introduced to determine $D_T(r)$ from (27). In this case (27) may be written in the form

$$2\chi D_T'(r) + |S'|\sqrt{D_{rr}(r)}\,D_T(r) = \frac{4}{3}Nr, \tag{37}$$

where $|S'| = -S'$.

Equation (37) is a linear equation for $D_T(r)$. If $D_{rr}(r)$ is a known function, the solution of (37), as can be easily verified, has the form

$$D_T(r) = \frac{2N}{3\chi}\int_0^r x e^{-\frac{|S'|}{2\chi}\int_x^r \sqrt{D_{rr}(\rho)}\,d\rho}\,dx. \tag{38}$$

Taking $D_{rr}(\rho) = \frac{1}{15}\frac{\varepsilon}{\nu}\rho^2$, we can derive (28) from (38). Putting $D_{rr}(\rho) = C^2\varepsilon^{2/3}\rho^{2/3}$ and using the asymptotic expansion of the resulting integral, we can recover expression (33), in which a^2 is related to S' by (36).

The inner scale of turbulence for temperature fluctuations is the l_1 defined by (31). It differs from the scale $l_0 = \sqrt[4]{\frac{\nu^3}{\varepsilon}}$ in that ν has been replaced by χ. Since ν and χ in the atmosphere and in the ocean are of the same order of magnitude, in practice we have $l_1 \approx l_0$.

Let us also find the spectral density corresponding to the structure function (33). Using the results of example 2 from §5, we obtain for the three-dimensional spectral density of the temperature field

$$\Phi_T(\varkappa) = \frac{\Gamma\left(\frac{8}{3}\right)\sin\frac{\pi}{3}}{4\pi^2}\frac{a^2N}{\varepsilon^{1/3}}\varkappa^{-11/3} = 0.033\,C_T^2\varkappa^{-11/3}, \tag{39}$$

where as before $C_T^2 = a^2N\varepsilon^{-1/3}$. A spectral density of the form (39), like the corresponding expression for $E(\varkappa)$, is applicable only for $2\pi L_0^{-1} \ll \varkappa \ll 2\pi l_1^{-1}$. For $\varkappa \gtrsim \varkappa_m \sim 2\pi l_1^{-1}$ the function $\Phi_T(\varkappa)$ should decay rapidly. An expression analogous to (12.17) can be used to approximate $\Phi_T(\varkappa)$ for all $\varkappa \gg 2\pi L_0^{-1}$:

$$\Phi_T(\varkappa) = \frac{AN}{\varepsilon^{1/3}}\varkappa^{-11/3}\exp(-\gamma^2 l_1^2\varkappa^2), \tag{40}$$

where A and γ are numerical constants. For $\varkappa l_1 \ll 1$, expression (40) reduces to (39), and for $\varkappa l_1 \gg 1$, (40) decays rapidly with increasing $\varkappa$. Equation (29) imposes a certain constraint on the constants A and γ. Indeed, by (5.8)

$$D_T(r) = 8\pi\int_0^\infty \left(1 - \frac{\sin\varkappa r}{\varkappa r}\right)\Phi_T(\varkappa)\,\varkappa^2\,d\varkappa. \tag{41}$$

Differentiating this expression and inserting the result in (29), we find

$$D_T''(0) = \frac{8\pi}{3}\int_0^\infty \Phi_T(\varkappa)\,\varkappa^4\,d\varkappa = \frac{2}{3}\frac{N}{\chi}. \tag{42}$$

Inserting (40) in (42) and integrating, we obtain

$$\frac{2N}{3\chi} = 2\pi\Gamma\left(\frac{5}{3}\right) AN\varepsilon^{-1/3}(\gamma l_1)^{-4/3}.$$

Putting $l_1^{4/3} = \chi\,\varepsilon^{-1/3}$, we obtain the equality

$$\gamma^{4/3} = 3\pi\Gamma\left(\frac{5}{3}\right) A = 0.851A, \tag{43}$$

relating γ to A.

Let us now calculate the structure function $D_T(r)$. Inserting (40) in (41) we integrate the expression after expanding $\sin \varkappa r$ in a series. Term-by-term integration gives the confluent hypergeometric function, so that

$$D_T(r) = 18\pi\Gamma\left(\frac{5}{3}\right)\frac{AN}{\varepsilon^{1/3}}(\gamma l_1)^{2/3}\left[{}_1F_1\left(-\frac{1}{3},\ \frac{3}{2},\ -\frac{r^2}{4\gamma^2 l_1^2}\right)-1\right]. \tag{44}$$

Using the asymptotic expansion of the function ${}_1F_1$ for large negative values of the argument (12.20)

$${}_1F_1\left(-\frac{1}{3},\ \frac{3}{2},\ -\frac{r^2}{4\gamma^2 l_1^2}\right) \approx \frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{11}{6}\right)}\left(\frac{r^2}{4\gamma^2 l_1^2}\right)^{1/3},$$

we obtain

$$D_T(r) = a^2 N\varepsilon^{-1/3} r^{2/3} = C_T^2 r^{2/3}, \tag{45}$$

where

$$a^2 = 9\pi\sqrt{\pi}\,\Gamma\left(\frac{5}{3}\right)\left[2^{2/3}\Gamma\left(\frac{11}{6}\right)\right]^{-1} A = \frac{54}{5}\pi\Gamma\left(\frac{4}{3}\right) A = \frac{A}{0.033}.$$

Thus, for $r \gg l_1$ we recover the previous expression for $D_T(r)$. For $r \ll l_1$ the first two terms in the series for ${}_1F_1$ also lead to the previous expression for $D_T(r)$:

$$D_T(r) = \frac{1}{3}\frac{N}{\chi} r^2 + \cdots$$

We now introduce a scale λ_0, defined as the intersection point of the asymptotic expansions $D_T(r) = C_T^2 r^{2/3}$ and $D_T(r) = \frac{Nr^2}{3\chi}$:

$$C_T^2 \lambda_0^{2/3} = \frac{1}{3} N\chi^{-1}\lambda_0^2,$$

from which $\lambda_0^{4/3} = 3\chi N^{-1} C_T^2$.

Substituting $C_T^2 = a^2 N\varepsilon^{-1/3}$, we obtain

$$\lambda_0^{4/3} = 3a^2 l_1^{4/3}, \text{ or } a^2 = \frac{1}{3}(\lambda_0 / l_1)^{4/3}.$$

All the numerical parameters A, a^2, γ related by (43) and (45), are thus expressed in terms of the ratio of two inner scales λ_0 and $l_1 = \chi^{3/4}\varepsilon^{-1/4}$:

$$a^2 = \frac{1}{3}\left(\frac{\lambda_0}{l_1}\right)^{4/3}; \quad A = 0.011\left(\frac{\lambda_0}{l_1}\right)^{4/3}; \quad \gamma = 0.169\frac{\lambda_0}{l_1}. \tag{46}$$

$\Phi_T(\varkappa)$ can now be expressed in terms of the more convenient parameters C_T^2 and λ_0:

$$\Phi_T(\varkappa) = 0.033\, C_T^2 \varkappa^{-11/3} e^{-\frac{\varkappa^2}{\varkappa_m^2}}, \tag{47}$$

where

$$\varkappa_m^2 = (\gamma l_1)^{-2} = (0.169\lambda_0)^{-2},$$

i.e.,

$$\varkappa_m \lambda_0 = \frac{1}{0.169} = 5.92. \tag{48}$$

In concluding this section, let us consider the microstructure of the concentration field of a conservative passive additive. An additive is said to be conservative if its concentration in the process of turbulent mixing is changed only by molecular diffusion (thermal conduction) and not by any other process (chemical reactions, phase transitions, recombination, etc.). The concentration θ of the conservative additive thus satisfies the equation of diffusion

$$\frac{\partial\theta}{\partial t} + \frac{\partial\theta v_k}{\partial x_k} = D\Delta\theta, \tag{49}$$

where D is the coefficient of molecular diffusion of the given additive.

An additive is said to be passive if θ does not affect the turbulent motion (θ does not enter the equations of motion). Examples of conservative passive additives are provided by water vapor (when condensation can be ignored), by electrons in the ionosphere, etc.

Since equation (49) is identical to the equation of heat conduction (1), all the statistical relations derived for the temperature field can be extended without the slightest modification to the concentration θ. Strictly speaking, the temperature T is not a passive additive, since deviations of T from the mean give rise to Archimedes forces (buoyancy forces) which affect the turbulent conditions.

However, if the temperature deviations from equilibrium are small, this effect is negligible. Therefore, the analysis of this section is even more applicable to the concentration of a conservative passive additive than to temperature. In waht follows we will discuss the modifications introduced in the theory by the fact that the temperature is not a passive additive.

§ 14. The relation of the microstructure characteristics of the velocity and temperature fields to the mean field characteristics

The parameters ε and N which determine the intensity of the velocity and temperature fluctuations in a turbulent flow can be related to the parameters characterizing the mean velocity and temperature fields.

First consider the simpler case of the temperature field. As we have shown in the previous section, continuously acting heat sources are needed to maintain a statistically stationary regime. In this case the equation of heat conduction takes the form

$$\frac{\partial T}{\partial t}+\frac{\partial v_k T}{\partial x_k}-\chi\Delta T=Q(\boldsymbol{r}). \qquad (1)$$

Here $T=T_0+T'$, where $T_0(\boldsymbol{r})=\langle T\rangle$ and $\langle T'\rangle=0$. Averaging equation (1) and taking into account that $\langle v_k\rangle=0$, we find

$$-\chi\Delta T_0+\frac{\partial}{\partial x_k}\langle v_k T'\rangle=Q(\boldsymbol{r}), \qquad (2)$$

where we made use of the fact that under stationary conditions $\frac{\partial T_0}{\partial t}=0$.

The equality $\langle v_k T'\rangle\equiv 0$, often used before, no longer applies in this case, since the presence of heat sources produces a certain preferred direction in space, characterized by the vector ∇T_0.

Subtracting (2) from (1) we obtain

$$\frac{\partial T'}{\partial t}+\frac{\partial}{\partial x_k}\left[v_k T_0+v_k T'-\langle v_k T'\rangle-\chi\frac{\partial T'}{\partial x_k}\right]=0. \qquad (3)$$

Multiplying (3) by T' and averaging (remember that $\frac{\partial}{\partial t}\langle T'^2\rangle=0$ under stationary conditions), we obtain

$$\left\langle T'\frac{\partial}{\partial x_k}(v_k T_0)\right\rangle+\left\langle T'\frac{\partial}{\partial x_k}v_k T'\right\rangle-\left\langle T'\frac{\partial}{\partial x_k}\chi\frac{\partial T'}{\partial x_k}\right\rangle=0, \qquad (4)$$

where we used the equality

$$\left\langle T'\frac{\partial}{\partial x_k}\langle v_k T'\rangle\right\rangle=\langle T'\rangle\frac{\partial}{\partial x_k}\langle v_k T'\rangle=0.$$

Let us transform the first term in (4) using the equality $\frac{\partial v_k}{\partial x_k}=0$:

$$\left\langle T'\frac{\partial v_k T_0}{\partial x_k}\right\rangle=\left\langle T' v_k\frac{\partial T_0}{\partial x_k}\right\rangle=\langle T' v_k\rangle\frac{\partial T_0}{\partial x_k}.$$

Similarly,

$$\left\langle T'\frac{\partial v_k T'}{\partial x_k}\right\rangle=\left\langle v_k T'\frac{\partial T'}{\partial x_k}\right\rangle=\left\langle v_k\frac{\partial}{\partial x_k}\left(\frac{T'^2}{2}\right)\right\rangle=\left\langle\frac{\partial}{\partial x_k}v_k\frac{T'^2}{2}\right\rangle,$$

$$\left\langle T'\frac{\partial}{\partial x_k}\chi\frac{\partial T'}{\partial x_k}\right\rangle=\left\langle\frac{\partial}{\partial x_k}\left(\chi T'\frac{\partial T'}{\partial x_k}\right)\right\rangle-\chi\left\langle\left(\frac{\partial T'}{\partial x_k}\right)^2\right\rangle.$$

Inserting all these equalities in (4), we obtain

$$\langle T' v_k\rangle\frac{\partial T_0}{\partial x_k}+\chi\left\langle\left(\frac{\partial T'}{\partial x_k}\right)^2\right\rangle-\left\langle\frac{\partial}{\partial x_k}\left[\chi T'\frac{\partial T'}{\partial x_k}-v_k\frac{T'^2}{2}\right]\right\rangle=0. \qquad (5)$$

But according to the previous section the mean of the divergence of a random field is zero.* Therefore the last term in (5) vanishes and we

* This proposition was proved for a homogeneous field. In the case on hand, however, the field T is no longer homogeneous, as T_0 is a function of $\boldsymbol{r}$. However, since equation (5) contains only $\frac{\partial T_0}{\partial x_k}$, then in the case of a locally homogeneous field, when $\frac{\partial T_0}{\partial x_k}=$ const, the proposition remains valid.

obtain the equality

$$N = \chi \left\langle \left(\frac{\partial T'}{\partial x_k} \right)^2 \right\rangle = - \langle T' v_k \rangle \frac{\partial T_0}{\partial x_k}, \tag{6}$$

which relates the parameter N to the mean temperature gradient. By (6) it follows that $N = 0$ when $\frac{\partial T_0}{\partial x_k} = 0$, i.e., when the mean temperature is constant, temperature fluctuations will be absent. This result is readily understood in the light of simple qualitative considerations. Turbulent mixing of a temperature-homogeneous medium does not give rise to temperature fluctuations, since the elements of fluid interchanged by the mixing processes all have the same temperature.

Besides the factor $\frac{\partial T_0}{\partial x_k}$, relation (6) also contains the function $\langle T' v_k \rangle = q_k$, which is proportional to the eddy heat flux.*

Using equation (2), we can express this quantity in terms of T_0 and Q. This is not very helpful, however, since Q is generally unknown. Therefore a different approach should be tried for the determination of q_k. The heat flux due to molecular diffusion, q^M, is proportional to $\frac{\partial T}{\partial x_k}$:

$$q_k^M = - \chi \frac{\partial T}{\partial x_k}.$$

By analogy with this relation, we introduce the eddy thermal diffusivity K_T, defined by

$$\langle T' v_k \rangle = - K_T \frac{\partial T_0}{\partial x_k}. \tag{7}$$

This is equivalent to a statement that the eddy heat flux is proportional to the mean temperature gradient and the two are in opposite directions. The eddy thermal diffusivity is greater by several orders of magnitude than χ. Using (7), we write (6) in the form

$$N = K_T \left(\frac{\partial T_0}{\partial x_k} \right)^2. \tag{8}$$

Relation (8) is useful for numerical estimates of N, since all the factors here can be fairly easily obtained from meteorological observations (see Part C). Structurally this relation is similar to $N = \chi \left\langle \left(\frac{\partial T'}{\partial x_k} \right)^2 \right\rangle$, but the true temperature gradient is replaced by the mean temperature gradient and the coefficient of molecular thermal diffusion is changed to the eddy thermal diffusivity.

Let us now consider the velocity field. Under statistically stationary conditions, we should consider motion due to constant external forces f_i. The equation of motion takes the form

$$\frac{\partial v_i}{\partial t} + v_k \frac{\partial v_i}{\partial x_k} + \frac{1}{\rho} \frac{\partial p}{\partial x_i} - \nu \Delta v_i = f_i, \tag{9}$$

where v_i is the full velocity:

$$v_i = u_i + v_i', \quad \langle v_i \rangle = u_i(\boldsymbol{r}), \qquad v_i' = v_i'(\boldsymbol{r}, t).$$

* The eddy heat flux is given by $Q_k = \rho C_p q_k$, where C_p is the specific heat per unit mass of the gas at constant pressure and ρ is the density.

Multiplying (9) by v_i and taking into account that

$$v_i \frac{\partial v_i}{\partial t} = \frac{\partial}{\partial t}\left(\frac{v_i}{2}\right)^2,$$

$$v_i v_k \frac{\partial v_i}{\partial x_k} = v_k \frac{\partial}{\partial x_k}\left(\frac{v_i}{2}\right)^2 = \frac{\partial}{\partial x_k}\left(\frac{1}{2} v_i^2 v_k\right),$$

$$v_i \frac{1}{\rho}\frac{\partial p}{\partial x_i} = \frac{\partial}{\partial x_k}\left(\frac{p v_k}{\rho}\right),$$

$$-\nu v_i \Delta v_i = \frac{\partial}{\partial x_k}\left(-\nu v_i \frac{\partial v_i}{\partial x_k}\right) + \nu\left(\frac{\partial v_i}{\partial x_k}\right)^2,$$

we obtain

$$\frac{\partial}{\partial t}\left(\frac{v_i}{2}\right)^2 + \nu\left(\frac{\partial v_i}{\partial x_k}\right)^2 + \frac{\partial}{\partial x_k}\left[\frac{v_i^2}{2} v_k + \frac{p v_k}{\rho} - \nu v_i \frac{\partial v_i}{\partial x_k}\right] = f_i v_i. \tag{10}$$

The right-hand side of (10) is the work done by the external force f_i per unit time. The force f_i can be expressed in terms of the mean velocity by averaging equation (9). Seeing that

$$\frac{\partial u_i}{\partial t} = 0, \quad \frac{\partial}{\partial x_i}\langle p\rangle = 0 \quad \text{and} \quad v_k \frac{\partial v_i}{\partial x_k} = \frac{\partial}{\partial x_k}(v_k v_i),$$

we obtain

$$\frac{\partial}{\partial x_k}\left(u_i u_k + \langle v_i' v_k'\rangle\right) - \nu \Delta u_i = f_i. \tag{11}$$

The term

$$\langle v_i' v_k'\rangle = \tau_{ik} \tag{12}$$

is proportional to the Reynolds stress tensor.* Multiplying equation (11) by v_i and averaging, we obtain

$$\langle f_i v_i\rangle = f_i u_i = u_i \frac{\partial u_i u_k}{\partial x_k} + u_i \frac{\partial \tau_{ik}}{\partial x_k} - \nu u_i \Delta u_i. \tag{13}$$

Equation (13) can also be transformed using the incompressibility equation $\frac{\partial u_k}{\partial x_k} = 0$:

$$u_i \frac{\partial u_i u_k}{\partial x_k} = \frac{\partial}{\partial x_k}\left(\frac{1}{2} u_i^2 u_k\right),$$

$$u_i \frac{\partial \tau_{ik}}{\partial x_k} = \frac{\partial}{\partial x_k}(u_i \tau_{ik}) - \tau_{ik}\frac{\partial u_i}{\partial x_k},$$

$$-\nu u_i \Delta u_i = -\frac{\partial}{\partial x_k}\left(\nu u_i \frac{\partial u_i}{\partial x_k}\right) + \nu\left(\frac{\partial u_i}{\partial x}\right)^2.$$

This gives

$$\langle f_i v_i\rangle = -\tau_{ik}\frac{\partial u_i}{\partial x_k} + \nu\left(\frac{\partial u_i}{\partial x_k}\right)^2 + \frac{\partial}{\partial x_k}\left(\frac{u_i^2}{2} u_k + u_i \tau_{ik} - \nu u_i \frac{\partial u_i}{\partial x_k}\right). \tag{14}$$

* The Reynolds stress tensor is $\langle \rho v_i' v_k'\rangle = \rho\tau_{ik}$.

Averaging equation (10) we substitute (14) in the right-hand side of the averaged expression. Divergence terms are omitted, as they vanish when averaged over the volume:

$$\frac{\partial}{\partial t}\left(\frac{\langle v_i'^2\rangle}{2}\right)+\nu\left(\frac{\partial u_i}{\partial x_k}\right)^2+\nu\left\langle\left(\frac{\partial v_i'}{\partial x_k}\right)^2\right\rangle=-\tau_{ik}\frac{\partial u_i}{\partial x_k}+\nu\left(\frac{\partial u_i}{\partial x_k}\right)^2.$$

In the left-hand side of the last equality we substituted

$$\left(\frac{\partial v_i}{\partial x_k}\right)^2=\left(\frac{\partial u_i}{\partial x}\right)^2+\left\langle\left(\frac{\partial v_i'}{\partial x_k}\right)^2\right\rangle.$$

We thus have the equality

$$\frac{\partial}{\partial t}\left(\frac{\langle v_i'^2\rangle}{2}\right)+\nu\left\langle\left(\frac{\partial v_i'}{\partial x_k}\right)^2\right\rangle=-\tau_{ik}\frac{\partial u_i}{\partial x_k}. \tag{15}$$

Under statistically stationary conditions $\langle v_i'^2\rangle$ is independent of time. According to equations (10.1) and (10.4') the second term in the left-hand side of (15) is the energy dissipation ε. Consequently, in the stationary case we have

$$\varepsilon=-\tau_{ik}\frac{\partial u_i}{\partial x_k}. \tag{16}$$

Equation (16) expresses ε in terms of the gradients of the mean flow velocity u_i and the Reynolds stresses τ_{ik}. This equation is analogous to (6), expressing N in terms of the mean temperature gradient $\frac{\partial T_0}{\partial x_k}$. The Reynolds stress τ_{ik} in (16) is the eddy momentum flux. Like the eddy heat flux, which is directed against the mean temperature gradient and is proportional to $\frac{\partial T_0}{\partial x_k}$, the eddy momentum flux is proportional to the mean velocity gradient, i.e.,

$$\tau_{ik}=-K\left(\frac{\partial u_i}{\partial x_k}+\frac{\partial u_k}{\partial x_i}\right), \tag{17}$$

where K is the eddy viscosity (the coefficient of eddy exchange). The expression on the right-hand side of (17) is symmetric with respect to the indices i and k, so as to yield a symmetric tensor τ_{ik}, in accordance with its definition (12). The eddy viscosity K, in general, is different from the eddy thermal diffusivity K_T introduced in (7).

Substituting (17) in (16), we obtain

$$\varepsilon=K\left(\frac{\partial u_i}{\partial x_k}+\frac{\partial u_k}{\partial x_i}\right)\frac{\partial u_i}{\partial x_k} \tag{18}$$

or, equivalently,

$$\varepsilon=\frac{1}{2}K\left(\frac{\partial u_i}{\partial x_k}+\frac{\partial u_k}{\partial x_i}\right)^2. \tag{18'}$$

Expressions (18) and (18') are identical in form with expression (10.1) for ε, but the true velocity gradients have been replaced by mean velocity gradients and the kinematic viscosity was changed to the eddy viscosity K.

The eddy coefficients K, K_T enable us to express the eddy "temperature flux" $\langle v_k' T' \rangle$ and the Reynolds stresses τ_{ik} in terms of the mean temperature and velocity gradients. In general, these quantities should be determined from the exact solutions of the corresponding nonlinear problems. However, since no such exact solutions are available, we are forced to consider coefficients of the form K for which various approximation formulas are used. This approach to the problem gives rise to the semiempirical theory of turbulence.

It will be useful to consider the analogy between molecular and eddy heat conduction. In the kinetic theory of gases (see, e.g., /26/), thermal diffusivity is expressed in the form $\chi \sim \lambda v$, where λ is the molecular free path and v is the rms velocity. A similar expression (with a different numerical coefficient) also holds true for the kinematic viscosity ν. In writing an analogous expression for the eddy viscosity K, we should remember that the analog of molecules in turbulent mixing is the individual moving inhomogeneities. Therefore, the rms velocity v of the molecules should be replaced by the rms velocity of the turbulent fluctuations and the molecular free path λ by the so-called mixing length (a concept introduced by Prandtl /27/), which is of the same order of magnitude as the velocity correlation radius L_0. The coefficient K is thus expressed in the form

$$K \sim L_0 u. \tag{19}$$

The proportionality coefficient in (19) is established for each particular type of motion from experimental data. In the following we will give an expression for K for one of the common types of turbulence in the boundary layer of the atmosphere.

In some cases the expression for the eddy viscosity K is derived from the following similarity considerations. Consider the expression of the eddy heat flux

$$\langle v_k' T' \rangle = -K_T \frac{\partial T_0}{\partial x_k}$$

(which is in fact the definition of K_T) for the particular case when $T_0 = T_0(z)$. In this case, the temperature fluctuations T' at a height z is caused by the displacement of an air volume from another height z', which is situated below z by a certain amount l':

$$z' = z - l'.$$

l' is the random "free path" of the moving volume element. In this case

$$T' = T_0(z - l') - T_0(z) \approx -\frac{\partial T_0}{\partial z} l' \;.^*$$

Inserting this expression in $\langle v_z' T' \rangle$, we obtain

$$\langle v_z' T' \rangle = -\frac{\partial T_0}{\partial z} \langle v_z' l' \rangle,$$

from which we get an expression for K_T:

$$K_T = \langle v_z' l' \rangle. \tag{19'}$$

* We ignore here the variation of temperature due to mixing (see § 15).

This expression of eddy viscosity is used by some authors who accept the concept of a "free path" or a "mixing length." Its derivation is not strictly related to the eddy heat flux, as temperature can be replaced by any other transportable additive.*

In the atmosphere, the mean characteristics of wind, temperature, humidity, etc., generally depend only on the height z above the ground. In this case

$$N = K_T \left(\frac{dT_0}{dz}\right)^2, \tag{20}$$

$$\varepsilon = K\left[\left(\frac{du_x}{dz}\right)^2 + \left(\frac{du_y}{dz}\right)^2\right] \doteq K\left(\frac{du}{dz}\right)^2. \tag{21}$$

Substituting these expressions in the relation $C_T^2 = a^2 \frac{N}{\varepsilon^{1/3}}$, which defines the coefficient in the "2/3 law" for temperature fluctuations and putting $\frac{K_T}{K} = \alpha$, we obtain

$$C_T^2 = a^2\alpha \left[\frac{K^2}{\left(\frac{du}{dz}\right)^2}\right]^{1/3} \left(\frac{dT_0}{dz}\right)^2. \tag{22}$$

Using expression (22), we can relate K to the outer scale of turbulence L_0. A gradient of mean temperatures will result in a systematic temperature difference between any two points at different heights. This temperature difference ΔT_0 is approximately given by $\Delta T_0 \approx \frac{dT_0}{dz}\Delta z$, and its square is $(\Delta T_0)^2 \approx \left(\frac{dT_0}{dz}\right)^2 (\Delta z)^2$. There is also a random temperature difference between these points, whose mean square value is $C_T^2(\Delta z)^{2/3}$. For small Δz, $C_T^2\,(\Delta z)^{2/3}$ is much greater than the gradient term $\left(\frac{dT_0}{dz}\right)^2 \Delta z^2$ (i.e., the random temperature differences are much greater than the systematic ones). There is however a certain Δz_0 when the two factors become comparable, and for $\Delta z > \Delta z_0$ the mean temperature difference is greater than the random difference. This Δz_0 is interpreted as the vertical mixing scale. Clearly the "2/3 law" is applicable only over distances r not greater than this mixing scale. Therefore Δz_0 may be taken equal to the outer scale of turbulence.

For determining Δz_0 we have the equation

$$C_T^2 \Delta z_0^{2/3} = \left(\frac{dT_0}{dz}\right)^2 \Delta z_0^2.$$

Substituting from (22), we obtain

$$\Delta z_0 = a^{3/2}\alpha^{3/4}\left[\frac{K}{(du/dz)}\right]^{1/2}.$$

The outer scale of turbulence L_0 differs from Δz_0 only by a numerical coefficient:

$$L_0 = \frac{\Delta z_0}{a^{3/2}\,\alpha^{3/4}}.$$

* Note, however, that (19') is not suitable for experimental determination of K_T, as the "mixing length" l' is a very vague quantity which can never be measured.

L_0 is related to K by the equality

$$K = L_0^2\left|\frac{du}{dz}\right|. \tag{23}$$

Relation (23) is only a slight modification of (19), since the factor u in (19) is proportional to $L_0\left|\frac{du}{dz}\right|$. Using (23), we rewrite (22) in the form

$$C_T^2 = \alpha a^2 L_0^{4/3}\left(\frac{dT_0}{dz}\right)^2. \tag{24}$$

There are no strong arguments in favor of either of the two relations (22) and (24), which express C_T in terms of the mean characteristics T_0 and u. The scale L_0 in the real atmosphere is apparently more stable and can be measured more readily.

Expressions (22) and (24) can be used to estimate C_T in the atmosphere. They are particularly effective in the ground layer, where extensive experimental material is available for immediate checking of the two relations and determination of the corresponding numerical coefficients. As regards the free atmosphere, the two relations give a correct order of magnitude for C_T, but more substantial experimental data are required for further checking.

§ 15. The microstructure of the refractive index in turbulent flow

The refractive index n of centimeter radio waves is a function of the absolute temperature T, the pressure p (in millibars), and the water vapor pressure e (in millibars):

$$n - 1 = 10^{-6}\frac{79}{T}\left(p + \frac{4800e}{T}\right). \tag{1}$$

The variables T and e entering this expression are clearly not conservative additives, in the strict sense of this term. Indeed, vertical mixing of small air volumes causes continuous equalization of pressure between these volumes and the ambient air at the same height. Pressure changes lead to temperature variation in accordance with the adiabatic equation

$$\frac{dT}{T} = \frac{\gamma - 1}{\gamma}\frac{dp}{p},$$

where $\gamma = \frac{C_p}{C_v}$ is Poisson's constant. The pressure increment dp is related to the height increment by the barometric equation $dp = -\rho g\, dz$, where ρ is the air density and g the acceleration of gravity.

Thus,

$$\frac{dT}{T} = -\frac{\gamma - 1}{\gamma}\frac{\rho g}{p}dz = -\frac{\gamma - 1}{\gamma}\frac{g}{RT}dz,$$

$$\frac{dT}{dz} = -\frac{\gamma - 1}{\gamma}\frac{g}{R} = -\frac{g}{C_p} = -\gamma_a. \tag{2}$$

Here $\gamma_a = 0.98\,\mathrm{deg}/100\,\mathrm{m}$ is called the adiabatic lapse rate (a rising air volume cools by 0.98 deg for each 100 m of ascent). Integration of (2) gives $T + \gamma_a z = \mathrm{const}$. The parameter

$$H = T + \gamma_a z, \tag{3}$$

which is approximately equal to the potential temperature used in meteorology,* thus remains constant for vertical displacements of air particles and it can be treated as a conservative additive.

The water vapor pressure e entering (1) is not a conservative additive, either, as it is a function of pressure. It can be expressed in terms of the so-called specific humidity q, which is the concentration of water vapor in air (i.e., the ratio of the mass of water vapor in a unit volume to the mass of humid air in that volume), using the approximate relation

$$e = 1.62\,pq. \tag{4}$$

Here q is a conservative additive (it is assumed that water vapor does not condense in the moving air masses). Substituting for T and e in (1) the corresponding expressions $H - \gamma_a z$ and $1.62\,pq$, we obtain a new relation

$$(n-1)\,10^6 = N = \frac{79p}{H-\gamma_a z}\left(1 + \frac{7800q}{H-\gamma_a z}\right), \tag{5}$$

which expresses the refractive index in terms of the conservative passive additives H and q. N is an explicit function of z and also depends on z implicitly through $p(z)$, $H(z)$, and $q(z)$, so that

$$N = N(z,\ p(z),\ H(z),\ q(z)).$$

Suppose that an air volume from the height z_1, characterized by the parameter

$$N_1 = N(z_1,\ p(z_1),\ H(z_1),\ q(z_1)),$$

is transferred by turbulent mixing to a height z_2. Since $H(z)$ and $q(z)$ do not change in mixing, whereas z and $p(z)$ take on the new values z_2 and $p(z_2)$, our air volume at the height z_2 is characterized by the parameter

$$N_1' = N(z_2,\ p(z_2),\ H(z_1),\ q(z_1)).$$

The value N_1' differs from the "local" value N at the height z_2 by an amount

$$\Delta N = N(z_2,\ p(z_2),\ H(z_1),\ q(z_1)) - N(z_2,\ p(z_2),\ H(z_2), q(z_2)) \approx \left(\frac{dN}{dH}\frac{dH}{dz} + \frac{dN}{dq}\frac{dq}{dz}\right)\Delta z.$$

In this case, the fluctuations in N are not proportional to the gradient of n, but rather to the quantity

$$M = \left(\frac{dN}{dH}\frac{dH}{dz} + \frac{dN}{dq}\frac{dq}{dz}\right)10^6 = -\frac{79\cdot 10^{-6}\,p}{T^2}\left(1 + \frac{15500q}{T}\right)\left(\frac{dH}{dz} - \frac{7800}{1+\frac{15500q}{T}}\frac{dq}{dz}\right) =$$

$$= -\frac{79\cdot 10^{-6}\,p}{T^2}\left(1 + \frac{15500q}{T}\right)\left(\frac{dT}{dz} + \gamma_a - \frac{7800}{1+\frac{15500q}{T}}\frac{dq}{dz}\right). \tag{6}$$

* The potential temperature θ is defined as $\theta = T(p_0/p)^{\alpha}$, where $p_0 = 1000\,\mathrm{mb}$ and $\alpha = (C_p - C_v)/C_v = 0.286$. Expanding θ in a series and using the barometric equation, we obtain (3).

Expression (6) should be used in computing the fluctuations in the refractive index n.

The structure function of the atmospheric refractive index can be expressed in the form

$$D_n(z) = \begin{cases} C_n^2 r^{2/3} & \text{for } r \gg \lambda_0, \\ C_n^2 \lambda_0^{2/3} \dfrac{r^2}{\lambda_0^2} & \text{for } r \ll \lambda_0. \end{cases} \tag{7}$$

The coefficient C_n^2 is defined by the relation

$$C_n^2 = a^2 \alpha' \left[\frac{K^2}{\left(\frac{du}{dz}\right)^2} \right]^{1/3} M^2 = a^2 \alpha' L_0^{4/3} M^2. \tag{8}$$

(The constant α' may slightly differ from the previously introduced α.)

The three-dimensional spectral density corresponding to (7) is

$$\Phi_n(\varkappa) = 0.033\, C_n^2 \varkappa^{-11/3} \exp\left(-\frac{\varkappa^2}{\varkappa_m^2}\right), \tag{9}$$

where $\varkappa_m$ is related to λ_0 by the equality $\varkappa_m \lambda_0 = 5.92$.

Note that if temperature variations due to vertical air motions are taken into account, the temperature fluctuations are found to be proportional to $(dT_0/dz + \gamma_a)^2$, and not to $(dT_0/dz)^2$ as in (14.22). Therefore, (14.22) should be replaced by

$$C_T^2 = a^2 \alpha \left[\frac{K^2}{\left(\frac{du}{dz}\right)^2} \right]^{1/3} \left(\frac{dT_0}{dz} + \gamma_a\right)^2. \tag{10}$$

In the lower (surface) layer of the atmosphere the observed temperature gradients are generally much greater than γ_a and the latter can therefore be neglected in (10). In the free atmosphere, however, the temperature gradients are comparable with γ_a and (10) must not be simplified.

This effect is readily allowed for if in all the expressions T_0 is replaced by the potential temperature $H = T_0 + \gamma_a z$. In view of this we never distinguish between T_0 and H in the following.

§ 16. Turbulence in the surface layer of the atmosphere

The surface layer, namely the first tens of meters above the surface of the earth, is structurally different from the free atmosphere. Air motion in this layer is mainly governed by friction against the underlying surface, and friction effects spread through the entire ground layer by turbulent mixing. In the absence of buoyancy forces, the friction force due to the earth is much greater than all the other forces (pressure gradient, coriolis force) in the surface layer. Under steady-state conditions frictional stresses are balanced by Reynolds stresses, which are thus constant within the entire surface layer.

Let us consider air motion over a surface which is, on the average, smooth.

The mean characteristics of motion in this case are only functions of the height z above the earth. If the axis x points in the direction of the mean wind velocity $\boldsymbol{u}$, we obtain $u_x = u(z)$, $u_y = u_z = 0$. A fundamental parameter of motion is the turbulent Reynolds stress per unit mass $\tau_{ik} = \langle v_i' v_k' \rangle$.

Symmetry considerations show that the only nonzero components of this tensor are τ_{11}, τ_{22}, τ_{33} (the mean square fluctuations of the velocity components) and $\tau_{13} = \tau_{31}$. The components $\tau_{12} = \tau_{21}$ and $\tau_{23} = \tau_{32}$ are zero, as the momentum flux is directed vertically. The tensor component $\tau_{13} = \tau_{31}$ — the Reynolds stress which balances the friction stresses — is constant in the surface layer. In place of τ_{13} we introduce a parameter v_* which is defined by the relation

$$\tau_{13} = -v_*^2. \tag{1}$$

Since τ_{ik} has the dimensions of velocity squared, v_* is regarded as some characteristic velocity in the surface layer, called the friction velocity.

The only meaningful scale length in the surface layer is the height above the surface.

The velocity component u_x is only a function of z. Its derivative may depend only on v_* and z.* From dimensional considerations we clearly have

$$\frac{du}{dz} = \frac{v_*}{\varkappa z}, \tag{2}$$

where $\varkappa$ is von Kármán's constant (not to be confused with the wavenumber!).

Using relation (14.17) between τ_{ik} and $\frac{\partial u_i}{\partial x_k}$ and setting $i = 1$ and $k = 3$, we obtain using (1) and (2)

$$-v_*^2 = -K\frac{v_*}{\varkappa z},$$

from which an expression for the coefficient of turbulent exchange is obtained:

$$K = \varkappa v_* z. \tag{3}$$

Let us compute the energy dissipation rate ε. Using (14.21), (2), and (3), we find

$$\varepsilon = \frac{v_*^3}{\varkappa z}. \tag{4}$$

From (14.23) we obtain an expression for the outer scale of turbulence L_0:

$$L_0 = \varkappa z, \tag{5}$$

which is simply proportional to height.

Integration of (2) also gives an expression for $u(z)$:

$$u(z) = \frac{v_*}{\varkappa} \ln \frac{z}{z_0}, \tag{6}$$

where z_0 is an integration constant. The constant z_0 is associated with the size of surface irregularities and therefore, in view of the footnote to equation (2), expression (6) is valid only for $z \gg z_0$. In this case the wind profile in the surface layer is described by a logarithmic function. The parameter z_0 in (6) (called the roughness height of the surface) depends

* This is obviously true only when the height z is large compared to the characteristic size of the surface irregularities.

on the structure of the underlying surface. Using (6) we can find the friction velocity v_* from fairly simple measurements of the mean wind speed at various heights. The numerical constant $\varkappa$ in these calculations is assumed to be approximately equal to 0.4 (see Part C).

In the surface layer ε is a function of height according to (4). Inserting (4) in expression (12.7) for D_{rr}, we obtain in the inertial range

$$D_{rr}(r) = C^2 v_*^2 \frac{r^{2/3}}{(\varkappa z)^{2/3}}. \qquad (7)$$

In this case the fluctuations in velocity difference diminish with height, i.e., $D_{rr} \sim z^{-2/3}$. The inner scale of turbulence $l_0 = \nu^{3/4}\varepsilon^{-1/4}$ is also dependent on height.

Inserting (4) in (7) gives $l_0 \sim z^{1/4}$. The "2/3 law" is valid only for $r \ll L_0$. Since in the surface layer $L_0 = \varkappa z$, (7) is applicable only when the distance r between observation points is small compared to the height. If r is comparable with or greater than z, the velocity difference is no longer an isotropic random variable and D_{rr} depends on the orientation of the vector $\boldsymbol{r}$.

Consider the structure of the temperature field in the surface layer. The previous expressions were obtained for a case when the effect of buoyancy forces on the flow dynamics could be ignored. The temperature lapse rate thus corresponded to neutral equilibrium. As we have established in the previous section, for a vertical adiabatic displacement of a small air volume the temperature varies as $T' = T_0 - \gamma_a \Delta z$. Therefore, in order for such displacements not to produce buoyancy forces, it is necessary for the temperature at the point to which the air parcel has moved also to be $T_0 - \gamma_a \Delta z$. Hence the conclusion that in case of neutral temperature stratification the temperature profile is a linear function of height $T(z) = T_0 - \gamma_a z$. No temperature fluctuations are observed in this case, since the factor $\frac{dT}{dz} + \gamma_a$ entering (15.10) is zero.

Consider a temperature stratification which is nearly neutral. In this case to a first approximation the effect of buoyancy forces on turbulence may be ignored, but $\frac{\partial T_0}{\partial z} + \gamma_a$ is no longer zero and temperature fluctuations develop in the medium.

Let us determine the mean temperature profile for this case. We start with the assumption that the eddy heat flux is constant, which is analogous to the assumption of constant turbulent stresses. (Practically all the heat flux released by the earth is transmitted without losses through the surface layer.) Then by (14.7)

$$-\langle T' v_z' \rangle = K_T \frac{dT_0}{dz} = \text{const} = \alpha \varkappa v_* T_*, \qquad (8)$$

where we have introduced a characteristic of the temperature field T_* with the dimension of temperature.

Relation (8), which is the definition of T_*, contains a coefficient $\alpha = K_T/K$ (α^{-1} is the turbulent Prandtl number). Since in the surface layer the eddy heat flux is independent of height, we have $\alpha \varkappa v_* T_* =$ const. We will take $T_* =$ const, so that $\alpha =$ const (this assumption is not inconsistent with experimental findings).

Inserting in (8) $K_T = \alpha K = \alpha\varkappa v_* z$, we obtain

$$\frac{dT_0}{dz} = \frac{T_*}{z} \tag{9}$$

and

$$T_0(z) = T_* \ln\frac{z}{z_0} + T(z_0). \tag{10}$$

Thus, in case of nearly neutral stratification of the atmosphere in the surface layer, the mean temperature profile and the mean wind velocity profile are logarithmic functions of height. T_* is readily determined from temperature measurements at several heights. Inserting (9) in (15.10), we obtain for the inertial range

$$D_T(r) = a^2\varkappa^{4/3}\alpha\frac{(T_* + \gamma_a z)^2}{z^{2/3}} r^{2/3}. \tag{11}$$

In this case the fluctuations of temperature difference in the surface layer decrease with height as $z^{-2/3}$ ($\gamma_a z$ is generally negligible).

We will now consider the case when the effect of buoyancy forces on the turbulence may not be neglected. In this case the equations of motion contain an additional term taking into account buoyancy forces. This term has the form gT'/T_0, where T' is the temperature fluctuation at the point $\boldsymbol{r}$. We are thus dealing with a new dimensional parameter $\beta = \frac{g}{T_0}$, which, together with v_* and T_*, describes the state of the turbulence. From the parameters v_*, T_* and β only one combination can be formed that gives a scale length (to within a numerical constant)

$$L = \frac{v_*^2}{\alpha\varkappa^2\beta T_*}, \tag{12}$$

which, first introduced by Obukhov and Monin /28–31/, will be used in the following treatment. The coefficient of turbulent exchange K, which in case of neutral stratification was given by $K = \varkappa v_* z$, now is written in the form

$$K = \frac{\varkappa v_* z}{\varphi\left(\frac{z}{L}\right)}, \tag{13}$$

where $\varphi(z/L)$ is a dimensionless function of the dimensionless argument z/L. If the eddy heat flux is zero, which corresponds to a neutral temperature stratification, $T_* = 0$ and $L = \infty$. In this limiting case, relation (13) should reduce to the previous expression (3), so that $\varphi(0) = 1$. For $z \ll |L|$ we have $\varphi \approx 1$; hence it follows that the effect of buoyancy forces may be neglected for $z \ll |L|$. In connection with this the scale L is called the thickness of the dynamic turbulence sublayer.

Inserting in (14.17) $\tau_{13} = -v_*^2$ and using expression (13) for K, we obtain (for $i = 1$, $k = 3$)

$$\frac{du}{dz} = \frac{v_*}{\varkappa z}\varphi\left(\frac{z}{L}\right). \tag{14}$$

Similarly, inserting in (8)

$$K_T = \alpha K = \frac{\alpha\varkappa v_* z}{\varphi},$$

we obtain

$$\frac{dT_0}{dz} = \frac{T_*}{z}\varphi\left(\frac{z}{L}\right). \tag{15}$$

Integrating equations (14) and (15), we find

$$u(z) - u(z_0) = \frac{v_*}{\varkappa}\left[f\left(\frac{z}{L}\right) - f\left(\frac{z_0}{L}\right)\right], \tag{16}$$

$$T(z) - T(z_0) = T_*\left[f\left(\frac{z}{L}\right) - f\left(\frac{z_0}{L}\right)\right], \tag{17}$$

where

$$f(\zeta) = \int^{\zeta}\varphi(\xi)\frac{d\xi}{\xi}. \tag{18}$$

From (16) and (17) it follows that wind velocity and temperature profiles are similar. This is a consequence of our assumption that $\alpha = K_T/K$ is constant. Since $\varphi(\xi) \approx 1$ for $\xi \ll 1$, we have

$$f(\zeta) \approx \ln\zeta + \text{const},$$

i.e., for $z \ll L$ the wind and temperature profiles are logarithmic. For large ζ, $f(\zeta)$ markedly departs from a logarithmic function. Its behavior for $\zeta > 0$ (which corresponds to $L > 0$, i.e., $T_* > 0$) and for $\zeta < 0$ ($T_* < 0$) is entirely different. Thus, for $\zeta \ll -1$ ($T_* < 0$, unstable stratification, convective conditions), $f(\zeta) = C_1\zeta^{-1/3} + \text{const}$. The form of the function $f(\zeta)$ (or of the corresponding function $\varphi(\zeta) = \zeta f'(\zeta)$) was studied experimentally by various authors (see Part C).

Let us now express the parameters ε and N in terms of φ or $f(\zeta)$.

In §14 we used the equations of motion to derive expression (14.21), $\varepsilon = K\left(\frac{du}{dz}\right)^2$, for the rate of dissipation of the turbulence energy. The equations of motion in this derivation did not contain the buoyancy term. This term enters the equation for the z-component and has the form $\beta T'$. In setting up the equation for ε in §14, we multiplied the equation of motion by the velocity and averaged the result. The additional term $+\beta T'$ thus gives a contribution $+\beta\langle T'v_z'\rangle = \beta q_z$ and the dissipation rate ε becomes

$$\varepsilon = -\tau_{ik}\frac{\partial u_i}{\partial x_k} + \beta q_z.$$

Inserting the expressions

$$-\tau_{ik}\frac{\partial u_i}{\partial x_k} = K\left(\frac{du}{dz}\right)^2,$$

$$q_z = -K_T\frac{dT_0}{dz} = -\alpha K\frac{dT_0}{dz},$$

and using (13), (14), and (15), we find

$$\varepsilon = K\left(\frac{du}{dz}\right)^2\left[1 - \alpha\beta\frac{dT_0}{dz} \Big/ \left(\frac{du}{dz}\right)^2\right], \tag{19}$$

or

$$\varepsilon = \frac{v_*^3}{\varkappa z}\varphi\left(\frac{z}{L}\right) - \alpha\beta\varkappa v_* T_* = \frac{v_*^3}{\varkappa z}\left[\varphi\left(\frac{z}{L}\right) - \frac{z}{L}\right]. \tag{19'}$$

The expression for N derived in §14 remains valid in this case too:

$$N = K_T\left(\frac{dT_0}{dz}\right)^2 = \frac{\varkappa\alpha v_* T_*^2}{z}\varphi\left(\frac{z}{L}\right). \tag{20}$$

Using (19) and (20), we can write expressions for the structure functions D_{rr} and D_T in the inertial range:

$$D_{rr}(r) = C^2\varepsilon^{2/3}r^{2/3} = C^2 v_*^2\left[\varphi\left(\frac{z}{L}\right) - \frac{z}{L}\right]^{2/3}\left(\frac{r}{\varkappa z}\right)^{2/3}, \tag{21}$$

$$D_T = a^2\frac{N}{\varepsilon^{1/3}}r^{2/3} = \varkappa^2 a^2\alpha T_*^2\frac{\varphi\left(\frac{z}{L}\right)}{\left[\varphi\left(\frac{z}{L}\right) - \frac{z}{L}\right]^{1/3}}\left(\frac{r}{\varkappa z}\right)^{2/3}. \tag{22}$$

Expressions (21) and (22) differ from the corresponding expressions in the case of neutral stratification in that they contain additional factors which depend on z/L and become unity for $L = \infty$.

An important characteristic of the temperature stratification of the atmosphere is the so-called Richardson number /32/

$$\mathrm{Ri} = \frac{g}{T_0}\frac{\frac{dT_0}{dz} + \gamma_a}{\left(\frac{du}{dz}\right)^2}, \tag{23}$$

which is defined as the ratio of the power expended to overcome buoyancy forces to the power generated by Reynolds stresses. The power to overcome buoyancy forces is (see derivation of (19))

$$\beta q_z = -\beta K_T\left(\frac{\partial T_0}{\partial z} + \gamma_a\right)$$

(the difference between the ordinary and potential temperature is explicitly introduced), and the power in the Reynolds stress term is $-K\left(\frac{du}{dz}\right)^2$. Their ratio (apart from a factor $-\alpha$) is equal to the Richardson number (23). In the case of stable temperature stratification $\mathrm{Ri} > 0$. For unstable stratification (convective conditions) $\mathrm{Ri} < 0$. As Ri increases, the fractional energy needed to overcome the buoyancy forces grows and the turbulence is progressively attenuated. (We recall that by (19) and (23) $\varepsilon = K\left(\frac{du}{dz}\right)^2 \times (1 - \alpha\,\mathrm{Ri})$.) In this case we expect the turbulence characteristics to depend on Ri.

In the surface layer the stratification parameter is related to Ri. Indeed, ignoring γ_a in (23) and inserting (14), (15), and (12), we obtain an expression relating Ri to $\varphi\left(\frac{z}{L}\right)$:

$$\mathrm{Ri}(z) = \frac{z/L}{\alpha\varphi\left(\frac{z}{L}\right)}. \tag{24}$$

Using the relation for a given L we can find the value of Ri at any height z. The parameters Ri and L are thus related by a single-valued expression, and instead of specifying L, we can give Ri for a certain height, from which L is then computed. For the surface layer L is the more convenient parameter, since in this case we are dealing with a simple numerical characteristic of the stratification, and not with the function Ri(z). The use of the Richardson number as a stratification parameter, however, has its advantages; in particular Ri is fairly easily determined not only for the surface layer but also for the free atmosphere, where universal wind and temperature profiles do not exist. Therefore, the experimental profiles obtained for the surface layer can be expressed in terms of Ri and then extrapolated to the free atmosphere.

If the function $\varphi(z/L)$ is known, we can establish a relation between Ri and $\zeta = \frac{z}{L}$. Then $\varphi(z/L)$ will be expressed as a function of Ri, denoted by

$$\varphi\left(\frac{z}{L}\right) = \psi(\mathrm{Ri}). \tag{25}$$

Then, $\frac{z}{L} = \alpha \mathrm{Ri}\cdot\psi(\mathrm{Ri})$. Expressing v_* and $\varkappa T_*$ from (14) and (15) in terms of the derivatives $\frac{du}{dz}, \frac{dT}{dz}$, inserting the results in (21), (22), and putting $L_0 = \varkappa z$, we obtain

$$D_{rr}(r) = C^2 \frac{(1-\alpha\,\mathrm{Ri})^{2/3}}{[\psi(\mathrm{Ri})]^{4/3}} \left(L_0 \frac{du}{dz}\right)^2 \left(\frac{r}{L_0}\right)^{2/3}, \tag{26}$$

$$D_T(r) = a^2\alpha \frac{1}{[\psi(\mathrm{Ri})]^{4/3}(1-\alpha\,\mathrm{Ri})^{1/3}} \left(L_0 \frac{dT_0}{dz}\right)^2 \left(\frac{r}{L_0}\right)^{2/3}, \tag{27}$$

These expressions are written in a convenient form for both the surface layer and the free atmosphere.* The function $\psi(\mathrm{Ri})$ entering (26) and (27) was determined experimentally by various independent methods and is given in §22.

Note that even for relatively small negative values of Ri, the turbulence regime corresponds to free convection conditions (the transition to free convection occurs at Ri < −0.05). In this case

$$\varphi(\zeta) \approx \frac{C_1}{3}(|\zeta|)^{-1/3}$$

and

$$\psi(\mathrm{Ri}) = \alpha^{-1/4}\left(\frac{C_1}{3}\right)^{3/4} |\mathrm{Ri}|^{-1/4}.$$

Substituting this $\varphi(\zeta)$ in (19), we obtain for $|\zeta| \gg 1$ (more precisely, for $\zeta \ll -1$)

$$\varepsilon = \frac{v_*^3}{\varkappa |L|},$$

i.e., for free convection and $z \gg |L|$, ε is independent of height. At the same time

$$N = \frac{\alpha C_1 \varkappa v_* T_*^2 |L|^{1/3}}{3 z^{4/3}},$$

* The applicability of these relations to the free atmosphere naturally requires experimental verification.

i.e., N decreases with height as $z^{4/3}$. In the case of free convection, velocity fluctuations are independent of height, whereas temperature fluctuations decrease with height as $z^{4/3}$, i.e., much faster than for pure mechanical turbulence. In free convection, both ε and N can be expressed in terms of the "temperature flux"

$$q = \langle T' v_z' \rangle = -\alpha \varkappa v_* T_*.$$

Inserting $L = v_*^2 / \alpha \varkappa^2 \beta T_*$ in the expressions for ε and N and expressing $v_* T_*$ in terms of q, we obtain

$$\varepsilon = \beta q, \qquad N = \frac{C_1}{3\varkappa^{4/3}\alpha} \frac{q^{5/3}}{z^{4/3}\beta^{1/3}}. \tag{28}$$

§ 17. The effect of buoyancy forces on the microstructure of the velocity and temperature fields

In the preceding section we considered the effect of buoyancy forces on the characteristics of the mean motion $u(z)$, $T(z)$. These forces, however, influence the microstructure of the wind and temperature fields, as well as their mean parameters. This problem was treated in some detail by A. M. Obukhov /33, 34/, R. Bolgiano /35/, and A. S. Monin /36/.

The microstructure of the velocity and temperature fields in the inertial range in the absence of buoyancy forces was determined by the parameters ε and N. If buoyancy forces are considered, a new parameter $\beta = g/T_0$ is added. The three parameter ε, N, and β can be combined to form quantities with the dimensions of length

$$L_k = \varepsilon^{5/4}\beta^{-3/2}N^{-3/4}, \tag{1}$$

velocity

$$v_k = \varepsilon^{3/4}\beta^{-1/2}N^{-1/4} \tag{2}$$

and temperature

$$T_k = \varepsilon^{1/4}\beta^{-1/2}N^{1/4}. \tag{3}$$

Using these scales, we can obtain an expression for the spectral densities $E(\varkappa)$ and $\Phi_T(\varkappa)$. Since the only nondimensional combination of ε, N, β and $\varkappa$ is $\varkappa L_k$, dimensional considerations lead to the expressions

$$E(\varkappa) = L_k v_k^2 f(\varkappa L_k) = \varepsilon^{11/4} N^{-5/4} \beta^{-5/2} f(\varkappa \varepsilon^{5/4} \beta^{-3/2} N^{-3/4}), \tag{4}$$

$$\Phi_T(\varkappa) = L_k^3 T_k^2 \varphi(\varkappa L_k) = \varepsilon^{17/4} \beta^{-11/2} N^{-7/4} \varphi(\varkappa \varepsilon^{5/4} \beta^{-3/2} N^{-3/4}). \tag{5}$$

As β approaches zero (no buoyancy forces), expressions (4) and (5) should become independent of this parameter. We thus obtain the asymptotic expressions

$$\lim_{x \to \infty} f(x) = C_1 x^{-5/3}, \qquad \lim_{x \to \infty} \varphi(x) = C_2 x^{-11/3},$$

which lead to the previous relations

$$E(\varkappa) = C_1 \varepsilon^{2/3} \varkappa^{-5/3}, \qquad \Phi_T(\varkappa) = C_2 N \varepsilon^{-1/3} \varkappa^{-11/3},$$

applicable in the inertial subrange. From these asymptotic expressions it is clear that for $\varkappa L_k \gg 1$, i.e., for $\varkappa \gg \frac{2\pi}{L_k}$, the spectra of velocity and temperature fluctuations have their previous form. Consequently, the effect of buoyancy forces is felt mainly in the range of fairly large scales, determined by the parameter L_k. The corresponding characteristic velocity and temperature scales are given by (2) and (3).

For wavenumbers in the range $\varkappa L_k \ll 1$, the effect of buoyancy forces is predominant. In the case of stable temperature stratification, the energy of large-scale motions is mainly expended as work against buoyancy forces, and is not transferred to the smaller scale components. This means that in the case of stable temperature stratification, relations (4) and (5) should be independent of ε for $\varkappa L_k \ll 1$. Thus, for $x \ll 1$,

$$f(x) \sim x^{-11/5}, \quad \varphi(x) \sim x^{-17/5}.$$

Consequently, for $\varkappa L_k \ll 1$, Bolgiano's expressions apply /35/:

$$E(\varkappa) = \text{const} \cdot N^{2/5} \beta^{4/5} \varkappa^{-11/5}, \tag{6}$$

$$\Phi_T(\varkappa) = \text{const} \cdot \beta^{-2/5} N^{4/5} \varkappa^{-17/5}. \tag{7}$$

In case of unstable temperature stratification, Monin /36/ showed that for $x \to 0$, $f(x) \sim x^5$ and $\varphi(x) \sim x^{-1}$. The function $f(x)$ has a maximum at $x_f = 1.01$ and $x^2 \varphi(x)$ at $x_\varphi = 0.46$.

Summarizing, we can say that in the range $\varkappa \gg \frac{2\pi}{L_k}$ the spectra of temperature and velocity fluctuations have the same form as without buoyancy forces (i.e., for $r \ll L_k$, the functions D_{rr} and D_T are expressed by the same relations as before). For $\varkappa \lesssim \frac{2\pi}{L_k}$, however, the effect of buoyancy forces is quite substantial (this is the range $r \gtrsim L_k$). In the case of stable temperature stratification, the spectral densities in this wavenumber range are expressed by (6) and (7). The corresponding structure functions are

$$\begin{aligned} D_{rr}(r) &\sim N^{2/5} \beta^{4/5} r^{6/5}, \\ D_T(r) &\sim N^{4/5} \beta^{-2/5} r^{2/5}. \end{aligned} \tag{8}$$

The scale L_k, which characterizes the size of the inhomogeneities markedly influenced by buoyancy forces, can be related to the outer scale of turbulence L_0. For this we use expressions (14.20), (16.19), and (16.23):

$$\begin{aligned} \varepsilon &= K \left(\frac{du}{dz}\right)^2 (1 - \alpha\,\mathrm{Ri}); \\ K &= L_0^2 \frac{du}{dz}; \quad N = K_T \left(\frac{dT_0}{dz}\right)^2. \end{aligned} \tag{9}$$

Substituting these quantities in (1) and using the definition of the Richardson number (16.23), which in our case (when γ_a is neglected) has the form

$$\mathrm{Ri} = \frac{\beta \frac{dT_0}{dz}}{\left(\frac{du}{dz}\right)^2},$$

we obtain

$$L_k = \frac{(1-\alpha\,\mathrm{Ri})^{5/4}}{\alpha^{3/4}\,|\,\mathrm{Ri}\,|^{3/2}} L_0. \quad (10)$$

We see that $L_k \gg L_0$ for $|\,\mathrm{Ri}\,| \ll 1$ and in this case the effect of buoyancy forces is felt only outside the inertial subrange of turbulence. If $|\,\mathrm{Ri}\,|$ is large, the scale L_k falls inside the inertial subrange dividing it into two parts: in the case of stable stratification, relations (8) and (9) apply in the subrange $L_0 \gg r \gg L_k$, and the previous expression for D_{rr} and D_T hold in the subrange $L_k \gg r \gg l_0$.

Monin /36/ computed the spectral densities $E(\varkappa)$ and $\Phi_T(\varkappa)$ for arbitrary values of the parameter $\varkappa L_k$ and various stratification conditions (stable, neutral, and unstable).

C. EXPERIMENTAL DATA ON ATMOSPHERIC TURBULENCE

§ 18. Measurements of the spatial structure functions of wind velocity and temperature in the surface layer of the atmosphere

The first measurements of the microstructure of wind velocity in the atmosphere were carried out by Gödecke /37/, Obukhov /38/, Krechmer /39/, and others /40, 41/. Hot-wire anemometers were used by these authors, which consist of a thin platinum filament (10 – 20 microns in diameter, some 1 cm long) heated by an electric current to a few hundred degrees centigrade. For a constant heating current, the temperature, and consequently the resistance, of the filament are highly sensitive to the velocity of the air stream past the filament. Given a calibration curve of the the hot-wire anemometer, one can find the wind velocity from resistance measurements. The hot-wire anemometer's time constant in air is generally a hundredth of a second (in water the time constant is much less). In measurements of the wind velocity structure function, two anemometers are coupled in opposite arms of a bridge, and velocity differences are measured.

Early measurements of the wind velocity structure functions revealed satisfactory agreement with the "2/3 law." Figure 7 shows some typical structure functions obtained by Obukhov /38/. The abscissa gives $r^{1/3}$ and the ordinate $\frac{\sqrt{D_{rr}(r)}}{\langle v \rangle}$. The "2/3 law" in these coordinates is represented by a straight line (which does not pass through the origin on account of the inertia of the gauge), and we see from the figure that the experimental data are indeed consistent with the "2/3 law." Obukhov /38/ also obtained some estimates of ε for the same times when $D_{rr}(r)$ was measured, using the mean wind velocity at several levels and the logarithmic boundary layer equations of §16. For the constant C^2 appearing in the equation $D_{rr} = C^2\varepsilon^{2/3}r^{2/3}$ he obtained a tentative value of $C^2 = 0.9$, which is somewhat less than the previous figure $C^2 = 1.5$, obtained in wind tunnel tests /42/.

The spatial structure function of the temperature field was measured in /43, 44/. Resistance thermometers with time constants $\tau \approx 0.01$ sec were used as gauges. Numerous measurements of D_T in the surface layer also show good agreement to the "2/3 law." A typical structure function for the temperature field is shown in Figure 8.

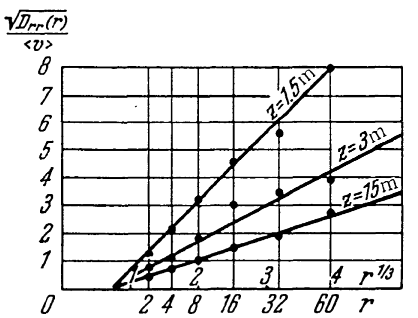

FIGURE 7. Typical empirical curves for the structure functions of the longitudinal velocity component obtained at heights of 1.5, 3, and 15 m:

The linear horizontal scale is that of $\sqrt[3]{r}$, and the linear vertical scale is $\frac{1}{\langle V \rangle}\sqrt{D_{rr}(r)}$.

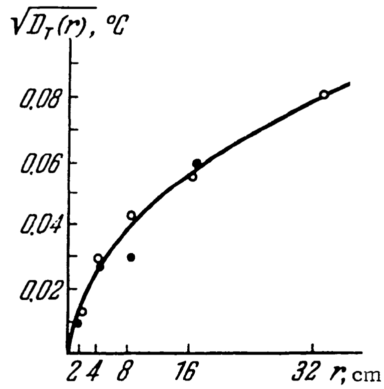

FIGURE 8. Empirical temperature structure function.

The early measurements of $D_{rr}(r)$ and $D_T(r)$ in the atmosphere provided the characteristic values of wind speed and temperature fluctuations in the surface layer. The random velocity difference Δv between two points at the same height z, separated by a distance $r \sim 0.5z$, has the characteristic value $\Delta v \sim v_*$. A crude estimate of v_* can be obtained using the relation $v_* \approx 0.1 u_{z=2\,\mathrm{m}}$, where $u_{z=2\,\mathrm{m}}$ is the mean wind speed at a height of 2 m. The characteristic difference Δv is thus of the order of a few tens of cm/sec. The analogous difference ΔT for the temperature field is highly sensitive to the temperature stratification (lapse rate) and may even reach 1 °C. The characteristic value of ε in the surface layer is a few hundreds of CGS units, and the inner scale $l_0 = \sqrt[4]{\frac{\nu^3}{\varepsilon}}$ is of the order of 1 mm. The outer scale of turbulence L_0 in the surface layer is $\varkappa z$, as we have noted before, where $\varkappa \approx 0.4$. The ratio L_0/l_0 characterizing the width of the inertial subrange (the "dynamic range" of turbulence) is thus of the order of $10^3 - 10^4$ in the surface layer.

§ 19. Equipment for measuring turbulent fluctuations of wind velocity and temperature in the atmosphere

The estimates of the preceding section throw some light on the fundamental requirements to be met by the equipment for measuring wind velocity and temperature fluctuations in the entire range of scales, starting with l_0. The accuracy and the measuring sensitivity of the velocity measurements should be of the order of a few centimeters per second (this is the order of magnitude of $v_0 = \sqrt[4]{\varepsilon\nu}$). The size of the probe, of course, should not be larger than the inner scale of turbulence, $l_0 \approx 1$ mm. The highest frequencies in the wind fluctuations are of the order of $\bar{u}/l_0$, where $\bar{u}$ is the mean wind velocity. Thus, for $\bar{u} = 5$ m/sec and $l_0 = 1$ mm, we obtain $f_0 \sim 5000$ Hz. The time constant of the instruments should thus be of the order of $f_0^{-1} = 2 \cdot 10^{-4}$ sec, and the pass band of the instrument of the order of 5 kHz. The temperature accuracy should be of the order of 10^{-2} deg.

If equipment meeting all these requirements were available, we could investigate the microstructure of turbulence down to its inner scale. Suitable equipment was designed only recently /22, 174/ and the measurements of turbulent spectra in the inertial and the viscous ranges gave exceedingly interesting results which provided direct confirmation of the Kolmogorov—Obukhov theory (see §12). The majority of the experimental data available, however, were obtained with relatively crude equipment suitable for measurements in the inertial subrange only.

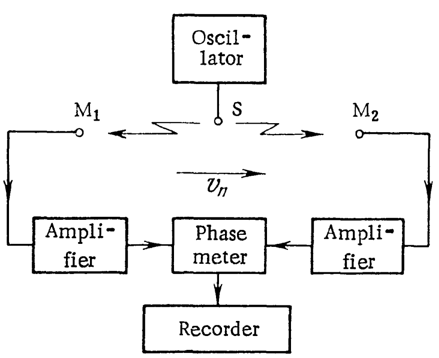

FIGURE 9. A block diagram of an acoustic anemometer.

An acoustic micro-anemometer was used in /45—49/ to measure the wind velocity components. Its principle of operation is based on the dependence of the velocity of sound propagation in a moving medium on the velocity of the propagating medium. Let c_0 be the velocity of sound in a static medium. The phase velocity of sound in the direction $\boldsymbol{n}$ is then $c = c_0 + \boldsymbol{nv}$, where $\boldsymbol{v}$ is the velocity of the medium relative to a stationary sound source. Let the sound source S (Figure 9) emit sound waves of frequency ω which are picked up by microphones M_1 and M_2. The velocity of sound over the distance SM_1 is $c_1 = c_0 - v_n$, and that over the distance SM_2 is $c_2 = c_0 + v_n$, where $v_n = \boldsymbol{nv}$. The phase change over the path $SM_1 = l$ is $\omega l/c_1$, and that over the path $SM_2 = l$ is $\omega l/c_2$. The phase difference of the sound wave reachings points M_1 and M_2 is thus

$$\varphi = \omega l\left(\frac{1}{c_1} - \frac{1}{c_2}\right) = \frac{2\omega l}{c_0^2}\left(1 - \frac{v_n^2}{c_0^2}\right)^{-1} v_n.$$

Since $v_n^2 \ll c_0^2$, we have with reasonable accuracy

$$\varphi = \frac{2\omega l}{c_0^2} v_n,$$

i.e., the phase difference of the oscillations picked up by microphones M_1 and M_2 is proportional to the wind velocity component along the base M_1M_2. An acoustic micro-anemometer is thus a linear instrument and is favorably different in this respect from the nonlinear hot-wire anemometer. In some acoustic anemometers (see /45/), the base $2l$ was about 2 cm long. The time constant of an acoustic anemometer is l/c_0, and the resulting instrumental averaging is negligible compared to the averaging over the base $2l$. Numerous measurements of the wind velocity microstructure with acoustic anemometers were carried out in /47, 49/.

Reduction of the results of measurements on the microstructure of turbulence involve lengthy calculations. Automatic equipment, such as multichannel frequency analyzers, correlators, etc., have recently found wide use in analyzing data /50/.

In connection with this, it is considerably more convenient to investigate the time microstructure of turbulence at one point or at a few points in space instead of the spatial structure. Application of new experimental techniques resulted in rapid acquisition and processing of experimental data corresponding to hundreds of observation hours.

§ 20. The relation between time and spatial structure of turbulence (the "frozen turbulence" hypothesis)

In Part A, in discussing the space-time spectra, we advanced G. Taylor's "frozen turbulence" hypothesis, according to which the entire spatial pattern of a random field $f(\boldsymbol{r})$ is transported with the mean wind velocity $\boldsymbol{u}$:

$$f(\boldsymbol{r},\ t+t') = f(\boldsymbol{r}-\boldsymbol{u}t',\ t).$$

Hence we derived a relation between the space-time structure functions $D(\boldsymbol{r}, \tau)$ and the pure spatial functions $D(\boldsymbol{r})$:

$$D(\boldsymbol{r},\ \tau) = D(\boldsymbol{r}-\boldsymbol{u}\tau). \tag{1}$$

We also derived an expression relating the time (frequency) spectrum of a random scalar field $W(\omega)$, defined by the equality

$$B(\tau) = 2\int_0^\infty W(\omega)\cos\omega\tau\, d\omega,$$

to its spatial spectral density $\Phi(\boldsymbol{\varkappa})$. For a locally isotropic field, when $\Phi(\boldsymbol{\varkappa}) = \Phi(\varkappa)$, this relation was expressed by

$$W(\omega) = \frac{2\pi}{v}\int_{|\omega|/v}^\infty \Phi(\varkappa)\varkappa\, d\varkappa \tag{2}$$

or

$$\Phi(\varkappa) = -\frac{v^2}{2\pi\varkappa} W'(\varkappa v). \tag{3}$$

If $\Phi(\varkappa)$ corresponds to the "2/3 law," i.e., has the form

$$\Phi(\varkappa) = A\,\varkappa^{-11/3}, \tag{4}$$

the frequency spectrum $W(\omega)$ is expressed by

$$W(\omega)=\frac{6\pi}{5}Av^{2/3}|\omega|^{-5/3}, \tag{5}$$

i.e., $W(\omega)$ is proportional to $\omega^{-5/3}$.

Comparing (3) with expression (4.13), i.e., $\Phi(\varkappa) = -(2\pi\varkappa)^{-1}V'(\varkappa)$, we readily obtain the relation

$$V(\varkappa) = vW(\varkappa v) \tag{6}$$

between the one-dimensional spatial spectrum $V(\varkappa)$ of a locally isotropic field and its time (frequency) spectrum $W(\omega)$ in the case of a "frozen" field.

Let us try to establish the conditions when Taylor's "frozen" field hypothesis is justified. Consider the motion of a velocity inhomogeneity of a characteristic size l. The time for this inhomogeneity to pass by the observation point is of the order of $T = l/\bar{u}$, where $\bar{u}$ is the mean wind velocity. On the other hand, the "intrinsic lifetime" of the inhomogeneity during which it appreciably changes (evolves) is of the order of

$$\tau \sim \frac{l}{v_l} \approx \frac{l}{(\varepsilon l)^{1/3}} = \frac{l^{2/3}}{\varepsilon^{1/3}}.$$

Clearly, the evolution of the inhomogeneity during the time it passes near the observation point is negligible only if $\tau \gg T$, i.e., $l^{2/3}/\varepsilon^{1/3} \gg l\bar{u}^{-1}$, hence

$$(\varepsilon l)^{1/3} \ll \bar{u}.$$

The last condition is always satisfied if $l \ll L_0$, where L_0 is the outer scale of turbulence. Indeed, $u \sim (\varepsilon L_0)^{1/3}$, which thus leads to the condition $l \ll L_0$. In this case the field cannot be regarded as "frozen" only for the large scales outside the inertial subrange. In the inertial and viscous ranges, on the other hand, Taylor's hypothesis is adequately satisfied.

Experimental verification of Taylor's "frozen field" hypothesis was carried out in /51, 52/. In /52/, the frequency spectra of temperature fluctuations at a static point at a height of 70 m (top of a tower) were measured simultaneously with the frequency spectra from an aircraft cruising at the same altitude (70 m) with a speed of v_c. Since the aircraft velocity is very large compared to the wind velocity, it crosses the inhomogeneities almost instantaneously, and the frequency spectrum $W_c(\omega)$ (after conversion to the spatial wavenumber $\varkappa = \frac{\omega}{v_c}$) virtually corresponds to the one-dimensional spatial spectrum of the random field. $V(\varkappa)$ can be computed from $W_c(\omega)$ using a relation similar to (6):

$$V(\varkappa) = v_c W_c(\varkappa v_c).$$

The frequency spectrum $W(\omega)$ measured at the static point also gives $V(\varkappa)$, using Taylor's hypothesis:

$$V(\varkappa) = \bar{u}W(\varkappa\bar{u}).$$

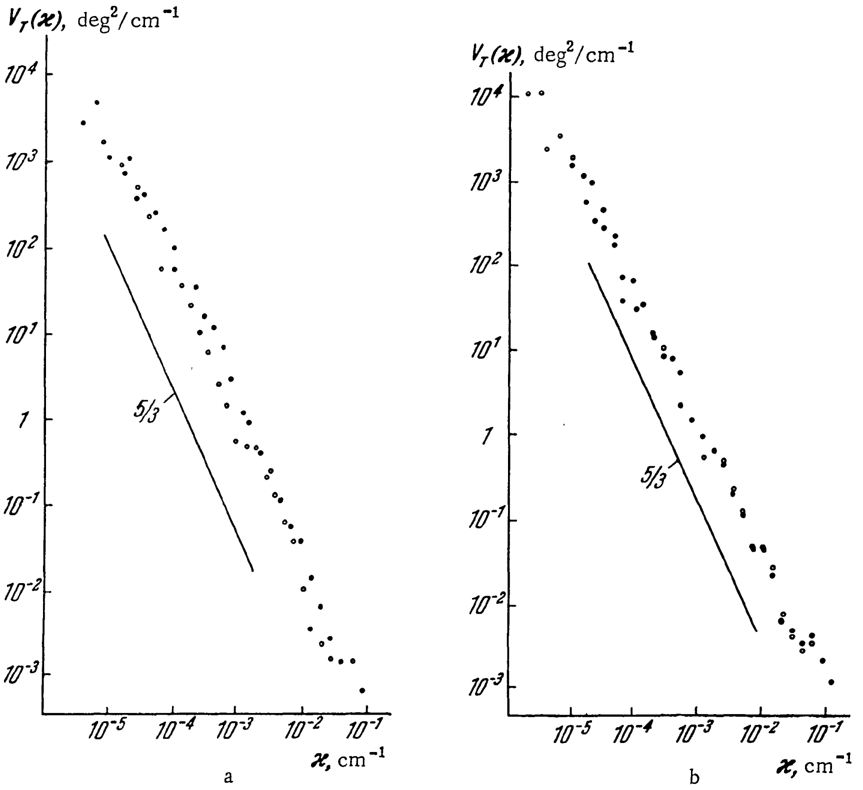

FIGURE 10. Comparison of "space" and "time" (frequency) spectra of temperature fluctuations:

The "spatial" spectra were obtained by measuring the time frequency spectrum with a rapidly moving instrument in an aircraft. The "time" spectra were obtained using a static temperature probe installed at a height of 70 m on the top of a tower. Aircraft velocities were 88 m/sec (a) and 55 m/sec (b), wind speed 9 m/sec. Dark circles correspond to the "time" spectrum, light circles to the "spatial" spectrum.

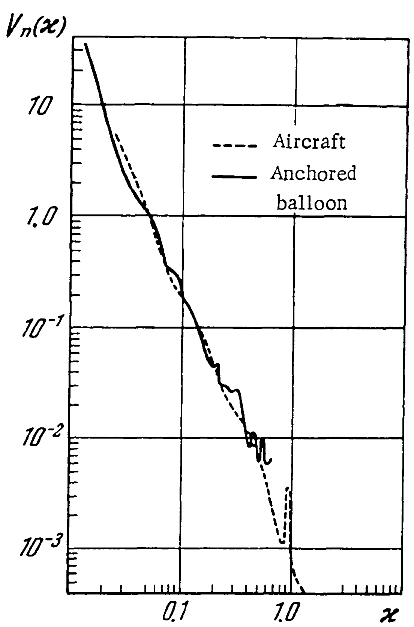

FIGURE 11. Comparison of "space" and "time" spectra of the refractive index.

If Taylor's hypothesis is applicable, the two expressions for $V(\varkappa)$ should coincide. Figure 10 presents two sets of data, which show that in the wavenumber range up to $10^{-5}\,\mathrm{cm}^{-1}$ the "frozen field" hypothesis is perfectly justified. The range of wavenumbers in which this hypothesis is experimentally confirmed is even much greater than what could be expected from the previous estimate $l \ll L_0$. Similar results were obtained in /51/ for the refractive index field (Figure 11).

Direct verification of Taylor's hypothesis for the velocity field was not attempted, but there is a wealth of indirect evidence supporting this hypothesis, obtained by comparisons with experimental data, assuming its validity.

§ 21. Measurement of the velocity spectrum in the inertial and viscous ranges

The frequency spectra of the longitudinal velocity component in the sea were studied in /22/.* A hot-wire anemometer was used, with a sensor measuring less than 0.5 mm. The probe was small enough to study both the inertial and the viscous wavenumber range.

Let us establish a relation between the spectrum of the longitudinal velocity component and the function $E(\varkappa)$. Using relation (9.3)

$$\Phi_{ik}(\varkappa) = \frac{1}{4\pi\varkappa^2}\left(\delta_{ik} - \frac{\varkappa_i\varkappa_k}{\varkappa^2}\right)E(\varkappa),$$

we write $D_{ik}(\boldsymbol{r})$ in the form

$$D_{ik}(\boldsymbol{r}) = 2\iiint\limits_{-\infty}^{\infty}(1 - \cos\boldsymbol{\varkappa r})\left(\delta_{ik} - \frac{\varkappa_i\varkappa_k}{\varkappa^2}\right)\frac{E(\varkappa)}{4\pi\varkappa^2}\,d^3\varkappa.$$

The x-axis is chosen along the vector $\boldsymbol{r}$, so that $\boldsymbol{r} = \{r,\ 0,\ 0\}$ and $D_{11}(\boldsymbol{r}) = D_{rr}(\boldsymbol{r})$. Then for $i = k = 1$ we have

$$D_{rr}(\boldsymbol{r}) = 2\iiint\limits_{-\infty}^{\infty}(1 - \cos\varkappa_1 r)\left(1 - \frac{\varkappa_1^2}{\varkappa^2}\right)\frac{E(\varkappa)}{4\pi\varkappa^2}\,d^3\varkappa =$$

$$= 2\int\limits_{-\infty}^{\infty}[1 - \cos\varkappa_1 r]\,d\varkappa_1 \iint\limits_{-\infty}^{\infty}(\varkappa_2^2 + \varkappa_3^2)\,\frac{E\left(\sqrt{\varkappa_1^2 + \varkappa_2^2 + \varkappa_3^2}\right)}{4\pi(\varkappa_1^2 + \varkappa_2^2 + \varkappa_3^2)^2}\,d\varkappa_2\,d\varkappa_3.$$

This expression has the form of a one-dimensional spectral expansion

$$D_{rr}(r) = 2\int\limits_{-\infty}^{\infty}(1 - \cos\varkappa_1 r)\,V_{rr}(\varkappa_1)\,d\varkappa_1,$$

where

$$V_{rr}(\varkappa_1) = \frac{1}{4\pi}\iint\limits_{-\infty}^{\infty}\frac{\varkappa_2^2 + \varkappa_3^2}{(\varkappa_1^2 + \varkappa_2^2 + \varkappa_3^2)^2}\,E\left(\sqrt{\varkappa_1^2 + \varkappa_2^2 + \varkappa_3^2}\right)d\varkappa_2\,d\varkappa_3.$$

* Similar tests were later carried out in the atmosphere (see /174, 183/). The results of these studies are in excellent agreement with the findings of /22/.

Substitution of the variables $\varkappa_2 = \lambda \cos \varphi$, $\varkappa_3 = \lambda \sin \varphi$ and integration over φ give

$$V_{rr}(\varkappa_1) = \frac{1}{2}\int_0^\infty \frac{E(\sqrt{\lambda^2+\varkappa_1^2})}{(\lambda^2+\varkappa_1^2)^2}\lambda^3 d\lambda,$$

and putting $\lambda^2 + \varkappa_1^2 = \varkappa'^2$ we obtain

$$V_{rr}(\varkappa_1) = \frac{1}{2}\int_{\varkappa_1}^\infty \frac{\varkappa'^2-\varkappa_1^2}{\varkappa'^3} E(\varkappa')\,d\varkappa'. \tag{1}$$

This relation expresses the one-dimensional spectral density $V_{rr}(\varkappa_1)$ in terms of $E(\varkappa)$. Differentiating, we obtain

$$\frac{dV_{rr}(\varkappa_1)}{d\varkappa_1} = -\varkappa_1\int_{\varkappa_1}^\infty \frac{E(\varkappa')}{\varkappa'^3}\,d\varkappa',$$

multiplying both sides by $\frac{1}{\varkappa_1}$ and again differentiating gives

$$\frac{d}{d\varkappa_1}\left[\frac{1}{\varkappa_1}\frac{dV_{rr}(\varkappa_1)}{d\varkappa_1}\right] = \frac{E(\varkappa_1)}{\varkappa_1^3}.$$

Consequently, if the function $V_{rr}(\varkappa)$ is known, $E(\varkappa)$ can be found from the relation

$$E(\varkappa) = \varkappa^3\frac{d}{d\varkappa}\left[\frac{1}{\varkappa}\frac{dV_{rr}(\varkappa)}{d\varkappa}\right]. \tag{2}$$

The rate of energy dissipation

$$\varepsilon = 2\nu\int_0^\infty \varkappa^2 E(\varkappa)\,d\varkappa$$

can be expressed directly in terms of the function $V_{rr}(\varkappa)$. Inserting the expression for $E(\varkappa)$ and integrating by parts twice, we obtain

$$\varepsilon = 30\nu\int_0^\infty \varkappa^2 V_{rr}(\varkappa)\,d\varkappa. \tag{3}$$

In /22/, seventeen measurements of the spectral function $V_{rr}(\varkappa)$ were carried out under various conditions (these authors used the relation $\varphi(\varkappa) = 2V_{rr}(\varkappa)$). Figure 12 plots on a log-log scale one of these spectra. The top left part of the graph is a straight line with a slope of −5/3, in accordance with the Kolmogorov–Okukhov theory. The marked deviation from linearity in the range of high wavenumbers is attributable to viscosity effects. Using the results of measurements for $V_{rr}(\varkappa)$, we can compute the energy dissipation ε. Figure 13 plots on a semilogarithmic scale the functions $\varkappa V_{rr}(\varkappa)$ and $\varkappa^3 V_{rr}(\varkappa)$. In the particular coordinates of the figure the area under the first curve is proportional to the energy of turbulence

$$T \sim \int_0^\infty V_{rr}(\varkappa)\,d\varkappa = \int_0^\infty \varkappa V_{rr}\,d\ln\varkappa,$$

and the area under the second curve is proportional to ε,

$$\varepsilon \sim \int_0^\infty \varkappa^2 V_{rr}(\varkappa)\, d\varkappa = \int_0^\infty \varkappa^3 V_{rr}(\varkappa)\, d\ln\varkappa.$$

The curves in Figure 13 clearly illustrate the separation of the energy-containing and the viscous ranges and the existence of a finite inertial subrange.

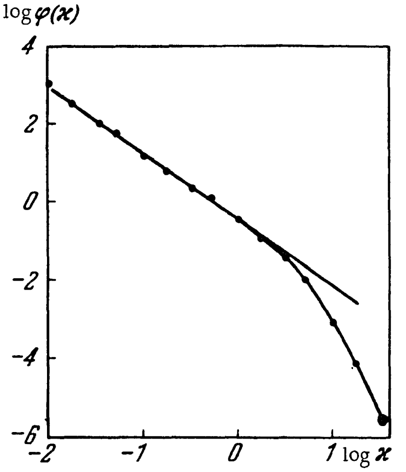

FIGURE 12. An empirical spectral density of the longitudinal velocity component in the inertial and viscous ranges $\varphi(\varkappa) = 2V_{rr}(\varkappa)$:

The straight line corresponds to the power law $\varkappa^{-5/3}$.

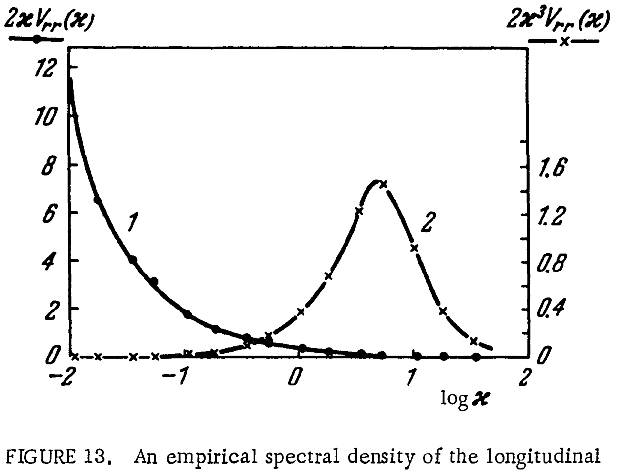

FIGURE 13. An empirical spectral density of the longitudinal velocity component (1) and the spectral density of the energy dissipation rate (2):

The abscissa is the log of the wavenumber; the ordinate gives $2\varkappa V_{rr}(\varkappa)$ (curve 1) and $2\varkappa^3 V_{rr}(\varkappa)$ (curve 2). In these coordinates the areas under curves 1 and 2 are proportional to the total turbulence energy (curve 1) and the rate of energy dissipation (curve 2). The curves illustrate the separation of the energy-containing and viscous ranges.

According to Kolmogorov's theory of turbulence, the function $V_{rr}(\varkappa)$ in the inertial and viscous ranges can be written in the form

$$V_{rr}(\varkappa) = (\varepsilon\nu^5)^{1/4} F(\varkappa l_0),$$

where $l_0 = (\nu^3\varepsilon^{-1})^{1/4}$ and $F(x)$ is a nondimensional function of a nondimensional argument. Thus, if for each measured spectrum $V_{rr}(\varkappa)$, we calculate ε from (3) and then plot a graph of $V_{rr}(\varkappa)/(\varepsilon\nu^5)^{1/4}$ versus $\varkappa l_0 = x$, the resulting plot will present us with the universal function $F(x)$. In Figure 14 all the 17 spectra of /22/ are assembled in log $V_{rr}(\varkappa)/(\varepsilon\nu^5)^{1/4}$ vs. log $\varkappa l_0$ coordinates. There is excellent agreement between all the data obtained for widely differing ε values, ranging from 0.0015 to 1.2 cm^2/sec^3. The curve of Figure 14 is a direct experimental confirmation of Kolmogorov's turbulence theory.*

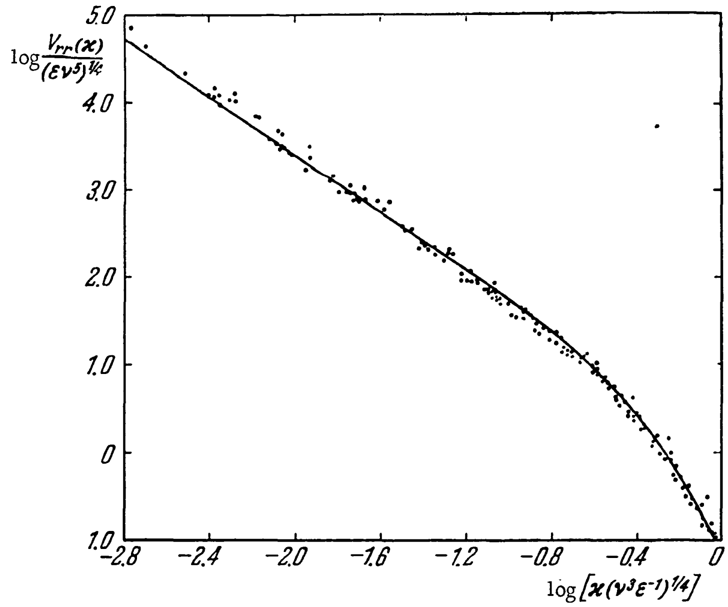

FIGURE 14. The spatial spectral density $V_{rr}(\varkappa)$ of the longitudinal velocity component in nondimensional coordinates:

The abscissa gives the logarithm of the wavenumber times the Kolmogorov scale length. The ordinate gives the logarithm of the spectral density normalized to $(\varepsilon\nu^5)^{1/4}$. The parameter ε needed for normalization was found for each of the 17 spectra assembled in this plot from the area under curve 2 in Figure 13. The data in this graph are a direct experimental confirmation of the universal character of the turbulence spectrum in the inertial and viscous ranges.

For $\varkappa \ll \varkappa_m \approx l_0^{-1}$ the measured spectra $V_{rr}(\varkappa)$ are adequately approximated by the relation

$$V_{rr}(\varkappa) = \frac{9}{55} A\varepsilon^{2/3}\varkappa^{-5/3}$$

* It also confirms Taylor's "frozen field" hypothesis, used to recover the spatial spectrum from the time spectrum.

(the factor 9/55 is introduced for the sake of convenience). Since ε is known from measurements of $V_{rr}(\varkappa)$ in the dissipation range, we can find the constant A. The values of A, as determined from each of the 17 spectra, proved to cluster quite closely. The average value of A was 1.35 ± 0.06. By (2) the function $V_{rr}(\varkappa) = \frac{9}{55} A \varepsilon^{2/3} \varkappa^{-5/3}$ corresponds to

$$E(\varkappa) = A \varepsilon^{2/3} \varkappa^{-5/3}.$$

For the structure function D_{rr} corresponding to this spectrum we find, as indicated before,

$$D_{rr}(r) = C^2 \varepsilon^{2/3} r^{2/3},$$

and C^2 is related to A by the equality

$$A = \frac{55\sqrt{3}}{36\pi} \Gamma\left(\frac{5}{3}\right) C^2$$

(see Part B). For C^2 we thus obtain

$$C^2 = 1.77 \pm 0.08.$$

The experimental data for all wavenumbers, including the viscous range, are adequately fitted by the approximate formula

$$E(\varkappa) = A \varepsilon^{2/3} \varkappa^{-5/3} e^{-\sqrt{\alpha l_0 \varkappa}}, \tag{4}$$

where α and A are related by the expression

$$A = \frac{3\alpha^{4/3}}{20\Gamma\left(\frac{5}{3}\right)},$$

which follows from the condition

$$\varepsilon = 2\nu \int_0^\infty E(\varkappa) \varkappa^2 \, d\varkappa.$$

The solid curve in Figure 14 was plotted from (1) using (4). We see from the graph that the spectral density given by (4) is in good agreement with the experimental findings for all wavenumbers.

The constant α corresponding to $A = 1.35 \pm 0.06$ is $\alpha = 4.78 \pm 0.17$. This constant can also be determined from the shape of the spectrum in the dissipation range by comparing the experimental data with the approximation for $V_{rr}(\varkappa)$ based on (4). This procedure gives $\alpha = 4.36$, which is close to the figure obtained for the inertial range. For $\alpha = 4.36$ we have

$$C^2 = 1.55.$$

§ 22. The microstructure of wind velocity and temperature fields in the surface layer of the atmosphere

Numerous measurements of the fundamental characteristics of turbulence were carried out in the surface layer of the atmosphere /47–49, 53–55/.

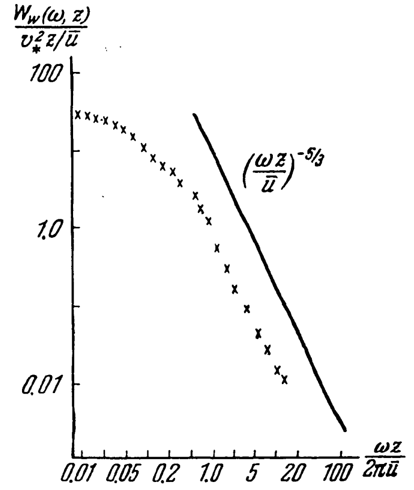

FIGURE 15. Empirical frequency spectrum of the vertical wind velocity component in the surface layer of the atmosphere in dimensionless log-log coordinates.

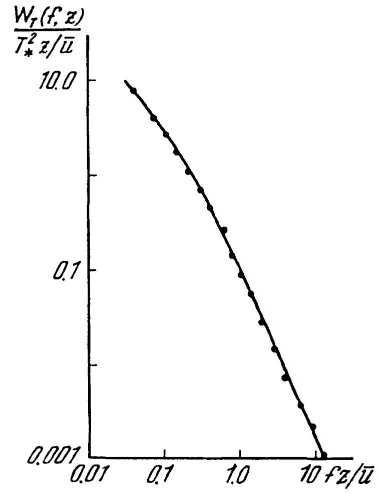

FIGURE 16. Empirical frequency spectrum of temperature fluctuations in the surface layer in dimensionless log-log coordinates.

A typical structure function of the velocity field was shown in Figure 7. Figure 15 plots in dimensionless coordinates the frequency spectrum of fluctuations in the vertical wind component, obtained in /49/ by averaging over a large volume of measurements under identical conditions (the Richardson number was close to zero, so that deviations from a neutral temperature stratification were insignificant). The spectrum, shown in semilogarithmic coordinates, has a linear section corresponding to a power law $E(\varkappa) \sim \varkappa^{-5/3}$ (the experimental exponent is -1.64, which is in satisfactory agreement with the theoretical value $-5/3 = -1.67$ and Taylor's hypothesis). In the large-scale region, corresponding to the outer scale of turbulence $L = \varkappa$, the spectrum departs from the power law (in the case of neutral stratification, the spectral density has a maximum near $\varkappa/2\pi \approx 0.06\, z^{-1}$).

Figure 16 plots the frequency spectrum of temperature fluctuations from /53/, also averaged over numerous measurements. Like the frequency spectrum of velocity fluctuations, this spectrum also has a linear section corresponding to a dependence of the form $f^{-5/3}$. The average value of the exponent obtained in these experiments is -1.67, which is in agreement with Obukhov's theory and Taylor's hypothesis of "frozen" turbulence.

The dependence of the intensity of fluctuations on external conditions was studied in /49, 54/. According to (16.26) and (16.27),

$$D_{rr}(r) = C^2 \frac{(1-\alpha\,\mathrm{Ri})^{2/3}}{[\psi(\mathrm{Ri})]^{4/3}} \left(L_0 \frac{du}{dz}\right)^2 \left(\frac{r}{L_0}\right)^{2/3},$$

$$D_T(r) = \alpha a^2 [\psi(\mathrm{Ri})]^{-4/3} (1-\alpha\,\mathrm{Ri})^{-1/3} \left(L_0 \frac{dT_0}{dz}\right)^2 \left(\frac{r}{L_0}\right)^{2/3}.$$

Figure 17 plots the function $f_1(\mathrm{Ri}) = (1-\alpha\,\mathrm{Ri})^{2/3}[\psi(\mathrm{Ri})]^{-4/3}$ obtained in /49/ from a comparison of the observed vertical wind velocity fluctuations with the gradients of u and T_0. The constant C^2, entering the expression for D_{rr}, was found to be 0.83 according to /49/ (in /49/, $(4/3)\; C^2 = 1.1$).

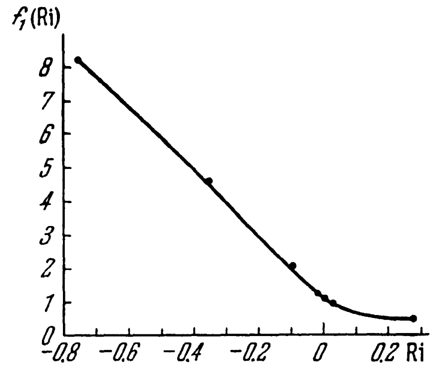

FIGURE 17. The function f_1 (Ri) characterizing the effect of temperature stratification on velocity fluctuations.

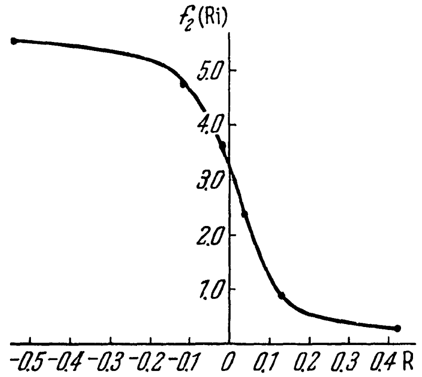

FIGURE 18. The function f_2 (Ri) characterizing the effect of temperature stratification on temperature fluctuations.

Figure 18 plots an analogous function

$$f_2(\mathrm{Ri}) = [\psi(\mathrm{Ri})]^{-4/3}(1-\alpha\,\mathrm{Ri})^{-1/3},$$

derived in /54/ from temperature fluctuation measurements. The constant αa^2, entering the expression for $D_T(r)$, was found to be $\alpha a^2 = 2.8$ /185/.*

For negative Ri < −0.05, the function ψ (Ri) is very close to the asymptote, shown by Priestley /56/ for the free convection regime (§ 16),

$$\psi(\mathrm{Ri}) = \alpha^{-1/4}\left(\frac{C_1}{3}\right)^{3/4}|\mathrm{Ri}|^{-1/4};$$

the numerical coefficient in this relation is found to be 0.37, thus:

$$\psi(\mathrm{Ri}) \approx 0.37\,|\mathrm{Ri}|^{-1/4}.$$

Let us note the work of Gurvich /48/, who measured the skewness $S = D_{rrr}/(D_{rr})^{3/2}$. For Reynolds numbers of the order of $10^5 - 10^6$ the skewness was found to be −0.42 and was independent of r within the margin of experimental error (the measurements were carried out for r = 25 cm and 50 cm).

From the measured skewness S we can compute the constant C^2 which is related to S by the expression $C^3 = \frac{4}{5}|S|^{-1}$ (see (12.11)). This gives $C^2 = 1.54$, which is close to the value obtained in turbulence measurements in the ocean. Also note that Townsend /42/ gives $C^2 = 1.60$. A detailed analysis of the results of numerous velocity field measurements will be found in Monin and Yaglom's book /185/. On the basis of this analysis the authors recommend as the most reliable value $C^2 = 1.9$, to an accuracy of about 10%. We will in fact use this figure in our numerical estimates.

§ 23. The structure of turbulence in the lower troposphere

Wind velocity, temperature, and refractive index fluctuations have been measured in recent years from tall masts /58/, anchored balloons /51/, and aircraft /51, 59, 60/. Refractive index fluctuations can also be

* Some data point to a dependence of α on Ri, which should lead to a lack of similarity of the $u(z)$ and $T_0(z)$ profiles. Observations indeed show a certain deviation from similarity in these profiles. Swinbank /57/ gives the dependence $\alpha = \alpha(\mathrm{Ri})$.

inferred indirectly from data on the propagation of radio waves and light in the atmosphere (the corresponding estimates will be found in Chapters 2 and 4).

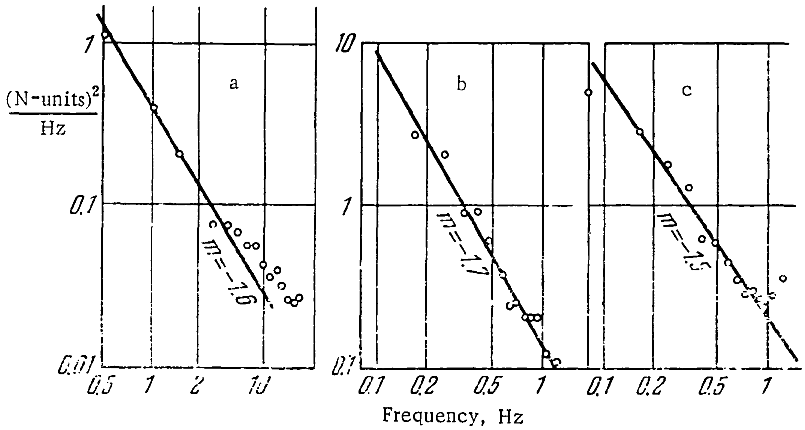

FIGURE 19. Examples of empirical frequency spectra of refractive-index fluctuations in the troposphere in log-log coordinates:

The straight lines correspond to a power law of the form f^{-m}; the exponent is marked in each case.

Figures 19 — 22 show some spectra of refractive index fluctuations obtained in /35, 61/ (aircraft data). These spectra are in good agreement with the theoretical law $W(\omega) \sim \omega^{-5/3}$. Figure 23 shows in relative units the one-dimensional spatial spectra of temperature fluctuations obtained by Tsvang /59/ from aircraft measurements. In the small-scale region, the $V(\varkappa) \sim \varkappa^{-5/3}$ law fits the experimental findings. In the large-scale region some deviations from this law are observed; these deviations are clearly dependent on the measurement height. Qualitatively these deviations are accounted for in a theory which introduces the effect of buoyancy on the spectrum of temperature fluctuations.

Figure 24 shows the spectra of the fluctuations of the vertical wind velocity component obtained in /60/. Here, as in the preceding cases, the results fit the $E(\varkappa) \sim \varkappa^{-5/3}$ law.

The vertical variation of the intensity of the fluctuations is of particular importance in practice. This dependence was studied in /59, 60/. Before proceeding with an analysis of the experimental data, we should establish the vertical variation of fluctuations in the surface layer. In the case of neutral temperature stratification $D_{rr}(r)$ and $D_T(r)$ vary with height as $z^{-2/3}$, while ε and N are proportional to z^{-1}. Therefore in this case we can expect a marked reduction in the fluctuations with height. In the case of an unstable temperature stratification, on the other hand, free convection sets in in practice already for Ri $\leqslant -0.05$. (Since the absolute value of the Richardson number in the surface layer increases monotonically with height, free convection is actually established at a relatively small height in case of unstable stratification.) In this case (see §16), ε is independent of height and $N \sim z^{-4/3}$, i.e., velocity fluctuations are constant with height and temperature fluctuations diminish as $z^{-4/3}$. The vertical dependence of the fluctuations is thus extremely sensitive to the temperature stratification.

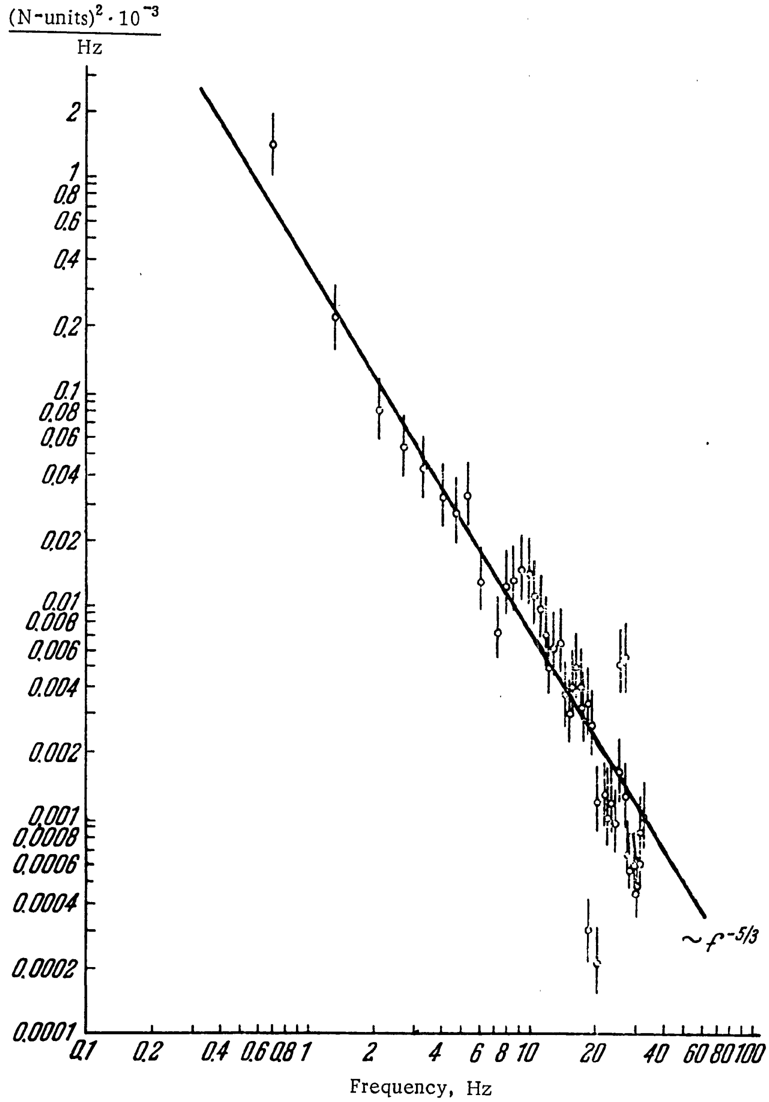

FIGURE 20. An example of an empirical frequency spectrum of refractive index fluctuations in the troposphere in log-log coordinates.

Figure 25 shows the vertical dependence of the coefficient $C_T^2 = a^2 N \varepsilon^{-1/3}$ entering the expression $D_T(z) = C_T^2 r^{2/3}$ obtained in /59/. C_T^2 was determined from the spectra of the temperature fluctuations.

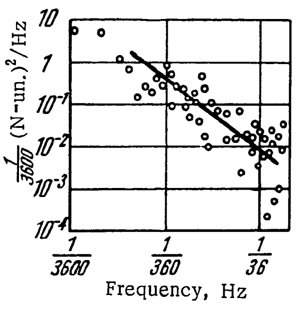

FIGURE 21. An empirical frequency spectrum of the refractive index in the surface layer of the atmosphere, $v = 18$ m/sec:

The straight line is the $f^{-5/3}$ law.

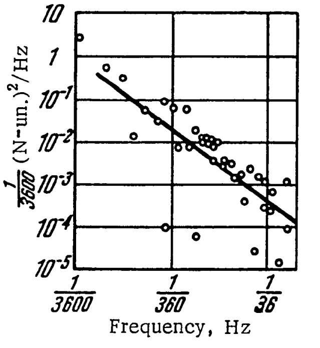

FIGURE 22. An empirical frequency spectrum of the refractive index in the surface layer, $v = 1.2$ m/sec:

The straight line is the $f^{-5/3}$ law.

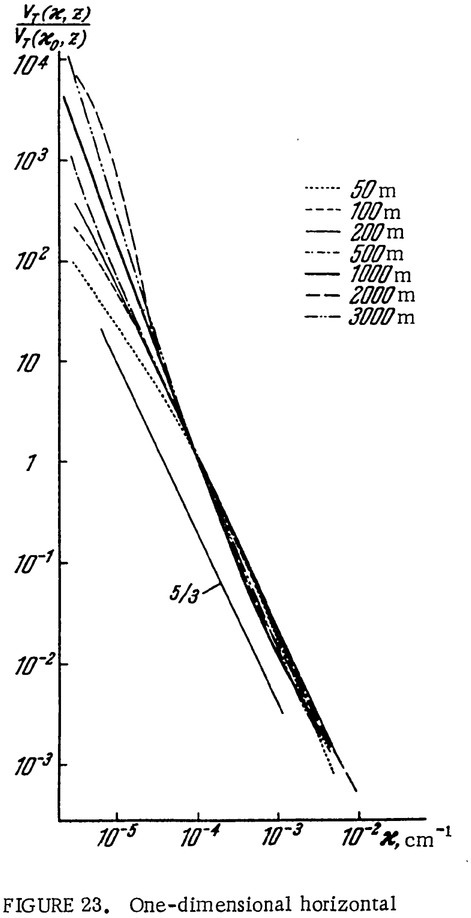

FIGURE 23. One-dimensional horizontal spatial spectra of temperature fluctuations at various heights in the troposphere in relative units $\varkappa_0 = 10^{-4}$ cm^{-1}.

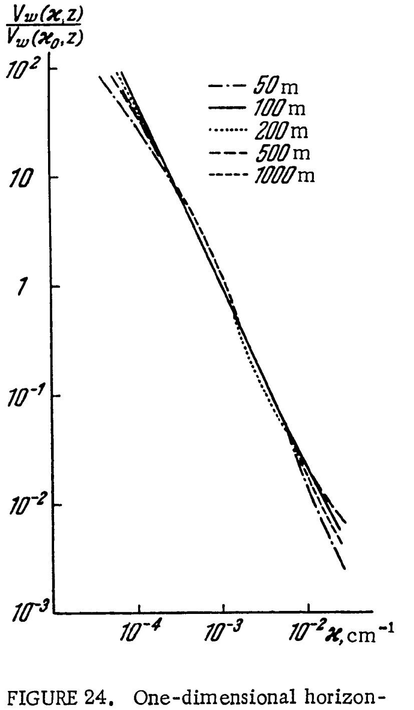

FIGURE 24. One-dimensional horizontal spatial spectra of the vertical wind velocity component at various heights in the troposphere in relative units $\varkappa_0 = 10^{-3}$ cm^{-1}.

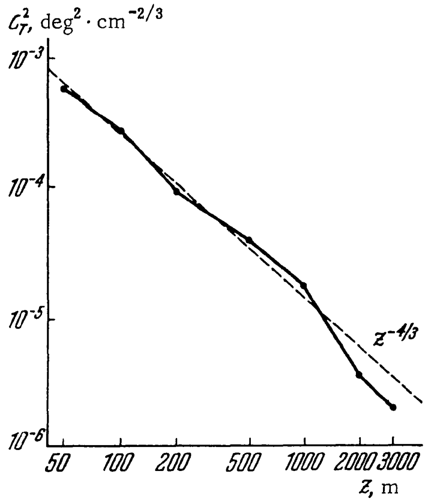

FIGURE 25. An example of the dependence of the structure function coefficient of tropospheric temperature fluctuations C_T^2 on height under conditions of free convection.

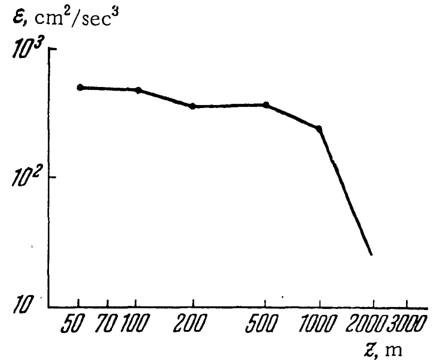

FIGURE 26. An example of the energy dissipation rate ε for tropospheric turbulence as a function of height under conditions of free convection.

The graph in Figure 25 was obtained by averaging ten different experimental profiles $C_T^2 = C_T^2(z)$ obtained under similar conditions corresponding to a convective regime (a hot summer day, 1200 to 1400 hrs). The dashed line in Figure 25 corresponds to $C_T^2(z) \sim z^{-4/3}$. It should be noted that each individual profile, as well as the averaged $C_T^2(z)$ curve, fits this power law.

Figure 26 gives the plot of $\varepsilon(z)$ obtained in the same series of airborne measurements (but at a different time). The plot was prepared from measurements of the frequency spectra of the horizontal wind velocity /60/. As we see from Figure 26, ε hardly changes with height up to $z \approx 1$ km. (At a height of about 1.5 km cloud strata were observed and the temperature stratification could not be regarded as unstable.)

Comparison of the experimental $C_T^2(z)$ and $\varepsilon(z)$ plots shows that the theoretical relations $\varepsilon(z) = \text{const}$ and $C_T^2(z) \sim z^{-4/3}$ derived for unstable temperature stratification in the surface layer of the atmosphere are applicable in the entire convection layer (up to altitudes of about 1 km). As we have previously shown (see (16.28)), in the case of free convection ε and $C_T^2 = \frac{a^2 N}{\varepsilon^{1/3}}$ are related by the equality

$$C_T^2 = \frac{a^2 C_1}{3\alpha\varkappa^{4/3}} \frac{\varepsilon^{4/3}}{\beta^2 z^{4/3}}$$

The numerical coefficient in this relation can be estimated from the results of /49, 54, 62/; it is found to be approximately 0.7 (this figure is not very reliable and requires further verification). Thus,

$$C_T^2 \approx \frac{0.7}{\beta^2}\left(\frac{\varepsilon}{z}\right)^{4/3}.$$

To check this relation, we will use the experimental data of Figures 25 and 26. For $z = 500$ m, $C_T^2 = 4\cdot 10^{-5}$ deg^2/cm$^{2/3}$ and $\varepsilon = 370$ cm^2 sec^{-3}.

Taking $\beta = g/T = 3.5\,\mathrm{cm\cdot sec^{-2}\cdot deg^{-1}}$, we find

$$C_T^2 = 0.7\,\frac{1}{(3.5)^2}\left(\frac{370}{5\cdot 10^4}\right)^{4/3} \approx 8\cdot 10^{-5}\ \mathrm{deg^2/cm^{2/3}},$$

i.e., the numerical values coincide apart from a factor of 2, which is perfectly satisfactory, as the comparison is made for averaged data obtained at different times and, possibly, for different Richardson numbers. A comparison of ε and C_T^2 values for other heights also gives satisfactory results.

The data of Figures 25 and 26 thus indicate that under convective conditions the "surface layer" theory can be applied to the entire convection layer, extending to heights of a few kilometers.

On the other hand, the measurements of ε from a 300-meter tower /58/, under conditions close to neutral stratification, reveal a rapid decrease of ε with height (ε falls from a few tens of $\mathrm{cm^2/sec^3}$ at a height of 25 m to a few units at a height of 300 m).

In conclusion, let us estimate the refractive index fluctuations in the optical range caused by the observed temperature fluctuations. By (15.1), in the radio-frequency range

$$n - 1 = \frac{79\cdot 10^{-6}}{T}\left(p + \frac{4800\,e}{T}\right),$$

where T is in °K, p and e in millibars. In the optical range, where humidity is of no consequence, $n - 1 = \frac{80\cdot 10^{-6}\,p}{T}$.* The fluctuations in n are related to the fluctuations in T by the expression $\delta n = -\frac{80\cdot 10^{-6}p}{T^2}\delta T$, and the structure characteristics C_n^2 and C_T^2 are related by the expression

$$C_n^2 = \left(\frac{80\cdot 10^{-6}p}{T^2}\right)^2 C_T^2.$$

According to Figure 25, C_T^2 ranges from 10^{-6} to $10^{-3}\,\mathrm{deg^2\,cm^{-2/3}}$. Putting $p = 850$ mb and $T = 280°$, we find that C_n^2 ranges from $5\cdot 10^{-19}$ to $5\cdot 10^{-16}\,\mathrm{cm^{-2/3}}$, or $C_n \sim 0.001 - 0.020$ N-units/$\mathrm{cm^{2/3}}$. (In radio meteorology the deviation of the refractive index from 1 is generally measured in so-called N-units: 1 N-unit is equal to 10^{-6}.) These C_n values are of the same order of magnitude as the C_n obtained from direct refractometer measurements /61/.

* The numerical coefficient in this equation refers to the middle of the visible optical range. More precise values are 82.9 for $\lambda = 0.3\,\mu$, 79.2 for $\lambda = 0.5\,\mu$, 77.4 for $\lambda = 0.7\,\mu$.

Chapter 2

SCATTERING OF ELECTROMAGNETIC AND SOUND WAVES IN A TURBULENT ATMOSPHERE

Scattering of waves by atmospheric turbulence attracted considerable attention following the experimental discovery of the long-range tropospheric propagation of ultrashort waves. The field strength values observed beyond the horizon were considerably greater than those predicted by diffraction of radio waves at the Earth's surface. Booker and Gordon /63/ suggested that this phenomenon could be associated with the scattering of radio waves by inhomogeneities in the dielectric constant of the atmosphere (see Figure 27). A long series of theoretical and experimental radio scattering studies then followed, and various alternative mechanisms explaining the long-range tropospheric propagation of ultrashort waves were advanced.

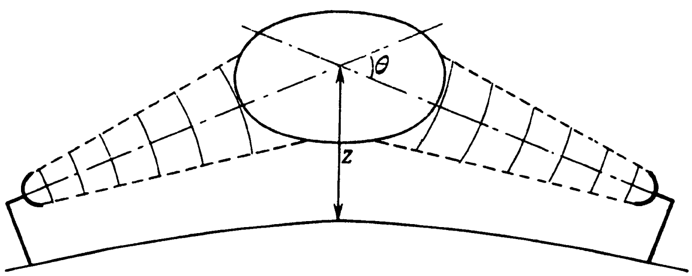

FIGURE 27. Scattering geometry for tropospheric propagation of radio waves beyond the horizon:

z — the height of the center of the scattering volume, θ — the scattering angle.

The phenomena of long-range tropospheric propagation are undoubtedly explained by a combination of numerous factors, with scattering playing a major role.

A. SCATTERING OF ELECTROMAGNETIC WAVES

§ 24. Wave propagation equations

The propagation of electromagnetic waves is described by Maxwell's equations

$$\operatorname{curl} \mathcal{E} = -\frac{1}{c}\frac{\partial \mathcal{H}}{\partial t}, \tag{1a}$$

$$\operatorname{curl} \mathcal{H} = \frac{1}{c}\frac{\partial \varepsilon\mathcal{E}}{\partial t}, \tag{1b}$$

$$\operatorname{div} \varepsilon\mathcal{E} = 0, \tag{1c}$$

where $\mathcal{E}$ and $\mathcal{H}$ are the electric and magnetic fields, and ε is the dielectric constant (the permittivity).

For the magnetic permeability we take $\mu = 1$ and for the conductivity $\sigma = 0$. The dielectric constant of the medium undergoes random fluctuations due to turbulence. The characteristic frequencies of these fluctuations are assumed to be small compared to the frequency of the electromagnetic wave. In this case it is useful to define new field variables $\boldsymbol{E}(\boldsymbol{r}, t)$ and $\boldsymbol{H}(\boldsymbol{r}, t)$ by the equalities

$$\mathcal{E}(\boldsymbol{r}, t) = \boldsymbol{E}(\boldsymbol{r}, t)\, e^{-i\omega t}, \tag{2a}$$

$$\mathcal{H}(\boldsymbol{r}, t) = \boldsymbol{H}(\boldsymbol{r}, t)\, e^{-i\omega t}, \tag{2b}$$

where $\boldsymbol{E}$ and $\boldsymbol{H}$ are the slowly varying complex field amplitudes. Inserting (2) in (1), we obtain

$$\operatorname{curl} \boldsymbol{E} = ik\boldsymbol{H} - \frac{1}{c}\frac{\partial \boldsymbol{H}}{\partial t}, \tag{3a}$$

$$\operatorname{curl} \boldsymbol{H} = -ik\varepsilon\boldsymbol{E} + \frac{1}{c}\frac{\partial \varepsilon\boldsymbol{E}}{\partial t}, \tag{3b}$$

$$\operatorname{div} \varepsilon\boldsymbol{E} = 0, \tag{3c}$$

where $k = \omega/c$.

Applying the curl operator to (3a) and using (3b), we obtain

$$\operatorname{curl}\operatorname{curl} \boldsymbol{E} = k^2\varepsilon\boldsymbol{E} + \frac{2ik}{c}\frac{\partial \varepsilon\boldsymbol{E}}{\partial t} - \frac{1}{c^2}\frac{\partial^2 \varepsilon\boldsymbol{E}}{\partial t^2}. \tag{4a}$$

Using the identity $\operatorname{curl}\operatorname{curl} \boldsymbol{E} = -\Delta\boldsymbol{E} + \operatorname{grad}\operatorname{div}\boldsymbol{E}$, we write this equation in the form

$$\Delta\boldsymbol{E} + k^2\varepsilon\boldsymbol{E} = \operatorname{grad}\operatorname{div}\boldsymbol{E} - \frac{2ik}{c}\frac{\partial \varepsilon\boldsymbol{E}}{\partial t} + \frac{1}{c^2}\frac{\partial^2 \varepsilon\boldsymbol{E}}{\partial t^2}. \tag{4b}$$

Equation (3c) may be rewritten as $\operatorname{div}\boldsymbol{E} + \boldsymbol{E}\operatorname{grad}\varepsilon = 0$, so that $\operatorname{div}\boldsymbol{E} = -\boldsymbol{E}\operatorname{grad}\ln\varepsilon$.
Inserting this expression in (4b) yields

$$\Delta\boldsymbol{E} + k^2\varepsilon\boldsymbol{E} = -\operatorname{grad}(\boldsymbol{E}\operatorname{grad}\ln\varepsilon) - \frac{2ik}{c}\frac{\partial \varepsilon\boldsymbol{E}}{\partial t} + \frac{1}{c^2}\frac{\partial^2 \varepsilon\boldsymbol{E}}{\partial t^2}. \tag{4c}$$

In the following we will consider wave propagation in a medium with small fluctuations in the dielectric constant. We take $\varepsilon = \langle\varepsilon\rangle + \varepsilon_1$, where $\langle\varepsilon\rangle$ is the mean value of ε and $\varepsilon_1 = \varepsilon - \langle\varepsilon\rangle$ is the fluctuation about the mean. By definition, $\langle\varepsilon_1\rangle = 0$. Sine the fluctuations are small, $\langle|\varepsilon_1|\rangle \ll \langle\varepsilon\rangle$. This condition is undoubtedly true for the troposphere, where $\langle\varepsilon\rangle$ is of the order of 1 and $\langle|\varepsilon_1|\rangle \sim 10^{-5}$–$10^{-6}$. The condition $\langle|\varepsilon_1|\rangle \ll \langle\varepsilon\rangle$ may not be true in the ionosphere, near the height where $\langle\varepsilon\rangle$ is zero, and this case requires special analysis (see, e.g., /64/). The mean dielectric constant $\langle\varepsilon\rangle$ may in general be a function of position and time. These changes in $\langle\varepsilon\rangle$ are responsible for the effects of systematic refraction (or superrefraction) and are beyond the scope of our treatment. Therefore we will consider only the case $\langle\varepsilon\rangle = \text{const}$.

In this case we may simply take $\langle\varepsilon\rangle = 1$. Putting $\varepsilon = 1 + \varepsilon_1$ and remembering that $\langle|\varepsilon_1|\rangle \ll 1$, we rewrite equation (4c) so that ε_1 appears only on the right.

Taking $\ln(1 + \varepsilon_1) \approx \varepsilon_1$, we obtain

$$\Delta \boldsymbol{E} + k^2 \boldsymbol{E} = -k^2 \varepsilon_1 \boldsymbol{E} - \operatorname{grad}(\boldsymbol{E} \operatorname{grad} \varepsilon_1) - \frac{2ik}{c}\frac{\partial \varepsilon \boldsymbol{E}}{\partial t} + \frac{1}{c^2}\frac{\partial^2 \varepsilon \boldsymbol{E}}{\partial t^2}. \tag{5}$$

The first term on the right in (5) is of the order of $\varepsilon_1 E/\lambda^2$; the second is estimated as $\varepsilon_1 E/\lambda l_0$ for $\lambda \ll l_0$ and $\varepsilon_1 E/l_0^2$ for $l_0 \ll \lambda$, where $\lambda = 2\pi/k$ is the wavelength and l_0 the inner scale of turbulence. The sum of the first plus the second term is thus of the order of $\varepsilon_1 E/\lambda^2$ for $\lambda \ll l_0$ and $\varepsilon_1 E/l_0^2$ for $l_0 \ll \lambda$.

Now consider the term $-\frac{2ik}{c}\frac{\partial \varepsilon E}{\partial t}$. Putting $\varepsilon = 1 + \varepsilon_1$, we have

$$\frac{\partial \varepsilon E}{\partial t} = E\frac{\partial \varepsilon_1}{\partial t} + (1 + \varepsilon_1)\frac{\partial E}{\partial t} \approx E\frac{\partial \varepsilon_1}{\partial t} + \frac{\partial E}{\partial t}.$$

But $\frac{\partial \varepsilon_1}{\partial t} \sim \frac{\varepsilon_1}{\tau}$, where τ is the characteristic time in which ε_1 changes significantly; $\tau \sim l_0/v$, where v is the transport velocity of the inhomogeneities (the wind speed). Thus,

$$E\frac{\partial \varepsilon_1}{\partial t} \sim \frac{\varepsilon_1 E v}{l_0}.$$

Let us now evaluate the term $\frac{\partial E}{\partial t}$. Two characteristic times which describe the variations in E may be defined. The first is associated with changes in ε_1 and is therefore of the order of τ, so that

$$\frac{\partial E}{\partial t} \sim \frac{E}{\tau} \sim \frac{Ev}{l_0}.$$

The second factor responsible for variations in E is the Doppler effect due to moving inhomogeneities. Here $\frac{\partial E}{\partial t} \sim \frac{v}{c}\omega E \sim \frac{vE}{\lambda}$.

Therefore,

$$\frac{\partial E}{\partial t} \sim \frac{Ev}{l_0} + \frac{Ev}{\lambda} = Ev\left(\frac{1}{l_0} + \frac{1}{\lambda}\right).$$

For $\lambda \ll l_0$

$$\frac{\partial E}{\partial t} \sim \frac{Ev}{\lambda}.$$

For $l_0 \ll \lambda$

$$\frac{\partial E}{\partial t} \sim \frac{Ev}{l_0}.$$

In either case, $\frac{\partial E}{\partial t}$ is large compared to $E\frac{\partial \varepsilon_1}{\partial t} \sim \frac{Ev}{l_0}\varepsilon_1$ for virtually all the wavelengths that need to be considered in practice. Thus,

$$\frac{\partial \varepsilon E}{\partial t} \sim \frac{Ev}{\lambda} \text{ for } \lambda \ll l_0$$

and

$$\frac{\partial \varepsilon E}{\partial t} \sim \frac{Ev}{l_0} \text{ for } l_0 \ll \lambda.$$

Let us now compare the orders of magnitude of the third term in the right-hand side of (5), $\frac{k}{c}\frac{\partial \varepsilon E}{\partial t}$, with the magnitude of the first two terms.

For $\lambda \ll l_0$ their ratio is of the order

$$\left[\frac{k}{c}\frac{\partial \varepsilon E}{\partial t}\right] \Big/ \left(\frac{\varepsilon_1 E}{\lambda^2}\right) \sim \frac{v}{\varepsilon_1 c}.$$

For $l_0 \ll \lambda$ we get

$$\left[\frac{k}{c}\frac{\partial \varepsilon E}{\partial t}\right] \Big/ \left(\frac{\varepsilon_1 E}{l_0^2}\right) \sim \frac{v}{\varepsilon_1 c}\frac{l_0}{\lambda} \ll \frac{v}{\varepsilon_1 c}.$$

Thus, if

$$\frac{v}{c} \ll \varepsilon_1, \tag{6}$$

the third term in equation (5) may be neglected compared to the first two terms. Condition (6) is readily satisfied in the atmosphere, where $\varepsilon_1 \sim 10^{-6} - 10^{-5}$ and $\frac{v}{c} \sim 10^{-8}$. The last term in the right-hand side of (5) is of the next higher order of smallness in v/c and practically always may be neglected. Assuming that (6) is satisfied, we obtain the equation

$$\Delta \boldsymbol{E} + k^2 \boldsymbol{E} = -k^2 \varepsilon_1 \boldsymbol{E} - \operatorname{grad}(\boldsymbol{E} \operatorname{grad} \varepsilon_1), \tag{7}$$

which contains time only as a parameter.

§ 25. The scattered field

Let a wave $\boldsymbol{E}_0(\boldsymbol{r})$ be incident on some volume element V filled by a random medium with variations in its dielectric constant ε_1. The field clearly satisfies the condition

$$\operatorname{div}\boldsymbol{E}_0 = 0. \tag{1}$$

The wave is scattered by the inhomogeneities in the scattering volume. We will consider the case of weak scattering, when the single-scattering approximation is applicable. This approximation corresponds to the solution of equation (24.7) which is obtained by the method of small perturbations, retaining only terms linear in ε_1. Let

$$\boldsymbol{E} = \boldsymbol{E}_0 + \boldsymbol{E}_s,$$

where $\boldsymbol{E}_s$ is the scattered field proportional to the fluctuations ε_1. Inserting this expression in (24.7), noting that $\Delta\boldsymbol{E}_0 + k^2\boldsymbol{E}_0 = 0$, and retaining in the right-hand side of the equation only the term with $\boldsymbol{E}_0$ (keeping $\boldsymbol{E}_s$ would result in terms with ε_1 squared), we obtain

$$\Delta\boldsymbol{E}_s + k^2\boldsymbol{E}_s = -k^2\varepsilon_1\boldsymbol{E}_0 - \operatorname{grad}(\boldsymbol{E}_0 \operatorname{grad}\varepsilon_1). \tag{2}$$

In solving equation (2), we will ignore the boundary conditions at the ground, and the scattering volume V will be regarded as situated in an infinite space. In this case the radiation condition is the only constraint imposed on the solution of equation (2). The known solution satisfying this condition is

$$\boldsymbol{E}_s(\boldsymbol{r}) = \frac{1}{4\pi}\int\limits_V \frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|}\{k^2\varepsilon_1(\boldsymbol{r}')\,\boldsymbol{E}_0(\boldsymbol{r}') + \operatorname{grad}(\boldsymbol{E}_0 \operatorname{grad}\varepsilon_1(\boldsymbol{r}'))\}\,d^3r'. \tag{3}$$

To transform the second term in this expression, we use the following form of Gauss's theorem:

$$\int\limits_V u \operatorname{grad} v\, dV = \iint\limits_S uv\, d\boldsymbol{\sigma} - \int\limits_V v \operatorname{grad} u\, dV. \tag{4}$$

Here $d\boldsymbol{\sigma}$ is a vector of magnitude $d\sigma$ directed along the outer normal to the surface S enclosing the volume V. Then

$$\begin{aligned}\boldsymbol{E}_s(\boldsymbol{r}) = \frac{1}{4\pi}\int\limits_V \Big[\frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|}k^2\varepsilon_1(\boldsymbol{r}')\,\boldsymbol{E}_0(\boldsymbol{r}')\\ -(\boldsymbol{E}_0 \operatorname{grad}\varepsilon_1(\boldsymbol{r}'))\operatorname{grad}_{r'}\Big(\frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|}\Big)\Big]\,d^3r' +\\ + \frac{1}{4\pi}\iint\limits_S \frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|}(\boldsymbol{E}_0 \operatorname{grad}\varepsilon_1)\,d\boldsymbol{\sigma}.\end{aligned} \tag{5}$$

The surface integral in (5) can be omitted, since for a sufficiently large scattering volume the contribution from surface effects is small compared with the volume effects (the former are proportional to L^2, whereas the latter increase as L^3, where L is the characteristic dimension of the scattering volume). Let us now compute the gradient term in (5).

$$\operatorname{grad}_{r'} \frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|} = -\left[\frac{ike^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|} - \frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|^2}\right] \frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}'-\boldsymbol{r}|}. \tag{6}$$

Here $|\boldsymbol{r}-\boldsymbol{r}'|$ is the distance from the scattering volume to the observation point. If $k|\boldsymbol{r}-\boldsymbol{r}'| \gg 1$ (the far-field condition), the second term in (6) can be dropped and (5) takes the form

$$\boldsymbol{E}_s(\boldsymbol{r}) \approx \frac{1}{4\pi}\int_V d^3r' \frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|}\Big\{k^2\varepsilon_1(\boldsymbol{r}')\boldsymbol{E}_0(\boldsymbol{r}') + $$

$$+ ik\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|}(\boldsymbol{E}_0(\boldsymbol{r}') \operatorname{grad} \varepsilon_1(\boldsymbol{r}'))\Big\}. \tag{7}$$

The second term in (7) also contains a gradient term and it is again transformed using Gauss' theorem, this time in the form

$$\int_V u_{il}\frac{\partial v}{\partial x_l}dV = \iint_S u_{il}v\,(d\sigma)_l - \int_V v\frac{\partial u_{il}}{\partial x_l}dV.$$

Transforming (7), we again ignore the surface integral; in carrying out the differentiation

$$\frac{\partial}{\partial x_l'}\left\{\frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|}\frac{x_i-x_i'}{|\boldsymbol{r}-\boldsymbol{r}'|}E_l^0(\boldsymbol{r}')\right\}$$

we differentiate only the exponential factor, since $\frac{\partial E_l^0(\boldsymbol{r}')}{\partial x_l'} \equiv 0$ due to the condition div $\boldsymbol{E}_0 = 0$, and the other factors in the far-field zone vary much more slowly than the exponential. After these transformations, we obtain the expression

$$\boldsymbol{E}_s(\boldsymbol{r}) = \frac{k^2}{4\pi}\int_V \frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|}\varepsilon_1(\boldsymbol{r}')\left\{\boldsymbol{E}_0(\boldsymbol{r}') - \frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|}\left(\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|}\boldsymbol{E}_0(\boldsymbol{r}')\right)\right\}dV. \tag{8}$$

The unit vector

$$\boldsymbol{n}(\boldsymbol{r}, \boldsymbol{r}') = \frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|} \tag{9}$$

is directed from the variable point of integration to the fixed observation point $\boldsymbol{r}$. Using the identity $\boldsymbol{E}_0 - \boldsymbol{n}(\boldsymbol{n}\,\boldsymbol{E}_0) = [\boldsymbol{n}\,[\boldsymbol{E}_0\,\boldsymbol{n}]]$, we rewrite (8) in the form

$$\boldsymbol{E}_s(\boldsymbol{r}) = \frac{k^2}{4\pi}\int_V \frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|}\varepsilon_1(\boldsymbol{r}')\,[\boldsymbol{n}\,[\boldsymbol{E}_0(\boldsymbol{r}')\boldsymbol{n}]]\,d^3r'. \tag{8a}$$

Expression (8a) for the scattered field is fairly general, as we did not introduce any restrictive assumptions concerning the geometry of the incident wave E_0, the size of the scattering volume V, or its distance from the observation point (except the requirement that this point should lie in the far-field zone).

§ 26. Mean scattered intensity

The mean scattered intensity is determined by the Poynting vector (the energy flux density vector)

$$\boldsymbol{S} = \frac{c}{4\pi}[\boldsymbol{E}_s\boldsymbol{H}_s].$$

If complex field amplitudes are used, the vector $\boldsymbol{S}$ averaged over one oscillation period is

$$\boldsymbol{S} = \frac{c}{8\pi}\operatorname{Re}[\boldsymbol{E}_s\boldsymbol{H}_s^*]. \tag{1}$$

To find $\boldsymbol{H}_s$, we use equation (24.3a) omitting the term $\frac{1}{c}\frac{\partial \boldsymbol{H}}{\partial t}$:

$$\boldsymbol{H}_s = \frac{1}{ik}\operatorname{curl}\boldsymbol{E}_s.$$

In calculating curl $\boldsymbol{E}_s$ from (25.8a), it is only necessary to differentiate the exponential $e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}$, since it is the most rapidly varying term in the far-field zone, while differentiation of the other factors gives terms of the order of $\frac{1}{kr}$. We recall that the origin is placed inside the scattering volume, so that r is the distance from the scattering volume to the observation point. All this gives

$$\boldsymbol{H}_s(\boldsymbol{r}) = \frac{k^2}{4\pi}\int\limits_V \frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|}\varepsilon_1(\boldsymbol{r}')[\boldsymbol{n}\boldsymbol{E}_0]\,d^3r'. \tag{2}$$

The expression for $\boldsymbol{S}$ takes the form

$$\boldsymbol{S} = \frac{c}{8\pi}\left(\frac{k^2}{4\pi}\right)^2 \operatorname{Re}\iint\limits_{VV}\frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|-ik|\boldsymbol{r}-\boldsymbol{r}''|}}{|\boldsymbol{r}-\boldsymbol{r}'|\cdot|\boldsymbol{r}-\boldsymbol{r}''|}\varepsilon_1(\boldsymbol{r}')\,\varepsilon_1(\boldsymbol{r}'')\times$$
$$\times[(\boldsymbol{E}_0'-\boldsymbol{n}'(\boldsymbol{E}_0'\boldsymbol{n}'))[\boldsymbol{n}''\ \boldsymbol{E}_0''^*]]\,d^3r'\,d^3r'', \tag{3}$$

where

$$\boldsymbol{E}_0' = \boldsymbol{E}_0(\boldsymbol{r}'),\ \boldsymbol{E}_0'' = \boldsymbol{E}_0(\boldsymbol{r}''),\quad \boldsymbol{n}' = \boldsymbol{n}(\boldsymbol{r},\ \boldsymbol{r}'),\quad \boldsymbol{n}'' = \boldsymbol{n}(\boldsymbol{r},\ \boldsymbol{r}'').$$

$\boldsymbol{S}$ is a random variable, as it is a function of the random variables $\varepsilon_1(\boldsymbol{r}')$, $\varepsilon_1(\boldsymbol{r}'')$. Let us calculate its mean value:

$$\langle \boldsymbol{S}(\boldsymbol{r})\rangle = \frac{c}{8\pi}\left(\frac{k^2}{4\pi}\right)^2 \mathrm{Re}\iint\limits_{VV} d^3r'\, d^3r'' \frac{e^{ik[|\boldsymbol{r}-\boldsymbol{r}'|-|\boldsymbol{r}-\boldsymbol{r}''|]}}{|\boldsymbol{r}-\boldsymbol{r}'|\cdot|\boldsymbol{r}-\boldsymbol{r}''|} B_\varepsilon(\boldsymbol{r}', \boldsymbol{r}'') \times$$
$$\times \{\boldsymbol{n}''[(\boldsymbol{E}_0'\boldsymbol{E}_0^{*''}) - (\boldsymbol{E}_0'\boldsymbol{n}')(\boldsymbol{E}_0^{*''}\boldsymbol{n}')] + \boldsymbol{E}_0^{*''}[(\boldsymbol{n}'\boldsymbol{n}'')(\boldsymbol{E}_0'\boldsymbol{n}') - (\boldsymbol{E}_0'\boldsymbol{n}'')]\}. \tag{4}$$

Here $B_\varepsilon(\boldsymbol{r}', \boldsymbol{r}'')$ is the correlation function of the dielectric constant fluctuations. We assume that the fluctuations are statistically homogeneous, i.e., $B_\varepsilon(\boldsymbol{r}', \boldsymbol{r}'') = B_\varepsilon(\boldsymbol{r}' - \boldsymbol{r}'')$.

Putting $\boldsymbol{\rho} = \boldsymbol{r}' - \boldsymbol{r}''$ and expanding $\boldsymbol{n}'' = \boldsymbol{n}(\boldsymbol{r}, \boldsymbol{r}'')$ in powers of $\boldsymbol{\rho}$, we obtain after simple manipulations

$$n_i'' = n_i' + \frac{\rho_i - n_i'(\rho_k n_k')}{|\boldsymbol{r}-\boldsymbol{r}'|} + \ldots \tag{5}$$

The variable $\boldsymbol{\rho}$ in (4) is at most of the order of magnitude of the correlation radius L_0 of the fluctuations in ε, since $B_\varepsilon(\boldsymbol{\rho})$ vanishes for larger ρ. Therefore, if the distance r from the scattering volume to the observation point is large compared with L_0, the second term in (5) is small (as it is of the order of L_0/r).

The scalar product $(\boldsymbol{n}'\boldsymbol{n}'')$ differs from unity only by terms of the order of L_0^2/r^2 (see (5)). Consequently, the second term in braces in (4) is of the order of $E_0^2L_0/r$, whereas the first term is of the order of E_0^2. If

$$L_0 \ll r, \tag{6}$$

the second term in (4) can be dropped.

With the same relative error, of the order of L_0/r, we can replace $\boldsymbol{n}''$ by $\boldsymbol{n}'$ in the first term in braces in (4) and $\boldsymbol{r}''$ by $\boldsymbol{r}'$ in the denominator in the integrand (but not in the phase factors). We thus obtain the expression

$$\langle \boldsymbol{S}(\boldsymbol{r})\rangle \approx \frac{c}{8\pi}\left(\frac{k^2}{4\pi}\right)^2 \mathrm{Re}\iint\limits_{VV} \frac{e^{ik[|\boldsymbol{r}-\boldsymbol{r}'|-|\boldsymbol{r}-\boldsymbol{r}''|]}}{|\boldsymbol{r}-\boldsymbol{r}'|^2} B_\varepsilon(\boldsymbol{r}'-\boldsymbol{r}'') \times$$
$$\times \boldsymbol{n}'[(\boldsymbol{E}_0'\boldsymbol{E}_0^{*''}) - (\boldsymbol{E}_0'\boldsymbol{n}')(\boldsymbol{E}_0^{*''}\boldsymbol{n}')]\, d^3r'\, d^3r''. \tag{7}$$

Expression (7) is further simplified if we assume that the incident field is generated by a source at a point $\boldsymbol{R}$. Then $\boldsymbol{E}_0(\boldsymbol{r})$ takes the form

$$\boldsymbol{E}_0(\boldsymbol{r}) = \boldsymbol{A}_0(\boldsymbol{r})\, e^{ik|\boldsymbol{R}-\boldsymbol{r}|}. \tag{8}$$

The complex function $\boldsymbol{A}_0(\boldsymbol{r})$ does not change much as $\boldsymbol{r}$ is varied by an amount on the order of a wavelength λ. It changes appreciably only over distances of the order of the size of the scattering volume L. The function $\boldsymbol{A}_0(\boldsymbol{r})$ depends on the transmitter beam pattern. Inserting (8) into (7), we can ignore the difference between $\boldsymbol{A}_0(\boldsymbol{r}')$ and $\boldsymbol{A}_0(\boldsymbol{r}'')$, since the relative magnitude of this difference does not exceed a value on the order of L_0/L, which is small whenever

$$L_0 \ll L. \tag{9}$$

In this case, we obtain

$$\boldsymbol{E}_0'\boldsymbol{E}_0^{*''} - (\boldsymbol{E}_0'\boldsymbol{n}')(\boldsymbol{E}_0^{*''}\boldsymbol{n}') \approx A_0(\boldsymbol{r}')A_0^*(\boldsymbol{r}')\sin^2\chi(\boldsymbol{r}')\exp\{ik[|\boldsymbol{R}-\boldsymbol{r}'| - |\boldsymbol{R}-\boldsymbol{r}''|]\}, \tag{10}$$

where $\chi(\boldsymbol{r})$ is the angle between the vectors $\boldsymbol{A}_0(\boldsymbol{r})$ and $\boldsymbol{n}(\boldsymbol{r})$.

Inserting (10) in (7), we obtain

$$\langle S(\boldsymbol{r})\rangle \approx \frac{c}{8\pi}\left(\frac{k^2}{4\pi}\right)^2 \mathrm{Re}\int\limits_V\int\limits_V \frac{n(\boldsymbol{r}')\,|A_0(\boldsymbol{r}')|^2 \sin^2\chi(\boldsymbol{r}')\,B_\varepsilon(\boldsymbol{r}'-\boldsymbol{r}'')}{|\boldsymbol{r}-\boldsymbol{r}'|^2}\times$$
$$\times \exp\{ik\,[|\boldsymbol{r}-\boldsymbol{r}'|-|\boldsymbol{r}-\boldsymbol{r}''|+|\boldsymbol{R}-\boldsymbol{r}'|-|\boldsymbol{R}-\boldsymbol{r}''|]\}\,d^3r'\,d^3r''. \qquad (11)$$

Series-expanding $|\boldsymbol{r}-\boldsymbol{r}''|=|\boldsymbol{r}-\boldsymbol{r}'+\boldsymbol{\rho}|$ and $|\boldsymbol{R}-\boldsymbol{r}''|=|\boldsymbol{R}-\boldsymbol{r}'+\boldsymbol{\rho}|$ in powers of $\boldsymbol{\rho}$, we obtain after simple manipulations

$$|\boldsymbol{r}-\boldsymbol{r}''|=|\boldsymbol{r}-\boldsymbol{r}'|+\boldsymbol{n}(\boldsymbol{r}')\boldsymbol{\rho}+\frac{1}{2|\boldsymbol{r}-\boldsymbol{r}'|}\{\rho^2-(\boldsymbol{\rho n}(\boldsymbol{r}'))^2\}+\ldots, \qquad (12)$$

$$|\boldsymbol{R}-\boldsymbol{r}''|=|\boldsymbol{R}-\boldsymbol{r}'|-\boldsymbol{m}(\boldsymbol{r}')\boldsymbol{\rho}+\frac{1}{2|\boldsymbol{R}-\boldsymbol{r}'|}\{\rho-(\boldsymbol{\rho m}(\boldsymbol{r}'))^2\}+\ldots \qquad (13)$$

where

$$\boldsymbol{m}(\boldsymbol{r}')=\frac{\boldsymbol{r}'-\boldsymbol{R}}{|\boldsymbol{r}'-\boldsymbol{R}|} \qquad (14)$$

is the unit vector directed from the source to the point $\boldsymbol{r}'$ (Figure 28).

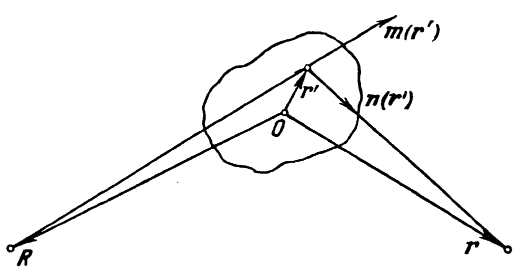

FIGURE 28. Coordinates for the determination of scattering. The origin is placed inside the scattering volume.

$\boldsymbol{R}$ and $\boldsymbol{r}$ are the coordinates of the source and the observation point, $\boldsymbol{r}'$ is an arbitrary point inside the scattering volume, $m(r')$ and $n(r')$ are unit vectors.

The last term in expansion (12) is of the order of magnitude of L_0^2/r, since in the important region of integration $\rho \lesssim L_0$. Since $|\boldsymbol{r}-\boldsymbol{r}''|$ is multiplied by k in the exponential in (11), the last term in (12) can be ignored if

$$\frac{kL_0^2}{r} \ll 1, \quad \text{or} \quad \frac{L_0^2}{\lambda r} \ll 1; \tag{15}$$

this condition signifies that the radius of the first Fresnel zone is large compared with the correlation radius of the fluctuations of ε. Similarly, if

$$\frac{L_0^2}{\lambda R} \ll 1 \tag{16}$$

we can ignore the last term in expansion (13). With these assumptions expression (11) takes the form

$$\langle \boldsymbol{S}(\boldsymbol{r}) \rangle = \frac{c}{8\pi}\left(\frac{k^2}{4\pi}\right)^2 \operatorname{Re} \int\limits_V \int\limits_V \frac{\boldsymbol{n}(\boldsymbol{r}') \mid A_0(\boldsymbol{r}') \mid^2 \sin^2 \chi(\boldsymbol{r}')}{\mid \boldsymbol{r} - \boldsymbol{r}' \mid^2} B_\varepsilon(\boldsymbol{\rho}) e^{i\boldsymbol{K}(\boldsymbol{r}')\boldsymbol{\rho}} d^3r' \, d^3r'', \tag{17}$$

where

$$\boldsymbol{K}(\boldsymbol{r}) = k\,[\boldsymbol{m}(\boldsymbol{r}) - \boldsymbol{n}(\boldsymbol{r})] \tag{18}$$

is the scattering vector, equal to the difference between the wave vector $\boldsymbol{k}_0 = k\boldsymbol{m}$ of the incident wave and the wave vector $\boldsymbol{k}_s = k\boldsymbol{n}$ of the scattered wave.

In (17), $\boldsymbol{r}''$ is replaced by a new variable $\boldsymbol{\rho}$. The inner integral over $\boldsymbol{\rho}$ takes the form

$$I(\boldsymbol{K}) = \int\limits_V B_\varepsilon(\boldsymbol{\rho}) e^{i\boldsymbol{K}\boldsymbol{\rho}} d^3\rho. \tag{19}$$

It can be expressed in simple form in terms of $\Phi_\varepsilon(\boldsymbol{\varkappa})$, the spectral density of the dielectric constant fluctuations, and is related to $B_\varepsilon(\boldsymbol{\rho})$ by the transform

$$B_\varepsilon = \iiint\limits_{-\infty}^{\infty} \Phi_\varepsilon e^{i\varkappa\rho} d^3\varkappa. \tag{20}$$

Inserting (20) in (19), we get

$$I(\boldsymbol{K}) = \iiint\limits_{-\infty}^{\infty} \Phi_\varepsilon(\boldsymbol{\varkappa}) \, d^3\varkappa \int\limits_V e^{i(\boldsymbol{K}+\boldsymbol{\varkappa})\boldsymbol{\rho}} d^3\rho. \tag{21}$$

Consider the function

$$\delta_V(\boldsymbol{\varkappa}) = \frac{1}{8\pi^3} \int\limits_V e^{i\varkappa\rho} d^3\rho.$$

If V is an infinite volume, $\delta_V(\boldsymbol{\varkappa}) = \delta(\boldsymbol{\varkappa})$, i.e., it reduces to a three-dimensional δ-function. If the volume is finite, $\delta_V(\boldsymbol{\varkappa})$ is a "smeared" δ-function, having the following properties:

$$\delta_V(0) = \frac{V}{8\pi^3},$$

$$\iiint\limits_{-\infty}^{\infty} \delta_V(\varkappa)\, d^3\varkappa = \int\limits_V d^3\rho \left[\frac{1}{8\pi^3} \iiint\limits_{-\infty}^{\infty} e^{i\varkappa\rho}\, d^3\varkappa\right] = \int d^3\rho\, \delta(\rho) = 1.$$

Consequently, $\delta_V(\varkappa)$ is markedly different from zero only within a region T in wavenumber space having the volume $T = \frac{8\pi^3}{V}$, which is concentrated near the point $\varkappa = 0$.

Hence, we obtain for the integral in (21)

$$I(\boldsymbol{K}) = 8\pi^3 \overline{\Phi}_\varepsilon(-\boldsymbol{K}), \tag{22}$$

where the bar denotes averaging in wavenumber space over a volume of the order $T = \frac{8\pi^3}{V}$. Indeed, using the mean value theorem, we get

$$\iiint\limits_{-\infty}^{\infty} \Phi_\varepsilon(\varkappa)\, \delta_V(\boldsymbol{K} + \varkappa)\, d^3\varkappa = \iiint\limits_{-\infty}^{\infty} \Phi_\varepsilon(-\boldsymbol{K} + \varkappa')\, \delta_V(\varkappa')\, d^3\varkappa' \approx$$

$$\approx \frac{V}{8\pi^3} \iiint\limits_{8\pi^3/V} \Phi_\varepsilon(-\boldsymbol{K} + \varkappa)\, d^3\varkappa = \overline{\Phi}_\varepsilon(-\boldsymbol{K}).$$

If the spectral density $\Phi_\varepsilon(\varkappa)$ does not change much near the point $\varkappa = -\boldsymbol{K}$, averaging does not markedly affect the function $\Phi_\varepsilon(\varkappa)$ and we may take $\overline{\Phi}_\varepsilon(-\boldsymbol{K}) \approx \Phi_\varepsilon(-\boldsymbol{K})$. (Note that averaging markedly alters the form of $\Phi_\varepsilon(\varkappa)$ near those points where this function has a sharp maximum. This effect may prove to be of considerable significance in forward scattering.)

Inserting (22) in (17) and omitting the symbol Re, since $\Phi_\varepsilon(-\boldsymbol{K})$ is a real (and positive) function, we obtain

$$\langle \boldsymbol{S}(\boldsymbol{r})\rangle = \frac{ck^4}{16} \int\limits_V \boldsymbol{n}(\boldsymbol{r}') \frac{|A_0(\boldsymbol{r}')|^2 \sin^2\chi(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|^2} \overline{\Phi}_\varepsilon(\boldsymbol{K}(\boldsymbol{r}'))\, d^3r' \tag{23}$$

(we made use of the equality $\Phi_\varepsilon(-\boldsymbol{K}) = \Phi_\varepsilon(\boldsymbol{K})$).

Relation (23) can be reduced to a more convenient form. Writing it in differential form, we get

$$\langle d\boldsymbol{S}(\boldsymbol{r})\rangle = \boldsymbol{n}(\boldsymbol{r}') \frac{ck^4 |A_0^2|}{16\, |\boldsymbol{r} - \boldsymbol{r}'|^2} \sin^2\chi(\boldsymbol{r}')\, \overline{\Phi}_\varepsilon(\boldsymbol{K}(\boldsymbol{r}'))\, dV'. \tag{24}$$

$\langle d\boldsymbol{S}(\boldsymbol{r})\rangle$ is the energy flux density scattered by a volume element $dV(\boldsymbol{r}')$ to the point $\boldsymbol{r}$. The factor

$$S_0(\boldsymbol{r}') = \frac{c\,|A_0^2|}{8\pi} \tag{25}$$

is the incident energy flux density.

We multiply (24) by $\boldsymbol{n}'|\boldsymbol{r} - \boldsymbol{r}'|^2\, d\Omega$; this gives the energy dE scattered in the direction $\boldsymbol{n}'$ in a solid angle $d\Omega$:

$$dE = S_0(\boldsymbol{r}')\, \sigma_0(\boldsymbol{r}')\, dV'\, d\Omega, \tag{26}$$

where

$$\sigma_0(\boldsymbol{r}) = \frac{\pi}{2} k^4 \overline{\Phi}_\varepsilon (\boldsymbol{K}(\boldsymbol{r})) \sin^2 \chi(\boldsymbol{r}) \qquad (27)$$

is the effective scattering cross section from a unit volume into a unit solid angle in the direction $\boldsymbol{m}$. Using the parameters $S_0(\boldsymbol{r}), \sigma_0(\boldsymbol{r})$, we write (23) in the form

$$\langle \boldsymbol{S}(\boldsymbol{r}) \rangle = \int\limits_V \boldsymbol{n}(\boldsymbol{r}') \frac{S_0(\boldsymbol{r}')\sigma_0(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^2} dV'. \qquad (28)$$

Relation (28) has an obvious physical interpretation: according to (26) the energy scattered in a solid angle $d\Omega$ is dE; the energy flux density associated with this scattered energy is

$$\frac{dE}{|\boldsymbol{r}-\boldsymbol{r}'|^2 d\Omega} = \frac{S_0(\boldsymbol{r}')\sigma_0(\boldsymbol{r}')dV}{|\boldsymbol{r}-\boldsymbol{r}'|^2},$$

where $|\boldsymbol{r}-\boldsymbol{r}'|^2 d\Omega$ is the area over which the scattered energy is distributed.

Note that in deriving (28) we did not impose any restrictions on the size of the scattering volume (except the weak constraint (9)). Expressions (27), (28) are valid if the correlation radius L_0 of the dielectric constant fluctuations is small compared with the radii of the first Fresnel zone $\sqrt{\lambda r}$ and $\sqrt{\lambda R}$ and to the distances r and R. This is the principal distinction of our derivation of (27), (28) from the usual derivation (see, e.g., /65/), when the scattering volume is assumed to be small compared to $\sqrt{\lambda r}$.

Note that the assumption of statistically homogeneous fluctuations, $B_\varepsilon(\boldsymbol{r}_1, \boldsymbol{r}_2) = B_\varepsilon(\boldsymbol{r}_1 - \boldsymbol{r}_2)$, is unnecessary and was introduced only so as not to digress from the main line of reasoning. The same results may be obtained if the mean square of the dielectric constant fluctuations is a function of position:

$$B_\varepsilon(\boldsymbol{r}_1, \boldsymbol{r}_2) = \overline{\varepsilon'^2}\left(\frac{\boldsymbol{r}_1+\boldsymbol{r}_2}{2}\right) b_\varepsilon(\boldsymbol{r}_1 - \boldsymbol{r}_2) \quad (b_\varepsilon(0) = 1). \qquad (29)$$

In this case the spectral density of the fluctuations is also a function of position (see Chapter 1) and has the form

$$\Phi_\varepsilon = C_\varepsilon^2(\boldsymbol{r}) \Phi_\varepsilon^{(0)}(\varkappa). \qquad (30)$$

Therefore expression (27) takes the form

$$\sigma_0(\boldsymbol{r}) = \frac{\pi k^4}{2} C_\varepsilon^2(\boldsymbol{r}) \overline{\Phi}_\varepsilon^{(0)}(\boldsymbol{K}(\boldsymbol{r})) \sin^2 \chi(\boldsymbol{r}), \qquad (31)$$

and the final result (28) does not change.

§ 27. Qualitative interpretation of scattering

Let us consider in more detail expression (26.27) for the effective scattering cross section. If the scattering volume is sufficiently large, i.e., $\overline{\Phi}_\varepsilon(\boldsymbol{K}) \approx \Phi_\varepsilon(\boldsymbol{K})$, it takes the form

$$\sigma_0(\boldsymbol{r}) = \frac{\pi k^4}{2} \sin^2 \chi(\boldsymbol{r}) \, \Phi_\varepsilon(\boldsymbol{k}_0 \boldsymbol{r}) - \boldsymbol{k}_s(\boldsymbol{r})), \tag{1}$$

where $\boldsymbol{k}_0$ and $\boldsymbol{k}_s$ are the incident and the scattered wave vectors.

The magnitude of the scattering vector $\boldsymbol{K} = \boldsymbol{k}_0 - \boldsymbol{k}_s$ is

$$K = 2k \sin \frac{\theta}{2}, \tag{2}$$

where $\theta = \arccos\left(\frac{\boldsymbol{k}_0 \boldsymbol{k}_s}{k^2}\right)$ is the angle between $\boldsymbol{k}_0$ and $\boldsymbol{k}_s$, i. e., the scattering angle. It follows from (1) that the intensity of the wave scattered at an angle θ is determined by one spectral component of inhomogeneities corresponding to the scale (spatial period).

$$l(\theta) = \frac{2\pi}{K} = \frac{\pi}{k \sin \frac{\theta}{2}} = \frac{\lambda}{2 \sin \frac{\theta}{2}}. \tag{3}$$

Relation (3) is the well-known Bragg condition for diffraction from spatial structures.

In order to interpret the physical meaning of (1), let us consider a model of dielectric constant inhomogeneities of the form

$$\varepsilon_1(\boldsymbol{r}) = d \cos\left(\frac{2\pi}{l} \boldsymbol{a}\boldsymbol{r}\right). \tag{4}$$

This is a spatial sinusoidal diffraction grating of period l in the direction of the unit vector $\boldsymbol{a}$. For simplicity, suppose that the scattering volume V is a right-angled parallelepiped with one of its axes along the vector $\boldsymbol{a}$. To simplify the mathematics, we further assume that the dimensions L_1, L_2, L_3 of the scattering volume satisfy the conditions $L \ll r$ and $L^2 \ll \lambda r$. In this case, the factor $|\boldsymbol{r} - \boldsymbol{r}'|$ in the exponential in (25.8a) can be expanded in a power series up to terms of second order, $|\boldsymbol{r} - \boldsymbol{r}'| \approx r - \boldsymbol{r}'\boldsymbol{n}$, where $\boldsymbol{n} = \boldsymbol{r}/r$, while in the denominator we may take $|\boldsymbol{r} - \boldsymbol{r}'| \approx r$. Moreover, an incident plane wave is assumed,

$$\boldsymbol{E}_0(\boldsymbol{r}) = \boldsymbol{A}_0 e^{i\boldsymbol{k}_0 \boldsymbol{r}},$$

and $\boldsymbol{n}(\boldsymbol{r}') \approx \boldsymbol{r}/r$. Expression (25.8a) thus gives

$$\boldsymbol{E}_s(\boldsymbol{r}) \approx \frac{dk^2 e^{ikr}}{4\pi r} [\boldsymbol{n}[\boldsymbol{A}_0 \boldsymbol{n}]] \int_V \cos\left(\frac{2\pi}{l} \boldsymbol{a}\boldsymbol{r}\right) e^{i(\boldsymbol{k}_0 - \boldsymbol{n}k)\boldsymbol{r}} \, dV. \tag{5}$$

Evaluating the integral, we obtain

$$\boldsymbol{E}_s(\boldsymbol{r}) = \frac{dk^2 e^{ikr}}{4\pi r} [\boldsymbol{n}[\boldsymbol{A}_0 \boldsymbol{n}]] \cdot \frac{L_1 L_2 L_3}{2} \frac{\sin\left(\frac{KL_1}{2} \sin\varphi\right)}{\frac{KL_1}{2} \sin\varphi} \times$$
$$\times \left\{ \frac{\sin\left(\frac{1}{2} KL_3 \cos\varphi + \frac{\pi L_3}{l}\right)}{\frac{1}{2} KL_3 \cos\varphi + \frac{\pi L_3}{l}} + \frac{\sin\left(\frac{1}{2} KL_3 \cos\varphi - \frac{\pi L_3}{l}\right)}{\frac{1}{2} KL_3 \cos\varphi - \frac{\pi L_3}{l}} \right\}, \tag{6}$$

where φ is the angle between $\boldsymbol{a}$ and $\boldsymbol{K} = \boldsymbol{k}_0 - k\boldsymbol{n}$, the axis z points along the vector $\boldsymbol{a}$, the plane xz contains the vectors $\boldsymbol{K}$, $\boldsymbol{a}$, and L_1, L_2, L_3 are the dimensions of the scattering volume along the axes x, y, z.

Consider the factor

$$\frac{\sin\left(\frac{KL_1}{2}\sin\varphi\right)}{\frac{KL_1}{2}\sin\varphi}=\frac{\sin\left(kL_1\sin\frac{\theta}{2}\sin\varphi\right)}{kL_1\sin\frac{\theta}{2}\sin\varphi}.$$

If $kL_1\sin\frac{\theta}{2}\gg 1$ (which is generally true, since the scattering volume is much greater than the wavelength), this factor is small for all φ, except those very close to zero, when φ satisfies the condition

$$|\varphi|\lesssim\frac{1}{kL_1}=\frac{\lambda}{2\pi L_1}, \tag{7}$$

and the factor is close to unity.

Substantial diffraction from the spatial periodic structure being considered is thus observed only for φ close to zero, i.e., when the vectors $\boldsymbol{K}$ and $\boldsymbol{a}$ are parallel (to within a small angle of the order of $1/kL_1$). Clearly, colinearity of the vectors $\boldsymbol{K}$ and $\boldsymbol{a}$ implies specular reflection, i.e., the incident and reflection angles relative to planes of equal ε_1 are equal.

However, the condition $\varphi\approx 0$ is not sufficient, since (6) contains a second factor,

$$\frac{\sin\left(kL_3\sin\frac{\theta}{2}\cos\varphi+\frac{\pi L_3}{l}\right)}{kL_3\sin\frac{\theta}{2}\cos\varphi+\frac{\pi L_3}{l}}+\frac{\sin\left(kL_3\sin\frac{\theta}{2}\cos\varphi-\frac{\pi L_3}{l}\right)}{kL_3\sin\frac{\theta}{2}\cos\varphi-\frac{\pi L_3}{l}},$$

which is very small everywhere except when the argument is close to zero. Suppose that condition (7) is satisfied; then taking $\cos\varphi=1$, we obtain new conditions in the form

$$\left|kL_3\sin\frac{\theta}{2}\pm\frac{\pi L_3}{l}\right|\ll 1,\quad\text{or}\quad\left|2\sin\frac{\theta}{2}\pm\frac{\lambda}{l}\right|\ll\frac{\lambda}{\pi L_3}. \tag{8}$$

Thus, significant diffraction is observed only for θ values satisfying the conditions

$$2\sin\frac{\theta}{2}\pm\frac{\lambda}{l}=0$$

(to within a small angle of the order of $\lambda/\pi L_3$).

In summary, we can say that diffraction from a sinusoidal spatial diffraction grating is observed only when two conditions are met, the condition of specular reflection and Bragg's condition /100/. If any of these conditions is not satisfied (with the accuracy indicated above), the incident electromagnetic wave passes through the grating "unimpeded," without undergoing substantial diffraction.

If we consider a whole set of spatial periodic diffraction gratings with various periods and various spatial orientations, then for given vectors $\boldsymbol{k}_0$ and $\boldsymbol{k}_s$, the wave is diffracted only by that particular grating which has its vector $\boldsymbol{a}$ parallel to $\boldsymbol{k}_0-\boldsymbol{k}_s$ and whose l satisfies condition (8). All the other gratings have no effect on the scattering.

The elementary calculations above show why scattering at a certain angle described by relation (1) depends only on one spectral component of the inhomogeneities.

In reality (26.27) contains $\Phi_\varepsilon(\boldsymbol{K})$, and not $\overline{\Phi}_\varepsilon(\boldsymbol{K})$, i.e., a spectrum averaged over some region of volume $8\pi^3/V$ in wavenumber space. Due to this averaging $\overline{\Phi}_\varepsilon(\boldsymbol{K})$ clearly incorporates spectral components close to $\boldsymbol{K}$. In the elementary treatment of this section, a similar situation arose when we required that the condition of specular reflection and Bragg's condition be satisfied to within angles of the order of λ/L. Thus, nearby spectral components always contribute to scattering at a given angle.

This result is readily understood from the following considerations. Diffraction from an infinite sinusoidal grating gives an infinitesimally narrow diffracted beam in the direction satisfying Bragg's condition. If the grating dimension L is finite, the angular extent of the diffracted beam is of the order of λ/L. As a result, the scattering observed in the presence of a whole range of diffraction gratings is determined not only by that diffraction grating for which Bragg's condition is exactly satisfied, but also by other gratings of approximately the same dimensions and orientation, whose diffraction maxima do not quite coincide with the selected direction but nevertheless overlap in view of the finite width of the diffracted beams.

In concluding this section, let us briefly consider the problem of the spatial correlation of the scattered fields. In Chapter 1 we derived the following relation for statistically homogeneous turbulence:

$$\langle Z_\varepsilon(d^3\varkappa)\, Z_\varepsilon^*(d^3\varkappa')\rangle = \Phi_\varepsilon(\varkappa)\,\delta(\varkappa-\varkappa')\,d^3\varkappa\, d^3\varkappa', \tag{9}$$

where $Z_\varepsilon(d^3\varkappa)$ is the (random) spectral amplitude of the ε fluctuations, corresponding to the wavenumber $\varkappa$. It follows from (9) that the spectral components corresponding to different $\varkappa$ are uncorrelated.

Consider the fields scattered in two directions θ_1 and θ_2. For an infinite scattering volume*, the scattering at each angle is determined precisely by one single component of the turbulence spectrum. By virtue of (9), different spectral components are uncorrelated, so that the fields scattered at two different directions by an infinite scattering volume are uncorrelated also. Now if the scattering volume is finite, the field scattered at a certain direction is produced by a group of spectral components bunched around the wavenumber $\varkappa = \boldsymbol{K}$, the distance between any two of them not exceeding $\sqrt[3]{\frac{8\pi^3}{V}} = \frac{2\pi}{L}$ (in order of magnitude). Hence, fields scattered in two different directions are correlated if the corresponding volumes in wavenumber space intersect. In this case, we have the relation

$$|\boldsymbol{K}_1 - \boldsymbol{K}_2| < \frac{2\pi}{L}. \tag{10}$$

* An infinite scattering volume is of course not permitted, since in this case the fundamental requirement of our approximation $\langle |E_s^2|\rangle \ll E_0$ breaks down.

If the same incident wave is scattered in both directions, $\boldsymbol{K}_1 = \boldsymbol{k}_0 - \boldsymbol{k}_{s1}$, $\boldsymbol{K}_2 = \boldsymbol{k}_0 - \boldsymbol{k}_{s2}$, and we have

$$|\boldsymbol{k}_{s1} - \boldsymbol{k}_{s2}| < \frac{2\pi}{L}.$$

If $\Delta\theta$ is the angle between $\boldsymbol{k}_{s1}$ and $\boldsymbol{k}_{s2}$, the last condition reduces to the form

$$2\sin\frac{\Delta\theta}{2} < \frac{\lambda}{L}$$

or (since generally $\Delta\theta \ll 1$),

$$\Delta\theta < \frac{\lambda}{L}. \qquad (11)$$

In this case the angular correlation radius of the scattered field is of the order of λ/L.

If the two observation points are at the same distance r from the scattering volume, the distance Δr between them is related to $\Delta\theta$ by the formula $\Delta r = r\Delta\theta$, so that the distance over which the scattered fields remain correlated is restricted by the inequality

$$\Delta r < \frac{\lambda r}{L}. \qquad (12)$$

In this case the linear correlation radius for transverse separation of observation points is of the order of $\lambda r/L$.

Let us also consider the correlation between two fields of different frequency scattered in the same direction. In this case $\boldsymbol{K}_1 = k_1(\boldsymbol{m} - \boldsymbol{n})$, $\boldsymbol{K}_2 = k_2(\boldsymbol{m} - \boldsymbol{n})$, and

$$|\boldsymbol{K}_1 - \boldsymbol{K}_2| = (k_1 - k_2)\,2\sin\frac{\theta}{2} = \frac{2\pi\Delta f}{c}\,2\sin\frac{\theta}{2}.$$

Inserting this expression in (10), we obtain the condition for two fields with a frequency difference Δf scattered in the same direction to remain correlated:

$$\Delta f < \frac{c}{2L\sin\frac{\theta}{2}}. \qquad (13)$$

Here L is the dimension of the volume V in the direction of the vector $(\boldsymbol{m} - \boldsymbol{n})$.

If scattering is used to transmit a signal occupying a finite frequency band, undistorted transmission requires complete correlation between the fluctuations of the various spectral components of the signal. This requirement is satisfied if the width of the signal spectrum meets condition (13). The parameter $\frac{c}{2L\sin\frac{\theta}{2}}$ can thus be called the passband of a scattering transmission channel.*

* In practice, this parameter is sufficiently large to permit scattering transmission even of TV broadcasts, which require passbands of a few MHz.

§ 28. The effective scattering volume

In the previous sections the scattering volume V was assumed to be given and we did not particularize its geometry. We will now consider this problem in greater detail. The most important case is when the scattering volume is formed by the intersecting beams of the transmitting and the receiving antenna. We will start with expression (25.8a) for the scattered field at the point $\boldsymbol{r}$. The function $\boldsymbol{E}_0(\boldsymbol{r}')$ (the incident wave field) can be written in the form

$$\boldsymbol{E}_0(\boldsymbol{r}') = \boldsymbol{e}_0(\boldsymbol{r}')\, f_0(\boldsymbol{m}(\boldsymbol{r}')), \tag{1}$$

where $\boldsymbol{m}(\boldsymbol{r}') = (\boldsymbol{r}' - \boldsymbol{R})/|\boldsymbol{r}' - \boldsymbol{R}|$ is the unit vector directed from the source to the point $\boldsymbol{r}'$, and $\boldsymbol{e}_0(\boldsymbol{r}')$ is the field of a dipole with the long axis of its lobe pointing in the direction of maximum emission of our source. The dipole strength is chosen so that the beam pattern $f_0(\boldsymbol{m})$ is 1 in this direction.

Let us determine the mean field seen by the receiving antenna. It can be found by integrating the field $\boldsymbol{E}_s(\boldsymbol{r})$ (the scattered field at the point $\boldsymbol{r}$) over the plane of the receiving aperture Σ centered at the point $\boldsymbol{r}_0$:

$$\mathscr{E}_s(\boldsymbol{r}_0) = \frac{1}{\Sigma}\iint\limits_{\Sigma} \boldsymbol{E}_s(\boldsymbol{r}_0 + \boldsymbol{\rho})\, d\xi\, d\eta = \\ = \frac{k^2}{4\pi\Sigma}\int\limits_V d^3r'\varepsilon_1(\boldsymbol{r}')\,[\boldsymbol{n}(\boldsymbol{r}')\,[\boldsymbol{E}_0(\boldsymbol{r}')\,\boldsymbol{n}(\boldsymbol{r}')]]\iint\limits_{\Sigma}\frac{e^{ik|\boldsymbol{r}_0-\boldsymbol{r}'+\boldsymbol{\rho}|}}{|\boldsymbol{r}_0-\boldsymbol{r}'+\boldsymbol{\rho}|}\,d\xi\, d\eta. \tag{2}$$

Expanding $|\boldsymbol{r}_0 - \boldsymbol{r}' + \boldsymbol{\rho}|$ in a series, we obtain the equality

$$\frac{e^{ik|\boldsymbol{r}_0-\boldsymbol{r}'+\boldsymbol{\rho}|}}{|\boldsymbol{r}_0-\boldsymbol{r}'+\boldsymbol{\rho}|} \approx \frac{e^{ik|\boldsymbol{r}_0-\boldsymbol{r}'|-ik\boldsymbol{n}(\boldsymbol{r}')\boldsymbol{\rho}}}{|\boldsymbol{r}_0-\boldsymbol{r}'|}, \tag{3}$$

which is valid when $k\rho^2/|\boldsymbol{r}_0 - \boldsymbol{r}'| \ll 1$, so that the next term in the expansion can be safely omitted.

Here ρ is of the order of the antenna size D. Consequently, this condition can be written in the form

$$r_0 \gg \frac{D^2}{\lambda}. \tag{4}$$

Condition (4) indicates that the scattering volume is located in the Fraunhofer diffraction region, where the antenna pattern $f_1(\boldsymbol{n})$ is applicable. The antenna pattern can be expressed in the form

$$\frac{1}{\Sigma}\iint\limits_{\Sigma} e^{-ik\boldsymbol{n}\boldsymbol{\rho}}\, d\xi\, d\eta = f_1(\boldsymbol{n}). \tag{5}$$

Using (5) and (1), we write

$$\mathscr{E}_s(\boldsymbol{r}_0) = \frac{k^2}{4\pi}\int \varepsilon_1(\boldsymbol{r}')\,[\boldsymbol{n}\,[\boldsymbol{e}_0\boldsymbol{n}]]\, f_0(\boldsymbol{m})\, f_1(\boldsymbol{n})\,\frac{e^{ik|\boldsymbol{r}_0-\boldsymbol{r}'|}}{|\boldsymbol{r}_0-\boldsymbol{r}'|}\, d^3r'. \tag{6}$$

This expression differs from (25.8a) only in that the domain of integration is infinite and the integrand contains two additional factors $f_0(\boldsymbol{m}(\boldsymbol{r}'))$ and $f_1(\boldsymbol{n}(\boldsymbol{r}'))$, characterizing the beam patterns of the transmitting and the receiving antennas. These factors take their maximum value on the beam axes, rapidly falling off with angular distance from the axes. The integral in (6), therefore, actually is taken over the intersection of the two beam patterns of the receiving and the transmitting antennas.

If using (6) we compute the scattered flux density, the result will be similar to (26.28), but the integrand will contain an additional factor $|f_1(\boldsymbol{n})|^2$ characterizing the power pattern of the receiving antenna and the integral will be taken over an infinite region:

$$\langle \boldsymbol{S}(\boldsymbol{r})\rangle = \int \boldsymbol{n}(\boldsymbol{r}') \frac{S_0(\boldsymbol{r}')\,\sigma_0(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^2} |f_1(\boldsymbol{n})|^2\, dV'. \tag{7}$$

This expression can be further clarified if we write

$$S_0(\boldsymbol{r}') = s_0(\boldsymbol{r}')\,|f_0(\boldsymbol{m}(\boldsymbol{r}'))|^2, \tag{8}$$

where $s_0(\boldsymbol{r}')$ is the energy flux density created by the dipole $\boldsymbol{e}_0$ (see (1)) and $|f_0(\boldsymbol{m})|^2$ is the normalized power pattern of the source (in the direction of maximum emission $f_0 = 1$). Then

$$\langle \boldsymbol{S}(\boldsymbol{r})\rangle = \int \boldsymbol{n}(\boldsymbol{r}') \frac{s_0(\boldsymbol{r}')\,\sigma_0(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^2} |f_0(\boldsymbol{m}(\boldsymbol{r}'))|^2 \cdot |f_1(\boldsymbol{n}(\boldsymbol{r}'))|^2\, dV'. \tag{9}$$

Two different cases may arise depending on the rate of change of the functions $s_0(\boldsymbol{r}')\,\sigma_0(\boldsymbol{r}')/|\boldsymbol{r}-\boldsymbol{r}'|^2$ and $|f_0f_1|^2$. Let the beam patterns f_0 and f_1 be "narrow," i.e., they change much faster than the function $s_0(\boldsymbol{r}')\,\sigma_0(\boldsymbol{r}')/|\boldsymbol{r}-\boldsymbol{r}'|^2$ does (in the following we will consider in some detail the conditions when this is indeed so). In this case the integral (9) can be approximately calculated by factoring out of the integral $\boldsymbol{n}(\boldsymbol{r}')\,s_0(\boldsymbol{r}')\,\sigma_0(\boldsymbol{r}')/|\boldsymbol{r}-\boldsymbol{r}'|^2$ evaluated at the intersection point of the two antenna beams. Denoting this point as $\boldsymbol{r}_1$, we have

$$\langle \boldsymbol{S}(\boldsymbol{r})\rangle \approx \boldsymbol{n}(\boldsymbol{r}_1) \frac{s_0(\boldsymbol{r}_1)\,\sigma(\boldsymbol{r}_1)}{|\boldsymbol{r}-\boldsymbol{r}_1|^2} V_{\text{ef}}, \tag{10}$$

where the effective scattering volume V_{ef} is

$$V_{\text{ef}} = \int |f_0(\boldsymbol{m}(\boldsymbol{r}'))\, f_1(\boldsymbol{n}(\boldsymbol{r}'))|^2\, d^3r'. \tag{11}$$

In some simple cases V_{ef} can be computed in explicit form. If the apertures of the receiving and the transmitting antenna are equal rectangles, we have

$$V_{\text{ef}} = \frac{d^3 \sin^2\gamma_1 \sin\gamma_2}{12\cos^3(\theta/2)\sin\theta} \approx \frac{d^3\gamma_1^2\gamma_2}{12\theta} \quad (\gamma_i \ll \theta \ll 1). \tag{12}$$

Here d is the distance between the two antennas (along a straight line), γ_1 is the angle between the main axis of the antenna beam and the first minimum of the antenna pattern in the vertical plane (which is the plane through the vectors k_0 and k_s), and γ_2 is the same angle for the antenna pattern in the horizontal plane (Figure 29).

For an arbitrary antenna aperture, V_{ef}(for $\gamma < \theta_0$) is expressed by the approximate relation

$$V_{ef} \approx \frac{d^3\gamma_{1ef}^2\gamma_{2ef}}{8\theta} \ (\theta \ll 1), \tag{12'}$$

where, in contrast to (12), γ_{1ef} and γ_{2ef} are the effective beam widths between half-power points in the vertical and horizontal plane (in (12), γ_1 and γ_2 are the half-widths between the first minima).

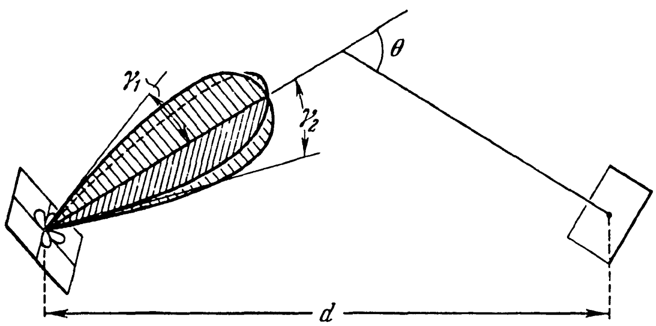

FIGURE 29. Illustrating the calculation of the effective scattering volume for rectangular-aperture antennas.

Consider the case when $K = 2k \sin\frac{\theta}{2}$ lies inside the inertial subrange of the spectrum of the fluctuations in ε. Then

$$\Phi_\varepsilon(K) = 0.033\, C_\varepsilon^2 (K)^{-11/3}$$

and by (26.27)

$$\sigma_0(\boldsymbol{r}') \approx 0.052 C_\varepsilon^2 k^{1/3} \sin^2\chi \left(2\sin\frac{\theta(r')}{2}\right)^{-11/3}. \tag{13}$$

Let $\boldsymbol{r}'$ be changed by $\Delta\boldsymbol{r}'$ so that the scattering angle θ_0 changes by an amount of the order of γ, where γ is the beam width. In this case, $f_0(\boldsymbol{m})$ and $f_1(\boldsymbol{n})$ drop from their maximum to their minimum values.

Let us estimate the resulting change in σ_0. The relative change is of the order

$$\left|\frac{\Delta\sigma_0}{\sigma_0}\right| \sim \frac{11}{3}\frac{\Delta\theta}{\theta_0} = \frac{11}{3}\frac{\gamma}{\theta_0}. \tag{14}$$

Thus, if

$$\gamma \ll \theta_0, \tag{15}$$

the change in σ_0 is insignificant and we may use (11). If, however,

$$\gamma \gg \theta_0, \tag{16}$$

the integrand in (9) is affected by the change in $\sigma_0(\mathbf{r}')$ more than by the change in f_0/f_1. In this case, the effective scattering volume is no longer limited by the beam width γ but by the decrease in scattered intensity with increasing angle θ.

Additional variation in $\sigma_0(\mathbf{r}')$ may be caused by the variation with height of C_ε^2. For example, in case of free convection, $C_\varepsilon^2(z)$ decreases as $z^{-4/3}$ with the height z above the earth's surface (see Chapter 1, Part B). In this case,

$$\sigma_0(\mathbf{r}') \sim z^{-5} \sim \theta^{-5}, \tag{17}$$

since $\theta(\mathbf{r}') \approx 4z/d$.

Therefore, for wide beam antennas, the scattered intensity drops markedly as the scattering angle θ (initially having the minimum value θ_0) is increased by an amount of the order of θ_0. This signifies that the effective scattering volume in this case is limited by the scattering angle θ_0, and not by the beam width γ.

Substituting θ_0 for γ in (12), we get

$$V_э \sim d^3\theta_0^2 \text{ for } \theta_0 \ll \gamma;\ \theta_0 \ll 1. \tag{18}$$

The scattering volume, as we have already established, determines the correlation properties of the scattered field. Consequently, the correlation radii of the scattered fields, and the pass band Δf, are expressed by different formulas for narrow-beam and wide-beam antennas. For example, the angular correlation of fields scattered in two different directions is preserved up to angles of the order of

$$\Delta\theta \sim \frac{\lambda}{\gamma d} \quad \text{for } \gamma \ll \theta_0, \qquad \Delta\theta \sim \frac{\lambda}{\theta_0 d} \quad \text{for } \gamma \gg \theta_0, \tag{19}$$

whence it follows that as the observation points are moved away from one another at right angles to the direction to the source, the correlation radii are of the order of magnitude of

$$a_{\text{cor}} \sim \frac{\lambda}{\gamma} \quad \text{for } \gamma \ll \theta_0, \qquad a_{\text{cor}} \sim \frac{\lambda}{\theta_0} \quad \text{for } \gamma \gg \theta_0. \tag{20}$$

The expressions for the passband Δf are

$$\Delta f \sim \frac{c}{\gamma\theta_0 d} = \frac{1}{\gamma\theta T_0} \quad \text{for } \gamma \ll \theta_0,$$
$$\Delta f \sim \frac{c}{\theta_0^2 d} = \frac{1}{\theta_0^2 T_0} \quad \text{for } \gamma \gg \theta_0, \tag{21}$$

where $T_0 = d/c$ is the time for the wave to propagate from the transmitter to the receiver.

In (19) – (21) we invariably omitted the particular numerical coefficients, and some of these expressions will be improved in the following.

§ 29. The frequency spectrum of the scattered field

So far we have considered only the mean scattered intensity. Let us now turn to the time autocorrelation function and the frequency spectrum of the scattered field fluctuations. To simplify the mathematics, note that expression (26.27) for the effective scattering cross section derived assuming that $L_0 \ll \sqrt{\lambda r}$ (L_0 is the correlation radius of the inhomogeneities in the medium, r is the distance to the source or to the observation point) can be derived with much less difficulty if the much stronger condition $L \ll \sqrt{\lambda r}$ is satisfied (L is the size of the scattering volume). Since the results are identical in both cases, we will adopt the condition $L \ll \sqrt{\lambda r}$ in deriving the expression for the time autocorrelation function. In this case, using the expansion

$$\exp\{ik\,|\boldsymbol{r} - \boldsymbol{r}'|\} \approx \exp\{ikr - ik\boldsymbol{n}_0\boldsymbol{r}'\} \tag{1}$$

for the exponential in (25.8a) (here $\boldsymbol{n}_0 = \boldsymbol{r}/r$ and $\boldsymbol{r}'$ goes over the entire scattering volume) and in the denominator putting $|\,\boldsymbol{r} - \boldsymbol{r}'\,| \approx r$, we find

$$\boldsymbol{E}_s(\boldsymbol{r}) \approx \frac{k^2 e^{ikr}}{4\pi r} \int_V \varepsilon_1(\boldsymbol{r}')\,[\boldsymbol{n}\,[\boldsymbol{E}_0(\boldsymbol{r}')\,\boldsymbol{n}]]\,e^{-ik\boldsymbol{n}_0\boldsymbol{r}'}\,d^3r'. \tag{2}$$

Assuming an incident plane wave $\boldsymbol{E}_0(\boldsymbol{r}')$ (this implies that $L \ll \sqrt{\lambda R}$, where R is the distance to the source), we write

$$\boldsymbol{E}_0(\boldsymbol{r}) = \boldsymbol{A}_0 e^{i\boldsymbol{k}_0\boldsymbol{r}}.$$

Moreover, in the integrand we take $\boldsymbol{n}(\boldsymbol{r}') = \boldsymbol{n}_0$ (this is legitimate for $L \ll r$). Putting $[\boldsymbol{n}[\boldsymbol{A}_0\boldsymbol{n}]] \equiv \boldsymbol{a}$ and $\boldsymbol{k}_0 = k\boldsymbol{m}_0$, we obtain

$$\boldsymbol{E}_s(\boldsymbol{r}) \approx \frac{k^2\boldsymbol{a}e^{ikr}}{4\pi r} \int_V \varepsilon_1(\boldsymbol{r}')\,e^{ik(\boldsymbol{m}_0 - \boldsymbol{n}_0)\boldsymbol{r}'}\,d^3r'. \tag{3}$$

$\boldsymbol{E}_s$ is the complex amplitude of the scattered field, which is related to the field $\mathscr{E}_s$ by the expression

$$\mathscr{E}_s(\boldsymbol{r}, t) = \boldsymbol{E}_s(\boldsymbol{r}, t)\,e^{-i\omega t}.$$

So far, we have ignored the explicit dependence of ε_1 and $\boldsymbol{E}_s$ on time. When transforming to the fields $\mathscr{E}_s$, we write the time dependence in explicit form:

$$\mathscr{E}_s(\boldsymbol{r}, t) \approx \frac{k^2 a e^{i(kr-\omega t)}}{4\pi r} \int\limits_V \varepsilon_1(\boldsymbol{r}, t)\, e^{ik(\boldsymbol{m}_0-\boldsymbol{n}_0)\boldsymbol{r}'}\, d^3r'. \tag{4}$$

Now consider the time autocorrelation function of the scattered field

$$B_{\mathscr{E}}(\tau) = \langle \mathscr{E}_s(\boldsymbol{r}, t)\, \mathscr{E}_s^*(\boldsymbol{r}, t+\tau)\rangle. \tag{5}$$

Inserting (4) and noting that for statistically homogeneous and stationary turbulence

$$\langle \varepsilon_1(\boldsymbol{r}', t)\, \varepsilon_1(\boldsymbol{r}'', t+\tau)\rangle = B_\varepsilon(\boldsymbol{r}' - \boldsymbol{r}'', \tau), \tag{6}$$

we find

$$B_{\mathscr{E}}(\tau) = \frac{k^4 a^2}{(4\pi r)^2}\, e^{i\omega\tau} \iint\limits_{VV} B_\varepsilon(\boldsymbol{r}' - \boldsymbol{r}'', \tau)\, e^{ik(\boldsymbol{m}_0-\boldsymbol{n}_0)(\boldsymbol{r}'-\boldsymbol{r}'')}\, d^3r'\, d^3r''. \tag{7}$$

In (7), $\boldsymbol{r}''$ is replaced by a new variable $\boldsymbol{\rho} = \boldsymbol{r}' - \boldsymbol{r}''$. The integrand is then a function of $\boldsymbol{\rho}$ only and the integration over $\boldsymbol{r}'$ can be carried out explicitly,* giving the scattering volume V. Now putting $a^2 = A_0^2 \sin^2\chi$, where χ is the angle between the vectors $\boldsymbol{A}_0$ and $\boldsymbol{n}_0$, we find

$$B_{\mathscr{E}}(\tau) = \frac{k^4 A_0^2 V \sin^2\chi}{(4\pi r)^2}\, e^{i\omega\tau} \int\limits_V B_\varepsilon(\boldsymbol{\rho}, \tau)\, e^{ik(\boldsymbol{m}_0-\boldsymbol{n}_0)\boldsymbol{\rho}}\, d^3\rho. \tag{8}$$

We now use the spectral expansion of the space-time correlation function introduced in Chapter 1:

$$B_\varepsilon(\boldsymbol{\rho}, \tau) = \iiiint\limits_{-\infty}^{\infty} e^{-i(\boldsymbol{\varkappa}\boldsymbol{\rho}+\Omega\tau)}\, u_\varepsilon(\boldsymbol{\varkappa}, \Omega)\, d^3\varkappa\, d\Omega, \tag{9}$$

where $u_\varepsilon(\boldsymbol{\varkappa}, \Omega)$ is the space-time spectrum of the inhomogeneities (since u_ε is even both with respect to $\boldsymbol{\varkappa}$ and Ω, we replace the cosine by an exponential function). Inserting (9) in (8), recalling the definition of the function $\delta_V(\boldsymbol{\varkappa})$ and averaging over the volume $T = 8\pi^3/V$ in the $\boldsymbol{\varkappa}$ space, we obtain

$$B_{\mathscr{E}}(\tau) = \frac{\pi k^4 A_0^2 V \sin^2\chi}{2r^2}\, e^{i\omega\tau} \int\limits_{-\infty}^{\infty} e^{-i\Omega\tau}\, \bar{u}_\varepsilon(\boldsymbol{K}, \Omega)\, d\Omega, \tag{10}$$

where as before $\boldsymbol{K} = k(\boldsymbol{m}_0 - \boldsymbol{n}_0)$ is the difference between the incident and the scattered wave vectors.

Substitution of the variable $\Omega' = \omega - \Omega$ in (10) gives for $B_{\mathscr{E}}(\tau)$ a Fourier integral in the time domain:

$$B_{\mathscr{E}}(\tau) = \int\limits_{-\infty}^{\infty} e^{i\Omega\tau}\, \frac{\pi k^4 A_0^2 V \sin^2\chi}{2r^2}\, \bar{u}_\varepsilon(\boldsymbol{K}, \omega - \Omega)\, d\Omega. \tag{11}$$

* In general, the limits of integration over ρ depend on $\boldsymbol{r}'$, but the corresponding correction is insignificant.

Hence for the frequency spectrum of the scattered field, $W_{\mathscr{E}}(\Omega)$, we have

$$W_{\mathscr{E}}(\Omega)=\frac{\pi k^4 A_0^2 V \sin^2\chi}{2r^2}\bar{u}_\varepsilon(\boldsymbol{K},\omega-\Omega), \tag{12}$$

i. e., the function $W_{\mathscr{E}}(\Omega)$ is proportional to the space-time spectrum of the dielectric-constant inhomogeneities. Here the spatial spectral decomposition is produced by the scattering process itself, whereas the time decomposition is performed by an instrument which analyzes the frequency spectrum of the scattered field.

Note that the function $\bar{u}_\varepsilon(\boldsymbol{K},\omega-\Omega)$ differs from zero only when its argument is small, $|\omega-\Omega|<\Delta\Omega$ ($\Delta\Omega$ is generally of the order of 10^3 Hz).

Since the electromagnetic field frequency ω is much higher than the frequency of the dielectric constant fluctuations, the function (12) differs from zero only when

$$|\omega-\Omega|\lesssim\Delta\Omega. \tag{13}$$

Therefore, the scattered field spectrum is concentrated around the carrier frequency ω in a frequency band of the order of $\Delta\Omega$.

The autocorrelation function of the field envelope $\boldsymbol{E}$ coincides with (10) except for the factor $\exp(i\omega\tau)$, so that the spectrum of the envelope is given by

$$W_E(\Omega)=\frac{\pi k^4 A_0^2 V\sin^2\chi}{2r^2}\bar{u}_\varepsilon(\boldsymbol{K},\Omega) \tag{14}$$

and, unlike $W_{\mathscr{E}}(\Omega)$, the envelope's spectrum $W_E(\Omega)$ is concentrated near zero frequency (its shape, however, is identical to that of the spectrum of the field $\mathscr{E}$).

Let us consider in more detail the case of "frozen" turbulence, when by (6.10)

$$u_\varepsilon(\boldsymbol{K},\Omega)=\Phi_\varepsilon(\boldsymbol{K})\,\delta(\Omega+\boldsymbol{K}\boldsymbol{v}), \tag{15}$$

where $\boldsymbol{v}$ is the transport velocity of the inhomogeneities (the wind velocity). What we have in (14), however, is not the function $u_\varepsilon(\boldsymbol{K},\Omega)$ itself, but the averaged value

$$\bar{u}_\varepsilon(\boldsymbol{K},\Omega)\approx\frac{1}{T}\int\limits_T u_\varepsilon(\boldsymbol{K}+\boldsymbol{\varkappa},\Omega)\,d^3\varkappa, \tag{16}$$

where $T=\frac{8\pi^3}{V}$ is a volume in wavenumber space near the point $\varkappa=0$.

In evaluating (16), the function $\Phi_\varepsilon(\boldsymbol{K})$ is fairly smooth, so that we may take it outside the integral, and therefore

$$\bar{u}_\varepsilon(\boldsymbol{K},\Omega)\approx\Phi_\varepsilon(\boldsymbol{K})\cdot\frac{1}{T}\int\limits_T\delta(\Omega+\boldsymbol{K}\boldsymbol{v}+\boldsymbol{\varkappa}\boldsymbol{v})\,d^3\varkappa. \tag{16'}$$

The axis $\varkappa_1$ is directed along the vector $\boldsymbol{v}$, so that $\boldsymbol{v} = \{v, 0, 0\}$. Then $\boldsymbol{\varkappa v} = \varkappa_1 v$. To simplify the calculations, let the volume T be a right parallelepiped with one of its sides along the vector $\boldsymbol{v}$. Then $T = \frac{8\pi^3}{L_1 L_2 L_3}$ and

$$\bar{u}_\varepsilon(\boldsymbol{K}, \Omega) \approx \Phi_\varepsilon(\boldsymbol{K}) \frac{L_1 L_2 L_3}{8\pi^3} \int\limits_{-\pi/L_2}^{\pi/L_2} d\varkappa_2 \int\limits_{-\pi/L_3}^{\pi/L_3} d\varkappa_3 \int\limits_{-\pi/L_1}^{\pi/L_1} \delta(\Omega + \boldsymbol{Kv} + \varkappa_1 v) d\varkappa_1. \qquad (16'')$$

Since

$$\delta(\Omega + \boldsymbol{Kv} + v\varkappa_1) = \frac{1}{v}\delta\left(\frac{\Omega + \boldsymbol{Kv}}{v} + \varkappa_1\right),$$

the inner integral in (16'') is readily evaluated:

$$\int\limits_{-\pi/L_1}^{\pi/L_1} \delta(\Omega + \boldsymbol{Kv} + \varkappa_1 v)\, d\varkappa_1 = \frac{1}{v} \cdot \theta\left(\frac{\pi v}{L_1} - |\Omega + \boldsymbol{Kv}|\right), \qquad (17)$$

where

$$\theta(x) = \begin{cases} 1 & \text{for } x > 0, \\ 0 & \text{for } x < 0, \end{cases}$$

so that

$$\bar{u}_\varepsilon(\boldsymbol{K}, \Omega) \approx \Phi_\varepsilon(\boldsymbol{K}) \frac{L_1}{2\pi v}\, \theta\left(\frac{\pi v}{L_1} - |\Omega + \boldsymbol{Kv}|\right).$$

Let $\tau = L_1/v$ be the time for a moving inhomogeneity to traverse the scattering volume. Then insertion of $\bar{u}_\varepsilon$ in (14) gives

$$W_E(\Omega) = \frac{\pi k^4 A_0^2 V \sin^2\chi}{2r^2}\, \Phi_\varepsilon(\boldsymbol{K})\, \frac{\tau}{2\pi} \theta\left(\frac{\pi}{\tau} - |\Omega + \boldsymbol{Kv}|\right). \qquad (18)$$

The function $\theta\left(\frac{\pi}{\tau} - |\Omega + \boldsymbol{Kv}|\right)$ is zero for $|\Omega + \boldsymbol{Kv}| > \frac{\pi}{\tau}$, i.e., for $\Omega + \boldsymbol{Kv} > \frac{\pi}{\tau}$ and for $\Omega + \boldsymbol{Kv} < -\frac{\pi}{\tau}$, or in other words for $\Omega > -\boldsymbol{Kv} + \frac{\pi}{\tau}$ and for $\Omega < -\boldsymbol{Kv} - \frac{\pi}{\tau}$.
Inside this interval, i.e., for

$$-\boldsymbol{Kv} - \frac{\pi}{\tau} < \Omega < -\boldsymbol{Kv} + \frac{\pi}{\tau}, \qquad (19)$$

the function $\theta\left(\frac{\pi}{\tau} - |\Omega + \boldsymbol{Kv}|\right)$ is unity.

Therefore, in the case of "frozen" turbulence the frequency spectrum is concentrated around the frequency

$$\Omega_0 = -\boldsymbol{Kv} \qquad (20)$$

in a frequency band

$$\Delta\Omega_0 = \frac{2\pi}{\tau} = \frac{2\pi v}{L_1}; \tag{21}$$

Ω_0 is the Doppler frequency shift due to the motion of the inhomogeneities.
If $L_1 \to \infty$,

$$\frac{\tau}{2\pi}\,\theta\left(\frac{\pi}{\tau} - |\,\Omega + \boldsymbol{Kv}\,|\right) \to \delta(\Omega + \boldsymbol{Kv}),$$

since the integral of this function is always unity, and its maximum value, $\tau/2\pi$, approaches infinity as $L_1 \to \infty$. For a finite L_1, we have a "smeared" δ-function which gives a frequency band of finite width. The form of the spectrum is described by a single function $\theta\left(\frac{\pi}{\tau} - |\,\Omega + \boldsymbol{Kv}|\right)$ in our treatment, since we used a right parallelepiped to approximate the volume T. In fact T is a region with "smeared" boundaries, so that the spectrum is by no means as simple as implied by (18). However, relations (20) and (21) remain valid, and (21) gives the effective rather than the exact bandwidth.

If, in addition to the general transport with velocity v the various scattering elements also move with certain relative velocities (averaging out to zero), the frequency band is additionally broadened by $(\Delta\Omega)' \sim \omega\frac{\delta v}{c}$, where δv is the order of magnitude of the relative velocities inside the scattering volume. The ratio of these bandwidths is of the order

$$\frac{(\Delta\Omega)'}{\Delta\Omega_0} \sim \frac{Lk\delta v}{v}. \tag{22}$$

In practice $kL \gg 1$, and $\frac{\delta v}{v} \ll 1$, but for turbulent velocity fluctuations $\delta v/v$ is generally of the order of $0.1 - 0.3$, so that the ratio (22) may be quite large. This means that in practice the spectrum width is determined not by the finite size of the scattering volume but rather by the relative velocities of the scattering elements inside that volume. (This effect, of course, is also described by relation (14).)

The overall shift of the spectrum by $\Omega_0 = -\boldsymbol{Kv}$ is observed always (in the presence of random velocity fluctuations $\boldsymbol{v}$ is to be interpreted as the mean velocity of motion inside the scattering volume).

The frequency spectrum of a signal scattered off wandering inhomogeneities is considered in more detail in /66 – 70/. We will approach the problem on the basis of the general expression (12).

To compute the space-time spectral density $u(\boldsymbol{\varkappa}, \Omega)$, we will use a method similar to that advanced in /70/. We will show in the following that the time autocorrelation function of the envelope of the scattered field falls off to a negligibly small value in such a short time that the velocities of the individual inhomogeneities (which are now regarded as different at different points) can be treated as constant. Thus, for sufficiently small times τ, we have

$$\varepsilon_1(\boldsymbol{r}, t+\tau) = \varepsilon_1(\boldsymbol{r} - \boldsymbol{v}(\boldsymbol{r}, t)\,\tau, t), \tag{23}$$

where $\boldsymbol{v}(\boldsymbol{r}, t)$ is the (Lagrange) velocity of that element which at the time $(t+\tau)$ passes through the point $\boldsymbol{r}$.

Since τ is sufficiently small, the velocity $\boldsymbol{v}$ hardly changes as t is changed by an amount τ (suitable estimates will be given below). Therefore in (23) we do not distinguish between $\boldsymbol{v}(\boldsymbol{r}, t+\tau)$ and $\boldsymbol{v}(\boldsymbol{r}, t)$, and write $\boldsymbol{v}\tau$ instead of $\int \boldsymbol{v}\, d\tau$. Condition (23) indicates that the transportable characteristic of the liquid — the dielectric constant in this case — is a conservative quantity. We can thus ignore the smoothing of inhomogeneities by molecular diffusion and heat conduction processes in a short time (the corrections contributed by this effect are of the same order of smallness as the relative variations of the local transport velocity $\boldsymbol{v}$).

Condition (23) is superficially similar to the condition of frozen turbulence, but unlike the latter, the velocity $\boldsymbol{v}$ in (23) changes from point to point and does not remain constant for long. Condition (23) can be called the condition of "locally frozen" turbulence.

In computing the space-time correlation function, we will assume the fluctuations in ε_1 and $\boldsymbol{v}$ to be statistically independent. Indeed, these functions for a locally isotropic turbulent flow are uncorrelated (see Chapter 1). If we further assume a normal distribution for ε_1 and $\boldsymbol{v}$, they of necessity will be independent.* To obtain the space-time correlation function for ε_1, we multiply the equality

$$\varepsilon_1(\boldsymbol{r}+\boldsymbol{\rho}, t+\tau) = \varepsilon_1(\boldsymbol{r}+\boldsymbol{\rho} - \boldsymbol{v}(\boldsymbol{r}+\boldsymbol{\rho}, t)\,\tau, t)$$

by $\varepsilon_1(\boldsymbol{r}, t)$ and average over the fluctuations in ε_1 and over the fluctuations in $\boldsymbol{v}$. Since ε_1 and $\boldsymbol{v}$ are assumed independent, the two averaging processes can be carried out separately. Averaging the expression

$$\varepsilon_1(\boldsymbol{r}+\boldsymbol{\rho}, t+\tau)\, \varepsilon_1(\boldsymbol{r}, t) = \varepsilon_1(\boldsymbol{r}+\boldsymbol{\rho} - \boldsymbol{v}(\boldsymbol{r}+\boldsymbol{\rho}, t)\tau, t)\varepsilon_1(\boldsymbol{r}, t)$$

over ε_1, we obtain (by definition) the correlation function B_ε, which however contains an additional random parameter $\boldsymbol{v}$:

$$\langle \varepsilon_1(\boldsymbol{r}+\boldsymbol{\rho}, t+\tau)\, \varepsilon_1(\boldsymbol{r}, t)\rangle_{\varepsilon_1} = B_\varepsilon(\boldsymbol{\rho} - \boldsymbol{v}(\boldsymbol{r}+\boldsymbol{\rho}, t)\,\tau). \tag{24}$$

This expression should be additionally averaged over $\boldsymbol{v}$:

$$B_\varepsilon(\boldsymbol{\rho}, \tau) = \langle\langle \varepsilon_1(\boldsymbol{r}+\boldsymbol{\rho}, t+\tau)\, \varepsilon_1(\boldsymbol{r}, t)\rangle_{\varepsilon_1}\rangle_v = \langle B_\varepsilon(\boldsymbol{\rho} - \boldsymbol{v}(\boldsymbol{r}+\boldsymbol{\rho}, t)\tau)\rangle_v. \tag{25}$$

$B_\varepsilon(\boldsymbol{\rho})$ can be represented as a Fourier integral:

$$B_\varepsilon(\boldsymbol{\rho}, \tau) = \left\langle \int e^{i\varkappa[\rho - v(r+\rho, t)\tau]}\, \Phi_\varepsilon(\varkappa)\, d^3\varkappa \right\rangle_v = \int e^{i\varkappa\rho} \langle e^{-i\tau\varkappa v(r+\rho, t)}\rangle_v\, \Phi_\varepsilon(\varkappa)\, d^3\varkappa. \tag{26}$$

By definition,

$$\langle e^{i\boldsymbol{a}\boldsymbol{v}(r, t)}\rangle_v = \chi_{r, t}(\boldsymbol{a}) \tag{27}$$

* The actual distribution of ε_1 and $\boldsymbol{v}$ is nearly normal. The distribution of the two-point velocity difference, however, is a priori not normal, since $\langle(\Delta v_i)^3\rangle \neq 0$ (see Chapter 1).

is the characteristic function of the probability distribution of the velocity v at point r at time t. Since the field v is assumed to be statistically homogeneous, χ is virtually independent of r and t. Hence

$$B_\varepsilon(\rho, \tau) = \int e^{i\varkappa\rho}\chi(-\varkappa\tau)\,\Phi_\varepsilon(\varkappa)\,d^3\varkappa. \tag{28}$$

Taking the Fourier transform of (28) with respect to ρ and τ, we obtain

$$u_\varepsilon(\varkappa, \Omega) = \frac{1}{2\pi}\Phi_\varepsilon(\varkappa)\int\limits_{-\infty}^{\infty} e^{-i\Omega\tau}\chi(-\varkappa\tau)\,d\tau. \tag{29}$$

Expression (29) gives the space-time spectrum of the dielectric constant for "locally frozen" turbulence. Taking the Fourier transform of (29) with respect to Ω we obtain

$$\chi(-a) = \frac{1}{\Phi_\varepsilon(a\tau^{-1})}\int u_\varepsilon(a\tau^{-1}, \Omega)\,e^{i\Omega\tau}\,d\Omega,$$

which permits determining the function $\chi(a)$ if $u_\varepsilon(\varkappa, \Omega)$ is known from experiment.

As an example, consider the case when $v(r, t)$ is normally distributed with mean v_0 and variance

$$\langle (v_i - v_{0_i})^2\rangle = \frac{1}{3}\langle(\delta v)^2\rangle = \frac{1}{3}\sigma_v^2$$

for each component (summation over i is not implied). In this case,

$$\chi(a) = e^{iav_0 - \frac{1}{2}\langle(a(v-v_0))^2\rangle} = e^{iav_0 - \frac{1}{6}a^2\sigma_v^2}. \tag{30}$$

Inserting (30) into (29) and carrying out the integration gives

$$u_\varepsilon(\varkappa, \Omega) = \Phi_\varepsilon(\varkappa)\frac{1}{\sqrt{2\pi\varkappa^2\sigma_v^2/3}}\exp\left(-\frac{(\Omega + \varkappa v_0)^2}{2\varkappa^2\sigma_v^2/3}\right). \tag{31}$$

Expression (31) is a generalization of the previous expression

$$u_\varepsilon(\varkappa, \Omega) = \Phi_\varepsilon(\varkappa)\,\delta(\Omega + \varkappa v_0), \tag{31'}$$

derived for pure "frozen" turbulence to the case of "locally frozen" turbulence. If $\sigma_v^2 = 0$, (31) reduces to (31').

From (31), the effective width of the spectrum is of the order

$$\sqrt{\frac{\varkappa^2\sigma_v^2}{3}} \sim \Delta\Omega.$$

The correlation time τ_0 is estimated from the relation $\tau_0 \cdot \Delta\Omega \sim 1$, which gives $\tau_0 \sim (\varkappa\sigma_v)^{-1}$. The wavenumber $\varkappa$ should be replaced by L_0^{-1}, where L_0 is the outer scale of turbulence. Thus,

$$\tau_0 \sim \frac{L_0}{\sigma_v}.$$

In deriving (29) we assumed that the velocity of a fluid element could be regarded as constant during a time τ. Since the correlation time is of the order $\tau_0 \sim L_0/\sigma_v$, the velocity of a fluid element should in fact change very little during the time τ_0. The change in the velocity of an element during the time τ_0 is of the order /13/

$$\Delta v_{\tau_0} \sim \sqrt{\varepsilon\tau_0} \sim \sqrt{\frac{\varepsilon L_0}{\sigma_v}},$$

where ε is the rate of dissipation of the turbulent energy (not to be confused with the dielectric constant!). On the other hand, from the "2/3 law" we have $(\varepsilon L_0)^{2/3} \sim \sigma_v^2$, therefore $\Delta v_{\tau_0} \sim \sigma_v$. Thus the requirement that the change in velocity during the time τ_0 be small compared to the mean velocity v_0 can be expressed in the form

$$\sigma_v \ll v_0. \tag{32}$$

This condition is definitely true for the real atmosphere. The assumption of constant $\boldsymbol{v}$ during the time τ introduced in the derivation of (29) is valid when condition (32) is satisfied.

Inserting (31) in (14) (and disregarding the difference between u and $\bar{u}$, which has been analyzed in some detail previously), we obtain

$$W_E(\Omega) = \frac{\pi k^4 A_0^2 V \sin^2\chi}{2r^2} \Phi_\varepsilon(\boldsymbol{K}) \frac{1}{\sqrt{2\pi K^2\sigma_v^2/3}} \exp\left(-\frac{(\Omega + \boldsymbol{K}\boldsymbol{v}_0)^2}{2K^2\sigma_v^2/3}\right). \tag{33}$$

It thus follows that the peak of the spectrum is at the Doppler frequency $-\boldsymbol{K}\boldsymbol{v}_0$ associated with the mean motion of the scattering elements, whereas the shape of the frequency spectrum follows the probability distribution curve of the fluctuations in the velocity component directed along the scattering vector $\boldsymbol{K}$.

Let us now consider further the frequency spectrum of the intensity fluctuations of the scattered field. The intensity of the scattered field is proportional to $I = \mathscr{E}\mathscr{E}^* = \boldsymbol{E}\boldsymbol{E}^*$. Using (3), we get

$$I(t) = \frac{k^4 A_0^2 \sin^2\chi}{(4\pi r)^2} \iint\limits_{VV} \varepsilon_1(\boldsymbol{r}_1, t)\, \varepsilon_1(\boldsymbol{r}_2, t)\, e^{i\boldsymbol{K}(\boldsymbol{r}_1 - \boldsymbol{r}_2)} d^3r_1\, d^3r_2. \tag{34}$$

Consider the function

$$B_I(\tau) = \langle (I(t+\tau) - \langle I(t+\tau)\rangle)(I(t) - \langle I(t)\rangle)\rangle = \langle I(t+\tau)\, I(t)\rangle - (\langle I\rangle)^2 \tag{35}$$

(in the last equality we made use of the assumed stationarity of the turbulence). As we see from (34), $\langle I(t+\tau)\, I(t)\rangle$ contains fourth-order moments of the field $\varepsilon_1(\boldsymbol{r}, t)$:

$$\langle I(t+\tau)I(t)\rangle=\left(\frac{k^2A_0\sin\chi}{4\pi r}\right)^4\iiiint\limits_{VVVV}\langle\varepsilon_1(\boldsymbol{r}_1,t+\tau)\,\varepsilon_1(\boldsymbol{r}_2,t+\tau)\,\varepsilon_1(\boldsymbol{r}_3,t)\,\varepsilon_1(\boldsymbol{r}_4,t)\rangle\times$$
$$\times e^{i\boldsymbol{K}(\boldsymbol{r}_1-\boldsymbol{r}_2+\boldsymbol{r}_3-\boldsymbol{r}_4)}\,d^3r_1\,d^3r_2\,d^3r_3\,d^3r_4. \qquad (36)$$

To evaluate (36), we assume that the fourth-order moments of the field ε_1 are related to the second-order moments by the same expressions as in case of a normal distribution (Millionshchikov's hypothesis). If a_1, a_2, a_3, a_4 are random variables from four measurements of a normal probability distribution and $\langle a_i\rangle=0$, then

$$\langle a_1a_2a_3a_4\rangle=\langle a_1a_2\rangle\langle a_3a_4\rangle+\langle a_1a_3\rangle\langle a_2a_4\rangle+\langle a_1a_4\rangle\langle a_2a_3\rangle.$$

Using this relation, we find

$$\langle\varepsilon_1(\boldsymbol{r}_1,t+\tau)\,\varepsilon_1(\boldsymbol{r}_2,t+\tau)\,\varepsilon_1(\boldsymbol{r}_3,t)\,\varepsilon_1(\boldsymbol{r}_4,t)\rangle\equiv A(\boldsymbol{r}_1,\boldsymbol{r}_2,\boldsymbol{r}_3,\boldsymbol{r}_4,t,\tau)=$$
$$=\langle\varepsilon_1(\boldsymbol{r}_1,t+\tau)\,\varepsilon_1(\boldsymbol{r}_2,t+\tau)\rangle\cdot\langle\varepsilon_1(\boldsymbol{r}_3,t)\,\varepsilon_1(\boldsymbol{r}_4,t)\rangle+$$
$$+\langle\varepsilon_1(\boldsymbol{r}_1,t+\tau)\,\varepsilon_1(\boldsymbol{r}_3,t)\rangle\cdot\langle\varepsilon_1(\boldsymbol{r}_2,t+\tau)\,\varepsilon_1(\boldsymbol{r}_4,t)\rangle+$$
$$+\langle\varepsilon_1(\boldsymbol{r}_1,t+\tau)\,\varepsilon_1(\boldsymbol{r}_4,t)\rangle\cdot\langle\varepsilon_1(\boldsymbol{r}_2,t+\tau)\,\varepsilon_1(\boldsymbol{r}_3,t)\rangle. \qquad (37)$$

Using (23), we obtain

$$A=B_\varepsilon(\boldsymbol{r}_1-\boldsymbol{r}_2)B_\varepsilon(\boldsymbol{r}_3-\boldsymbol{r}_4)+B_\varepsilon(\boldsymbol{r}_1-\boldsymbol{r}_3-\boldsymbol{v}(\boldsymbol{r}_1,t)\tau)\times$$
$$\times B_\varepsilon(\boldsymbol{r}_2-\boldsymbol{r}_4-\boldsymbol{v}(\boldsymbol{r}_2,t)\tau)+$$
$$+B_\varepsilon(\boldsymbol{r}_1-\boldsymbol{r}_4-\boldsymbol{v}(\boldsymbol{r}_1,t)\tau)\,B_\varepsilon(\boldsymbol{r}_2-\boldsymbol{r}_3-\boldsymbol{v}(\boldsymbol{r}_2.\,t)\tau). \qquad (38)$$

Inserting (38) in (36), we see that the first term in (38) gives $\langle I\rangle^2$; subtracting this quantity from the two sides of (36), we obtain

$$B_I(\tau)=\langle I(t+\tau)I(t)\rangle-\langle I\rangle^2=$$
$$=\left(\frac{k^2A_0\sin\chi}{4\pi r}\right)^4\left\langle\left[\iint\limits_{VV}B_\varepsilon(\boldsymbol{r}_1-\boldsymbol{r}_3-\boldsymbol{v}(\boldsymbol{r}_1,t)\tau)\,e^{i\boldsymbol{K}(\boldsymbol{r}_1+\boldsymbol{r}_3)}\,d^3r_1\,d^3r_3\right]\times\right.$$
$$\times\left[\iint\limits_{VV}B_\varepsilon(\boldsymbol{r}_2-\boldsymbol{r}_4-\boldsymbol{v}(\boldsymbol{r}_2,t)\tau)\,e^{-i\boldsymbol{K}(\boldsymbol{r}_2+\boldsymbol{r}_4)}\,d^3r_2\,d^3r_4\right]+$$
$$+\left[\iint\limits_{VV}B_\varepsilon(\boldsymbol{r}_1-\boldsymbol{r}_4-\boldsymbol{v}(\boldsymbol{r}_1,t)\tau)\,e^{i\boldsymbol{K}(\boldsymbol{r}-\boldsymbol{r}_4)}\,d^3r_1\,d^3r_4\right]\times$$
$$\left.\times\left[\iint\limits_{VV}B_\varepsilon(\boldsymbol{r}_2-\boldsymbol{r}_3-\boldsymbol{v}(\boldsymbol{r}_2,t)\tau)\,e^{-i\boldsymbol{K}(\boldsymbol{r}_2-\boldsymbol{r}_3)}\,d^3r_2\,d^3r_3\right]\right\rangle_v, \qquad (39)$$

where the brackets $\langle\ \rangle_v$ denote averaging over the random field $\boldsymbol{v}$.

Before we carry out this averaging, note that the first term in (39) vanishes only if $\boldsymbol{K}\neq 0$. Actually, introducing new variables $(\boldsymbol{r}_1-\boldsymbol{r}_3)$ and $(\boldsymbol{r}_1+\boldsymbol{r}_3)$ in the first integral and integrating over the variable $(\boldsymbol{r}_1+\boldsymbol{r}_3)$, which appears only in the $\exp(i\boldsymbol{K}(\boldsymbol{r}_1+\boldsymbol{r}_3))$ term, we obtain the function $\delta_V(\boldsymbol{K})$. However, for $\boldsymbol{K}\neq 0$, $\delta_V(\boldsymbol{K})\approx 0$, so that the first term in (39) may indeed be neglected. In the second term, $\boldsymbol{r}_4$ is replaced by a new variable $\boldsymbol{\rho}_1=\boldsymbol{r}_1-\boldsymbol{r}_4$, and $\boldsymbol{r}_3$ by $\boldsymbol{\rho}_2=\boldsymbol{r}_2-\boldsymbol{r}_3$. This gives

$$B_I(\tau)=\left(\frac{k^2A_0\sin\chi}{4\pi r}\right)^4\int\limits_V d^3r_1\int\limits_V d^3r_2\left\langle\int\limits_V B_\varepsilon(\boldsymbol{\rho}_1-\boldsymbol{v}(\boldsymbol{r}_1,t)\tau)\,e^{i\boldsymbol{K}\boldsymbol{\rho}_1}d^3\rho_1\times\right.$$
$$\left.\times\int B_\varepsilon(\boldsymbol{\rho}_2-\boldsymbol{v}(\boldsymbol{r}_2,t)\tau)\,e^{-i\boldsymbol{K}\boldsymbol{\rho}_2}d^3\rho_2\right\rangle_v. \qquad (40)$$

Consider the integrals over ρ:

$$\int_V B_\varepsilon(\rho - v_1\tau)\, e^{iK\rho}\, d^3\rho = \int_V B_\varepsilon(\rho - v_1\tau)\, e^{iK(\rho - v_1\tau)} e^{iKv_1\tau}\, d^3\rho = e^{iKv_1\tau}\, 8\pi^3 \overline{\Phi}_\varepsilon(K),$$

where $\overline{\Phi}$ is the spectrum averaged (according to (26.22)) over some volume in wavenumber space. Similarly

$$\int_V B_\varepsilon(\rho_2 - v_2\tau)\, e^{-iK\rho_2}\, d^3\rho_2 = e^{-iKv_2\tau}\, 8\pi^3 \overline{\Phi}_\varepsilon(K),$$

and (40) takes the form

$$B_I(\tau) = \left(\frac{k^2 A_0 \sin\chi}{4\pi r}\right)^4 (2\pi)^6\, [\overline{\Phi}_\varepsilon(K)]^2 \int_V d^3r_1 \int_V d^3r_2 \langle e^{iK[v(r_1, t) - v(r_2, t)]\tau} \rangle_v. \qquad (41)$$

Unlike the expressions for the spectrum of the field, the spectrum of the intensity fluctuations contains only two-point velocity differences. Let $v_K(r, t)$ be the projection of the velocity at the point r on the direction of K:

$$Kv = Kv_K \quad \text{and} \quad \Delta v_K(r_1, r_2) = v_K(r_1, t) - v_K(r_2, t).$$

In (41) introduce the expression

$$\langle e^{iK\tau\Delta v_K(r_1, r_2)} \rangle_v = \chi_{\Delta v_K(r_1, r_2)}(K\tau), \qquad (42)$$

where

$$\chi_{\Delta v_K(r_1, r_2)}(\mu) \equiv \langle e^{i\mu(v_K(r_1, t) - v_K(r_2, t))} \rangle_v$$

is the characteristic function of the probability distribution of the K-component of the velocity difference at the two points r_1 and r_2. Thus,

$$B_I(\tau) = \left(\frac{k^2 A_0 \sin\chi}{4\pi r}\right)^4 (2\pi)^6\, [\overline{\Phi}_\varepsilon(K)]^2 \int_V \int_V d^3r_1\, d^3r_2\, \chi_{\Delta v_K(r_1, r_2)}(K\tau). \qquad (43)$$

Consider the frequency spectrum of the intensity fluctuations. Taking the Fourier transform of (43) with respect to τ and noting that the characteristic function of the random variable Δv_K is related to the probability density $P_{\Delta v_K}(u)$ by the expression

$$\chi_{\Delta v_K}(\mu) = \langle e^{i\mu\Delta v_K} \rangle = \int_{-\infty}^{\infty} e^{i\mu u} P_{\Delta v_K}(u)\, du,$$

we obtain

$$W_I(\Omega) = \left(\frac{k^2 A_0 \sin\chi}{4\pi r}\right)^4 (2\pi)^6\, [\overline{\Phi}_\varepsilon(K)]^2 \frac{1}{K} \iint_{VV} P_{\Delta v_K(r_1, r_2)}\left(\frac{\Omega}{K}\right) d^3r_1\, d^3r_2. \qquad (44)$$

In a locally homogeneous turbulent flow the probability distribution of the two-point velocity difference depends only on $\boldsymbol{r}_1 - \boldsymbol{r}_2$. This is clearly also applicable to the characteristic function of this distribution. Therefore in (43) and (44) we may introduce new integration variables $\boldsymbol{r}_1$ and $\boldsymbol{\rho} = \boldsymbol{r}_1 - \boldsymbol{r}_2$ and carry out the integration over $\boldsymbol{r}_1$, which gives the scattering volume V. In this case we obtain

$$B_I(\tau) = \left(\frac{k^2 A_0 \sin\chi}{4\pi r}\right)^4 (2\pi)^6 [\overline{\Phi}_\varepsilon(\boldsymbol{K})]^2 V \int_V \chi_{\Delta v_K(\rho)}(K\tau)\, d^3\rho, \qquad (43')$$

$$W_I(\Omega) = \left(\frac{k^2 A_0 \sin\chi}{4\pi r}\right)^4 (2\pi)^6 [\overline{\Phi}_\varepsilon(\boldsymbol{K})]^2 V \frac{1}{K} \int_V P_{\Delta v_K(\rho)}\left(\frac{\Omega}{K}\right) d^3\rho. \qquad (44')$$

Moreover, putting $\tau = 0$ in (10) and using the relations

$$B_{\mathscr{E}}(0) = \langle I \rangle, \quad \int_{-\infty}^{\infty} u_\varepsilon(\boldsymbol{K}, \Omega)\, d\Omega = \Phi_\varepsilon(\boldsymbol{K}),$$

we get

$$\langle I \rangle = \left(\frac{k^2 A_0 \sin\chi}{4\pi r}\right)^2 (2\pi)^3 \overline{\Phi}_\varepsilon(\boldsymbol{K})\, V. \qquad (45)$$

Expressions (43') and (44') can therefore be written in the form

$$B_I(\tau) = \langle I \rangle^2 \frac{1}{V} \int_V \chi_{\Delta v_K(\rho)}(K\tau)\, d^3\rho, \qquad (43'')$$

$$W_I(\Omega) = \langle I \rangle^2 \frac{1}{KV} \int_V P_{\Delta v_K(\rho)}\left(\frac{\Omega}{K}\right) d^3\rho. \qquad (44'')$$

Putting $\tau = 0$ in (43'') and noting that $\chi_{\Delta v_K(\rho)}(0) = 1$, we obtain $B_I(0) = \langle I \rangle^2$, hence for the correlation coefficient $b_I(\tau)$ we have

$$b_I(\tau) = \frac{B_I(\tau)}{B_I(0)} = \frac{1}{V} \int_V \chi_{\Delta v_K(\rho)}(K\tau)\, d^3\rho. \qquad (45')$$

Note that Ω/K is due to the K-component of the velocity difference corresponding to a difference of Ω in the Doppler frequencies.

As a crude example, consider the model of a normal probability distribution for the velocity difference. As we have noted before, the actual probability distribution for this function is definitely not normal, and our example will only provide the orders of magnitude of the various factors in (43'') and (44'').

For a normal probability distribution

$$\chi_{\Delta v_K(\rho)}(\mu) = e^{-\frac{1}{2}\mu^2 \langle [\Delta v_K(\rho)]^2 \rangle}. \qquad (46)$$

But

$$\Delta v_K(\boldsymbol{\rho}) = \frac{1}{K} \boldsymbol{K} \Delta \boldsymbol{v}(\boldsymbol{\rho}) = \frac{1}{K} K_i \Delta v_i(\boldsymbol{\rho}),$$

so that

$$\langle [\Delta v_K(\boldsymbol{\rho})]^2 \rangle = \frac{1}{K^2} K_i K_j D_{ij}(\boldsymbol{\rho}).$$

Substituting

$$D_{ij}(\boldsymbol{\rho}) = D_{tt}(\rho)\,\delta_{ij} + (D_{rr}(\rho) - D_{tt}(\rho))\, n_i n_j,$$

where $\boldsymbol{n} = \boldsymbol{\rho}/\rho$ (see Chapter 1), we obtain

$$\langle [\Delta v_K(\boldsymbol{\rho})]^2 \rangle = D_{tt}(\rho) + [D_{rr}(\rho) - D_{tt}(\rho)]\cos^2\vartheta = D_{tt}(\rho)\sin^2\vartheta + D_{rr}(\rho)\cos^2\vartheta,$$

where ϑ is the angle between $\boldsymbol{K}$ and $\boldsymbol{\rho}$. Also putting

$$D_{rr}(\rho) = C^2 \varepsilon^{2/3} \rho^{2/3} \quad \text{and} \quad D_{tt} = \frac{4}{3} D_{rr}$$

(these relations are applicable in the inertial subrange), we obtain

$$\chi_{\Delta v_K(\rho)}(\mu) = e^{-\frac{\mu^2}{2} C^2 (\varepsilon\rho)^{2/3} \left[1 + \frac{1}{3}\sin^2\vartheta\right]}. \tag{46'}$$

Inserting (46') in (45'), we introduce spherical coordinates with the center inside the scattering volume and the polar axis pointing along the vector $\boldsymbol{K}$. Then

$$b_I(\tau) = \frac{1}{V} \int_V e^{-\frac{1}{2}\tau^2 K^2 C^2 (\varepsilon\rho)^{2/3} \left[1 + \frac{1}{3}\sin^2\vartheta\right]} \rho^2 \sin\vartheta \, d\rho\, d\varphi\, d\vartheta. \tag{47}$$

Let the characteristic dimension of the scattering volume V be $V^{1/3} = L$, so that $\rho \lesssim V^{1/3}$. In this case, if $\tau^2 K^2 C^2 (\varepsilon V^{1/3})^{2/3} \ll 1$, or $\tau^2 \ll \tau_0^2$, where

$$\tau_0 = \frac{1}{CK(\varepsilon)^{1/3} V^{1/9}}, \tag{48}$$

we have $b_I(\tau) \approx 1$. If, however, $\tau^2 \gg \tau_0^2$ or $\tau^2 K^2 C^2 (\varepsilon L)^{2/3} \gg 1$, the integral over ρ in (47) may be taken from 0 to ∞. In this case simple, but fairly tedious computations give

$$b_I(\tau) = \left(\frac{N\tau_0}{\tau}\right)^9, \text{ where } N = \sqrt[9]{\frac{83 \cdot 3^3 \pi^{3/2}}{2^{5/2}}} = 2.35. \tag{49}$$

τ_0 is thus the correlation time of fluctuations in the intensity of the scattered signal. Unlike the correlation time for field fluctuations (which was found to be of the order of L_0/σ_v), τ_0 is determined only by the local characteristics of the fluctuations (namely, the energy dissipation rate ε). This is so because the spectrum of intensity fluctuations is determined only by the difference of Doppler frequencies of the various scattering elements, and consequently only by the difference in their velocities. Equations (43) and (44) may thus prove to be applicable for a wider range of conditions than the corresponding relations (28) and (31) for the spectrum of the field fluctuations.

Indeed, τ_0 is of the order of $\lambda/\Delta v(L)$, where $\Delta v(L)$ is the order of magnitude of the velocity difference at the ends of the scattering volume. Since for sufficiently large volumes V, $\Delta v(L) \sim \sigma_v$, the correlation time of intensity fluctuations is a factor λ/L_0 less than the correlation time of field fluctuations (remember that L_0 is the correlation distance of velocity fluctuations). The condition of constant velocity for a fluid element during the time τ_0, on which our entire derivation is based, is thus more readily satisfied than condition (32).

The change in velocity during the time τ_0 is of the order $\sqrt{\varepsilon\tau_0}$. This change is now required to be small compared to the characteristic difference in the velocities of the scattering elements, and not compared to their velocity. The velocity difference is of the order of $(\varepsilon L)^{1/3}$. If we require that $\sqrt{\varepsilon\tau_0} \ll (\varepsilon L)^{1/3}$ and insert τ_0 for (48), we obtain the condition

$$(KL)^{1/2} \gg 1 \quad \text{or} \quad \theta^{1/2} \gg (kL)^{-1/2} \tag{50}$$

(θ is the scattering angle), which if satisfied ensures "locally frozen" turbulence. Condition (50) does not include any turbulence parameters and is fulfilled in almost all practical experiments.

Note that in turbulence studies, (48) and (49) can be used to determine the energy dissipation rate ε from observations of the spectrum of the scattered intensity /69/, although it would be better to start with (43″) and (44″), which are not based on the assumption of a normal distribution for the velocity difference.

Note that expression (48) for the correlation time of intensity fluctuations is independent (apart from a numerical factor) of the particular velocity difference distribution used. On the other hand, the asymptotic behavior of the function $b_I(\tau)$ for $\tau \gg \tau_0$, expressed by (49), is a consequence of assuming in the derivation that Δv obeys a normal distribution.

§ 30. Scattering of pulse waveforms

Pulsed signals are often used in modern communication systems, and the scattering of pulses is therefore a highly topical problem. We will consider the scattering of a rectangular pulse with an envelope given by

$$A(t) = \begin{cases} A_0 & \text{for } |t| < \frac{T}{2}, \\ 0 & \text{for } |t| > \frac{T}{2}. \end{cases} \tag{1}$$

To simplify the problem, we will consider only the scalar equation in the Fraunhofer diffraction approximation. At a later stage the results will be extended to a more general case.

The scattering equation in the scalar approximation is

$$\Delta\mathscr{E}_1 - \frac{1}{c^2}\frac{\partial^2\mathscr{E}_1}{\partial t^2} = -k^2\varepsilon_1(\boldsymbol{r})\mathscr{E}_0(\boldsymbol{r}, t), \tag{2}$$

where

$$\mathscr{E}_0(\boldsymbol{r}, t) = A\left(t - \frac{\boldsymbol{m}\boldsymbol{r}}{c}\right)e^{i(kr - \omega t)}, \tag{3}$$

and the function $A(t)$ is defined by (1). In (3), $\boldsymbol{m} = \boldsymbol{k}/k$ is the unit vector in the direction of propagation of the incident plane wave. The function $\mathscr{E}_0(\boldsymbol{r}, t)$ is zero except in a layer given by

$$ct - \frac{cT}{2} \leqslant \boldsymbol{mr} \leqslant ct + \frac{cT}{2}$$

whose thickness is cT and which moves in the direction $\boldsymbol{m}$ with velocity c. The solution of (3), as is well known, is the function

$$\mathscr{E}_1(\boldsymbol{r}, t) = \frac{k^2}{4\pi} \int_V \varepsilon_1(\boldsymbol{r}') \mathscr{E}_0\left(\boldsymbol{r}', t - \frac{|\boldsymbol{r} - \boldsymbol{r}'|}{c}\right) \frac{d^3r'}{|\boldsymbol{r} - \boldsymbol{r}'|}. \tag{4}$$

In the Fraunhofer diffraction region we again take $|\boldsymbol{r} - \boldsymbol{r}'| = r - \boldsymbol{nr}' + \ldots$, where $\boldsymbol{n} = \boldsymbol{r}/r$ is the unit vector directed from the center of the scattering volume to the observation point, and r is the distance from the center of the scattering volume to that point. In the denominator in (4) we may take $|\boldsymbol{r} - \boldsymbol{r}'| \approx r + \ldots$ so that

$$\mathscr{E}_1(\boldsymbol{r}, t) = \frac{k^2}{4\pi r} \int_V \varepsilon_1(\boldsymbol{r}') \mathscr{E}_0\left(\boldsymbol{r}', t - \frac{r}{c} + \frac{\boldsymbol{nr}'}{c}\right) d^3r', \tag{5}$$

and inserting (3) we get

$$\mathscr{E}_1(\boldsymbol{r}, t) = \frac{k^2}{4\pi r} e^{-i\omega\left(t - \frac{r}{c}\right)} \int_V \varepsilon_1(\boldsymbol{r}') e^{ik(\boldsymbol{m}-\boldsymbol{n})\boldsymbol{r}'} A\left[t - \frac{r}{c} - \frac{(\boldsymbol{m}-\boldsymbol{n})\boldsymbol{r}'}{c}\right] d^3r'. \tag{6}$$

Let $V'(t)$ be the common part (the intersection) of the scattering volume V and the moving plane layer

$$-r + ct - \frac{cT}{2} < (\boldsymbol{m} - \boldsymbol{n})\boldsymbol{r} < ct + \frac{cT}{2} - r;$$

$V'(t) = 0$ if the two do not overlap and $V'(t) = V$ if the scattering volume fits completely inside the propagating layer. In general $V'(t)$ is equal to that part of the scattering volume which falls inside the layer.

Using the new notation $V'(t)$, we may write (6) in the form

$$\mathscr{E}_1(\boldsymbol{r}, t) = \frac{k^2 A_0}{4\pi r} e^{-i\omega\left(t - \frac{r}{c}\right)} \int_{V'(t)} \varepsilon_1(\boldsymbol{r}') e^{ik(\boldsymbol{m}-\boldsymbol{n})\boldsymbol{r}'} d^3r'. \tag{7}$$

The envelope of the scattered field (which now is a function of t) is given by

$$E_1(\boldsymbol{r}, t) = \frac{k^2 A_0}{4\pi r} \int_{V'(t)} \varepsilon_1(\boldsymbol{r}') e^{ik(\boldsymbol{m}-\boldsymbol{n})\boldsymbol{r}'} d^3r'. \tag{8}$$

This expression differs from the analogous relation (29.3) only in that $V'(t)$ is substituted for V.

Now consider the mean scattered energy flux, which is proportional to $\overline{E_1 E_1^*}$. Since in the case of a monochromatic wave this expression was proportional to V, in the present case it is proportional to $V'(t)$. Thus, the time dependence of the intensity of the scattered wave is determined by this factor alone (since all the other factors are independent of time).* Substituting $V'(t)$ for V in expression (29.45) for $I=\overline{E_1E_1^*}$ and introducing the factor $\sin^2\chi$, which appears when transforming from the case of a scalar wave to the electromagnetic field, we obtain

$$I(t)=\left(\frac{k^2A_0\sin\chi}{4\pi r}\right)^2(2\pi)^3\overline{\Phi}_\varepsilon(\boldsymbol{K})V'(t). \tag{9}$$

Let us consider in some detail the function $V'(t)$. We introduce a coordinate system inside the scattering volume with the z-axis directed along the vector $\boldsymbol{m}-\boldsymbol{n}$. Then $\boldsymbol{m}-\boldsymbol{n}=\left\{0,\,0,\,2\sin\frac{\theta}{2}\right\}$ and the boundaries of the layer

$$ct-r-\frac{cT}{2}<(\boldsymbol{m}-\boldsymbol{n})\boldsymbol{r}'<ct-r+\frac{cT}{2}$$

are described by the equations

$$z_1'=\frac{ct-r+\frac{1}{2}cT}{2\sin\frac{\theta}{2}},\quad z_2'=\frac{ct-r-\frac{1}{2}cT}{2\sin\frac{\theta}{2}}. \tag{10}$$

The thickness of this layer is

$$h=z_1'-z_2'=\frac{cT}{2\sin\frac{\theta}{2}}, \tag{11}$$

and its velocity is

$$v=\frac{dz_1'}{dt}=\frac{c}{2\sin\frac{\theta}{2}}. \tag{12}$$

Thus, in order to find the average shape of the received signal, we should consider the uniform motion of a plane layer of thickness h in the direction $\boldsymbol{m}-\boldsymbol{n}$ with constant velocity $v=\frac{c}{2\sin\frac{\theta}{2}}$. The average received signal power is proportional (if we ignore the overall retardation of the signal) to the part of the scattering volume V which lies at the given instant of time inside the moving layer. If the layer thickness h is greater than the extent H of the scattering volume V (in the direction $\boldsymbol{m}-\boldsymbol{n}$), at some time the entire volume will lie inside the layer and the received signal will

* There is a certain conceptual difference in the averaging of the function over the wavenumber space, which transforms Φ to $\overline{\Phi}$, since the relevant volume of the wavenumber space is variable in the present case. However, since $\overline{\Phi}\approx\Phi$, the averaging does not greatly affect the function $\Phi(\boldsymbol{K})$, so that this difference is not significant.

have a flat top (on the average). The signal rise time (and fall time) are determined by the parameter H and the velocity v in the z_2' plane, i. e., by the time it takes the leading edge of the pulse to cross the scattering volume,

$$\tau_{tr} = \frac{H}{v} = \frac{2H}{c} \sin \frac{\theta}{2}. \tag{13}$$

This transit time decreases as the scattering angle θ is decreased. Clearly (Figure 30) the duration of the flat part of the pulse is $T - \tau_{tr}$, and the entire pulse duration is $T + \tau_{tr}$.

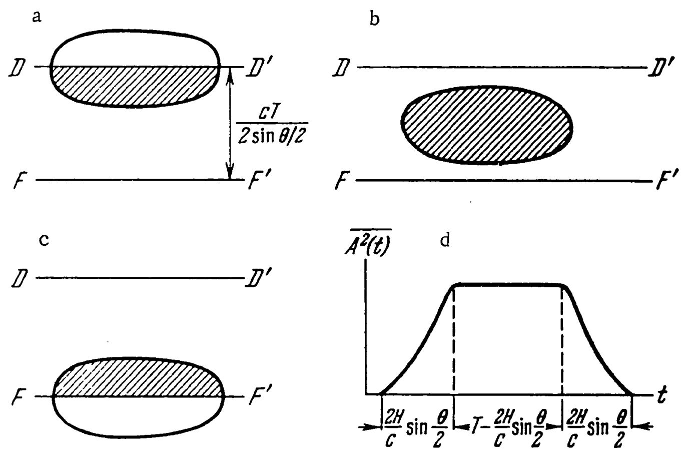

FIGURE 30. Three successive stages showing the pulse crossing the scattering volume (a, b, c) and the average shape of the scattered signal (d) when the vertical extent of the scattering volume is less than $cT/2\sin\frac{\theta}{2}$.

Figure 30 illustrates schematically the construction of the shape of the pulse for $h > H$. Figure 31 corresponds to the case $h < H$, when the scattering volume is never completely utilized as a scatterer and the amplitude of the received signal is correspondingly smaller than in the previous case. The total pulse duration is again $T + \tau_{tr}$, but it is possible that it is much longer than the initial duration T. Pulse distortion is insignificant if $\tau_{tr} \ll T$; considerable distortion occurs when τ_{tr} and T are comparable or $\tau_{tr} > T$. We can therefore naturally identify $1/\tau_{tr}$ as the "passband" of a scattering communication channel,

$$\Delta f = \frac{c}{2H \sin \frac{\theta}{2}}. \tag{14}$$

Note that Δf is the maximum frequency difference at which the intensities of two scattered signals of frequencies f and $f + \Delta f$ still show correlation.

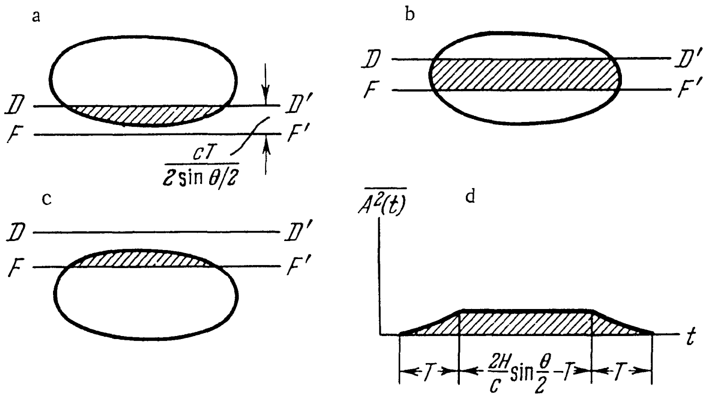

FIGURE 31. Three successive stages showing the pulse crossing the scattering volume (a, b, c) and the average shape of the scattered signal (d) when the vertical extent of the scattering volume is greater than $cT/2\ \sin\frac{\theta}{2}$.

For "narrow" antenna patterns $H \approx \alpha R = {}^1/_2\,\alpha d$ (α is the angular beam width) and

$$\Delta f = \frac{c}{\alpha d \sin\frac{\theta}{2}} \approx \frac{2c}{\alpha\theta d}. \tag{15}$$

For "broad" antenna patterns $H \approx \theta R = \frac{1}{2}\theta d$ and

$$\Delta f = \frac{c}{\theta d \sin\frac{\theta}{2}} \approx \frac{2c}{\theta^2 d}. \tag{16}$$

For common values of α, θ and d, the passband Δf may reach a few MHz, which is adequate even for TV broadcasts.

Let us now consider the physical interpretation of the above relations. Suppose that Π_1 (Figure 32) is the transmitting antenna and Π_2 is the receiving antenna. If at some time t an "instantaneous" signal is emitted by Π_1, at time $t+\tau$, Π_2 will receive signals scattered by all the scattering elements the sum of whose distances from Π_1 and Π_2 is $c\tau$. All these scattering elements lie on the surface of an ellipsoid of revolution with foci at Π_1 and Π_2, its semimajor axis is $a=\frac{1}{2}c\tau$ (the axis through Π_1 and Π_2) and its semiminor axis is $b=\frac{1}{2}\sqrt{c^2\tau^2-d^2}$ (d is the distance between Π_1 and Π_2). Consider a scattering volume (small compared to d) located near the end of the semiminor axis of the ellipsoid. The scattering angle θ is related to d and τ by the equality $c\tau=\frac{d}{\cos\frac{\theta}{2}}$. At time $\tau'>\tau$, Π_2 receives waves scattered by elements on the surface of a larger ellipsoid with semiaxes $a'=\frac{1}{2}c\tau'$ and $b'=\frac{1}{2}\sqrt{c^2\tau'^2-d^2}$. Consequently, the surface of the ellipsoid crossing the scattering volume will have moved upward with a velocity

$$v=\frac{db}{d\tau}=\frac{c}{2}\frac{c\tau}{\sqrt{c^2\tau^2-d^2}}=\frac{c}{2\sin\frac{\theta}{2}},$$

which agrees with (12).

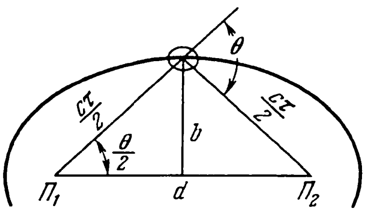

FIGURE 32. Illustrating the construction of the scattered signal.

If the volume V is sufficiently small, the surface of the ellipsoid within the scattering volume can be approximately represented by its tangent plane, and we return to our previous results. This treatment clearly shows what alterations are to be introduced in our results if the approximation of Fraunhofer diffraction is dropped. In this case, the above pulse shapes remain applicable if the plane-parallel moving layer is replaced by the surfaces of an ellipsoid which expands with time.

§ 31. Correlation functions of the scattered field

The correlation of fields scattered at various angles has been considered qualitatively in the preceding. We now proceed with a more exact treatment of this problem /71, 72/. The scattered field at the point $\boldsymbol{r}_1$ is given by (25.8a):

$$\boldsymbol{E}_s(\boldsymbol{r}_1)=\frac{k^2}{4\pi}\int\limits_V\frac{e^{ik|\boldsymbol{r}_1-\boldsymbol{r}'|}}{|\boldsymbol{r}_1-\boldsymbol{r}'|}\varepsilon_1(\boldsymbol{r}')\left[\boldsymbol{n}(\boldsymbol{r}_1,\boldsymbol{r}')\left[\boldsymbol{E}_0(\boldsymbol{r}')\boldsymbol{n}(\boldsymbol{r}_1,\boldsymbol{r}')\right]\right]d^3r', \qquad (1)$$

where $\boldsymbol{n}(\boldsymbol{r}_1,\boldsymbol{r}')=(\boldsymbol{r}_1-\boldsymbol{r}')/|\boldsymbol{r}_1-\boldsymbol{r}'|$. The scattered field at the point $\boldsymbol{r}_2$ is

$$\boldsymbol{E}_s(\boldsymbol{r}_2)=\frac{k^2}{4\pi}\int\limits_V\frac{e^{ik|\boldsymbol{r}_2-\boldsymbol{r}''|}}{|\boldsymbol{r}_2-\boldsymbol{r}''|}\varepsilon_1(\boldsymbol{r}'')\left[\boldsymbol{n}(\boldsymbol{r}_2,\boldsymbol{r}'')\left[\boldsymbol{E}_0(\boldsymbol{r}'')\boldsymbol{n}(\boldsymbol{r}_2,\boldsymbol{r}'')\right]\right]d^3r''. \qquad (2)$$

Taking the mean value of (1) multiplied by the complex conjugate of (2), we obtain

$$\langle\boldsymbol{E}_s(\boldsymbol{r}_1)\boldsymbol{E}_s^*(\boldsymbol{r}_2)\rangle=\left(\frac{k^2}{4\pi}\right)^2\iint\limits_{VV}\frac{e^{ik[|\boldsymbol{r}_1-\boldsymbol{r}'|-|\boldsymbol{r}_2-\boldsymbol{r}''|]}}{|\boldsymbol{r}_1-\boldsymbol{r}'||\boldsymbol{r}_2-\boldsymbol{r}''|}B_\varepsilon(\boldsymbol{r}'-\boldsymbol{r}'')\times$$
$$\times\left[\boldsymbol{n}(\boldsymbol{r}_1,\boldsymbol{r}')\left[\boldsymbol{E}_0(\boldsymbol{r}')\boldsymbol{n}(\boldsymbol{r}_1,\boldsymbol{r}')\right]\right]\left[\boldsymbol{n}(\boldsymbol{r}_2,\boldsymbol{r}'')\left[\boldsymbol{E}_0^*(\boldsymbol{r}'')\boldsymbol{n}(\boldsymbol{r}_2,\boldsymbol{r}'')\right]\right]d^3r'\,d^3r''. \qquad (3)$$

As in the derivation of the average intensity of the scattered field, we take $\boldsymbol{E}_0(\boldsymbol{r})$ in the form

$$\boldsymbol{E}_0(\boldsymbol{r}) = \boldsymbol{A}_0(\boldsymbol{r})\, e^{ik\,|\boldsymbol{R}_0-\boldsymbol{r}\,|}, \tag{4}$$

where $\boldsymbol{R}_0$ is the position of the source, and $\boldsymbol{A}_0(\boldsymbol{r})$ is the amplitude, which depends on the beam pattern of the transmitting antenna. Expression (3) takes the form

$$\langle \boldsymbol{E}_s(\boldsymbol{r}_1)\,\boldsymbol{E}_s^*(\boldsymbol{r}_2)\rangle =$$
$$= \left(\frac{k^2}{4\pi}\right)^2 \iint\limits_{VV} \frac{\exp\{ik\,[|\boldsymbol{r}_1-\boldsymbol{r}'| - |\boldsymbol{r}_2-\boldsymbol{r}''| + |\boldsymbol{R}_0-\boldsymbol{r}'| - |\boldsymbol{R}_0-\boldsymbol{r}''|]\}\, B_\varepsilon(\boldsymbol{r}'-\boldsymbol{r}'')}{|\boldsymbol{r}_1-\boldsymbol{r}'|\cdot|\boldsymbol{r}_2-\boldsymbol{r}''|} \times$$
$$\times \{\boldsymbol{A}_0(\boldsymbol{r}') - \boldsymbol{n}(\boldsymbol{r}_1, \boldsymbol{r}')(\boldsymbol{A}_0(\boldsymbol{r}')\,\boldsymbol{n}(\boldsymbol{r}_1, \boldsymbol{r}'))\} \times$$
$$\times \{\boldsymbol{A}_0(\boldsymbol{r}'') - \boldsymbol{n}(\boldsymbol{r}_2, \boldsymbol{r}'')(\boldsymbol{A}_0(\boldsymbol{r}'')\,\boldsymbol{n}(\boldsymbol{r}_2, \boldsymbol{r}''))\}\, d^3r'\, d^3r''. \tag{5}$$

We introduce new variables of integration $\boldsymbol{x} = \frac{1}{2}(\boldsymbol{r}' + \boldsymbol{r}'')$ and $\boldsymbol{\rho} = \boldsymbol{r}' - \boldsymbol{r}''$ and take $\boldsymbol{R} = \frac{1}{2}(\boldsymbol{r}_1 + \boldsymbol{r}_2)$ as the midpoint of the segment joining the observation points $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$; also $\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$.

All the factors in (5) can now be expressed in terms of the new variables:

$$\boldsymbol{r}_1 - \boldsymbol{r}' = \boldsymbol{R} - \boldsymbol{x} + \frac{1}{2}(\boldsymbol{r} - \boldsymbol{\rho}); \quad \boldsymbol{r}_2 - \boldsymbol{r}'' = \boldsymbol{R} - \boldsymbol{x} - \frac{1}{2}(\boldsymbol{r} - \boldsymbol{\rho}),$$
$$|\boldsymbol{r}_1 - \boldsymbol{r}'| = \sqrt{(\boldsymbol{R}-\boldsymbol{x})^2 + (\boldsymbol{R}-\boldsymbol{x})(\boldsymbol{r}-\boldsymbol{\rho}) + \frac{1}{4}(\boldsymbol{r}-\boldsymbol{\rho})^2}.$$

Expanding the magnitudes of these vectors in binomial series, we obtain

$$|\boldsymbol{r}_1 - \boldsymbol{r}'| = |\boldsymbol{R}-\boldsymbol{x}| + \frac{\boldsymbol{R}-\boldsymbol{x}}{|\boldsymbol{R}-\boldsymbol{x}|}\cdot\frac{\boldsymbol{r}-\boldsymbol{\rho}}{2} +$$
$$+ \frac{1}{2|\boldsymbol{R}-\boldsymbol{x}|}\left[\left(\frac{\boldsymbol{r}-\boldsymbol{\rho}}{2}\right)^2 - \left(\frac{\boldsymbol{R}-\boldsymbol{x}}{|\boldsymbol{R}-\boldsymbol{x}|}\cdot\frac{\boldsymbol{r}-\boldsymbol{\rho}}{2}\right)^2\right] + \ldots, \tag{6}$$

$$|\boldsymbol{r}_2 - \boldsymbol{r}''| = |\boldsymbol{R}-\boldsymbol{x}| - \frac{\boldsymbol{R}-\boldsymbol{x}}{|\boldsymbol{R}-\boldsymbol{x}|}\,\frac{\boldsymbol{r}-\boldsymbol{\rho}}{2} +$$
$$+ \frac{1}{2|\boldsymbol{R}-\boldsymbol{x}|}\left[\left(\frac{\boldsymbol{r}-\boldsymbol{\rho}}{2}\right)^2 - \left(\frac{\boldsymbol{R}-\boldsymbol{x}}{|\boldsymbol{R}-\boldsymbol{x}|}\cdot\frac{\boldsymbol{r}-\boldsymbol{\rho}}{2}\right)^2\right] + \ldots, \tag{7}$$

$$|\boldsymbol{r}_1 - \boldsymbol{r}'| - |\boldsymbol{r}_2 - \boldsymbol{r}''| = \frac{\boldsymbol{R}-\boldsymbol{x}}{|\boldsymbol{R}-\boldsymbol{x}|}(\boldsymbol{r}-\boldsymbol{\rho}) + O\left(\frac{|\boldsymbol{r}-\boldsymbol{\rho}|^3}{|\boldsymbol{R}-\boldsymbol{x}|^2}\right), \tag{8}$$

where $O(x)$ are terms of order not greater than x. Similarly,

$$|\boldsymbol{R}_0 - \boldsymbol{r}'| = \left|\boldsymbol{R}_0 - \boldsymbol{x} - \frac{1}{2}\boldsymbol{\rho}\right| =$$
$$= |\boldsymbol{R}_0-\boldsymbol{x}| - \frac{\boldsymbol{R}_0-\boldsymbol{x}}{|\boldsymbol{R}_0-\boldsymbol{x}|}\cdot\frac{\boldsymbol{\rho}}{2} + \frac{1}{2|\boldsymbol{R}_0-\boldsymbol{x}|}\left[\left(\frac{\boldsymbol{\rho}}{2}\right)^2 - \left(\frac{\boldsymbol{R}_0-\boldsymbol{x}}{|\boldsymbol{R}_0-\boldsymbol{x}|}\cdot\frac{\boldsymbol{\rho}}{2}\right)^2\right] + \ldots,$$

$$|\boldsymbol{R}_0 - \boldsymbol{r}''| = \left|\boldsymbol{R}_0 - \boldsymbol{x} + \frac{1}{2}\boldsymbol{\rho}\right| = \tag{9}$$
$$= |\boldsymbol{R}_0-\boldsymbol{x}| + \frac{\boldsymbol{R}_0-\boldsymbol{x}}{|\boldsymbol{R}_0-\boldsymbol{x}|}\,\frac{\boldsymbol{\rho}}{2} + \frac{1}{2|\boldsymbol{R}_0-\boldsymbol{x}|}\left[\left(\frac{\boldsymbol{\rho}}{2}\right)^2 - \left(\frac{\boldsymbol{R}_0-\boldsymbol{x}}{|\boldsymbol{R}_0-\boldsymbol{x}|}\,\frac{\boldsymbol{\rho}}{2}\right)^2\right] + \ldots,$$

$$|\boldsymbol{R}_0 - \boldsymbol{r}'| - |\boldsymbol{R}_0 - \boldsymbol{r}''| = -\frac{\boldsymbol{R}_0-\boldsymbol{x}}{|\boldsymbol{R}_0-\boldsymbol{x}|}\boldsymbol{\rho} + O\left(\frac{\rho^3}{|\boldsymbol{R}-\boldsymbol{x}|^2}\right). \tag{10}$$

Let

$$m(R_0, x) = \frac{x - R_0}{|x - R_0|} \tag{11}$$

be the unit vector in the direction from R_0 to x. We will now establish under what conditions only the first terms in (8) and (10) need to be retained.

The expressions (8) and (10), when multiplied by k, enter (5) in the exponent. The quantity ρ is limited by the inequality $\rho \lesssim L_0$ (L_0 is the correlation radius of the dielectric constant fluctuations) since for large ρ the integrand in (5) is close to zero due to the factor $B_\varepsilon(\rho)$.

$|R - x|$ is of the order of R. Therefore, the condition under which only the first term need be retained in (10) is

$$\frac{kL_0^3}{R^2} \ll 1 \quad \text{or} \quad L_0^3 \ll \lambda R^2. \tag{12}$$

Condition (12) is readily satisfied in practice for long-range tropospheric radio propagation.

Expression (8), in addition to (12), gives a constraint on the maximum permissible separation r of the observation points:

$$r^3 \ll \lambda R^2. \tag{13}$$

In the qualitative analysis, however, we found that for a transverse separation of the observation points the correlation radius r_0 is of the order $r_0 \sim \lambda R/L$, where L is the dimension of the scattering volume. Inserting for r in (13) this expression for r_0, we obtain the constraint

$$\lambda^2 R \ll L^3, \tag{14}$$

which is also quite insignificant in practice.

In the denominator in (5) $|r_1 - r'|$ and $|r_2 - r''|$ can be replaced by the first terms in expansions (6) and (7) if $r \ll R$ and $L_0 \ll R$. In this case we may also take $n(r_1, r') \approx n(r_2, r'') \approx n(R, x)$. Furthermore, if the function $A_0(r)$ does not change much as r is changed by L_0, we may take $A_0(r') \approx A_0(r'') = A_0(x)$. Since the function A_0 only changes appreciably over distances of the order of L (the size of the scattering volume is in fact defined as the region where A_0 does not vanish), then the latter condition is of the form

$$L_0 \ll L. \tag{15}$$

If all the above constraints are satisfied, we may write (5) in the form

$$\langle E_s(r_1) E_s^*(r_2) \rangle = \left(\frac{k^2}{4\pi}\right)^2 \iint\limits_{VV} \frac{e^{ik((m(R,x)-n(R,x))\rho + ikn(R,x)r}}{|R - x|^2} \times$$
$$\times A_0^2(x) \sin^2 \chi(x) B_\varepsilon(\rho)\, d^3\rho\, d^3x. \tag{16}$$

Expression (16) differs from the analogous expression (26.17) in a numerical factor $c/8\pi$ and, more significantly, in an exponential factor $\exp(ik\boldsymbol{nr})$ which is responsible for the dependence of the correlation function of the scattered field on $\boldsymbol{r}$. Integrating over $\boldsymbol{\rho}$ in (16) and again putting

$$\boldsymbol{K}(\boldsymbol{x}) = k(\boldsymbol{m}(\boldsymbol{R}, \boldsymbol{x}) - \boldsymbol{n}(\boldsymbol{R}, \boldsymbol{x})),$$

we obtain

$$\langle \boldsymbol{E}_s(\boldsymbol{r}_1)\boldsymbol{E}_s^*(\boldsymbol{r}_2)\rangle \equiv B_E(\boldsymbol{r}) = (2\pi)^3\left(\frac{k^2}{4\pi}\right)^2 \int\limits_V \frac{e^{ik\boldsymbol{n}(\boldsymbol{R},\boldsymbol{x})\boldsymbol{r}}}{|\boldsymbol{R}-\boldsymbol{x}|^2} \times$$
$$\times A_0^2(\boldsymbol{x})\sin^2\chi(\boldsymbol{x})\overline{\Phi}_\varepsilon(\boldsymbol{K}(\boldsymbol{x}))\,d^3x. \quad (17)$$

In the derivation of (17) we assumed that the dielectric-constant fluctuations were statistically homogeneous, i.e., $B_\varepsilon(\boldsymbol{r}', \boldsymbol{r}'') = B_\varepsilon(\boldsymbol{r}'-\boldsymbol{r}'')$. This assumption is not absolutely essential. All the calculations can be repeated assuming that the statistical characteristics of the fluctuations are smoothly varying, i.e., a correlation function of the form (see Part A)

$$B_\varepsilon(\boldsymbol{r}'; \boldsymbol{r}'') = \overline{\varepsilon_1^2}\left(\frac{\boldsymbol{r}'+\boldsymbol{r}''}{2}\right) b_\varepsilon(\boldsymbol{r}'-\boldsymbol{r}''), \quad (18)$$

where $b_\varepsilon(\boldsymbol{\rho})$ is the normalized ($b_\varepsilon(0) = 1$) correlation function of the fluctuations of ε_1, which depends only on the spatial separation $\boldsymbol{\rho} = \boldsymbol{r}' - \boldsymbol{r}''$, and $\overline{\varepsilon_1^2}(\boldsymbol{r})$ the variance of the fluctuations, which depends on position. The spectral density of the fluctuations in this case is also a function of position:

$$\Phi_\varepsilon(\boldsymbol{r}, \boldsymbol{\varkappa}) = C_\varepsilon^2(\boldsymbol{r})\,\Phi_\varepsilon^{(0)}(\boldsymbol{\varkappa}). \quad (19)$$

The final result (17) remains unchanged, and only $\overline{\Phi}_\varepsilon(\boldsymbol{K}(\boldsymbol{x}))$ should be replaced by expression (19):

$$B_E(\boldsymbol{r}) = (2\pi)^3\left(\frac{k^2}{4\pi}\right)^2 \int\limits_V \frac{e^{ik\boldsymbol{n}(\boldsymbol{R},\boldsymbol{x})\boldsymbol{r}}}{|\boldsymbol{R}-\boldsymbol{x}|^2} A_0^2(\boldsymbol{x})\sin^2\chi(\boldsymbol{x}) \times$$
$$\times C_\varepsilon^2(\boldsymbol{x})\overline{\Phi}_\varepsilon^{(0)}(\boldsymbol{K}(\boldsymbol{x}))\,d^3x. \quad (20)$$

To elucidate the physical meaning of (20), we expand the vector $\boldsymbol{n}(\boldsymbol{R}, \boldsymbol{x})$ in power of $\boldsymbol{x}/R$:

$$\boldsymbol{n}(\boldsymbol{R}, \boldsymbol{x}) = \boldsymbol{n}_0 - \frac{1}{R}(\boldsymbol{x} - \boldsymbol{n}_0(\boldsymbol{n}_0\boldsymbol{x})) + O\left(\frac{x^2}{R^2}\right). \quad (21)$$

Here $\boldsymbol{n}_0 = \boldsymbol{R}/R$ is the unit vecotr directed from the center of the scattering volume to the midpoint of the segment joining the two observation points. The last term in (21) may be neglected if

$$\frac{L^2 r}{R^2\lambda} \ll 1 \quad (22)$$

(this is a more exacting constraint than (13)). In this case

$$\boldsymbol{n}(\boldsymbol{R}, \boldsymbol{x})\boldsymbol{r} \cong \boldsymbol{n}_0\boldsymbol{r} - \frac{1}{R}(\boldsymbol{x}\boldsymbol{r} - (\boldsymbol{n}_0\boldsymbol{r})(\boldsymbol{n}_0\boldsymbol{x})) = \boldsymbol{n}_0\boldsymbol{r} - \boldsymbol{x}\Delta\boldsymbol{n}, \tag{23}$$

where

$$\Delta\boldsymbol{n} = \frac{\boldsymbol{r} - \boldsymbol{n}_0(\boldsymbol{n}_0\boldsymbol{r})}{R} = \frac{[\boldsymbol{n}_0[\boldsymbol{r}\boldsymbol{n}_0]]}{R} \tag{24}$$

is the difference of the two unit vecotrs pointing from the center of the scattering volume to the two observation points.

As in the treatment of the average value of the scattered intensity, we take

$$A_0^2(\boldsymbol{x}) = A_0^2 f_0^2(\boldsymbol{m}(\boldsymbol{x})),$$

where $f(\boldsymbol{m}(\boldsymbol{x}))$ describes the beam pattern of the transmitter. Moreover, we introduce in the integrand in (20) the function $f_1^2(\boldsymbol{n}(\boldsymbol{x}))$ which characterizes the beam patterns of the receiving antennas,* and the integration is extended to cover an infinite region. This gives

$$B_E(\boldsymbol{r}) = (2\pi)^3\left(\frac{k^2A_0}{4\pi}\right)^2 e^{ik\boldsymbol{n}_0\boldsymbol{r}}\int e^{-ik\Delta\boldsymbol{n}\boldsymbol{x}}\,\frac{f_0^2(\boldsymbol{m}(\boldsymbol{x}))\,f_1^2(\boldsymbol{n}(\boldsymbol{x}))}{|\boldsymbol{R}-\boldsymbol{x}|^2}\times$$
$$\times\, C_\varepsilon^2(\boldsymbol{x})\,\overline{\Phi}_\varepsilon^{(0)}(\boldsymbol{K}(\boldsymbol{x}))\,d^3x. \tag{25}$$

The factor $\exp(ik\boldsymbol{n}_0\boldsymbol{r})$ before the integral is associated with the systematic phase difference between the scattered fields at the two points $\boldsymbol{R}\pm\frac{1}{2}\boldsymbol{r}$ when these points are at different distances from the scattering volume (it is only in this case that $\boldsymbol{r}\boldsymbol{n}_0 \neq 0$). The integral in (25) is the Fourier transform (for $\varkappa = k\Delta\boldsymbol{n}$) of the product

$$F(\boldsymbol{x}) = f_0^2(\boldsymbol{m}(\boldsymbol{x}))\,f_1^2(\boldsymbol{n}(\boldsymbol{x}))\,C_\varepsilon^2(\boldsymbol{x})\,\overline{\Phi}_\varepsilon^{(0)}(\boldsymbol{K}(\boldsymbol{x}))\,|\boldsymbol{R}-\boldsymbol{x}|^{-2}. \tag{26}$$

Note that this integral depends only on $\Delta\boldsymbol{n}(\boldsymbol{r})$, and not on $\boldsymbol{r}$ itself. The correlation of the scattered fields at two points is thus not affected (except for the phase factor $\exp(ik\boldsymbol{n}_0\boldsymbol{r})$) when these points are displaced along the rays through the center of the scattering volume (this displacement does not change $\Delta\boldsymbol{n}(\boldsymbol{r})$). In other words, the correlation of the scattered fields depends only on the angle at which the observation points are viewed from the scattering center. This is true, however, only when inequality (22) is satisfied; the situation is entirely different when (23) is inapplicable.

The correlation distance of the fluctuations is determined by the dimensions of the region where the function $F(\boldsymbol{x})$, given by (26), is appreciably different from zero. Let the dimensions of this region be of the order a. In this case, the difference $\Delta\boldsymbol{n}$ for which the scattered fields are still correlated is related to a by the relation

* In a more rigorous treatment, each of the receiving antenna patterns should be introduced separately. If, however, we assume that the antenna patterns are identical and do not change much as the angle is varied by L_0/R (L_0 is the correlation radius of the fluctuations of the dielectric constant), we obtain the result given in the text.

$$k\Delta n_0 \sim \frac{2\pi}{a}, \text{ i. e., } \Delta n_0 \sim \frac{\lambda}{a}.$$

This relation coincides with the result given in § 28.

The parameter a is mainly determined by the beam pattern (the case of "narrow-beam" antennas considered in § 28), by the decrease of the factor $C_\varepsilon^2(\boldsymbol{x})$ with height, or by the decrease of $\Phi_\varepsilon^0(\boldsymbol{K}(\boldsymbol{x}))$ with increasing scattering angle. In general, all these factors act jointly.

The correlation distance increases as the effective scattering volume is decreased, i. e., as the antenna pattern becomes narrower or as the rate of decrease of dielectric constant fluctuations with height becomes steeper.

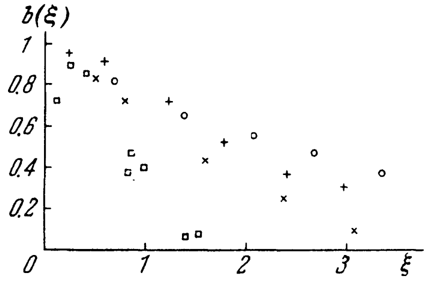

FIGURE 33. Correlation functions of the scattered field for observation points moved transverse to the beam in the case of wide beam antennas.

The distance between the observation points divided by $\lambda/\pi\theta$ (where λ is the wavelength and θ the scattering angle) is shown on the horizontal axis. The plot summarizes the results of various experiments:

○ $\lambda = 3$ m, 6 m, $d = 1200$ km;
+ $\lambda = 3$ cm, $d = 120$ km; × $\lambda = 10$ cm, $d = 190$ km; □ $\lambda = 30$ cm, 3 m, $d = 365$ km.

Figure 33 plots some experimental data on the scattered field correlation at observation points moved perpendicular to the path. Only the results of experiments with angular beam widths greater than the scattering angle are shown. In this case, the dimension of the effective scattering volume perpendicular to the path is of the order of $a \sim \theta R$, from which $\Delta n_0 \sim \frac{\lambda}{\theta R}$. But $\Delta n \approx \frac{S}{R}$, where S is the distance between observation points. Hence, the correlation radius S_0 is of the order of λ/θ, and the ratio $\frac{S}{S_0} = \frac{\theta S}{\lambda}$ or the proportional parameter $\xi = \frac{1}{2} kS\theta$ can be regarded as a dimensionless universal argument of the correlation function. If the correlation coefficient $b(\xi)$ is plotted as a function of ξ, the result will not depend on the particular values of k and θ used in the experiment. This is confirmed by the results shown in Figure 33.

Particular calculations using (25) were carried out in /71, 72/ for different models of the dielectric constant fluctuations and their dependence on height.

Note that the correlation functions are highly sensitive to the vertical variation of the intensity of the dielectric constant fluctuations. Therefore, comparison of experimental correlation functions with theoretical expressions will hardly yield any definite information on the spectra of dielectric constant fluctuations. Moreover, since in the real atmosphere the intensity distribution of dielectric constant fluctuations with height need not follow any regular pattern and may change considerably from one case to the next, the experimentally measured correlation functions need not have much in common. The only firm conclusion is that for "wide" beam antennas the correlation distance for transverse displacements of the antennas is of the order of λ/θ, while the correlation distance for antennas displaced along the beam is λ/θ^2.

§ 32. Probability distributions for the scattered field

The scattered field (in the single-scattering approximation used in our treatment) is an integral of a determinate function multiplied by a random function $\varepsilon_1(\boldsymbol{r}')$. The scattering volume is much larger than the correlation radius L_0 of the fluctuations in ε_1. The probability distribution of the scattered field in this case is nearly normal by virtue of the central limit theorem of probability theory.* Moreover, $E_s(\boldsymbol{r})$ may be regarded as a Gaussian random field.

To simplify further treatment, we consider the case of a scalar equation in the Fraunhofer diffraction approximation. The scattered field is given by (29.2)

$$E_s(\boldsymbol{r}) = \frac{k^2 A_0 e^{ikr}}{4\pi r} \int\limits_V \varepsilon_1(\boldsymbol{r}') e^{i\boldsymbol{K}\boldsymbol{r}'} d^3r' \qquad (1)$$

(the origin is at the center of the scattering volume). We are interested in establishing the probability distributions for the scattered field at two points $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$. Since E_s is a complex variable, we are dealing in fact with two random variables at each of the points $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$:

$$C(\boldsymbol{r}_i) + iD(\boldsymbol{r}_i) = E_s(\boldsymbol{r}_i).$$

Since C_1, D_1, C_2, D_2 follow a four-point normal distribution, it suffices to find the matrix of the second moments for C_i, D_i in order to obtain all the distribution parameters.

Let us first consider $\langle E_s(\boldsymbol{r}_1) E_s(\boldsymbol{r}_2)\rangle$. Using (1) and assuming the dielectric constant fluctuations to be statistically homogeneous, we obtain

$$\langle E_s(\boldsymbol{r}_1) E_s(\boldsymbol{r}_2)\rangle = \left(\frac{k^2 A_0}{4\pi r}\right)^2 e^{2ikr} \iint\limits_{VV} B_\varepsilon(\boldsymbol{r}' - \boldsymbol{r}'') e^{i\boldsymbol{K}(\boldsymbol{r}'+\boldsymbol{r}'')} d^3r' \, d^3r''. \qquad (2)$$

* Application of the central limit theorems implies that certain special conditions are satisfied (e.g., the correlation function of the ε_1 fluctuations should fall off sufficiently rapidly). Without going into this problem in any detail, we assume that the necessary conditions are indeed satisfied.

In (2) we introduce new variables $(\boldsymbol{r}'-\boldsymbol{r}'')$ and $(\boldsymbol{r}'+\boldsymbol{r}'')$. Integration over $(\boldsymbol{r}'+\boldsymbol{r}'')$ gives the function $\delta_V(\boldsymbol{K})$, which for sufficiently large V is close to zero for all $K \neq 0$. We may thus take

$$\langle E_s(\boldsymbol{r}_1)\,E_s(\boldsymbol{r}_2)\rangle = 0. \tag{3}$$

Inserting $E_s(\boldsymbol{r}_i) = C(\boldsymbol{r}_i) + iD(\boldsymbol{r}_i)$, we obtain

$$\begin{aligned}&\langle [C(\boldsymbol{r}_1)+iD(\boldsymbol{r}_1)]\,[C(\boldsymbol{r}_2)+iD(\boldsymbol{r}_2)]\rangle = \\ &= [\langle C(\boldsymbol{r}_1)\,C(\boldsymbol{r}_2)\rangle - \langle D(\boldsymbol{r}_1)\,D(\boldsymbol{r}_2)\rangle] + i\,[\langle C(\boldsymbol{r}_1)\,D(\boldsymbol{r}_2)\rangle + \langle C(\boldsymbol{r}_2)\,D(\boldsymbol{r}_1)\rangle] = 0,\end{aligned}$$

hence

$$\langle C(\boldsymbol{r}_1)\,C(\boldsymbol{r}_2)\rangle = \langle D(\boldsymbol{r}_1)\,D(\boldsymbol{r}_2)\rangle, \tag{4}$$

$$\langle C(\boldsymbol{r}_1)\,D(\boldsymbol{r}_2)\rangle = -\langle C(\boldsymbol{r}_2)\,D(\boldsymbol{r}_1)\rangle. \tag{5}$$

Let us now calculate $\langle E_s(\boldsymbol{r}_1)E_s^*(\boldsymbol{r}_2)\rangle = B_E(\boldsymbol{r}_1, \boldsymbol{r}_2)$, as was done in the previous section. Putting $E_s = C + iD$ and using (4) and (5), we get

$$\begin{aligned}B_E(\boldsymbol{r}_1, \boldsymbol{r}_2) &= \langle [C(\boldsymbol{r}_1)+iD(\boldsymbol{r}_1)]\,[C(\boldsymbol{r}_2)-iD(\boldsymbol{r}_2)]\rangle = \\ &= 2\langle C(\boldsymbol{r}_1)\,C(\boldsymbol{r}_2)\rangle + 2i\langle D(\boldsymbol{r}_1)\,C(\boldsymbol{r}_2)\rangle.\end{aligned}$$

Therefore

$$\alpha = \langle C(\boldsymbol{r}_1)\,C(\boldsymbol{r}_2)\rangle = \langle D(\boldsymbol{r}_1)\,D(\boldsymbol{r}_2)\rangle = \frac{1}{2}\,\mathrm{Re}\,B_E(\boldsymbol{r}_1, \boldsymbol{r}_2), \tag{6}$$

$$\beta = \langle D(\boldsymbol{r}_1)\,C(\boldsymbol{r}_2)\rangle = -\langle C(\boldsymbol{r}_1)\,D(\boldsymbol{r}_2)\rangle = \frac{1}{2}\,\mathrm{Im}\,B_E(\boldsymbol{r}_1, \boldsymbol{r}_2). \tag{7}$$

First we consider the probability distribution of the field at one point $\boldsymbol{r}$. Leeting $\boldsymbol{r}_1 = \boldsymbol{r}_2 = \boldsymbol{r}$ in (4) and (5) we obtain

$$\langle C^2\rangle = \langle D^2\rangle, \quad \langle C(\boldsymbol{r})\,D(\boldsymbol{r})\rangle = 0.$$

From (6) for $\boldsymbol{r}_1 = \boldsymbol{r}_2$ we have

$$\langle C^2\rangle = \langle D^2\rangle = \frac{1}{2}\langle |E|^2\rangle.$$

The random variables C and D are thus normally distributed with equal variances and are uncorrelated. Consequently, the probability density function $P(C, D)$ is given by

$$P(C, D) = \frac{1}{2\pi\sigma^2}\,e^{-\frac{1}{2\sigma^2}(C^2+D^2)}, \tag{8}$$

where

$$\sigma^2 = \langle C^2\rangle = \frac{1}{2}\langle |E|^2\rangle.$$

Let us now consider the distribution of the amplitude of the scattered field $A = \sqrt{C^2 + D^2}$ and the phase $\varphi = \tan^{-1}(D/C)$:

$$P(A, \varphi) = P(C, D)\frac{d(C, D)}{d(A, \varphi)} = \frac{A}{2\pi\sigma^2}e^{-\frac{A^2}{2\sigma^2}}. \tag{9}$$

From (9) it follows that the phase of the scattered field is uniformly distributed over the interval $(-\pi, \pi)$, and the amplitude follows a Rayleigh distribution:

$$P(A) = \frac{A}{\sigma^2}e^{-\frac{A^2}{2\sigma^2}}. \tag{10}$$

We will use (10) to compute the fluctuations in the scattered power. The scattered power is proportional to $I = A^2$. Using (10), it is easy to show that $\langle A^2\rangle = 2\sigma^2$ and $\langle A^4\rangle = 8\sigma^4$. Therefore,

$$\langle[I - \langle I\rangle]^2\rangle = \langle A^4\rangle - \langle A^2\rangle^2 = 4\sigma^4 = \langle A^2\rangle^2,$$

or

$$\langle[I - \langle I\rangle]^2\rangle = \langle I\rangle^2. \tag{11}$$

In this case the mean square of the fluctuations of the power in a scattered signal is equal to the average power, i.e., the entire received power is of a "fluctuating" character.

Now we proceed with the probability distribution for the four variables $C(\boldsymbol{r}_1), D(\boldsymbol{r}_1), C(\boldsymbol{r}_2), D(\boldsymbol{r}_2)$. Suppose that the scattered intensity is the same at the two points, i.e.,

$$\langle C^2(\boldsymbol{r}_1)\rangle = \langle C^2(\boldsymbol{r}_2)\rangle = \sigma^2,$$

then we obtain for the matrix of the second moments

$$\begin{pmatrix} \langle C(\boldsymbol{r}_1)C(\boldsymbol{r}_1)\rangle & \langle C(\boldsymbol{r}_1)D(\boldsymbol{r}_1)\rangle & \langle C(\boldsymbol{r}_1)C(\boldsymbol{r}_2)\rangle & \langle C(\boldsymbol{r}_1)D(\boldsymbol{r}_2)\rangle \\ \langle D(\boldsymbol{r}_1)C(\boldsymbol{r}_1)\rangle & \cdots & \cdots & \cdots \\ \langle C(\boldsymbol{r}_2)C(\boldsymbol{r}_1)\rangle & \cdots & \cdots & \cdots \\ \langle D(\boldsymbol{r}_2)C(\boldsymbol{r}_1)\rangle & \cdots & \cdots & \cdots \end{pmatrix} =$$

$$= \begin{pmatrix} \sigma^2 & 0 & \alpha & -\beta \\ 0 & \sigma^2 & \beta & \alpha \\ \alpha & \beta & \sigma^2 & 0 \\ -\beta & \alpha & 0 & \sigma^2 \end{pmatrix} = (B_{ik}). \tag{12}$$

The characteristic function of the probability distribution for C_1, D_1, C_2, D_2 is

$$\chi(u_1, u_2, u_3, u_4) = e^{-\frac{1}{2}\Sigma\Sigma B_{ik}u_iu_k}, \tag{13}$$

and the probability density is determined by the inverse quadratic form.

If in the probability distribution for C_1, D_1, C_2, D_2 we introduce new variables the amplitudes $A(r_j) = \sqrt{C^2(r_j) + D^2(r_j)}$ and the phases $\varphi(r_j) = \tan^{-1} D(r_j)/C(r_j)$, and integrate the joint probability distribution of amplitudes and phases over φ_1, φ_2, we obtain a joint probability distribution for $A(r_1), A(r_2)$. This is the well-known result /73/

$$P(A_1, A_2) = \frac{A_1 A_2}{\sigma^4(1-p^2)} I_0\left(\frac{pA_1A_2}{\sigma^2(1-p^2)}\right) \exp\left(-\frac{A_1^2 + A_2^2}{2\sigma^2(1-p^2)}\right), \tag{14}$$

where $p = \frac{\sqrt{\alpha^2+\beta^2}}{\sigma^2}$, $I_0(x)$ is the modified Bessel function of zero order.

Knowing $P(A_1, A_2)$, we can find the covariance function of the amplitude fluctuations at points r_1 and r_2 /73/:

$$\langle A(r_1) A(r_2)\rangle = \int_0^\infty\int_0^\infty A_1 A_2 P(A_1, A_2)\, dA_1\, dA_2 =$$
$$= \sigma^2 [2E(p) - (1-p^2) K(p)]. \tag{15}$$

Here $K(p)$ and $E(p)$ are complete elliptical integrals of the first and second kind.

The correlation function of amplitude fluctuations is

$$b_A(r_1, r_2) = \frac{\langle A_1 A_2\rangle - \langle A\rangle^2}{\langle A^2\rangle - \langle A\rangle^2}.$$

Using the series expansions of E and K, we readily find

$$b_A = \frac{\pi}{4-\pi}\left[\left(\frac{1}{2}\right)^2 p^2 + \left(\frac{1}{2\cdot 4}\right)^2 p^4 + \ldots\right] =$$
$$= 0.921p^2 + 0.058p^4 + 0.014p^6 + \ldots \tag{16}$$

The correlation function of the amplitude fluctuations is thus expressed in terms of the square of the modulus of the field correlation function

$$p^2 = \frac{B_E(r) B_E^*(r)}{B_E(0) B_E^*(0)} = b_E(r) b_E^*(r). \tag{17}$$

Given particular expressions for $b_E(r)$, from (16) we can find the corresponding expressions for $b_A(r)$.

§ 33. Tropospheric propagation of ultrashort radio waves

In this section, the experimental data on long-range tropospheric radio propagation are compared with the theory of scattering by turbulent inhomogeneities in the troposphere.

The first step is a comparison of the observed signal power with the results predicted by scattering theory. We will use the simplest form of (26.28), taking for the flux density of the scattered energy

$$S \approx \frac{S_0 \sigma_0 V}{r^2}, \tag{1}$$

where S_0 is the flux density of the incident energy, V is the scattering volume, r is the distance from the center of the scattering volume to the observation point,

$$\sigma_0 = \frac{\pi}{2} k^4 \overline{\Phi}_\varepsilon (\boldsymbol{K}) \sin^2 \chi \tag{2}$$

is the effective scattering cross section per unit scattering volume per unit solid angle.

Assuming that $K = 2k \sin \frac{\theta}{2}$ (θ is the scattering angle) lies inside the inertial subrange of turbulence, we find

$$\Phi_\varepsilon (K) = 0.033\, C_\varepsilon^2 K^{-11/3} \quad \left(\frac{2\pi}{L_0} < 2k \sin \frac{\theta}{2} < \frac{2\pi}{l_0}\right), \tag{3}$$

so that /74/

$$\sigma_0 (\theta) = 0.052 k^{1/3} C_\varepsilon^2 \left(2 \sin \frac{\theta}{2}\right)^{-11/3} \sin^2 \chi. \tag{4}$$

It is preferable not to compare the experimental data directly with the flux density S, but to compare it with the received scattered power instead,

$$P_s = AS = \frac{S_0 \sigma_0 V A}{r^2}, \tag{5}$$

where A is the effective aperture of the receiving antenna. Now,

$$S_0 = \frac{P_0 G_0}{4\pi R_0^2}, \tag{6}$$

where P_0 is the transmitted power, R_0 is the distance from the source to the center of the scattering volume, and G_0 is the (power) gain factor of the transmitting antenna. Inserting (6) in (5) gives

$$\frac{1}{N} \equiv \frac{P_s}{P_0} = \frac{\sigma_0 V A G_0}{4\pi r^2 R_0^2}. \tag{7}$$

N is called "the transmission loss."

N is sometimes replaced by the ratio of the received scattered power to the power ideally received in vacuum with the same antennas and equipment. If d is the distance between the receiving and the transmitting antennas (generally $d \approx r + R_0$), the received power in free space is

$$P_{fs} = \frac{P_0 G_0 A}{4\pi d^2}. \tag{8}$$

Comparison with (7) gives

$$\frac{P_s}{P_{fs}} = \frac{\sigma_0 V}{R_*^2}, \quad R_* = \frac{rR}{d} \approx \frac{rR_0}{r + R_0}. \tag{9}$$

Expression (9) is independent of antenna parameters, and therefore this ratio is very convenient for comparison with data from different experiments. However, the scattering volume V depends on the beam widths of the two antennas.

If the beam width is small compared with the scattering angle θ and $\theta \ll 1$,

$$V \approx \frac{d^3 \gamma_1^2 \gamma_2}{8\theta} \tag{10}$$

(see § 28), where γ_1 and γ_2 are the effective beam widths in the vertical and horizontal planes, respectively. Identical receiving and transmitting antennas are assumed.

The gain factor G for narrow beam antennas is related to γ_1, γ_2 by the approximate relation $G = 4\pi/\gamma_1\gamma_2$ (the ratio of the total solid angle 4π to the solid angle $\gamma_1\gamma_2$ in which the radiation is concentrated). Therefore, if we take $\gamma_1 = \gamma_2 = \gamma$, we obtain

$$V \approx 5.6 \frac{d^3}{\theta G^{3/2}} . \tag{10'}$$

The parameter A is related to G by the general expression (which follows from thermodynamic considerations /75/)

$$\lambda^2 G = 4\pi A .$$

Using this relation and (10'), we obtain for the numerator of (7)

$$VAG_0 \approx 0.44 d^3 \lambda^2 \sqrt{\frac{G}{\theta}} .$$

The ratio P_s/P_0 thus increases as $\sqrt{G}$ (the gain factor of the two antennas), whereas for free-space transmission $P_{fs}/P_0 \sim G^2$. This effect of antenna gain loss in long-range tropospheric propagation has been studied experimentally in considerable detail. It is a direct consequence of the fact that the scattered power is proportional to the volume contributing to the scattering.

Using (9), (10), and (4), we can use the experimentally measured value of P_s/P_{fs} and the known parameters in the experiment, θ, G, etc., to establish the value of the necessary parameter C_ε entering the expression for the spectral density of the dielectric constant fluctuations.

The results of these experiments (see, e. g., /76, 77/) give $C_\varepsilon = (4 \cdot 10^{-9} - 1.5 \cdot 10^{-7})$ cm$^{-1/3}$ as the value necessary to explain the actually observed field powers in long-range tropospheric propagation of ultrashort waves. If we introduce the parameter $C_n = 2C_\varepsilon$ characterizing the refractive index fluctuations ($n^2 = \varepsilon$), then the required values for C_n are $0.002 - 0.080$ N-units/cm$^{1/3}$ (1 N-unit $= 10^{-6}$). Direct measurements of refractive index fluctuations carried out with refractometers /35, 61/ give C_n on the order of 0.020 N-units/cm$^{1/3}$.

More numerous measurements of temperature fluctuations in the troposphere, when evaluated for C_n, give values of $0.001 - 0.020$ N-units/cm$^{1/3}$. Comparison of these C_n values with experiments on long-range tropospheric radio propagation paths leads to the conclusion that the scattering of radio waves from turbulent inhomogeneities accounts for the "weak component" of the received signal, which is observed most of the time. The occasional high-intensity fields in long-range propagation of ultrashort radio waves apparently is due to other mechanisms (atmospheric waveguides, reflection from atmospheric layers with high refractive index gradients, etc.).

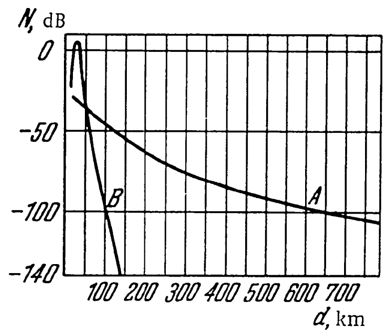

FIGURE 34. Signal intensity vs. distance in long-range tropospheric propagation (*A*) and the corresponding curve calculated from diffraction theory (*B*).

Let us consider the dependence of the scattered intensity on distance. Figure 34 gives the values of $N=10\log\frac{P_s}{P_{fs}}$ as a function of distance d /78/, obtained experimentally from various sources (curve *A*); curve *B* was computed from diffraction theory. Curve *A* corresponds approximately to the law $P_s/P_{fs}\sim d^{-6}$.

Let us establish the corresponding dependence proceeding from theoretical considerations. Inserting (4) and (10') in (9), we obtain

$$\frac{P_s}{P_{fs}}=\frac{0.29k^{1/3}}{G^{3/2}}\frac{d^3C_\varepsilon^2(z)}{R_*^2\theta^{14/3}} \tag{11}$$

R_* is proportional to d, so that $d^3/R_*^2\sim d$. The scattering angle θ is also related to d. If a is the radius of the Earth, then $\theta\approx d/a$. Thus, $d^3R_*^{-2}\theta^{-14/3}\sim d^{-11/3}$. However, $C_\varepsilon^2(z)$ is also a function of d. In fact, z (the height of the scattering volume) is related to d and a by

$$z\approx\frac{d^2}{8a}.$$

C_ε^2 is also a function of height. During daytime, when free convection conditions apply (see Chapter 1),

$$C_\varepsilon^2(z)\sim z^{-4/3}\sim d^{-8/3}.$$

Consequently,

$$\frac{P_s}{P_{fs}}\sim d^{-19/3}=d^{-6.33},$$

in good agreement with the experimental data for long-range tropospheric propagation of ultrashort radio waves.

Finally, let us consider the dependence of the scattered power on frequency. According to (4), it is determined only by the factor $k^{1/3}$, i. e., $P_s/P_{fs} \sim \lambda^{-1/3}$. Most experiments indicate, however, that the power ratio is probably directly proportional to the wavelength. In a special experiment described in /35/ signals scattered from the same scattering volume were received simultaneously at two frequencies (417 and 2290 MHz) (all antennas had identical beam patterns in these tests). The P_s/P_{fs} ratio obtained at two frequencies gives directly the dependence of the scattered intensity on frequency. In 99% of the cases the exponent in the expression $P_s/P_{fs} \sim \lambda^{\alpha}$ was found to be greater than $-1/3$, in 50% of the cases it was greater than $+1$, and in 1% of the cases greater than $+2$. The average value of the exponent is close to $+1$. The frequency dependence of the signal intensity in long-range tropospheric scattering is thus at variance with the prediction of the scattering theory based on turbulent inhomogeneities.

In summary, we can say that the scattering of radio waves by atmospheric turbulence clearly plays a definite role in explaining long-range tropospheric propagation, since the observed signal levels (or more precisely, the weak signal component received most of the time) are in good agreement with the experimentally measured refractive index fluctuations.

The scattering theory adequately accounts for such effect as the rapid fading of the signal, antenna gain loss, dependence on distance and some others. There are certain experimental facts, however, which cannot be explained within the framework of the scattering theory (dependence on frequency), so that at the same time other mechansims apparently also make a contribution.

B. SCATTERING OF SOUND IN A TURBULENT ATMOSPHERE

Scattering of sound waves in a turbulent flow has much in common with the scattering of electromagnetic waves. The propagation velocity of sound is a function of temperature and wind velocity. Both these quantities experience fluctuations caused by the atmospheric turbulence and therefore produce scattering. Scattering of sound in a turbulent atmosphere was investigated by Obukhov /79/ in 1941. Numerous publications on the subject followed /80 –86/, which approached the problem more rigorously.* Finally, this phenomenon was studied in detail experimentally /87, 88/.

§ 34. Derivation of the equations of sound propagation in a turbulent atmosphere

Propagation of sound in a turbulent medium is described by the equations of hydrodynamics. Ignoring dissipative processes, we write Euler's equation of motion

* Most of the early studies of the scattering of sound in a turbulent atmosphere proceeded from exceedingly crude starting equations. This problem was finally settled in the works of Kraichnan /85/ and Monin /86/.

$$\frac{\partial v_i}{\partial t} = -v_k \frac{\partial v_i}{\partial x_k} - \frac{1}{\rho}\frac{\partial p}{\partial x_i}. \tag{1}$$

Besides this equation, the equation of continuity is necessary,

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \boldsymbol{v}) = 0. \tag{2}$$

Propagation of sound is an adiabatic process, therefore the third equation is

$$\frac{dS}{dt} = \frac{\partial S}{\partial t} + \boldsymbol{v} \operatorname{grad} S = 0, \tag{3}$$

where S is the entropy. For an ideal gas

$$S = C_V \ln p - C_p \ln \rho, \tag{4}$$

where C_V and C_p are the specific heats for constant volume and constant pressure, respectively. Equation (2) can be written in the form

$$\frac{\partial \rho}{\partial t} + \boldsymbol{v} \operatorname{grad} \rho + \rho \operatorname{div} \boldsymbol{v} = 0$$

or, dividing through by ρ,

$$\frac{\partial \ln \rho}{\partial t} + \boldsymbol{v} \operatorname{grad} \ln \rho = \frac{d \ln \rho}{dt} = -\operatorname{div} \boldsymbol{v}. \tag{2a}$$

From equations (3) and (4) we have $(\gamma = C_p/C_V)$

$$\frac{d \ln \rho}{dt} = \frac{1}{\gamma}\frac{d \ln p}{dt}. \tag{3a}$$

Inserting (3a) in (2a), we find

$$\frac{1}{\gamma}\frac{d \ln p}{dt} = -\operatorname{div} \boldsymbol{v}. \tag{5}$$

The last term in the right-hand side of equation (1) can be transformed as follows:

$$\frac{1}{\rho}\frac{\partial p}{\partial x_i} = \frac{p}{\rho}\frac{\partial \ln p}{\partial x_i} = \frac{c^2}{\gamma}\frac{\partial \ln p}{\partial x_i},$$

where $c^2 = \gamma RT = \gamma \frac{p}{\rho}$ is the velocity of sound squared, which will be regarded as a known function of position and time. Equation (1) then takes the form

$$\frac{\partial v_i}{\partial t} = -v_k \frac{\partial v_i}{\partial x_k} - \frac{c^2}{\gamma}\frac{\partial \ln p}{\partial x_i}. \tag{6}$$

Equations (5) and (6) constitute a closed system for the functions p and $\boldsymbol{v}$.

Let $v_i = \xi_i + u_i$, where ξ_i is the acoustic velocity and u_i is the given velocity of turbulent motion. We take into account that div $\boldsymbol{u} = 0$, i.e., the turbulent motion is incompressible. Also let $p = p_0 + p_a$, where p_0 is the constant external pressure and p_a is the acoustic pressure. Then

$$\frac{1}{\gamma}\ln p = \frac{1}{\gamma}\ln\left[p_0\left(1+\frac{p_a}{p_0}\right)\right] = \frac{1}{\gamma}\ln p_0 + \frac{1}{\gamma}\ln\left(1+\frac{p_a}{p_0}\right).$$

Using the fact that $|p_a/p_0| \ll 1$, we write $\ln\left(1+\frac{p_a}{p_0}\right) \approx \frac{p_a}{p_0}$ and introduce a new variable

$$\Pi = \frac{p_a}{\gamma p_0} = \frac{p_a}{\rho_0 c_0^2}, \tag{7}$$

which is proportional to the acoustic pressure. Since we assumed that $p_0 = \text{const}$, we have

$$\frac{1}{\gamma}\frac{\partial \ln p}{\partial x_i} = \frac{\partial \Pi}{\partial x_i} \quad \text{and} \quad \frac{1}{\gamma}\frac{d\ln p}{dt} = \frac{d\Pi}{dt}.$$

Finally, taking into account that the temperature T fluctuates, we let $T = T_0 + T'$, where $T_0 = \text{const}$. Then

$$c^2 = c_0^2\left(1+\frac{T'}{T_0}\right), \text{ where } c_0^2 = \gamma R T_0 = \text{const}. \tag{8}$$

Substituting the above relations in (5) and (6) and linearizing the resulting equations with respect to the acoustic variables, we obtain*

$$\frac{\partial \xi_i}{\partial t} = -c_0^2\frac{\partial \Pi}{\partial x_i} - c_0^2\frac{T'}{T_0}\frac{\partial \Pi}{\partial x_i} - u_k\frac{\partial \xi_i}{\partial x_k} - \xi_k\frac{\partial u_i}{\partial x_k}, \tag{9}$$

$$\frac{\partial \Pi}{\partial t} = -\frac{\partial \xi_k}{\partial x_k} - u_k\frac{\partial \Pi}{\partial x_k}. \tag{10}$$

In the following the time dependence of the acoustic variables Π and ξ_i is described solely by an exponential factor $\exp(-i\omega t)$. The functions $\boldsymbol{u}$ and T' also depend on time, but we assume that all the characteristic frequencies in their spectrum are small compared to ω. In this case, $\boldsymbol{u}$, T' may be regarded as time-independent for our purposes, and only the final result is adjusted by the introduction of an appropriate time factor (this problem has been discussed in some detail in Part A in connection with electromagnetic waves).

Substituting $(-i\omega)$ for $\frac{\partial}{\partial t}$ in (9) and (10), we obtain

$$i\omega\xi_i = c_0^2\frac{\partial \Pi}{\partial x_i} + c_0^2\frac{T'}{T_0}\frac{\partial \Pi}{\partial x_i} + u_k\frac{\partial \xi_i}{\partial x_k} + \xi_k\frac{\partial u_i}{\partial x_k}, \tag{11}$$

$$i\omega\Pi = \frac{\partial \xi_k}{\partial x_k} + u_k\frac{\partial \Pi}{\partial x_k}. \tag{12}$$

* Terms containing $\boldsymbol{u}$ only are omitted; they cancel out as a result of the equations of motion for $\boldsymbol{u}$.

Taking the divergence of (11) and substituting in (12), we obtain

$$i\omega\Pi = \frac{1}{i\omega}\left[c_0^2\Delta\Pi + c_0^2\frac{\partial}{\partial x_i}\left(\frac{T'}{T_0}\frac{\partial\Pi}{\partial x_i}\right) + \frac{\partial}{\partial x_i}\left(u_k\frac{\partial\xi_i}{\partial x_k} + \xi_k\frac{\partial u_i}{\partial x_k}\right)\right] + u_k\frac{\partial\Pi}{\partial x_k}. \tag{13}$$

To eliminate the acoustic velocity ξ from (13), we again use equation (11). It suffices to take only the first term on the right-hand side of (11), since all the other terms when substituted in (13) give terms of second order of smallness with respect to u and T'. Furthermore multiplying (13) by $(-i\omega)$, we obtain

$$c_0^2\Delta\Pi + \omega^2\Pi = -i\omega u_k\frac{\partial\Pi}{\partial x_k} - c_0^2\frac{\partial}{\partial x_i}\left(\frac{T'}{T_0}\frac{\partial\Pi}{\partial x_i}\right) + \\ + \frac{ic_0^2}{\omega}\frac{\partial}{\partial x_i}\left[u_k\frac{\partial^2\Pi}{\partial x_i\partial x_k} + \frac{\partial\Pi}{\partial x_k}\frac{\partial u_i}{\partial x_k}\right]. \tag{14}$$

Let us examine the terms containing u in the right member of (14). Since $\partial u_k/\partial x_k = 0$, we may put the u_k inside the derivatives:

$$A = -i\omega u_k\frac{\partial\Pi}{\partial x_k} + \frac{ic_0^2}{\omega}\frac{\partial}{\partial x_i}\left[u_k\frac{\partial^2\Pi}{\partial x_i\partial x_k} + \frac{\partial\Pi}{\partial x_k}\frac{\partial u_i}{\partial x_k}\right] = \frac{ic_0^2}{\omega}\frac{\partial}{\partial x_i}\left\{\left[\frac{\partial}{\partial x_k}\left(u_k\frac{\partial\Pi}{\partial x_i}\right) + \frac{\partial\Pi}{\partial x_k}\frac{\partial u_i}{\partial x_k}\right] - \frac{\omega^2}{c_0^2}(u_i\Pi)\right\}.$$

Using the identity

$$\frac{\partial\Pi}{\partial x_k}\frac{\partial u_i}{\partial x_k} = \frac{\partial}{\partial x_k}\left(u_i\frac{\partial\Pi}{\partial x_k}\right) - u_i\Delta\Pi,$$

we obtain

$$A = \frac{ic_0^2}{\omega}\frac{\partial^2}{\partial x_i\partial x_k}\left[u_k\frac{\partial\Pi}{\partial x_i} + u_i\frac{\partial\Pi}{\partial x_k}\right] - \frac{ic_0^2}{\omega}\frac{\partial}{\partial x_i}\left[u_i\left(\Delta\Pi + \frac{\omega^2}{c_0^2}\Pi\right)\right].$$

But as is evident from (14), $\Delta\Pi + \frac{\omega^2}{c_0^2}\Pi$ is a linear function of the small variables u and T', so that the last term in the expression for A is of second order of smallness and can be omitted.

Seeing that

$$\frac{\partial^2}{\partial x_i\partial x_k}\left[u_k\frac{\partial\Pi}{\partial x_i} + u_i\frac{\partial\Pi}{\partial x_k}\right] = 2\frac{\partial^2}{\partial x_i\partial x_k}\left(u_i\frac{\partial\Pi}{\partial x_k}\right),$$

we finally write equation (14) in the form $(k = \frac{\omega}{c_0})$

$$\Delta\Pi + k^2\Pi = -\frac{\partial}{\partial x_i}\left(\frac{T'}{T_0}\frac{\partial\Pi}{\partial x_i}\right) - \frac{2}{i\omega}\frac{\partial^2}{\partial x_i\partial x_k}\left(u_i\frac{\partial\Pi}{\partial x_k}\right). \tag{15}$$

Equation (15), where the right-hand side is correct only to terms of first order in u and T', describes the propagation of sound in an atmosphere with variations in its temperature and wind velocity /86/. Since (15) is a linear equation, the acoustic pressure p_a, which differs from Π only by a constant factor, also satisfies this equation.

Note that having found Π from equation (15), we can determine the acoustic velocity ξ from (11), in which only the first term on the right-hand side needs to be retained (the other terms give contributions of second order of smallness in $\boldsymbol{u}$ and T').

The vector of the energy flux density is defined in terms of the complex variables p_a, ξ:

$$\boldsymbol{S} = \operatorname{Re} p_a \cdot \operatorname{Re} \xi. \tag{16}$$

Expressing p_a and ξ in terms of Π, it is easy to show that

$$\boldsymbol{S} = \frac{\rho_0 c_0^3}{2k} \operatorname{Im} [\Pi \nabla \Pi + \Pi^* \nabla \Pi]. \tag{17}$$

We will be mainly concerned with the flux density averaged over one period of oscillation T,

$$\boldsymbol{S}_{\text{av}} = \frac{1}{T} \int_0^T \boldsymbol{S}(t)\, dt.$$

Since $\Pi \sim e^{-i\omega t}$, the factor $\Pi \nabla \Pi$ contains a factor $e^{-2i\omega t}$, which averages out to zero. On the other hand, $\Pi^* \nabla \Pi$ is time-independent and is thus not affected by averaging. Consequently,

$$\boldsymbol{S}_{\text{av}} = \frac{\rho_0 c_0^3}{2k} \operatorname{Im} (\Pi^* \nabla \Pi). \tag{18}$$

In the following the subscript av is omitted.

§ 35. The effective scattering cross section

To derive an expression for the scattered field, we put in (34.15)

$$\Pi = \Pi_0 + \Pi_s,$$

where Π_0 is the incident wave satisfying the equation $\Delta \Pi_0 + k^2 \Pi_0 = 0$, and Π_s is the scattered field. Since the scattered field Π_s is a linear function of $\boldsymbol{u}$ and T', we may take $\Pi = \Pi_0$ on the right-hand side of (34.15). In this case, we obtain the equation

$$\Delta \Pi_s + k^2 \Pi_s = -\frac{\partial}{\partial x_i} \left(\frac{T'}{T_0} \frac{\partial \Pi_0}{\partial x_i} \right) - \frac{2}{i\omega} \frac{\partial^2}{\partial x_i \partial x_j} \left(u_i \frac{\partial \Pi_0}{\partial x_j} \right), \tag{1}$$

which describes single scattering of sound. The solution of (1) is

$$\Pi_s(\boldsymbol{r}) = \frac{1}{4\pi} \int_V d^3 r' \frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|} \frac{\partial}{\partial x_i'} \left[\frac{T'(\boldsymbol{r}')}{T_0} \frac{\partial \Pi_0(\boldsymbol{r}')}{\partial x_i'} + \frac{2}{i\omega} \frac{\partial}{\partial x_j'} \left(u_i(\boldsymbol{r}') \frac{\partial \Pi_0(\boldsymbol{r}')}{\partial x_j'} \right) \right]. \tag{2}$$

As we saw in the previous section, an expression for the effective scattering cross section can be obtained by carrying out the integration in (2) in the Fraunhofer diffraction approximation (the necessary conditions for the applicability of this procedure are $L \ll r$, $\lambda r \gg L^2$, where L is the size of the scattering volume and r is the distance from the scattering volume to the observation point). However, the result obtained in this way, as we established in the previous section, remains valid under the weaker condition $\lambda r \gg L_0^2$, where L_0 is the correlation distance of the fluctuations. We take in (2)

$$\frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|} \approx \frac{e^{ikr}}{r} e^{-ik\boldsymbol{n}\boldsymbol{r}'},$$

where $\boldsymbol{n}$ is the unit vector along $\boldsymbol{r}$. In the case where Π_0 is a plane wave

$$\Pi_0(\boldsymbol{r}') = Ae^{i\boldsymbol{k}\boldsymbol{r}'}. \tag{3}$$

For Π_s we then obtain

$$\Pi_s(\boldsymbol{r}) = \frac{iA_0 e^{ikr}}{4\pi r} \int\limits_V d^3r' e^{-ik\boldsymbol{n}\boldsymbol{r}'} \frac{\partial}{\partial x_i'} \left[\frac{T'(\boldsymbol{r}')}{T_0} k_i e^{i\boldsymbol{k}\boldsymbol{r}'} + \frac{2}{i\omega} \frac{\partial}{\partial x_j'} \left(u_i(\boldsymbol{r}') k_j e^{i\boldsymbol{k}\boldsymbol{r}'} \right) \right]. \tag{4}$$

Applying Gauss' theorem to the integral in (4), we ignore the surface integral, as surface effects are small compared to volume effects. We thus obtain

$$\Pi_s(\boldsymbol{r}) = -\frac{A_0 e^{ikr}}{4\pi r} kn_i \left[\int\limits_V e^{i(\boldsymbol{k}-k\boldsymbol{n})\boldsymbol{r}'} \frac{T'(\boldsymbol{r}')}{T_0} k_i d^3r' + \right.$$
$$\left. + \frac{2}{i\omega} \int\limits_V e^{-ik\boldsymbol{n}\boldsymbol{r}'} \frac{\partial}{\partial x_j'} (u_i(\boldsymbol{r}') k_j e^{i\boldsymbol{k}\boldsymbol{r}'}) d^3r' \right]. \tag{5}$$

The second integral in (5) is also transformed using Gauss' theorem and again the surface integral is dropped. The result gives

$$\Pi_s(\boldsymbol{r}) = -\frac{A_0 e^{ikr}}{4\pi r} k \left[n_i k_i \int\limits_V e^{i\boldsymbol{K}\boldsymbol{r}'} \frac{T'(\boldsymbol{r}')}{T_0} d^3r' + \right.$$
$$\left. + \frac{2}{i\omega} ikn_j k_j n_i \int\limits_V e^{i\boldsymbol{K}\boldsymbol{r}'} u_i(\boldsymbol{r}') d^3r' \right], \quad \boldsymbol{K} = \boldsymbol{k} - k\boldsymbol{n},$$

or changing the summation index in the first term from i to j,

$$\Pi_s(\boldsymbol{r}) = -\frac{A_0 e^{ikr}}{4\pi r} kn_j k_j \int\limits_V e^{i\boldsymbol{K}\boldsymbol{r}'} \left[\frac{T'(\boldsymbol{r}')}{T_0} + 2n_i \frac{u_i(\boldsymbol{r}')}{c_0} \right] d^3r'. \tag{6}$$

To find the flux density of the scattered energy, we need the gradient $\nabla\Pi_s$. When differentiating (6) with respect to $\boldsymbol{r}$, it is only necessary to differentiate the factor $\exp(ikr)$, since for $kr \gg 1$ (the far-field zone) differentiation of the second factor r^{-1} gives a contribution of the order $(kr)^{-1}$

$$\frac{\partial \Pi_s(\boldsymbol{r})}{\partial x_l} \approx ikn_l \Pi_s(\boldsymbol{r}); \tag{7}$$

hence,

$$S_l = \frac{\rho_0 c_0^3}{2k} \operatorname{Im}(\Pi^* \nabla \Pi) = \frac{1}{2} \rho_0 c_0^3 n_l \Pi_s(\boldsymbol{r}) \Pi_s^*(\boldsymbol{r}). \tag{8}$$

For this case the flux of the scattered energy is directed along the vector $\boldsymbol{n}$. Its average value is

$$\langle \boldsymbol{S} \rangle = \frac{\rho_0 c_0^3}{2} \boldsymbol{n} \langle \Pi_s(\boldsymbol{r}) \Pi_s^*(\boldsymbol{r}) \rangle. \tag{9}$$

Using (6), we obtain

$$\langle \boldsymbol{S} \rangle = \boldsymbol{n} \frac{\rho_0 c_0^3}{2} \frac{k^2 A_0^2}{16\pi^2 r^2} (\boldsymbol{kn})^2 \iint\limits_{VV} e^{i\boldsymbol{K}(\boldsymbol{r}_1 - \boldsymbol{r}_2)} \times$$
$$\times \left\langle \left[\frac{T'(\boldsymbol{r}_1)}{T_0} + 2n_i \frac{u_i(\boldsymbol{r}_1)}{c_0} \right] \left[\frac{T'(\boldsymbol{r}_2)}{T_0} + 2n_j \frac{u_j(\boldsymbol{r}_2)}{c_0} \right] \right\rangle d^3 r_1 d^3 r_2. \tag{10}$$

In case of isotropic turbulence, we have (see Chapter 1)

$$\langle T'(\boldsymbol{r}_1) u_i(\boldsymbol{r}_2) \rangle = 0,$$
$$\langle T'(\boldsymbol{r}_1) T'(\boldsymbol{r}_2) \rangle = B_T(\boldsymbol{r}_1 - \boldsymbol{r}_2), \quad \langle u_i(\boldsymbol{r}_1) u_j(\boldsymbol{r}_2) \rangle = B_{ij}(\boldsymbol{r}_1 - \boldsymbol{r}_2).$$

Introducing the coordinates $\boldsymbol{\rho} = \boldsymbol{r}_1 - \boldsymbol{r}_2$, we carry out the integration over $\boldsymbol{r}_2$, which gives the scattering volume V:

$$\langle \boldsymbol{S} \rangle = \boldsymbol{n} \frac{\rho_0 c_0^3}{2} \frac{k^2 A_0^2}{16\pi^2 r^2} (\boldsymbol{kn})^2 V \int\limits_V e^{i\boldsymbol{K}\boldsymbol{\rho}} \left[\frac{B_T(\boldsymbol{\rho})}{T_0^2} + 4 n_i n_j \frac{B_{ij}(\boldsymbol{\rho})}{c_0^2} \right] d^3\rho. \tag{11}$$

As in the previous section

$$\int\limits_V e^{i\boldsymbol{K}\boldsymbol{\rho}} B_T(\boldsymbol{\rho}) \, d^3\rho = (2\pi)^3 \overline{\Phi}_T(\boldsymbol{K}), \tag{12}$$

$$\int\limits_V e^{i\boldsymbol{K}\boldsymbol{\rho}} B_{ij}(\boldsymbol{\rho}) \, d^3\rho = (2\pi)^3 \overline{\Phi}_{ij}(\boldsymbol{K}), \tag{13}$$

and we obtain

$$\langle \boldsymbol{S} \rangle = \boldsymbol{n} \frac{\rho_0 c_0^3}{2} \frac{k^2 A_0^2}{16\pi^2 r^2} (\boldsymbol{kn})^2 V \cdot 8\pi^3 \left[\frac{\overline{\Phi}_T(\boldsymbol{K})}{T_0^2} + 4 n_i n_j \frac{\overline{\Phi}_{ij}(\boldsymbol{K})}{c_0^2} \right]. \tag{14}$$

For isotropic turbulence, $\Phi_{ij}(\boldsymbol{K})$ has the form (9.3),

$$\Phi_{ij}(\boldsymbol{K}) = \frac{1}{4\pi K^2} \left(\delta_{ij} - \frac{K_i K_j}{K^2} \right) E(K), \tag{15}$$

where $E(K)$ is the spectral energy density of the turbulence.

Let θ be the scattering angle. Then $\boldsymbol{kn} = k \cos\theta$ and

$$n_i n_j \Phi_{ij}(\boldsymbol{K}) = \frac{1}{4\pi K^2} \left(1 - \frac{(\boldsymbol{nK})^2}{K^2} \right) E(K).$$

If now $\boldsymbol{k} = \boldsymbol{m}k$, we have

$$\frac{\boldsymbol{K}}{K} = \frac{\boldsymbol{m}-\boldsymbol{n}}{|\boldsymbol{m}-\boldsymbol{n}|} \quad \text{and} \quad \boldsymbol{n}\frac{\boldsymbol{K}}{K} = \frac{\boldsymbol{mn}-1}{|\boldsymbol{m}-\boldsymbol{n}|} = \frac{\cos\theta - 1}{2\sin\frac{\theta}{2}} = -\sin\frac{\theta}{2}.$$

Hence,

$$n_i n_j \Phi_{ij}(\boldsymbol{K}) = \frac{\cos^2\frac{\theta}{2}}{4\pi K^2} E(K), \tag{16}$$

and expression (14) takes the form

$$\langle \boldsymbol{S} \rangle = \boldsymbol{n}\cdot\frac{\pi}{4}\rho_0 c_0^3 \frac{k^4 A_0^2}{r^2} V \cos^2\theta \left[\frac{\overline{\Phi}_T(K)}{T_0^2} + \overline{\frac{\cos^2\frac{\theta}{2}}{\pi}\frac{E(K)}{c_0^2 K^2}}\right]. \tag{17}$$

We introduce the effective scattering cross section per unit scattering volume per unit solid angle,

$$\sigma_0(\theta) = \frac{\langle S \rangle r^2}{S_0 V}, \tag{18}$$

where S_0 is the flux density of the incident energy, which using (34.18) and (3) is given by

$$S_0 = \left|\frac{\rho_0 c_0^3}{2k}\,\mathrm{Im}\,(\Pi_0^* \nabla \Pi_0)\right| = \frac{1}{2}\rho_0 c_0^3 A_0^2. \tag{19}$$

Inserting (17) and (19) in (18), we obtain

$$\sigma_0(\theta) = \frac{\pi}{2} k^4 \cos^2\theta \left[\frac{\overline{\Phi}_T\left(2k\sin\frac{\theta}{2}\right)}{T_0^2} + \overline{\frac{\cos^2\frac{\theta}{2}}{\pi}\frac{E\left(2k\sin\frac{\theta}{2}\right)}{c_0^2\left(2k\sin\frac{\theta}{2}\right)^2}}\right]. \tag{20}$$

We see from (20) that there is no scattering at right angles (the factor $\cos^2\theta$ vanishes for $\theta = \frac{\pi}{2}$) regardless of the particular form of the spectral functions $\Phi_T(\varkappa)$ and $E(\varkappa)$. Backward scattering ($\theta = \pi$), on the other hand, is produced predominantly by temperature inhomogeneities, since the factor $\cos^2\frac{\theta}{2}$ is zero in this case (but this effect is partly compensated by averaging over wavenumber space). That velocity fluctuations do not contribute to backward scattering is a direct consequence of the incompressibility of turbulent flow. No scattering at right angles is readily interpreted using Fresnel's relations.*

Indeed, for incidence at an angle α on the interface of two media characterized by the parameters ρ_1, c_1 and ρ_2, c_2, Fresnel's reflection coefficient R is given by

* The argument that follows is due to V. M. Bovsheverov.

$$R = \frac{\rho_2 c_2 \cos\alpha - \rho_1 c_1 \sqrt{1 - \left(\frac{c_2}{c_1}\sin\alpha\right)^2}}{\rho_2 c_2 \cos\alpha + \rho_1 c_1 \sqrt{1 - \left(\frac{c_2}{c_1}\sin\alpha\right)^2}}$$

(we are dealing with the case of temperature fluctuations only). It vanishes for $\alpha = \alpha_0$, where

$$\cos^2\alpha_0 = \frac{\rho_1^2 (c_1^2 - c_2^2)}{c_2^2 (\rho_2^2 - \rho_1^2)}.$$

For an ideal gas, $c_1^2 = \frac{\gamma p_0}{\rho_1}$, $c_2^2 = \frac{\gamma p_0}{\rho_2}$, where p_0 is the hydrostatic pressure, which is equal on both sides of the interface. Substituting these expressions, we obtain

$$\cos^2\alpha_0 = \frac{\rho_1}{\rho_1 + \rho_2}.$$

In the case of weak temperature fluctuations $\rho_1 \approx \rho_2$, so that $\cos^2\alpha_0 = \frac{1}{2}$ and $\alpha_0 = \frac{\pi}{4}$. As a result, there is no scattering at $\theta_0 = \pi - 2\alpha_0 = \frac{\pi}{2}$. A similar though somewhat more involved argument will show that a weak velocity inhomogeneity does not reflect sound at $\theta_0 = \frac{\pi}{2}$. In § 24 we showed that scattering at a given angle is determined by a single spectral component of the inhomogeneities, corresponding to spatial sinusoidal diffraction grating oriented so that the condition for mirror reflection is satisfied (as in Fresnel's formation). Such a diffraction grating, however, can be assembled from a set of small inhomogeneities, each satisfying Fresnel's formula for the reflection coefficient. Thus, no scattering should be observed at $\theta = \frac{\pi}{2}$ in the case of small temperature fluctuations.

Let us return to expression (20). if $2k \sin\frac{\theta}{2}$ lies inside the inertial subrange of the turbulence spectrum, i. e.,

$$\frac{2\pi}{L_0} < 2k \sin\frac{\theta}{2} < \frac{2\pi}{l_0},$$

we have (see Chapter 1)

$$\Phi_T(\varkappa) = 0.033\, C_T^2\, \varkappa^{-11/3}, \tag{21}$$

$$E(\varkappa) = 0.76 C^2 \varepsilon^{2/3} \varkappa^{-5/3} \tag{22}$$

($C^2 \approx 1.9$ is a numerical constant). Here C_T^2 is a characteristic of the temperature fluctuations, which enters the "2/3 law" for the temperature field, $\langle [T(\boldsymbol{r}_1) - T(\boldsymbol{r}_2)]^2 \rangle = C_T^2 |\boldsymbol{r}_1 - \boldsymbol{r}_2|^{2/3}$; ε is the rate of dissipation of turbulent energy. Inserting (21) and (22) in (20), we obtain

$$\sigma_0(\theta) = 0.38 k^{1/3} \cos^2\theta \left(2 \sin\frac{\theta}{2}\right)^{-11/3} \left[\frac{C^2 \varepsilon^{2/3}}{c_0^2} \cos^2\frac{\theta}{2} + 0.13 \frac{C_T^2}{T_0^2}\right]. \tag{23}$$

In the real atmosphere $C^2\varepsilon^{2/3}/c_0^2$ and C_T^2/T_0^2 are of the same order of magnitude. Since their ratio depends on meteorological conditions, the form of the function $\sigma_0 = \sigma_0(\theta)$ is sensitive to weather, and the experimental verification of expression (23) is difficult.

§ 36. Experiments with the scattering of sound in a turbulent atmosphere

An indirect indication of the existence of sound scattering effects in a turbulent atmosphere were first obtained back in 1940 by Sieg /89/, who discovered a relation between the observed attenuation of sound and wind speed. Comparison of Sieg's measurements with the results of scattering theory is hardly possible, however, since in order to calculate the attenuation it is necessary to take into account multiple scattering (see Chapters 3 and 5).

Sound scattering experiments carried out with the explicit purpose of comparing with theory were undertaken by Kallistratova /87, 88/.

Special electrostatic sources and microphones measuring $100 \times 120\,\text{cm}^2$ with an angular beam width of $+1°$ between half-power points at 11 kHz were developed for these experiments. The source and the microphone were mounted at a distance of a few tens of meters from each other. To eliminate the direct signal reaching the microphone without scattering (it is contributed by the side lobes of the radiation pattern of the source), the source operated in the pulse mode and not as a continuous signal. The direct and the scattered signal thus reached the microphone at different times, and this difference was quite sufficient for their complete resolution. A block diagram of the experiment is shown in Figure 35. Besides the main channel for scattering measurements at 11 kHz, $\lambda = 3$cm), it also shows an auxiliary 7 kHz channel for separating the direct from the scattered signal.

Figure 36 shows photographs of the radiated, direct, and scattered pulses, photographed from an oscilloscope screen for various scattering angles.

The equipment was used in repeated measurements of sound scattering for θ from 16° to 180° under a variety of meteorological conditions. Vertical wind velocity and temperature profiles were simultaneously measured.

To eliminate various effects due to the nonstationarity of the meteorological conditions, the measurements of $\sigma_0(\theta)$ were always accompanied by "standard" measurements of $\sigma_0(25°)$ (in some cases a different standard angle was chosen). The polar scattering diagrams were plotted using the ratios $\sigma_0(\theta)/\sigma_0(25°)$ independent of meteorological conditions for various θ. Figure 37 plots some experimental values of $\sigma_0(\theta)/\sigma_0(25°)$. The vertical bars mark the 5% confidence limits. The solid curve is drawn using the theoretical expression for scattering from inhomogeneities in the wind velocity. The data of Figure 37 are also shown in Table 1.

The graph reveals satisfactory agreement between theory and experiment. Moreover, no marked deviation from the theoretical features of the inertial subrange is observed. We thus conclude that the inner scale of turbulence in the ground layer is no more than a few millimeters.

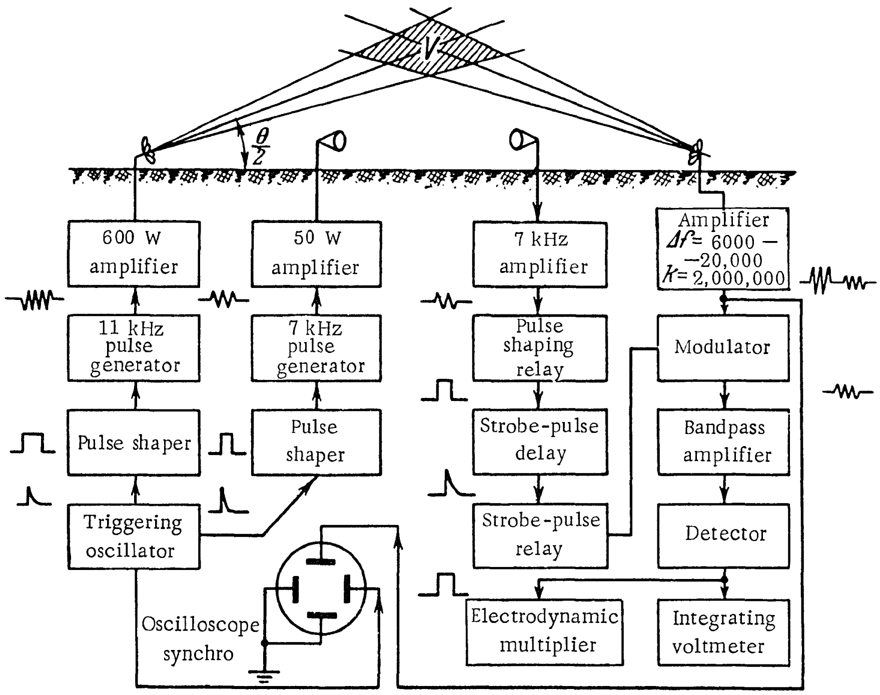

FIGURE 35. Block diagram of the equipment for sound scattering experiments.

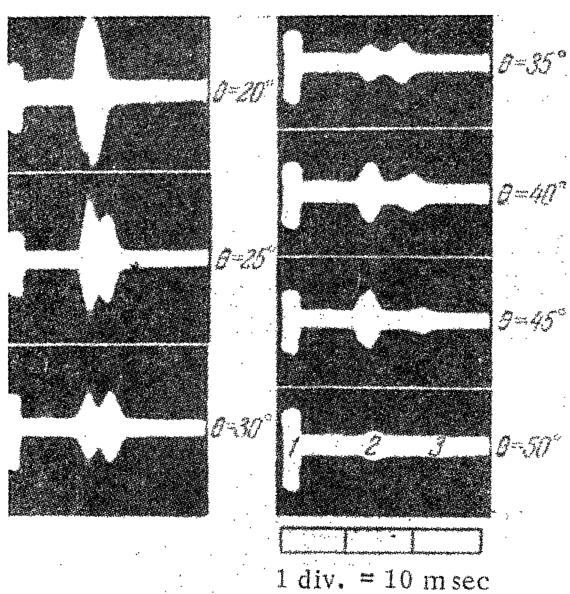

FIGURE 36. Photographs of radiated (1), direct (2), and scattered (3) pulses from an oscilloscope screen for various scattering angles.

Note the increasing delay between direct and scattered pulses and the attenuation of the scattered pulses at larger scattering angles.

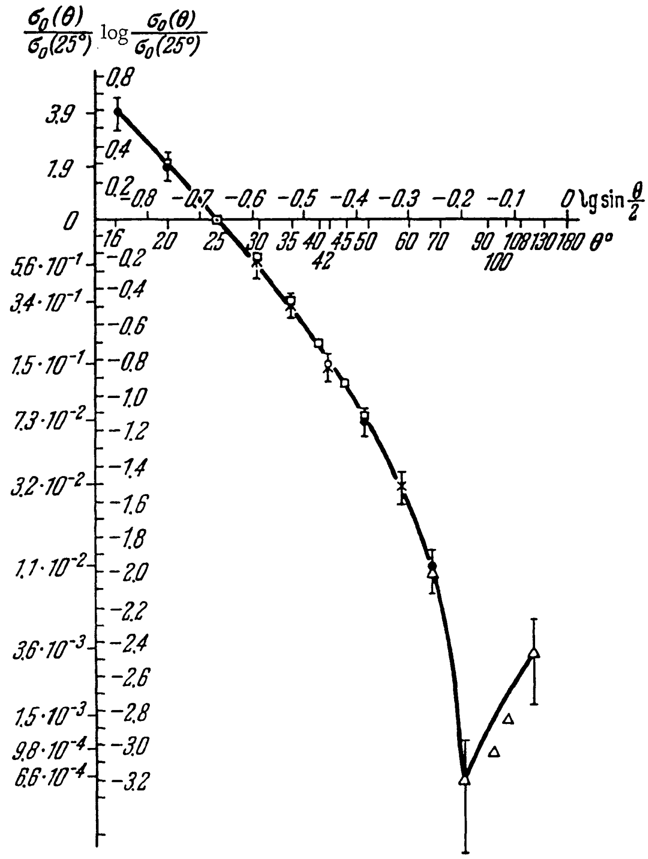

FIGURE 37. Empirical dependence of the effective sound scattering cross section (in relative units) on scattering angle.

● d = 140 m, o d = 80 m, × d = 40 m, Δ d = 20 m (1959 measurements),
□ d = 40 m (1958 measurements).

TABLE 1.

θ	16°	20°	25°	30°	35°	42°
$l = \frac{\lambda}{2\sin\frac{\theta}{2}}$, cm	11	8.6	6.9	5.8	5.0	4.2
$\sigma_0(\theta)/\sigma_0$ (25°), theor.	5.5	2.4	1	0.45	0.23	0.094
$\sigma_0(\theta)/\sigma_0$ (25°), exp.	4.8	2.2	1	0.49	0.25	0.086
Number of measurements	17	42	—	43	56	53

θ	50°	60°	70°	90°	130°
$l = \frac{\lambda}{2\sin\frac{\theta}{2}}$, cm	3.5	3.0	2.6	2.1	1.7
$\sigma_0(\theta)/\sigma_0$ (25°), theor.	0.035	$1\cdot10^{-2}$	$2.7\cdot10^{-3}$	0	$5\cdot10^{-4}$
$\sigma_0(\theta)/\sigma_0$ (25°), exp.	0.029	$0.76\cdot10^{-2}$	$1.3\cdot10^{-3}$	$0.9\cdot10^{-5}$	$2\cdot10^{-4}$
Number of measurements	50	25	23	4	4

The absolute value of the scattered signal was also compared with the theoretical results. The ratio of the scattered power to the power received when the microphone and the source are pointed directly at each other is given by (33.9)

$$\frac{P_s}{P_r} = \frac{\sigma_0(\theta)\, V}{R_*^2}, \quad R_* = \frac{rR}{d}, \tag{1}$$

where r and R are the distances from the center of the scattering volume to the microphone and the source, respectively; V is the distance from the microphone to the source. Experimentally the ratio $P_s(\theta)/P_r$ was measured directly for $\theta = 30°$ (as the theoretical scattering curve closely fits the experimental data, it suffices to check relation (1) for one angle only). The expression for $\sigma_0(\theta)$ in (1) is given by (35.23).

The quantities C_T^2 and ε entering (35.23) were computed from measurements of the vertical wind and temperature profiles using the relations of Chapter 1. The effect of temperature stratification on the turbulence, expressed by the functions $f_1(\mathrm{Ri})$ and $f_2(\mathrm{Ri})$ (see Figures 17, 18 on p. 97) was also taken into consideration.

Figure 38 plots the experimental ratio P_s/P_r vs. the theoretical ratio computed from measured wind and temperature profiles. The correlation coefficient between the theoretical and experimental values of P_s/P_r is 0.7. There is, however a systematic difference between the two quantitites (by a factor of 2), which is apparently due to errors in the calculation of the scattering volume and in the direct signal measurements. Despite the divergence in the numerical factor, the fit between theoretical and experimental figures is clearly satisfactory.

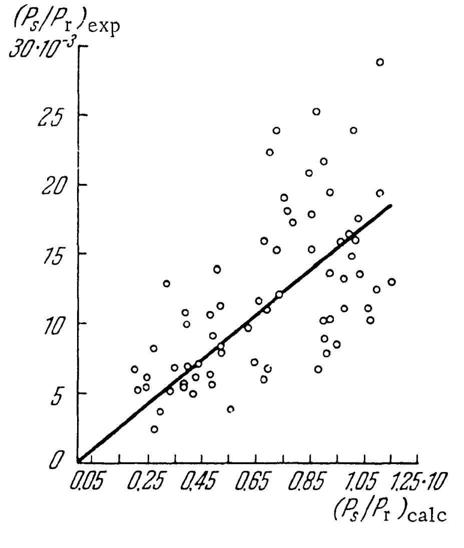

FIGURE 38. Scattered signal measurements ($\theta = 30°$) vs. the expected scattered signal computed from average wind velocity and temperature profiles in the ground layer.

The experiments of /87, 88/ thus provide a direct confirmation of the scattering theory. These experiments can be considered as a strong argument supporting our notions of tropospheric scattering of electromagnetic waves.

Chapter 3

LINE-OF-SIGHT PROPAGATION OF SHORT ELECTROMAGNETIC AND SOUND WAVES IN A TURBULENT ATMOSPHERE

In the previous chapter we considered in some detail single scattering of electromagnetic and sound waves in a turbulent medium. The single-scattering approximation is adequate for fields beyond the horizon. However, if the observation point intercepts the direct (incident) wave, multiple scattering effects cannot be ignored. Indeed, the polar diagram for the scattering of short waves, as we established before, is highly elongated in the forward direction. Multiple scattering is therefore possible when waves propagate over large distances in an inhomogeneous medium.

Interaction with inhomogeneities in the propagating medium causes fluctuations in phase, amplitude, frequency, direction of propagation (angle of arrival), and other parameters of the wave. These effects are significant in a wide range of applied problems associated with propagation of radio waves, light, and sound in the atmosphere (precision of navigation systems, atmospheric noise in communication channels, etc.).

In the present chapter we consider the propagation of short waves whose wavelength is small compared to the inner scale of turbulence ($\lambda \ll l_0$).*

Geometrical optics is successfully applied to the study of short wave propagation in a random medium if, in addition to the constraint $\lambda \ll l_0$, the radius of the first Fresnel zone is small compared to the inner scale of turbulence. If the latter condition is not satisfied, the method of smooth perturbations is used.

A. GEOMETRICAL OPTICS

§37. The effect of multiple scattering on the propagation of short waves in a random medium

Our treatment of geometrical optics starts with a discussion of multiple scattering by atmospheric inhomogeneities. The equations of geometrical optics are generally derived from the wave equation (see §38). Here we will obtain the corresponding solution by an approximate treatment of multiple scattering. The derivation that follows is far from being rigorous;

* The effect of multiple scattering on long-wave propagation is considered in Chapter 5.

its purpose is mainly to demonstrate how geometrical optics approximately takes into account multiple scattering at small angles.

We established in the previous chapter that small-angle forward scattering of electromagnetic waves can be considered with fair accuracy using the scalar wave equation, since the factor $\sin\chi$ associated with the wave polarization is then close to unity. We therefore start with the scalar equation

$$\Delta E + k^2 [1 + \varepsilon_1 (\boldsymbol{r}_1, t)] E = \delta (\boldsymbol{r} - \boldsymbol{R}) \tag{1}$$

in an unbounded medium. Here E is the complex amplitude of one of the electric field components, ε_1 is the deviation of the dielectric constant from its mean value (which is assumed equal to 1). For sufficiently small $\varepsilon_1 (\boldsymbol{r}, t)$ the solution of (1) can be written as a perturbation theoretical series.

Let

$$G_0 (\boldsymbol{r} - \boldsymbol{r}') = - \frac{\exp (ik |\boldsymbol{r} - \boldsymbol{r}'|)}{4\pi |\boldsymbol{r} - \boldsymbol{r}'|} \tag{2}$$

be the solution of the equation

$$\Delta G_0 (\boldsymbol{r}, \boldsymbol{r}') + k^2 G_0 (\boldsymbol{r}, \boldsymbol{r}') = \delta (\boldsymbol{r} - \boldsymbol{r}'),$$

satisfying the radiation conditions (we assume that $\operatorname{Im} k > 0$) in terms of which the solution of the equation

$$\Delta E + k^2 E = f (\boldsymbol{r})$$

is expressed in the form

$$E (\boldsymbol{r}) = \int G_0 (\boldsymbol{r} - \boldsymbol{r}') f (\boldsymbol{r}') \, d^3 r'. \tag{3}$$

We write equation (1) in the form

$$\Delta E + k^2 E = \delta (\boldsymbol{r} - \boldsymbol{R}) - k^2 \varepsilon_1 E$$

and apply to it (3):

$$E (\boldsymbol{r}) = G_0 (\boldsymbol{r} - \boldsymbol{R}) - k^2 \int G_0 (\boldsymbol{r} - \boldsymbol{r}') \varepsilon_1 (\boldsymbol{r}') E (\boldsymbol{r}') \, d^3 r'. \tag{4}$$

The unknown function E in (4) is on the right.

Inserting for $E(\boldsymbol{r}')$ on the right in (4) its expression from (4) (i.e., iterating equation (4)), we obtain

$$E (\boldsymbol{r}) = G_0 (\boldsymbol{r} - \boldsymbol{R}) - k^2 \int G_0 (\boldsymbol{r} - \boldsymbol{r}') \varepsilon_1 (\boldsymbol{r}') G_c (\boldsymbol{r}' - \boldsymbol{R}) \, d^3 r' +$$
$$+ k^4 \iint G_0 (\boldsymbol{r} - \boldsymbol{r}') \varepsilon_1 (\boldsymbol{r}') G_0 (\boldsymbol{r}' - \boldsymbol{r}'') \varepsilon_1 (\boldsymbol{r}'') E (\boldsymbol{r}'') \, d^3 r' d^3 r''.$$

Repeating the iteration, we obtain an infinite series

$$E (\boldsymbol{r}) = \sum_{n=0}^{\infty} E_n (\boldsymbol{r}), \tag{5}$$

where

$$E_0(\boldsymbol{r}) = G_0(\boldsymbol{r}-\boldsymbol{R}),$$
$$E_n(\boldsymbol{r}_0) = (-k^2)^n \underbrace{\int\ldots\int}_{(n)} G_0(\boldsymbol{r}_0-\boldsymbol{r}_1)\,G_0(\boldsymbol{r}_1-\boldsymbol{r}_2)\ldots$$
$$\ldots G_0(\boldsymbol{r}_{n-1}-\boldsymbol{r}_n)\,G_0(\boldsymbol{r}_n-\boldsymbol{R})\,\varepsilon_1(\boldsymbol{r}_1)\,\varepsilon_1(\boldsymbol{r}_2)\ldots\varepsilon_1(\boldsymbol{r}_n)\,d^3r_1\ldots d^3r_n \tag{6}$$

(here for the sake of uniformity we wrote $\boldsymbol{r}_0=\boldsymbol{r}$). Expression (6) for the n-th term in the series (5) represents a wave scattered n times and series (5) takes into account scattering effects of arbitrarily high orders.

Let us examine expression (6) for the case when the wavelength is much less than the characteristic size of the inhomogeneities. Consider the integral over the variable $\boldsymbol{r}_k$ in (6):

$$I_k = \int G_0(\boldsymbol{r}_{k-1}-\boldsymbol{r}_k)\,G_0(\boldsymbol{r}_k-\boldsymbol{r}_{k+1})\,\varepsilon_1(\boldsymbol{r}_k)\,d^3r_k =$$
$$= \frac{1}{16\pi^2}\int \frac{\exp\{ik[|\boldsymbol{r}_{k-1}-\boldsymbol{r}_k|+|\boldsymbol{r}_k-\boldsymbol{r}_{k+1}|]\}}{|\boldsymbol{r}_{k-1}-\boldsymbol{r}_k|\,|\boldsymbol{r}_k-\boldsymbol{r}_{k+1}|}\,\varepsilon_1(\boldsymbol{r}_k)\,d^3r_k. \tag{7}$$

Let us examine in more detail the structure of the exponential factor. If $\boldsymbol{r}_k$ lies on the line joining points $\boldsymbol{r}_{k-1}$ and $\boldsymbol{r}_{k+1}$, i.e.,

$$\boldsymbol{r}_k = \boldsymbol{r}_{k-1} + \alpha(\boldsymbol{r}_{k+1}-\boldsymbol{r}_{k-1}),\ \text{where}\ 0\leqslant\alpha\leqslant 1,$$

then $|\boldsymbol{r}_{k-1}-\boldsymbol{r}_k|+|\boldsymbol{r}_k-\boldsymbol{r}_{k+1}| = |\boldsymbol{r}_{k-1}-\boldsymbol{r}_{k+1}| =$ const and the exponential does not oscillate.

Consider another case. Find the locus of points for which $k[|\boldsymbol{r}_{k-1}-\boldsymbol{r}_k| + |\boldsymbol{r}_k-\boldsymbol{r}_{k+1}|]$ takes some constant a, i.e.,

$$|\boldsymbol{r}_{k-1}-\boldsymbol{r}_k|+|\boldsymbol{r}_k-\boldsymbol{r}_{k+1}| = \frac{a}{k}. \tag{8}$$

Since $|\boldsymbol{r}_{k-1}-\boldsymbol{r}_k|$ is the distance of $\boldsymbol{r}_k$ from $\boldsymbol{r}_{k-1}$ and $|\boldsymbol{r}_k-\boldsymbol{r}_{k+1}|$ is its distance from $\boldsymbol{r}_{k+1}$, the surface described by equation (8) is an ellipsoid of revolution with foci at $\boldsymbol{r}_{k-1}$, $\boldsymbol{r}_{k+1}$, a semimajor axis $a/2k$, and a semiminor axis $\sqrt{a^2-k^2|\boldsymbol{r}_{k-1}-\boldsymbol{r}_{k+1}|^2}/2k$. For $a=a_0=k|\boldsymbol{r}_{k-1}-\boldsymbol{r}_{k+1}|$, we obtain the straight line considered above. As a is changed by π, the exponential is multiplied by -1. The surface nearest to the axis is described by the equation $a_1=a_0+\pi$. The semiminor axis of this ellipsoid is

$$\frac{1}{2k}\sqrt{(a_0+\pi)^2-a_0^2} = \frac{1}{2k}\sqrt{2a_0\pi+\pi^2}.$$

The distance $|\boldsymbol{r}_{k-1}-\boldsymbol{r}_{k+1}|$ is of the same order of magnitude as the distance $|\boldsymbol{r}-\boldsymbol{R}|$ from the source to the observation point. If $k|\boldsymbol{r}-\boldsymbol{R}|\gg\pi$ (i.e., the observation point is situated in the far-field zone), $a_0\gg\pi$ and the semiminor axis of the first ellipsoid is approximately

$$\rho_1 \approx \sqrt{\frac{a_0\pi}{2k^2}} = \frac{1}{2}\sqrt{\lambda|\boldsymbol{r}_{k-1}-\boldsymbol{r}_{k+1}|}.$$

Similarly, if $a=a_0+m\pi$, the integrand in (7) acquires an additional factor $(-1)^m$ on the corresponding ellipsoidal surface. The semiminor axis is then

$$\rho_m \approx \frac{1}{2}\sqrt{m\lambda|\boldsymbol{r}_{k-1}-\boldsymbol{r}_{k+1}|}.$$

The succession of ellipsoids define three-dimensional shells (Fresnel zones), each with an exponential factor of constant sign in (7). The distance between successive surfaces near the midpoint of the segment from r_{k-1} to r_{k+1} is of the order of ρ_1, gradually decreasing for progressively larger Fresnel zones. Near the two ends of the major axis this distance is of the order of $\frac{\pi}{2k}=\frac{\lambda}{4}$.

Now suppose that the minimum size l_0 of the dielectric constant inhomogeneities satisfies the following two conditions:

$$l_0 \gg \lambda, \tag{9}$$

$$l_0 \gg \sqrt{\lambda L}, \text{ where } L=|\boldsymbol{r}-\boldsymbol{R}|. \tag{10}$$

Through r_{k-1} and r_{k+1} we draw two planes perpendicular to the vector $r_{k-1}-r_{k+1}$. The Fresnel ellipsoids meet these planes along circles of radii $\lambda n/2$ (for the n-th ellipsoid, when n is not excessively large). Thus, outside the layer between these planes, the exponential factor in (7) reverses its sign when $\boldsymbol{r}_k$ is incremented by $\lambda/2$. In virtue of condition (9), the remaining part of the integrand hardly changes over distances of the order of λ. Hence it follows that integration over the outer regions gives a contribution close to zero, since the integrand oscillates rapidly with almost equal positive and negative amplitudes.

Thus, when condition (9) is satisfied, the integration in (7) need be performed only over the region between the two planes, without leading to appreciable errors. Let the axis x_k point along the vector $r_{k+1}-r_{k-1}$. Then (7) is written as

$$I_k \approx \frac{1}{16\pi^2}\int_{x_{k-1}}^{x_{k+1}} dx_k \iint_{-\infty}^{\infty} dy_k dz_k \frac{\exp\{ik[|r_{k-1}-r_k|+|r_k-r_{k+1}|]\}}{|r_{k-1}-r_k||r_k-r_{k+1}|}\varepsilon_1(r_k). \tag{11}$$

The integral in (11) can be further simplified using condition (10). Note that I_k represents a singly scattered wave from a point source at r_{k-1} as observed at $\boldsymbol{r}_{k+1}$. Now, we know that the maximum scattering angle is of the order of $\theta\sim\frac{\lambda}{l_0}$. Since the distance $|x_{k+1}-x_{k-1}|$ between the source and the observation point is of the order L, scattered waves reach $\boldsymbol{r}_{k+1}$ only from those points whose distance from the x-axis does not excced $\theta L\sim\frac{\lambda L}{l_0}=$ $=\sqrt{\lambda L}\cdot\frac{\sqrt{\lambda L}}{l_0}\ll\sqrt{\lambda L}$ (since by (10), $\frac{\sqrt{\lambda L}}{l_0}\ll 1$).

Thus, a significant contribution in (11) comes only from the first ellipsoid (or the first few ellipsoids). The transverse dimension of this ellipsoid is much less than its longitudinal dimension, and therefore we may expand $|\boldsymbol{r}_{k-1}-\boldsymbol{r}_k|$ and $|\boldsymbol{r}_k-\boldsymbol{r}_{k+1}|$ in a series:

$$|\boldsymbol{r}_{k-1}-\boldsymbol{r}_k|=|x_{k-1}-x_k|+\frac{(y_{k-1}-y_k)^2+(z_{k-1}-z_k)^2}{2|x_{k-1}-x_k|}-$$
$$-\frac{[(y_{k-1}-y_k)^2+(z_{k-1}-z_k)^2]^2}{8|x_{k-1}-x_k|^3}+\dots \tag{12}$$

In the exponential term in (11) it is only necessary to retain the first two terms of this expansion. Since $[(y_{k-1}-y_k)^2+(z_{k-1}-z_k)^2]^2$ is of the order of $(\lambda L)^2$, this approximation is permissible when

$$\frac{k(\lambda L)^2}{L^3}\ll 1 \quad \text{or} \quad \lambda \ll L$$

(i.e., the observation point is located in the far-field zone). In the denominator in (11) we retain only the first term of expansion (12). By virtue of (10), we can also neglect the dependence of $\varepsilon_1(x, y, z)$ on the transverse coordinates y, z, since this function varies much more slowly than the exponential term. We thus obtain

$$I_k \approx \frac{e^{ik(x_{k+1}-x_{k-1})}}{16\pi^2}\int\limits_{x_{k-1}}^{x_{k+1}} \frac{\varepsilon_1(x_k, 0, 0)\,dx_k}{(x_k-x_{k-1})(x_{k+1}-x_k)} \iint\limits_{-\infty}^{\infty} dy_k\,dz_k \times$$
$$\times \exp\left\{\frac{ik}{2}\left[\frac{(y_{k+1}-y_k)^2+(z_{k+1}-z_k)^2}{(x_{k+1}-x_k)}+\frac{(y_k-y_{k-1})^2+(z_k-z_{k-1})^2}{(x_k-x_{k-1})}\right]\right\}. \tag{13}$$

The inner integral over y_k, z_k can be easily calculated,* and is equal to

$$\frac{2\pi i\,(x_{k+1}-x_k)(x_k-x_{k-1})}{k\,(x_{k+1}-x_{k-1})}\exp\left\{\frac{ik\,[(y_{k+1}-y_{k-1})^2+(z_{k+1}-z_{k-1})^2]}{2\,(x_{k+1}-x_{k-1})}\right\}$$

We thus obtain for I_k the expression

$$I_k \approx \frac{G_0(\boldsymbol{r}_{k-1}-\boldsymbol{r}_{k+1})}{2ik}\int\limits_{x_{k-1}}^{x_{k+1}}\varepsilon_1(x_k, 0, 0)\,dx_k, \tag{14}$$

where G_0 is to be interpreted not as the exact value of (2), but as the result obtained when expansion (12) is applied:

$$G_0(x, y, z)\approx -\frac{1}{4\pi|x|}\exp\left\{ik\left[|x|+\frac{y^2+z^2}{2|x|}\right]\right\}.$$

We will now use (14) in an approximate evaluation of E_n. First a remark concerning the direction of integration along the x_k axis. This axis was pointed from $\boldsymbol{r}_{k-1}$ to $\boldsymbol{r}_{k+1}$. The points $\boldsymbol{r}_1$ $\boldsymbol{r}_2,\ldots$ in general are arbitrarily scattered in space, as they are the integration variables. The x_k axes for different k therefore need not coincide. We thus have to introduce an additional assumption that all the x_k axes point along $\boldsymbol{r}-\boldsymbol{R}$. This choice of axes corresponds in the geometrical optics approximation to integration along a straight ray, which is legitimate only in the case of weak fluctuations in ε_1 (see §39).

We can now proceed with a computation of E_n. Integrating in (6) over y_1, z_1, we obtain, using (14)

$$E_n(\boldsymbol{r}_0)\approx(-k^2)^n\frac{1}{2ik}\underbrace{\int\ldots\int}_{(n-1)} d^3r_2\ldots d^3r_n G_0(\boldsymbol{r}_0-\boldsymbol{r}_2)\,G_0(\boldsymbol{r}_2-\boldsymbol{r}_3)\times\ldots$$
$$\ldots\times G_0(\boldsymbol{r}_n-\boldsymbol{R})\,\varepsilon_1(\boldsymbol{r}_2)\,\varepsilon_1(\boldsymbol{r}_3)\ldots\varepsilon_1(\boldsymbol{r}_n)\int\limits_{x_0}^{x_2}\varepsilon_1(x_1, 0, 0)\,dx_1. \tag{15}$$

Here the $(n-1)$-tuple integral over the variables $\boldsymbol{r}_2,\ldots,\boldsymbol{r}_n$ has precisely the same form as the n-tuple integral in E_n. The integration over y_1, z_1 thus

* In this integration we use the fact that Im $k>0$, i.e., the wave propagates in an absorbing medium.

lowered the multiplicity of the volume integral by one, without altering its general appearance, and introduced an additional factor

$$(2ik)^{-1}\int_{x_0}^{x_2}\varepsilon_1(x_1,0,0)\,dx_1$$

in the integrand.

Applying (14) to the integral over y_2, z_2, we obtain

$$E_n(\boldsymbol{r}_0)\approx(-k^2)^n\frac{1}{(2ik)^2}\int\limits_{(n-2)} d^3r_3\ldots d^3r_nG_0(\boldsymbol{r}_0-\boldsymbol{r}_3)\ldots G_0(\boldsymbol{r}_n-\boldsymbol{R})\times$$
$$\times\varepsilon_1(\boldsymbol{r}_3)\ldots\varepsilon_1(\boldsymbol{r}_n)\int_{x_0}^{x_3}\varepsilon_1(x_2,0,0)\,dx_2\int_{x_0}^{x_2}\varepsilon_1(x_1,0,0)\,dx_1. \tag{16}$$

The limits of integration over x_2 are from x_0 to x_3, since in (15) $\boldsymbol{r}_2$ enters the factor $G_0(\boldsymbol{r}_0-\boldsymbol{r}_2)\,G_0(\boldsymbol{r}_2-\boldsymbol{r}_3)$. Repeating the same argument n times, we finally get

$$E_n(\boldsymbol{r}_0)\approx(-k^2)^n\frac{G_0(\boldsymbol{r}_0-\boldsymbol{R})}{(2ik)^n}\int_{x_0}^{X}dx_n\varepsilon_1(x_n,0,0)\times$$
$$\times\int_{x_0}^{x_n}dx_{n-1}\varepsilon_1(x_{n-1},0,0)\ldots\int_{x_0}^{x_2}dx_1\varepsilon_1(x_1,0,0). \tag{17}$$

This expression can be considerably simplified. Consider the function

$$\theta(x)=\begin{cases}1 & \text{for } x>0,\\ 0 & \text{for } x<0.\end{cases}$$

We have the obvious equality

$$\int_{x_0}^{x_k}dx_{k-1}\varepsilon_1(x_{k-1},0,0)=\int_{x_0}^{X}\theta(x_k-x_{k-1})\,\varepsilon_1(x_{k-1},0,0)\,dx_{k-1},$$

and integral (17) can now be rewritten as

$$E_n(\boldsymbol{r}_0)=G_0(\boldsymbol{r}_0-\boldsymbol{R})\left(\frac{ik}{2}\right)^n\int_{x_0}^{X}\ldots\int_{x_0}^{X}\theta(x_n-x_{n-1})\,\theta(x_{n-1}-x_{n-2})\ldots$$
$$\ldots\theta(x_2-x_1)\,\varepsilon_1(x_n,0,0)\ldots\varepsilon_1(x_1,0,0)\,dx_1\ldots dx_n. \tag{18}$$

We introduce new variables in (18), $x_1=x_k'$, $x_k=x_1'$, and drop the primes. The variables x_1 and x_k are thus interchanged. The expression $\varepsilon_1(x_n,0,0)\ldots$ $\ldots\varepsilon_1(x_1,0,0)$ is not affected, as it is symmetric in all the variables. The factor with the function θ is the only one affected by this transformation. There is a total of $n!$ possible substitutions of this type, corresponding to all the different permutations of $x_1,\ldots,x_n$. Making in (18) all the possible substitutions, we take the arithmetic mean of the resulting expressions. This gives

$$E_n(\boldsymbol{r}_0)=G_0(\boldsymbol{r}_0-\boldsymbol{R})\left(\frac{ik}{2}\right)^n\frac{1}{n!}\int_{x_0}^{X}\ldots\int_{x_0}^{X}\varepsilon_1(x_1,0,0)\ldots\varepsilon_1(x_n,0,0)\times$$
$$\times[\theta(x_n-x_{n-1})\,\theta(x_{n-1}-x_{n-2})\ldots\theta(x_2-x_1)+$$
$$+\theta(x_1-x_{n-1})\ldots\theta(x_2-x_n)+\ldots]\,dx_1\ldots dx_n. \tag{19}$$

In square brackets we have the sum of the products of θ corresponding to all the different arrangements of the variables $x_1, \ldots, x_n$. We will show that this sum is equal to unity. At each point of the n-dimensional cube over which the integration in (19) is carried out, the variables $x_1, \ldots, x_n$ can be arranged in a descending order, for example $x_{k_1}, x_{k_2}, \ldots, x_{k_n}$. The corresponding term

$$\theta(x_{k_1} - x_{k_2})\,\theta(x_{k_2} - x_{k_3})\ldots\theta(x_{k_{n-1}} - x_{k_n}) \tag{20}$$

in the sum in (19) is equal to unity, since all the differences $x_{k_1} - x_{k_2}, \ldots$ are positive. All the other terms in that sum are zero, as they are obtained from (20) by a certain permutation of coordinates, and each permutation of this kind involves the appearance of at least one negative coordinate difference. The sum in (19) is thus equal to unity (clearly, at different points of the n-dimensional cube the zero and nonzero terms of the sum in (19) are not the same).

In this case

$$E_n(\boldsymbol{r}_0) = G_0(\boldsymbol{r}_0 - \boldsymbol{R})\left(\frac{ik}{2}\right)^n \frac{1}{n!}\int\limits_{x_0}^{X}\ldots\int\limits_{x_0}^{X}\varepsilon_1(x_1, 0, 0)\ldots\varepsilon_1(x_n, 0, 0)\,dx_1\ldots$$

$$\ldots dx_n = G_0(\boldsymbol{r}_0 - \boldsymbol{R})\frac{1}{n!}\left[\frac{ik}{2}\int\limits_{x_0}^{X}\varepsilon_1(x, 0, 0)\,dx\right]^n, \tag{21}$$

where we expressed the n-tuple integral which separates into individual integral factors as a one-dimensional integral raised to the n-th power. Inserting (21) in (5), we obtain

$$E(\boldsymbol{r}_0) = G_0(\boldsymbol{r}_0 - \boldsymbol{R})\sum_{n=0}^{\infty}\frac{1}{n!}\left[\frac{ik}{2}\int\limits_{x_0}^{X}\varepsilon_1(x, 0, 0)\,dx\right]^n$$

or writing the appropriate exponential function for the series expansion,

$$E(\boldsymbol{r}_0) = G_0(\boldsymbol{r}_0 - \boldsymbol{R})\exp\left\{\frac{ik}{2}\int\limits_{x_0}^{X}\varepsilon_1(x, 0, 0)\,dx\right\}. \tag{22}$$

Inserting expression (2) for G_0 into (22) and seeing that $|\boldsymbol{r}_0 - \boldsymbol{R}| = X - x_0$, we may write (22) in the form

$$E(\boldsymbol{r}_0) = -\frac{1}{4\pi|\boldsymbol{r}_0 - \boldsymbol{R}|}\exp\left\{ik\left[(X - x_0) + \frac{1}{2}\int\limits_{x_0}^{X}\varepsilon_1(x, 0, 0)\,dx\right]\right\} =$$

$$= -\frac{1}{4\pi|\boldsymbol{r}_0 - \boldsymbol{R}|}\exp\left\{ik\int\limits_{x_0}^{X}\left[1 + \frac{1}{2}\varepsilon_1(x, 0, 0)\right]dx\right\}. \tag{23}$$

Expression (23) clearly differs from $G_0(\boldsymbol{r}_0 - \boldsymbol{R})$ in that the phase factor $k|\boldsymbol{r}_0 - \boldsymbol{R}|$ entering G_0 is replaced in (23) by the phase shift

$$S(X, 0, 0) = k\int\limits_{x_0}^{X}\left[1 + \frac{1}{2}\varepsilon_1(x, 0, 0)\right]dx \tag{24}$$

(the integration in (24) is carried out along the straight line between and $\boldsymbol{R}$). The phase shift S satisfies a first-order differential equation. To obtain this equation, we displace the points $\boldsymbol{r}_0$ and $\boldsymbol{R}$ away from the x-axis. Then

$$S(x, y, z) = k\int_{x_0}^{x}\left[1 + \frac{1}{2}\varepsilon_1(x', y, z)\right]dx'. \tag{24a}$$

Differentiating with respect to x, y, and z, we obtain

$$\frac{\partial S(x, y, z)}{\partial x} = k\left[1 + \frac{1}{2}\varepsilon_1(x, y, z)\right], \tag{25}$$

$$\frac{\partial S(x, y, z)}{\partial y} = \frac{1}{2}k\int_{x_0}^{x}\frac{\partial\varepsilon_1(x', y, z)}{\partial y}dx', \tag{26}$$

$$\frac{\partial S(x, y, z)}{\partial z} = \frac{1}{2}k\int_{x_0}^{x}\frac{\partial\varepsilon_1(x', y, z)}{\partial z}dx'. \tag{27}$$

As we have already noted, the approximate evaluation of E_n involves integration along a straight ray, which is justified only for small ε_1: $\varepsilon(\boldsymbol{r}) = 1 + \varepsilon_1(\boldsymbol{r})$, $\langle|\varepsilon_1|\rangle \ll 1$. Squaring (25), (26), and (27), then adding them, and retaining only terms which are linear in ε_1 (the squares of (26) and (27) drop out, as they are proportional to ε_1 squared), we obtain

$$[\nabla S(x, y, z)]^2 = k^2(1 + \varepsilon_1(x, y, z)) = k^2\varepsilon(x, y, z). \tag{28}$$

This equation is generally written in terms of the refractive index

$$n^2(\boldsymbol{r}) = \varepsilon(\boldsymbol{r}), \tag{29}$$

thus

$$[\nabla S(\boldsymbol{r})]^2 = k^2n^2(\boldsymbol{r}). \tag{30}$$

Equation (30) is known as the eikonal equation.

We derived the eikonal equation (30) by a somewhat unusual technique, the only purpose of which was to emphasize the close relation between geometrical optics and the multiple scattering of waves. Equation (30) can be derived directly from the wave equation; this is a more convenient approach, since it will enable us to investigate the geometrical optics method in more detail, to obtain higher approximations, and to assess the limits of its applicability. We now proceed with this alternative derivation.

§ 38. Derivation of the equations of geometrical optics

Consider a homogeneous wave equation in a medium with a given refractive index $n^2 \equiv \varepsilon$:

$$\Delta\Psi + k^2n^2(\boldsymbol{r})\Psi = 0. \tag{1}$$

If $n^2 =$ const, equation (1) has a solution of the form $\Psi = A \exp(iknmr)$, where m is a unit vector. If n is not a constant, and yet varies relatively slowly, it is reasonable to seek the wave function Ψ in the form

$$\Psi(r) = F(r) \exp(ik\theta(r)), \tag{2}$$

where $F(r)$ and $\theta(r)$ are slowly varying functions.

$F(r)$ can be sought as a series in powers of the small parameter $1/k$ (in fact the expansion is carried out in powers of the small parameter λ/l_0),

$$F(r) = F_0(r) + \frac{1}{k} F_1(r) + \frac{1}{k^2} F_2(r) + \ldots \tag{3}$$

Inserting (2), (3) in equation (1) and collecting equal powers of k, we set each coefficient equal to zero. The coefficient of k^2 gives the equation

$$(\nabla\theta)^2 - n^2(r) = \left(\frac{\partial\theta}{\partial x_1}\right)^2 + \left(\frac{\partial\theta}{\partial x_2}\right)^2 + \left(\frac{\partial\theta}{\partial x_3}\right)^2 - n^2(x_1, x_2, x_3) = 0. \tag{4}$$

The coefficient of k^1 gives the equation

$$F_0\Delta\theta + 2\nabla F_0 \nabla\theta = 0. \tag{5}$$

This relation is more convenient if written in a somewhat different form. Multiplying (5) by F_0 and seeing that $2F_0\nabla F_0 = \nabla(F_0^2)$, we obtain

$$F_0^2\Delta\theta + \nabla(F_0^2)\nabla\theta = 0,$$

which is equivalent to the equation $\nabla(F_0^2\Delta\theta) = 0$, i.e.,

$$\operatorname{div}(F_0^2 \operatorname{grad}\theta) = 0. \tag{6}$$

The equation for F_1 has the form

$$F_1\Delta\theta + 2\nabla F_1\nabla\theta = i\Delta F_0, \tag{7}$$

and in general the equation for F_l with $l \geqslant 1$ is

$$F_l\Delta\theta + 2\nabla F_l\nabla\theta = i\Delta F_{l-1}. \tag{8}$$

Equation (4) together with appropriate boundary conditions gives the function θ. Once θ has been found, equation (6) is used to find F_0, then (7) to find F_1, etc.

$S = k\theta$ is the phase of the solution (θ is the eikonal) only up to terms of the order $1/k$ in (3). Indeed, if $n^* = n$ (no absorption), we have from (4) that $\theta^* = \theta$, i.e., the eikonal is a real number. Then by (6) $F_0^* = F_0$, i.e., F_0 is also real. From (7) we then see that F_1 is a purely imaginary quantity. In general, $F_{2l}^* = F_{2l}$, $F_{2l+1}^* = -F_{2l+1}$. Thus all the odd terms in the expansion of F are purely imaginary numbers and consequently give a contribution to the phase of Ψ. Therefore F can be regarded as the amplitude only up to terms of the order $1/k$.

We can now proceed with a solution of equation (4). This is a nonlinear first-order partial differential equation of the type

$$f\left(x_1, x_2, x_3, \theta, \frac{\partial\theta}{\partial x_1}, \frac{\partial\theta}{\partial x_2}, \frac{\partial\theta}{\partial x_3}\right) = 0.$$

Equations of this kind are solved by the following method (see, e.g., /90/).

Let $p_i = \frac{\partial\theta}{\partial x_i}$, $P_i = \frac{\partial f}{\partial p_i}$, $X_i = \frac{\partial f}{\partial x_i}$, $z = \frac{\partial f}{\partial\theta}$. Then θ can be found as the solution of the following system of ordinary differential equations

$$\frac{dx_1}{P_1} = \frac{dx_2}{P_2} = \frac{dx_3}{P_3} = \frac{d\theta}{p_1P_1 + p_2P_2 + p_3P_3} = \frac{-dp_1}{X_1 + p_1z} = \frac{-dp_2}{X_2 + p_2z} = \frac{-dp_3}{X_3 + p_3z},$$

provided the integration constants are chosen so that they satisfy the equation

$$f(x_1, x_2, x_3, \theta, p_1, p_2, p_3) = 0. \tag{9}$$

In our case $P_i = 2p_i$, $X_i = -\frac{\partial n^2}{\partial x_i}$, $z = 0$ and the system of ordinary differential equations takes the form

$$\frac{dx_1}{2p_1} = \frac{dx_2}{2p_2} = \frac{dx_3}{2p_3} = \frac{d\theta}{2(p_1^2 + p_2^2 + p_3^2)} = \frac{dp_1}{\partial n^2/\partial x_1} = \frac{dp_2}{\partial n^2/\partial x_2} = \frac{dp_3}{\partial n^2/\partial x_3} = \frac{ds}{2n}. \tag{10}$$

The independent variable is s (the factor $(2n)^{-1}$ is introduced for the sake of convenience).

We should first find one of the integrals of the set in (10). We write the equations $\frac{dx_i}{2p_i} = \frac{dp_i}{\partial n^2/\partial x_i}$ (no summation over i is implied) in the form

$$\frac{\partial n^2}{\partial x_1}dx_1 = 2p_1dp_1, \quad \frac{\partial n^2}{\partial x_2}dx_2 = 2p_2dp_2, \quad \frac{\partial n^2}{\partial x_3}dx_3 = 2p_3dp_3$$

and add them term by term. This gives

$$dn^2 = d(p_1^2 + p_2^2 + p_3^2),$$

from which

$$p_1^2 + p_2^2 + p_3^2 = n^2 + \text{const}.$$

The value of the constant is determined from condition (9). It is obviously zero and so

$$p_1^2 + p_2^2 + p_3^2 = n^2(x_1, x_2, x_3). \tag{11}$$

Equating each of the terms in (10) to the last term, we write the system in the form

$$\frac{dx_i}{ds} = \frac{p_i}{n} \qquad (i = 1, 2, 3), \tag{12}$$

$$\frac{dp_i}{ds} = \frac{1}{2n}\frac{\partial n^2}{\partial x_i} = \frac{\partial n}{\partial x_i}, \tag{13}$$

$$\frac{d\theta}{ds} = n, \tag{14}$$

where in writing (14) we made use of (11).

We now replace the p_i by new unknowns l_i defined by $p_i = nl_i$. Then by (11)

$$l_1^2 + l_2^2 + l_3^2 = 1. \tag{15}$$

Equation (12) is written in the form

$$\frac{dx_i}{ds} = l_i, \tag{16}$$

and equation (13) takes the form

$$\frac{dnl_i}{ds} = \frac{\partial n}{\partial x_i}. \tag{13a}$$

But

$$\frac{dnl_i}{ds} = n\frac{dl_i}{ds} + l_i\frac{\partial n}{\partial x_j}\frac{\partial x_j}{\partial s} = n\frac{dl_i}{ds} + l_i l_j\frac{\partial n}{\partial x_j}.$$

Inserting the above expression in (13a), we obtain

$$n\frac{dl_i}{ds} = \frac{\partial n}{\partial x_i} - l_i l_j\frac{\partial n}{\partial x_j}. \tag{17}$$

Equation (14) is not changed.

Inserting (16) in (15), we obtain

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2.$$

For this case, the parameter s introduced from purely formal considerations turns out to be the length of the curve $\boldsymbol{x}(s)$. From (16) we see that $\boldsymbol{l} = \frac{d\boldsymbol{x}}{ds}$ is the unit vector tangent to the curve $\boldsymbol{x}(s)$. Equation (17) can be written in the vector form

$$n\frac{d\boldsymbol{l}}{ds} = \nabla n - \boldsymbol{l}(\boldsymbol{l}\nabla n) = [\boldsymbol{l}[\nabla n\,\boldsymbol{l}]]. \tag{17a}$$

Equations (16) and (17) with appropriate boundary conditions define the curve $\boldsymbol{x} = \boldsymbol{x}(s)$. Then the function θ can be found from equation (4) in the form of a line integral

$$\theta = \int n(\boldsymbol{r}(s))\,ds \tag{18}$$

along the curve $\boldsymbol{r}(s)$ $(\boldsymbol{r} = \boldsymbol{x})$.

The vector $\nabla\theta$ is normal to the surfaces $\theta =$ const (to surfaces of equal phase). Since $p_i = \frac{\partial\theta}{\partial x_i}$, we have $\boldsymbol{l} = \frac{1}{n}\nabla\theta$. Consequently, the curves $\boldsymbol{x}(s)$ are normal to surfaces of equal phase or, in other words, they are rays. Thus, θ is found by integrating n along a ray. Equation (17) can also be derived from Fermat's variational principle, which requires minimizing the integral (18) along the path of the ray (see, e.g., /91, 92/).

Let us now return to equation (6), where $\nabla\theta$ is replaced by $n(\boldsymbol{r})\,\boldsymbol{l}$:

$$\operatorname{div}(nF_0^2\boldsymbol{l}) = 0. \tag{6a}$$

Consider some surface $\theta(\boldsymbol{r}) =$ const on which a closed contour is drawn, enclosing a small surface element $d\sigma$. Through each point of this contour we draw rays which will intersect with some other surface $\theta(\boldsymbol{r}) =$ const. On this surface the ray tube forms another closed contour which in general encloses a different surface area $d\sigma_1$. Let us integrate equation (6a) over the entire volume inside the ray tube between the two surfaces $\theta =$ const. Using Gauss' theorem, we transform the volume integral to a surface integral

$$\oiint nF_0^2\boldsymbol{l}\boldsymbol{m}\,d\sigma = 0. \tag{19}$$

Here $\boldsymbol{m}$ is the unit vector in the direction of the outer normal to the integration surface. On the lateral surface of the ray tube $\boldsymbol{lm} = 0$. On the initial $\theta = \text{const}$ surface, $\boldsymbol{lm} = -1$, and on the other "end-face" of the tube $\boldsymbol{lm} = +1$.

Equation (19) thus gives for the ray tube

$$nF_0^2 d\sigma = \text{const.} \tag{20}$$

Here nF_0^2 is proportional (to the first approximation) to the energy flux density vector. $nF_0^2\, d\sigma$ represents the power of the wave propagating in the given ray tube. The variation of the cross sectional area $d\sigma$ along the ray tube, which is responsible for the changes in the amplitude F_0, is determined by the ray equations.

Both the amplitude and the phase of the wave can be found easily once the ray equations (17) have been solved.

Equations (17) are solved fairly easily for a layered inhomogeneous medium, when n is a function of one coordinate only. In this case we have

$$\Psi = F_0 \exp(ik\theta)$$

for any function $n(z)$ (see, e.g., /93, 94/). We are interested, however, in the much more complex case, where $n(\boldsymbol{r})$ is a random function of three variables, i.e., essentially an arbitrary function from some sample space. In this general formulation of the problem, the ray equations cannot be integrated and approximate methods have to be used.

§ 39. Solution of the equations of geometrical optics by the perturbation technique

We will now make use of the fact that the refractive index fluctuations are small in order to proceed with an approximate integration of the ray equation. Dividing (38.17a) by n, we write it in the form

$$\frac{d\boldsymbol{l}}{ds} = [\boldsymbol{l}\,[(\nabla \ln n)\,\boldsymbol{l}]]. \tag{1}$$

As before, we set $n^2 = \varepsilon = 1 + \varepsilon_1$. Then

$$\ln n = \frac{1}{2} \ln(1 + \varepsilon_1) \approx \frac{1}{2}\varepsilon_1$$

and equation (1) takes the form

$$\frac{d\boldsymbol{l}}{ds} = \frac{1}{2}\,[\boldsymbol{l}\,[\nabla\varepsilon_1\,\boldsymbol{l}]]. \tag{2}$$

To simplify the mathematics, we take $\varepsilon_1 = \nu\frac{\varepsilon_1}{\nu}$, where $\nu = \sqrt{\langle\varepsilon_1^2\rangle} \ll 1$ is the small parameter in which we carry out the perturbation expansion. The function $\alpha(\boldsymbol{r}) = \frac{1}{\nu}\varepsilon_1(\boldsymbol{r})$ is of the order of unity.

We seek $\boldsymbol{l}$ as a series in powers of ν:

$$\boldsymbol{l} = \boldsymbol{l}_0 + \nu\boldsymbol{l}_1 + \nu^2\boldsymbol{l}_2 + \ldots \tag{3}$$

Correspondingly

$$x = x_0 + \nu x_1 + \nu^2 x_2 + \dots \tag{4}$$

Inserting (3) in equation (2) and equating to zero the coefficients of the successive powers of ν, we obtain a chain of equations

$$\frac{dl_0}{ds} = 0, \quad \frac{dx_0}{ds} = l_0; \tag{5}$$

$$\frac{dl_1}{ds} = \frac{1}{2}[l_0[\nabla\alpha\, l_0]], \quad \frac{dx_1}{ds} = l_1; \tag{6}$$

$$\frac{dl_2}{ds} = \frac{1}{2}\{[l_0[\nabla\alpha\, l_1]] + [l_2[\nabla\alpha\, l_0]]\}, \quad \frac{dx_2}{ds} = l_2. \tag{7}$$

By (5),

$$l_0 = \text{const}, \; x_0 = l_0 s, \tag{8}$$

i.e., to a first approximation the ray is a straight line propagating in the original direction. Equation (6) is integrated using the first approximation (8), i.e., taking $\nabla\alpha = \nabla\alpha(l_0 s)$. Then

$$l_1 = \frac{1}{2}\int_0^s [l_0[\nabla\alpha(l_0 s')\, l_0]]\, ds' = \left[\frac{1}{2}\, l_0\left[\left(\int_0^s \nabla\alpha(l_0 s')\, ds'\right) l_0\right]\right]. \tag{9}$$

From (9) it follows that

$$l_1 l_0 = 0. \tag{10}$$

Integrating the equation $\frac{dx_1}{ds} = l_1$, we find

$$x_1(s) = \frac{1}{2}\left[l_0\left[\left(\int_0^s ds_2 \int_0^{s_2} ds_1 \nabla\alpha\,(l\, s_1)\right) l_0\right]\right] = \frac{1}{2}\left[l_0\left[\left(\int_0^s (s-s')\,\nabla\alpha(l_0 s')\, ds'\right) l_0\right]\right]. \tag{11}$$

(On passing from (10) to (11), we changed the order of integration over s_1, s_2.) Inserting (9) and (11) in expansions (3), (4) and substituting ε_1 for $\nu\alpha$, we obtain

$$l(s) = l_0 + \frac{1}{2}\left[l_0\left[\left(\int_0^s \nabla\varepsilon_1(l_0 s')\, ds'\right) l_0\right]\right] + \dots = l_0 + \delta l_1 + \dots, \tag{12}$$

$$x(s) = l_0 s + \frac{1}{2}\left[l_0\left[\int_0^s (s-s')\,\nabla\varepsilon_1(l_0 s')\, ds',\; l_0\right]\right] + \dots = l_0 s + \delta x_1(s) + \dots \tag{13}$$

We now prove that the quantity $l_0 = |l_0|$ is equal to unity. In general, we should have the equality

$$l^2 = (l_0 + \nu l_1 + \nu^2 l_2 + \dots)^2 = 1. \tag{14}$$

Developing (14), we obtain the normalization condition

$$l_0^2 + 2\nu l_0 l_1 + \nu^2[l_1^2 + 2l_0 l_2] + 2\nu^3[l_0 l_3 + l_1 l_2] + \dots = 1. \tag{15}$$

Here $l_0 l_1 = 0$. It remains to show that $u_2 = l_1^2 + 2l_0 l_2 = 0$, $u_3 = 2(l_0 l_3 + l_1 l_2) = 0$, etc. Consider the expression

$$\frac{du_2}{ds} = 2\left[l_1 \frac{dl_1}{ds} + l_0 \frac{dl_2}{ds}\right].$$

Inserting (6) and (7) (and using the equality $l_0 l_1 = 0$), we obtain $du_2/ds = 0$. 23
For $s = 0$, $u_2 = 0$, hence $u_2 = 0$. Similarly, it can be shown that $u_3 = 0, \ldots$, from which it follows that

$$|l_0| = 1 \tag{16}$$

(note that for arbitrary ν, (16) is the only way to satisfy equality (15)).

Expression (13) is the equation of a ray emerging from the point $x = 0$ in the direction of the unit vector l_0. To a first approximation this is a straight line. The second approximation gives a certain transverse shift of the ray relative to this line.

Using (38.18), we find θ:

$$\theta(x(s)) = \int_0^s n(x(s))\,ds = \int_0^s \left[1 + \frac{1}{2}\varepsilon_1(l_0 s + \delta x(s))\right] ds. \tag{17}$$

Expanding ε_1 in a series up to terms of second order, we obtain

$$\varepsilon_1(l_0 s + \delta x(s)) = \varepsilon_1(l_0 s) + \nabla\varepsilon_1(l_0 s)\,\delta x(s) + \ldots =$$

$$= \varepsilon_1(l_0 s) + \frac{1}{2}\nabla\varepsilon_1(l_0 s)\left[l_0\left[\int_0^s (s - s_1)\ \nabla\varepsilon_1(l_0 s_1)\,ds_1,\ l_0\right]\right] + \ldots \tag{18}$$

θ is thus expressed by the series

$$\theta = s + \frac{1}{2}\int_0^s \varepsilon_1(l_0 s)\,ds + \frac{1}{4}\int_0^s ds_1 \int_0^{s_1} ds_2\,(s_1 - s_2)\ \{\nabla\varepsilon_1(l_0 s_1)\ \nabla\varepsilon_1(l_0 s_2) -$$

$$- (l_0 \nabla\varepsilon_1(l_0 s_1))\,(l_0 \nabla\varepsilon_1(l_0 s_2))\} + \ldots \tag{19}$$

If we retain in (19) only the term linear in ε_1, this is equivalent to neglecting δx in (17), i.e., to integration over the straight line $x = l_0 s$ instead of the curved ray. The value of θ from (19) corresponds to the point $x(s)$ given by (13). The function $\theta(x)$ is thus written in parametric form as

$$\theta(l_0 s + \delta x(s)) = s + \frac{1}{2}\int_0^s \varepsilon_1(l_0 s)\,ds + O(\nu^2). \tag{20}$$

It is of course much more convenient to have an explicit expression for $\theta(x)$. To obtain this expression, we expand $\theta(l_0 s)$ in a series in δx:

$$\theta(l_0 s) = \theta((l_0 s + \delta x) - \delta x) = \theta(l_0 s + \delta x) - \delta x(s)\,\nabla\theta(l_0 s + \delta x) + \ldots$$

By (38.12)

$$\nabla\theta(x(s)) = n(x(s))\,l(x(s)).$$

Consequently,

$$\nabla\theta(l_0 s + \delta x) = \left(1 + \frac{1}{2}\varepsilon_1\right)(l_0 + \nu l_1(s) + \ldots)$$

and

$$\delta \boldsymbol{x}(s) \cdot \nabla \theta\,(\boldsymbol{l}_0 s + \delta \boldsymbol{x}) = \boldsymbol{l}_0 \delta \boldsymbol{x}\,(s)\left(1 + \frac{1}{2}\,\varepsilon_1\right) + \nu \boldsymbol{l}_1\,(s)\,\delta \boldsymbol{x}\,(s) + \ldots \tag{21}$$

But $\boldsymbol{l}_0 \delta \boldsymbol{x}\,(s) = 0$, and $\delta \boldsymbol{x}\,(s)$ is of the order of ν. Hence, expression (21) is of the order of ν^2, i.e.,

$$\theta\,(\boldsymbol{l}_0 s) = \theta\,(\boldsymbol{l}_0 s + \delta \boldsymbol{x}) + O\,(\nu^2). \tag{22}$$

We thus have up to terms of the second order of smallness

$$\theta\,(\boldsymbol{l}_0 s) = s + \frac{1}{2}\int_0^s \varepsilon_1\,(\boldsymbol{l}_0 s)\,ds, \tag{23}$$

and this is an explicit expression of $\theta\,(\boldsymbol{x})$ as an integral along a straight ray to the observation point.

Since in solving our equations for $\theta\,(\boldsymbol{r})$ we invariably assumed $\theta\,(0) = 0$, expression (23) in fact gives only the change in the eikonal along the path s. Also note that it coincides with expression (37.24) derived directly from the solution of the wave equation.

We now proceed with a treatment of amplitude fluctuations. From equation (38.6a) we have

$$\operatorname{div}\,(nF_0^2 \boldsymbol{l}) = nF_0^2 \operatorname{div} \boldsymbol{l} + \boldsymbol{l} \operatorname{grad} nF_0^2 = 0$$

or

$$\boldsymbol{l} \operatorname{grad}\,(\ln nF_0^2) = -\operatorname{div} \boldsymbol{l}. \tag{24}$$

But $\boldsymbol{l} \operatorname{grad} f = \frac{df}{ds}$, so that

$$\frac{d \ln\,(nF_0^2)}{ds} = -\operatorname{div} \boldsymbol{l}. \tag{24a}$$

Integrating this equation along the ray gives /92/

$$\ln\,[n\,(\boldsymbol{x}\,(s_1))\,F_0^2\,(\boldsymbol{x}\,(s_1))] - \ln\,[n\,(\boldsymbol{x}\,(s_0))\,F_0^2\,(\boldsymbol{x}\,(s_0))] = -\int_{s_0}^{s_1} \operatorname{div} \boldsymbol{l}\,(s)\,ds. \tag{25}$$

Equation (25) specifies the change in the log amplitude along the path of the ray. The factor div $\boldsymbol{l}$ entering (25) and the change in the cross sectional area of the ray tube are not completely determined by the ray equation. Indeed, expression (12) for $\boldsymbol{l}\,(s)$ only gives the change in $\boldsymbol{l}$ along the given ray. To find div $\boldsymbol{l}$, we require the derivatives of $\boldsymbol{l}$ not only along the ray but also in the transverse direction, i.e., we need the change in $\boldsymbol{l}$ on passing to the neighboring rays. Thus, in order to be able to compute div $\boldsymbol{l}$, we should specify the variation of $\boldsymbol{l}_0$ on some initial surface. For example, if the incident wave is plane, $\boldsymbol{l}_0$ is constant for all rays, and not only along one ray. For a spherical wave, $\boldsymbol{l}_0 = \frac{\boldsymbol{r} - \boldsymbol{r}_0}{|\boldsymbol{r} - \boldsymbol{r}_0|}$, where $\boldsymbol{r}_0$ is the location of the source.

Calculations for the cases of plane and spherical waves will be carried out in the following sections.

§ 40. Fluctuations of phase, angle of arrival, and amplitude for a plane wave

Let a medium with random refractive index inhomogeneities fill the half-space $x > 0$. Consider a plane monochromatic incident wave

$$\Psi_0 = A_0 e^{ikx}. \tag{1}$$

We will concentrate on the field at some point $\boldsymbol{r}$ inside the random medium.

First we calculate the fluctuations of the optical path (eikonal) θ at the point $\boldsymbol{r}$. We will use expression (39.23) for θ,

$$\theta(x, y, z) = x + \frac{1}{2}\int_0^x \varepsilon_1(\xi, y, z)\, d\xi. \tag{2}$$

The mean value of θ is $\langle\theta\rangle = x$, and the deviation from the mean is

$$\theta(\boldsymbol{r}) - \langle\theta(\boldsymbol{r})\rangle = \frac{1}{2}\int_0^x \varepsilon_1(\xi, y, z)\, d\xi. \tag{3}$$

The mean square fluctuations in θ are expressed by

$$\sigma_\theta^2 = \langle[\theta - \langle\theta\rangle]^2\rangle = \frac{1}{4}\int_0^x d\xi_1 \int_0^x d\xi_2 \langle\varepsilon_1(\xi_1, y, z)\, \varepsilon_1(\xi_2, y, z)\rangle. \tag{4}$$

Assuming statistically homogeneous fluctuations in ε, we get

$$\langle\varepsilon_1(\xi_1, y, z)\, \varepsilon_1(\xi_2, y, z)\rangle = B_\varepsilon(\xi_1 - \xi_2, 0, 0) \tag{5}$$

and

$$\sigma_\theta^2 = \frac{1}{4}\int_0^x d\xi_1 \int_0^x d\xi_2 B_\varepsilon(\xi_1 - \xi_2, 0, 0). \tag{6}$$

The double integral in (6) can be transformed into a single integral. Since in the following we will often use similar transformations, we do it here in a quite general form.

Let $f(x) = f(-x)$. Then

$$I(x) = \int_0^x d\xi_1 \int_0^x d\xi_2 f(\xi_1 - \xi_2) = \int_0^x d\xi_1 \int_{-\xi_1}^{x-\xi_1} f(-\eta)\, d\eta = \int_0^x d\xi_1 \int_{-\xi_1}^{x-\xi_1} f(\eta)\, d\eta.$$

Changing the order of integration over η and ξ_1, we get

$$I(x) = \int_{-x}^0 f(\eta)\, d\eta \int_{-\eta}^x d\xi_1 + \int_0^x f(\eta)\, d\eta \int_0^{x-\eta} d\xi =$$

$$= \int_0^x (x - \eta) f(\eta)\, d\eta + \int_{-x}^0 f(\eta)(x + \eta)\, d\eta = 2\int_0^x (x - \eta) f(\eta)\, d\eta.$$

Thus, if $f(x) = f(-x)$, we have

$$\int_0^x d\xi_1 \int_0^x d\xi_2 f(\xi_1 - \xi_2) = 2\int_0^x (x - \xi) f(\xi)\, d\xi. \tag{7}$$

Applying this relation to (6), we find

$$\sigma_\theta^2 = \frac{1}{2}\int_0^x (x - \xi) B_\varepsilon(\xi, 0, 0)\, d\xi = \frac{x}{2}\int_0^x \left(1 - \frac{\xi}{x}\right) B_\varepsilon(\xi, 0, 0)\, d\xi. \tag{8}$$

Suppose that the path x of the wave in the random medium is much greater than L_0, the correlation radius of the fluctuations. In this case ξ/x is negligible compared to unity, and the upper limit of integration can be changed to ∞:

$$\sigma_\theta^2 \approx \frac{x}{2}\int_0^\infty B_\varepsilon(\xi, 0, 0)\,d\xi. \tag{9}$$

We define a one-dimensional integral scale of the fluctuations by

$$L_1 = \frac{1}{\sigma_\varepsilon^2}\int_0^\infty B_\varepsilon(\xi, 0, 0)\,d\xi, \tag{10}$$

where $\sigma_\varepsilon^2 = B_\varepsilon(0) = \langle\varepsilon_1^2\rangle$. Then (9) can be written as

$$\sigma_\theta^2 = \frac{1}{2}\sigma_\varepsilon^2 L_1 x. \tag{11}$$

We see from this relation that σ_θ^2 increases linearly with increasing x. This is a consequence of the assumption $x \gg L$ made in passing from (8) to (9). In this case, a great many uncorrelated inhomogeneities are encountered along the path of the ray and, as is well known, the mean square fluctuations are proportional to the number of inhomogeneities, i.e., x. Also note that σ_θ^2 is proportional to the integral scale of the inhomogeneities, i.e., it is mainly determined by the large-scale components of the spectrum. If the dielectric constant fluctuations are described by the "2/3 law," σ_θ^2 turns out to be infinite. This is of course not the actual value of the mean square fluctuations, it only implies that σ_θ^2 is not determined by the local properties of the inhomogeneities which follow the "2/3 law."

Let us express σ_θ^2 in terms of the spectrum $\Phi_\varepsilon(\varkappa)$. Assuming statistically isotropic fluctuations, we have

$$\sigma_\varepsilon^2 L_1 = \int_0^\infty B_\varepsilon(\xi, 0, 0)\,d\xi = \frac{1}{2}\int_{-\infty}^\infty B_\varepsilon(\xi, 0, 0)\,d\xi =$$
$$= \frac{1}{2}\iiint_{-\infty}^\infty \Phi_\varepsilon(\boldsymbol{\varkappa})\,d^3\varkappa \int_{-\infty}^\infty e^{i\varkappa_1\xi}d\xi = \pi\iiint_{-\infty}^\infty \Phi_\varepsilon(\boldsymbol{\varkappa})\,\delta(\varkappa_1)\,d^3\varkappa =$$
$$= \pi\iint_{-\infty}^\infty \Phi_\varepsilon\left(\sqrt{\varkappa_2^2+\varkappa_3^2}\right)d\varkappa_2 d\varkappa_3 = 2\pi^2\int_0^\infty \varkappa\Phi_\varepsilon(\varkappa)\,d\varkappa.$$

Hence,

$$\sigma_\theta^2 = \pi^2 x\int_0^\infty \Phi_\varepsilon(\varkappa)\,\varkappa\,d\varkappa. \tag{12}$$

Assuming a "2/3 law," $\Phi_\varepsilon(\varkappa) \sim \varkappa^{-11/3}$ and the integral in (12) diverges for small $\varkappa$. This means that departures from the "2/3 law" in the large-scale region must be taken into consideration.

Let us now consider the fluctuations of the optical path difference between two points located in the plane x = const at a distance ρ from one another:

$$\theta(x, y, z) - \theta(x, y', z') = \frac{1}{2}\int_0^x [\varepsilon_1(\xi, y, z) - \varepsilon_1(\xi, y', z')]\,d\xi. \tag{13}$$

Squaring (13) and averaging, we find

$$D_\theta(y-y',\ z-z') = \langle[(\theta(x,\ y,\ z) - \theta(x,\ y',\ z')]^2\rangle = \\ = \frac{1}{4}\int_0^x d\xi \int_0^x d\xi' \langle[\varepsilon_1(\xi, y, z) - \varepsilon_1(\xi, y', z')]\,[\varepsilon_1(\xi', y, z) - \varepsilon_1(\xi', y', z')]\rangle. \tag{14}$$

We use the identity

$$(a-b)(c-d) = \frac{1}{2}[(a-d)^2 + (b-c)^2 - (a-c)^2 - (b-d)^2]$$

to express the integrand in (14) in terms of the structure functions of ε, which are assumed to depend on the difference of the coordinates only (statistical homogeneity):

$$\langle[\varepsilon_1(\boldsymbol{r}_1) - \varepsilon_1(\boldsymbol{r}_2)]^2\rangle = D_\varepsilon(\boldsymbol{r}_1 - \boldsymbol{r}_2). \tag{15}$$

We obtain

$$D_\theta(y-y',\ z-z') = \frac{1}{8}\int_0^x d\xi \int_0^x d\xi' [D_\varepsilon(\xi-\xi', y-y', z-z') + \\ + D_\varepsilon(\xi-\xi',\ y'-y,\ z'-z) - D_\varepsilon(\xi-\xi', 0, 0) - D_\varepsilon(\xi-\xi',\ 0, 0)]. \tag{16}$$

Seeing that

$$D_\varepsilon(\xi-\xi',\ y-y',\ z-z') = D_\varepsilon(\xi-\xi',\ y'-y,\ z'-z)$$

(this equality is satisfied for locally isotropic fluctuations), we obtain

$$D_\theta(\eta,\ \zeta) = \frac{1}{4}\int_0^x d\xi \int_0^x d\xi' [D_\varepsilon(\xi-\xi',\ \eta,\ \zeta) - D_\varepsilon(\xi-\xi',\ 0,\ 0)]. \tag{17}$$

The integrand in (17) is an even function of the difference $(\xi - \xi')$. Using (7), we obtain

$$D_\theta(\eta,\ \zeta) = \frac{1}{2}\int_0^x (x-\xi)\,[D_\varepsilon(\xi,\ \eta,\ \zeta) - D_\varepsilon(\xi,\ 0,\ 0)]\,d\xi. \tag{18}$$

The difference $D_\varepsilon(\xi,\ \eta,\ \zeta) - D_\varepsilon(\xi,\ 0,\ 0)$ is small for $\xi \gg \sqrt{\eta^2+\zeta^2}$.* Therefore, if $x \gg \sqrt{\eta^2+\zeta^2}$ we can ignore ξ compared to x in the difference $(x-\xi)$ in (18) and extend the integration over ξ to infinity:

$$D_\theta(\eta,\ \zeta) \approx \frac{x}{2}\int_0^\infty [D_\varepsilon(\xi,\ \eta,\ \zeta) - D_\varepsilon(\xi,\ 0,\ 0)]\,d\xi. \tag{19}$$

* If, for example, $D_\varepsilon(r) = C_\varepsilon^2 r^\mu$, then

$$D_\varepsilon(\xi,\ \eta,\ \zeta) - D_\varepsilon(\xi,\ 0,\ 0) = C_\varepsilon^2\{(\xi^2+\eta^2+\zeta^2)^{\mu/2} - \xi^\mu\} = C_\varepsilon^2\xi^\mu\left\{\left(1+\frac{\eta^2+\zeta^2}{\xi^2}\right)^{\mu/2} - 1\right\}.$$

For $n^2+\zeta^2 \ll \xi^2$ we expand in a series and obtain

$$D_\varepsilon(\xi,\ \eta,\ \zeta) - D_\varepsilon(\xi, 0, 0) = C_\varepsilon^2\xi^\mu\left\{1+\mu\frac{\eta^2+\zeta^2}{2\xi^2} + \ldots - 1\right\} = \frac{1}{2}\mu C_\varepsilon^2\xi^{\mu-2}(\eta^2+\zeta^2) + \ldots \to 0$$

for $\xi \to \infty$ and $\mu < 2$. If $\mu < 1$, the integral (19) converges at its upper limit.

In the case of locally isotropic turbulence the integrand is an even function of ξ, and the integration may therefore be extended over the entire interval $(-\infty, +\infty)$:

$$D_\theta(\eta, \zeta) = \frac{x}{4}\int_{-\infty}^{\infty}[D_\varepsilon(\xi, \eta, \zeta) - D_\varepsilon(\xi, 0, 0)]\,d\xi. \qquad (19a)$$

Like σ_θ^2 in (9), D_θ is proportional to x. Unlike σ_θ^2, however, $D_\theta(\eta, \zeta)$ is determined by the structure function of the dielectric constant fluctuations. For $\xi \gg \sqrt{\eta^2+\zeta^2}$, the behavior of the structure function has very little influence on the integral in (19), since in this region the integrand is small. This means that $D_\theta(\eta, \xi)$ is determined by inhomogeneities with scales of the order of $\sqrt{\eta^2+\zeta^2}$, i.e., inhomogeneities whose diameter is comparable with the distance between observation points.

Let us write equation (19) in spectral representation. Inserting

$$D_\varepsilon(\xi, \eta, \zeta) = 2\iiint_{-\infty}^{\infty}[1 - e^{i(\varkappa_1\xi+\varkappa_2\eta+\varkappa_3\zeta)}]\,\Phi_\varepsilon(\boldsymbol{\varkappa})\,d^3\varkappa,$$

$$D_\varepsilon(\xi, 0, 0) = 2\iiint_{-\infty}^{\infty}[1 - e^{i\varkappa_1\xi}]\,\Phi_\varepsilon(\boldsymbol{\varkappa})\,d^3\varkappa$$

in (19a), we get

$$D_\theta(\eta, \zeta) = \frac{x}{2}\iiint_{-\infty}^{\infty}[1 - e^{i(\varkappa_2\eta+\varkappa_3\zeta)}]\,\Phi_\varepsilon(\boldsymbol{\varkappa})\,d^3\varkappa\int_{-\infty}^{\infty}e^{i\varkappa_1\xi}d\xi =$$

$$= \pi x\iiint_{-\infty}^{\infty}[1 - e^{i(\varkappa_2\eta+\varkappa_3\zeta)}]\,\Phi_\varepsilon(\varkappa_1, \varkappa_2, \varkappa_3)\,\delta(\varkappa_1)\,d^3\varkappa =$$

$$= \pi x\iint_{-\infty}^{\infty}[1 - e^{i(\varkappa_2\eta+\varkappa_3\zeta)}]\,\Phi_\varepsilon(0, \varkappa_2, \varkappa_3)\,d\varkappa_2 d\varkappa_3. \qquad (20)$$

The last equality is in the form of a two-dimensional Fourier expansion of the structure function $D_\theta(\eta, \zeta)$, where the spectral density corresponding to $D_\theta(\eta, \zeta)$ is

$$F_\theta(\varkappa_2, \varkappa_3) = \frac{1}{2}\pi x\Phi_\varepsilon(0, \varkappa_2, \varkappa_3). \qquad (21)$$

For a statistically isotropic medium

$$\Phi_\varepsilon(0, \varkappa_2, \varkappa_3) = \Phi_\varepsilon(\sqrt{\varkappa_2^2+\varkappa_3^2}) \text{ and } D_\theta(\eta, \zeta) = D_\theta(\sqrt{\eta^2+\zeta^2}) = D_\theta(\rho).$$

In this case, introduction of polar coordinates in (20) and integration over the angular variable gives

$$D_\theta(\rho) = 2\pi^2 x\int_0^\infty[1 - J_0(\varkappa\rho)]\,\Phi_\varepsilon(\varkappa)\,\varkappa\,d\varkappa, \qquad (22)$$

where $J_0(x)$ is Bessel function of zero order. The integral in (22) differs from that in (12) by the additional factor $[1 - J_0(\varkappa\rho)]$ in the integrand. This factor approaches zero for $\varkappa\rho \to 0$ as $(\varkappa\rho)^2$. Therefore, the large-scale components of the spectrum $\Phi_\varepsilon(\varkappa)$, which correspond to small $\varkappa$, are definitely suppressed. In case of a power spectrum $\Phi_\varepsilon(\varkappa) \sim \varkappa^{-p}$, the contribution to the integral in (22) from $\varkappa$ less than ρ^{-1} is approximately equal to the contribution from $\varkappa$ greater than ρ^{-1}. This means that the major influence is due to spectral components corresponding to the base length ρ.

If we change over from the eikonal θ to the phase $S = k\theta$, the expressions for the phase structure function are obtained from the corresponding expressions for D_θ by multiplying by k^2:

$$D_S(\eta, \zeta) = \frac{k^2 x}{4} \int_{-\infty}^{\infty} [D_\varepsilon(\xi, \eta, \zeta) - D_\varepsilon(\xi, 0, 0)]\, d\xi, \tag{23}$$

$$D_S(\rho) = 2\pi^2 k^2 x \int_0^{\infty} [1 - J_0(\varkappa\rho)]\, \Phi_\varepsilon(\varkappa)\, \varkappa\, d\varkappa. \tag{24}$$

Let us now find the mean square and the correlation function for the fluctuations of the direction of propagation. In the previous section we derived an expression for the unit vector $\boldsymbol{l}$ tangent to the ray:

$$\boldsymbol{l}(s) = \boldsymbol{l}_0 + \frac{1}{2}\left[\boldsymbol{l}_0\left[\left(\int_0^s \nabla\varepsilon_1(\boldsymbol{l}_0 s')\, ds'\right)\boldsymbol{l}_0\right]\right] = \boldsymbol{l}_0 + \delta\boldsymbol{l}_1.$$

Since $\boldsymbol{l}_0 \delta\boldsymbol{l}_1 = 0$, i.e., the vector $\delta\boldsymbol{l}_1$ is perpendicular to $\boldsymbol{l}_0$, we have $|\delta\boldsymbol{l}_1| = \tan\alpha$, where α is the angle between $\boldsymbol{l}$ and $\boldsymbol{l}_0$. Since $|\delta\boldsymbol{l}_1| \ll 1$, i.e., $\alpha \ll 1$, replacing $\tan\alpha$ by α we obtain $\alpha \approx |\delta\boldsymbol{l}_1|$. In calculating the fluctuations in the angle of arrival, however, we may also start with equation (23), which proves to be somewhat more convenient for our purposes.

Let the two observation points be located at a small distance η from one another in a common plane $x = \text{const}$. Then the phase difference δS between these two points is related to the rotation angle α of the wave front (which is assumed to be small) by the expression

$$\alpha = \frac{\delta S}{k\eta} = \frac{\delta\theta}{\eta}. \tag{25}$$

Let $\eta \to 0$. Then in this limit we obtain the exact angle of arrival α. Hence, the mean square fluctuation of the angle of arrival (in one plane) can be computed from

$$\langle\alpha^2\rangle = \lim_{\eta\to 0} \frac{D_S(\eta, 0)}{k^2\eta^2} = \lim_{\eta\to 0} \frac{D_\theta(\eta, 0)}{\eta^2}, \tag{26}$$

which is obtained from (25) by squaring and subsequent averaging.

Expanding the structure function $D_S(\eta, 0)$ in powers of η and seeing that $D'_S(0, 0) = 0$ ($D_S(\eta, 0)$ is an even function of η), we obtain

$$D_S(\eta, 0) = \frac{1}{2} D''_S(0, 0)\, \eta^2 + \ldots,$$

so that

$$\langle\alpha^2\rangle = \frac{1}{2k^2} \left.\frac{\partial^2 D_S(\eta, 0)}{\partial\eta^2}\right|_{\eta=0} = \frac{1}{2} \left.\frac{\partial^2 D_\theta(\eta, 0)}{\partial\eta^2}\right|_{\eta=0}. \tag{27}$$

If we further introduce the angle β in the (x, z) plane, we obtain a similar relation

$$\langle\beta^2\rangle = \frac{1}{2k^2} \left.\frac{\partial^2 D_S(0, \zeta)}{\partial\zeta^2}\right|_{\zeta=0} = \frac{1}{2} \left.\frac{\partial^2 D_\theta(0, \zeta)}{\partial\zeta^2}\right|_{\zeta=0}, \tag{28}$$

and for the sum of the mean square of these angles we get

$$\langle\alpha^2\rangle + \langle\beta^2\rangle = \frac{1}{2k^2}\Delta_2 D_S(0,\ 0) = \frac{1}{2}\Delta_2 D_0(0,\ 0), \qquad (29)$$

where

$$\Delta_2 = \frac{\partial^2}{\partial\eta^2} + \frac{\partial^2}{\partial\zeta^2} \qquad (30)$$

is the transverse Laplacian. Using (19), we find

$$\langle\alpha^2\rangle = \frac{1}{2}\cdot\frac{x}{2}\int_0^\infty \frac{\partial^2 D_\varepsilon(\xi,\ \eta,\ 0)}{\partial\eta^2}\bigg|_{\eta=0} d\xi. \qquad (31)$$

In the case of isotropic fluctuations

$$D_\varepsilon(\xi,\ \eta) = D_\varepsilon(\sqrt{\xi^2+\eta^2})$$

and

$$\frac{\partial^2 D_\varepsilon(\sqrt{\xi^2+\eta^2})}{\partial\eta^2}\bigg|_{\eta=0} = \frac{D'_\varepsilon(\xi)}{\xi}.$$

Thus,

$$\langle\alpha^2\rangle = \frac{x}{4}\int_0^\infty \frac{D'_\varepsilon(\xi)}{\xi}\, d\xi. \qquad (32)$$

For small ξ, $D_\varepsilon(\xi) \sim \xi^2$, so that $D'_\varepsilon(\xi)/\xi$ does not have a singularity at the origin. For $\xi\to\infty$, $D'_\varepsilon(\xi)$ falls off to zero, so that the integral in (32) converges at both limits. In evaluating this integral, however, we must remember that for small ξ, $D_\varepsilon(\xi)$ behaves as a quadratic, since otherwise a diverging integral is obtained. The main contribution to $\langle\alpha^2\rangle$ is thus from small-scale inhomogeneities.

Let us find the correlation function $B_\alpha(\rho)$. It can be derived using (25). Consider the eikonal θ at four points in the plane x = const:

$$M_1(x,\ y_1,\ 0),\quad M_2(x,\ y_1+\eta,\ 0),$$
$$M_3(x,\ y_2,\ 0),\quad M_4(x,\ y_2+\eta,\ 0).$$

Then

$$\alpha_1 = \lim_{\eta\to 0}\frac{\theta_1-\theta_2}{\eta},\qquad \alpha_2 = \lim_{\eta\to 0}\frac{\theta_3-\theta_4}{\eta}$$

and

$$\langle\alpha_1\alpha_2\rangle = \lim_{\eta\to 0}\frac{1}{\eta^2}\langle(\theta_1-\theta_2)(\theta_3-\theta_4)\rangle =$$
$$= \lim_{\eta\to 0}\frac{1}{2\eta^2}[\langle(\theta_1-\theta_4)^2\rangle + \langle(\theta_2-\theta_3)^2\rangle - \langle(\theta_1-\theta_3)^2\rangle - \langle(\theta_2-\theta_4)^2\rangle]. \qquad (33)$$

But

$$\langle(\theta_1-\theta_4)^2\rangle = D_\theta(y_2-y_1+\eta),\quad \langle(\theta_2-\theta_3)^2\rangle = D_\theta(y_2-y_1-\eta),$$
$$\langle(\theta_1-\theta_3)^2\rangle = \langle(\theta_2-\theta_4)^2\rangle = D_\theta(y_2-y_1).$$

Putting $y_2 - y_1 = y$, and using (33) we obtain

$$\langle\alpha_1\alpha_2\rangle = B_\alpha(y) = \lim_{\eta\to 0}\frac{1}{2\eta^2}\left[D_\theta(y+\eta) + D_\theta(y-\eta) - 2D_\theta(y)\right]. \tag{34}$$

Expanding $D_\theta\ (y \pm \eta)$ in a series

$$D_\theta(y\pm\eta) = D_\theta(y) \pm \eta D_\theta'(y) + \frac{1}{2}\eta^2 D_\theta''(y) + \ldots,$$

substituting this expansion in (34), and taking the limit, we obtain

$$B_\alpha(y) = \frac{1}{2}D_\theta''(y). \tag{35}$$

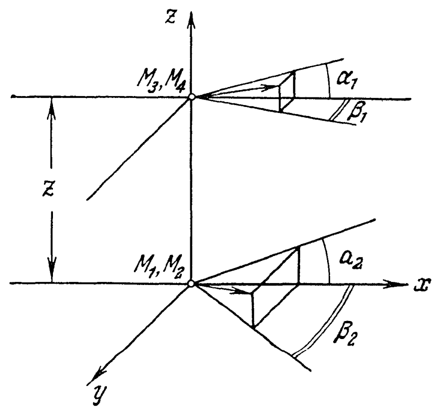

FIGURE 39. Illustrating the calculation of the correlation function for fluctuations in the angle of arrival:

To find $\langle\alpha_1\alpha_2\rangle$, each of the observation points is split into two infinitesimally close points $(M_1,\ M_2)$ and $(M_3,\ M_4)$, the components of each pair being moved away from one another along the z axis. To find $\langle\beta_1\beta_2\rangle$, the components of the points $(M_1,\ M_2)$ and $(M_3,\ M_4)$ are displaced infinitesimally over the y axis.

For $y = 0$, (35) reduces to (27). Expression (35) gives the corrrelation of the angle of arrival in the plane through both rays. We can also consider the correlation of the angle of arrival in the two planes which pass through each of the two rays and which are perpendicular to the plane through both rays (Figure 39). If the axis y is directed from $M_1 = (x,\ y,\ 0)$ to $M_2 = (x, y+\eta, 0)$, and the observation points are moved transversally along the z axis ($M_3 = (x,\ y,\ z)$, $M_4 = (x, y+\eta_1, z)$), we obtain for the angles of arrival

$$\beta_1 = \lim_{\eta\to 0}\frac{\theta_1-\theta_2}{\eta}, \quad \beta_2 = \lim_{\eta\to 0}\frac{\theta_3-\theta_4}{\eta}$$

and

$$B_\beta = \langle \beta_1 \beta_2 \rangle = \lim_{\eta \to 0} \frac{1}{\eta^2} \langle (\theta_1 - \theta_2)(\theta_3 - \theta_4) \rangle. \tag{36}$$

This expression is formally similar to (33), but here

$$\langle (\theta_1 - \theta_4)^2 \rangle = D_\theta(\eta, z) = D_\theta(\sqrt{\eta^2 + z^2})$$

(the last equality is written for isotropic fluctuations),

$$\langle (\theta_2 - \theta_3)^2 \rangle = D_\theta(\eta, z), \quad \langle (\theta_1 - \theta_3)^2 \rangle = \langle (\theta_2 - \theta_4)^2 \rangle = D_\theta(0, z),$$

$$B_\beta(z) = \lim_{\eta \to 0} \frac{1}{2\eta^2} 2 [D_\theta(\eta, z) - D_\theta(0, z)] =$$

$$= \lim_{\eta \to 0} \frac{1}{\eta^2} [D_\theta(\sqrt{z^2 + \eta^2}) - D_\theta(z)] \tag{37}$$

(for the isotropic case). For $\eta \ll z$,

$$D_\theta(\sqrt{z^2 + \eta^2}) \approx D_\theta\left(z + \frac{\eta^2}{2z} + \ldots\right) = D_\theta(z) + \frac{\eta^2}{2z} D_\theta'(z).$$

Inserting this expression in (37) and passing to the limit $\eta \to 0$, we find

$$B_\beta(z) = \frac{D_\theta'(z)}{2z}. \tag{38}$$

For $z \to 0$, this expression approaches the limit $B_\beta(0) = \frac{1}{2} D_\theta''(0)$ in accordance with (28) (assuming isotropic fluctuations). For $z \neq 0$, however, expressions (35) and (38) do not coincide even for the case of isotropic turbulence.

$B_\alpha(y)$ can be called the longitudinal correlation function, and $B_\beta(z)$ the transverse correlation function of fluctuations in the angle of arrival.

Let us also compute the cross-correlation function

$$B_{\alpha\beta}(z) = \langle \alpha_1 \beta_2 \rangle. \tag{39}$$

Proceeding along the same lines as above, we obtain

$$B_{\alpha\beta}(z) = \lim_{\eta \to 0} \frac{1}{2\eta^2} \{D_\theta(z + \eta) + D_\theta(\sqrt{z^2 + \eta^2}) - D_\theta(z) - D_\theta(\sqrt{(z + \eta)^2 + \eta^2})\}. \tag{40}$$

Expanding in a series

$$D_\theta(\sqrt{(z + \eta)^2 + \eta^2}) = D_\theta(z + \eta) + \frac{\eta^2}{2(z + \eta)} D_\theta'(z + \eta) + O(\eta^3),$$

$$D_\theta(\sqrt{z^2 + \eta^2}) = D_\theta(z) + \frac{\eta^2}{2z} D_\theta'(z) + O(\eta^3),$$

inserting in (40), and passing to the limit $\eta \to 0$, we obtain

$$\langle \alpha_1 \beta_2 \rangle = 0, \tag{41}$$

i.e., the fluctuations of the arrival angle in two mutually perpendicular planes are uncorrelated.

Consider the sum

$$B_\alpha(\rho) + B_\beta(\rho) = \frac{1}{2}\left[D_\theta''(\rho) + \frac{D_\theta'(\rho)}{\rho}\right] = \frac{1}{2\rho} \frac{d}{d\rho} [\rho D_\theta'(\rho)].$$

We will express it in terms of the spectrum of the dielectric constant fluctuations. We start with expression (22). Using the well-known equality

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dJ_0(\varkappa\rho)}{d\rho}\right) = -\varkappa^2 J_0(\varkappa\rho),$$

we obtain

$$B_\alpha(\rho) + B_\beta(\rho) = \pi^2 x \int_0^\infty J_0(\varkappa\rho)\,\Phi_\varepsilon(\varkappa)\,\varkappa^3\,d\varkappa. \tag{42}$$

The factor $\varkappa^3$ in (42) rapidly increases and in order for the integral to converge at $\rho = 0$ and $\varkappa \to \infty$, the spectrum $\Phi_\varepsilon(\varkappa)$ should fall off faster than $\varkappa^{-4}$. If we put in (42) $\Phi_\varepsilon(\varkappa) \sim \varkappa^{-11/3}$, then for $\rho = 0$ the integral will diverge in the region of large $\varkappa$. Convergence of (42) can be ensured only with the aid of a "cutoff" factor. The value of the integral in (42) will then be determined by the corresponding scale, i.e., the inner scale of the inhomogeneities (particular calculations for a turbulent medium will be carried out later.)

Let us now consider amplitude fluctuations.

The wave amplitude (A in this section) satisfies the equation

$$\operatorname{div}(A^2 \operatorname{grad}\theta) = A^2\Delta\theta + \operatorname{grad}\theta \operatorname{grad} A^2 = 0 \tag{43}$$

or, dividing (43) by A^2 and putting

$$\chi = \ln A = \frac{1}{2}\ln A^2,$$

we obtain

$$\operatorname{grad}\theta\cdot\operatorname{grad}\chi + \frac{1}{2}\Delta\theta = 0. \tag{44}$$

Inserting for θ its expression from (2), we get

$$\operatorname{grad}\theta = \left\{1 + \frac{1}{2}\varepsilon_1(x, y, z),\ \frac{1}{2}\int_0^x \frac{\partial\varepsilon_1(\xi, y, z)}{\partial y}\,d\xi,\ \ \frac{1}{2}\int_0^x \frac{\partial\varepsilon_1(\xi, y, z)}{\partial z}\,d\xi\right\},$$

$$\Delta\theta = \frac{1}{2}\frac{\partial\varepsilon_1(x, y, z)}{\partial x} + \frac{1}{2}\int_0^x \Delta_\perp\varepsilon_1(\xi, y, z)\,d\xi,$$

$$\Delta_\perp = \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

Substituting this expression in (44), we make use of the fact that the first component of grad θ (the one containing 1) is large compared to the other components ($\nabla\chi$ is of the order of ε_1, since for a plane wave in a homogeneous medium $\nabla\chi = 0$). Therefore, retaining in the product $\nabla\theta\nabla\chi$ only the term which is linear in ε_1, we obtain

$$\frac{\partial\chi(x, y, z)}{\partial x} + \frac{1}{4}\frac{\partial\varepsilon_1(x, y, z)}{\partial x} + \frac{1}{4}\int_0^x \Delta_\perp\varepsilon_1(\xi, y, z)\,d\xi = 0. \tag{45}$$

Integration of (45) over x from 0 to x, taking into account that $\chi(0, y, z) = 0$, $\varepsilon_1(0, y, z) = 0$ gives

$$\chi(x, y, z) = -\frac{1}{4}\left\{\varepsilon_1(x, y, z) + \int_0^x dx' \int_0^{x'} d\xi \Delta_\perp \varepsilon_1(\xi, y, z)\right\}.$$

Changing the order of integration over ξ and x' and integrating over x', we get

$$\chi(x, y, z) = -\frac{1}{4}\left\{\varepsilon_1(x, y, z) + \int_0^x (x-\xi)\, \Delta_\perp \varepsilon_1(\xi, y, z)\, d\xi\right\}. \qquad (46)$$

Let us find the correlation function of fluctuations in χ for the case when the two observation points are situated at the same distance x from the boundary of the random medium:

$$\begin{aligned}\langle\chi(x, y_1, z_1)\,\chi(x, y_2, z_2)\rangle = B_\chi(y_1-y_2, z_1-z_2) = \frac{1}{16}\Big\{\langle\varepsilon_1(x, y_1, z_1)\,\varepsilon_1(x, y_2, z_2)\rangle + \\ + \int_0^x (x-\xi)\,\langle\varepsilon_1(x, y_1, z_1)\,\Delta_\perp\varepsilon_1(\xi, y_2, z_2)\rangle\, d\xi + \\ + \int_0^x (x-\xi)\,\langle\varepsilon_1(x, y_2, z_2)\,\Delta_\perp\varepsilon_1(\xi, y_1, z_1)\rangle\, d\xi + \\ + \int_0^x\int_0^x (x-\xi_1)(x-\xi_2)\,\langle\Delta_\perp\varepsilon_1(\xi_1, y_1, z_1)\,\Delta_\perp\varepsilon_1(\xi_2, y_2, z_2)\rangle\, d\xi_1 d\xi_2\Big\}.\end{aligned} \qquad (47)$$

But

$$\langle\varepsilon_1(x, y_1, z_1)\,\varepsilon_1(x, y_2, z_2)\rangle = B_\varepsilon(y_1-y_2, z_1-z_2),$$

$$\langle\varepsilon_1(x, y_1, z_1)\,\Delta_\perp\varepsilon_1(\xi, y_2, z_2)\rangle = \Delta_\perp B_\varepsilon(x-\xi, y_1-y_2, z_1-z_2) = = -\frac{1}{2}\,\Delta_\perp D_\varepsilon(x-\xi, y_1-y_2, z_1-z_2).$$

(In the last equality, on passing from B_ε to D_ε, we used the relation $D_\varepsilon(r) = 2B_\varepsilon(0) - 2B_\varepsilon(r)$.) Similarly

$$\langle\Delta_\perp\varepsilon_1(\xi_1, y_1, z_1)\,\Delta_\perp\varepsilon_1(\xi_2, y_2, z_2)\rangle = -\frac{1}{2}\,\Delta_\perp^2 D_\varepsilon(\xi_1-\xi_2, y_1-y_2, z_1-z_2).$$

Inserting these expressions in (47) and putting $y_1 - y_2 = y$, , we find

$$B_\chi(y, z) = \frac{1}{16}\left\{B_\varepsilon(y, z) - \int_0^x (x-\xi)\,\Delta_\perp D_\varepsilon(x-\xi, y, z)\, d\xi - -\frac{1}{2}\int_0^x\int_0^x (x-\xi_1)(x-\xi_2)\,\Delta_\perp^2 D_\varepsilon(\xi_1-\xi_2, y, z)\, d\xi_1 d\xi_2\right\}. \qquad (48)$$

In the first integral we change to a new variable by taking $x - \xi = \xi'$, and in the second integral we put $\xi = \xi_1 - \xi_2$, $\eta = x - \frac{\xi_1+\xi_2}{2}$. Integration over η can then be carried out explicitly, and (48) takes the form

$$B_\chi(y, z) = \frac{1}{16}\left\{B_\varepsilon(y, z) - \int_0^x \xi\Delta_\perp D_\varepsilon(\xi, y, z)\, d\xi - \frac{1}{6}\int_0^x (2x+\xi)(x-\xi)^2\,\Delta_\perp^2 D_\varepsilon(\xi, y, z)\, d\xi\right\}. \qquad (49)$$

Let us estimate the various terms in (49). The function $\Delta_\perp D_\varepsilon(\xi, y, z)$ changes appreciably when its argument is changed by an amount of the order of the inner turbulence scale l_0, which is the characteristic scale of this function (this is due to the fact that the operator $\Delta_\perp$ acting on the function D_ε emphasizes the small scales). Also $\Delta_\perp D_\varepsilon(0) \sim \frac{\sigma_\varepsilon^2}{l_0^2}$, where $\sigma_\varepsilon^2 = \langle \varepsilon_1^2 \rangle$. Therefore the first integral in (49) is of the order

$$\frac{\sigma_\varepsilon^2}{l_0^2}\int_0^{l_0} \xi\, d\xi \sim \sigma_\varepsilon^2.$$

The characteristic scale of $\Delta_\perp^2 D_\varepsilon$ is also l_0, but its value at the origin is of the order of $\sigma_\varepsilon^2/l_0^4$. The second integral is therefore of the order

$$\left(\frac{\sigma_\varepsilon^2}{l_0^4}\right) x^3 \int_0^{l_0} d\xi \sim \sigma_\varepsilon^2 \frac{x^3}{l_0^3}$$

(in the integrand ξ can be ignored compared to x). Therefore if $\frac{x^3}{l_0^3} \gg 1$, we need only retain the last term in (49), which is equivalent to dropping the first term in (46). Then

$$B_\chi(y, z) = -\frac{1}{48}\int_0^x \left(x + \frac{\xi}{2}\right)(x - \xi)^2 \Delta_\perp^2 D_\varepsilon(\xi, y, z)\, d\xi. \tag{50}$$

Expression (50) can be further simplified. Indeed, since the characteristic scale of $\Delta_\perp^2 D_\varepsilon$ is $l_0 \ll x$, we can neglect ξ in comparison to x in (50) and the integration over x can be extended to ∞. Then

$$B_\chi(y, z) = -\frac{x^3}{48}\int_0^\infty \Delta_\perp^2 D_\varepsilon(\xi, y, z)\, d\xi. \tag{51}$$

Taking $y = z = 0$, we obtain for the mean square fluctuations of the log amplitude

$$\langle \chi^2 \rangle = -\frac{x^3}{48}\int_0^\infty \Delta_\perp^2 D_\varepsilon(\xi, 0, 0)\, d\xi. \tag{51a}$$

Formula (51) is the final expression for the correlation function of a plane wave in the geometrical optics approximation /92/.

Let us derive the expression for the spectrum of the correlation function $B_\chi(y, z)$. In the right-hand side of (51) we substitute

$$\Delta_\perp^2 D_\varepsilon(\xi, y, z) = \Delta_\perp^2 2 \iiint_{-\infty}^{\infty} [1 - e^{i(\varkappa_1\xi + \varkappa_2 y + \varkappa_3 z)}]\, \Phi_\varepsilon(\varkappa)\, d^3\varkappa =$$

$$= -2 \iiint_{-\infty}^{\infty} e^{i(\varkappa_1\xi + \varkappa_2 y + \varkappa_3 z)} (\varkappa_2^2 + \varkappa_3^2)^2\, \Phi_\varepsilon(\varkappa_1, \varkappa_2, \varkappa_3)\, d^3\varkappa.$$

We integrate this expression over ξ. Seeing that $\Delta_\perp^2 D_\varepsilon(\xi, y, z)$ is an even function of ξ, we obtain

$$\int_0^\infty \Delta_\perp^2 D_\varepsilon(\xi, y, z)\, d\xi = \frac{1}{2}\int_{-\infty}^{\infty} \Delta_\perp^2 D_\varepsilon(\xi, y, z)\, d\xi =$$

$$= -\iiint_{-\infty}^{\infty} e^{i(\varkappa_2 y+\varkappa_3 z)}(\varkappa_2^2+\varkappa_3^2)^2\,\Phi_\varepsilon(\varkappa_1, \varkappa_2, \varkappa_3)\, d^3\varkappa \int_{-\infty}^{\infty} e^{i\varkappa_1\xi}\, d\xi =$$

$$= -2\pi \iiint_{-\infty}^{\infty} e^{i(\varkappa_2 y+\varkappa_3 z)}(\varkappa_2^2+\varkappa_3^2)^2\,\Phi_\varepsilon(\varkappa_1, \varkappa_2, \varkappa_3)\,\delta(\varkappa_1)\, d^3\varkappa =$$

$$= -2\pi \iint_{-\infty}^{\infty} e^{i(\varkappa_2 y+\varkappa_3 z)}(\varkappa_2^2+\varkappa_3^2)^2\,\Phi_\varepsilon(0, \varkappa_2, \varkappa_3)\, d\varkappa_2\, d\varkappa_3.$$

Insertion in (51) gives a formula of the type

$$B_\chi(y, z) = \iint_{-\infty}^{\infty} e^{i(\varkappa_2 y+\varkappa_3 z)} F_\chi(\varkappa_2, \varkappa_3)\, d\varkappa_2\, d\varkappa_3, \tag{52}$$

where

$$F_\chi(\varkappa_2, \varkappa_3) = \frac{\pi x^3}{24}(\varkappa_2^2+\varkappa_3^2)^2\,\Phi_\varepsilon(0, \varkappa_2, \varkappa_3). \tag{53}$$

Relation (53) is the spectral equivalent of (51).

As follows from (51), the log amplitude correlation function is proportional to the cube of the path length traversed in the random medium. In the case of locally isotropic fluctuations in ε,

$$\Phi_\varepsilon(0, \varkappa_2, \varkappa_3) = \Phi_\varepsilon\left(\sqrt{\varkappa_2^2+\varkappa_3^2}\right) = \Phi_\varepsilon(\varkappa),$$

where $\varkappa = \sqrt{\varkappa_2^2+\varkappa_3^2}$. In this case F_χ also depends on $\varkappa$ alone:

$$F_\chi(\varkappa) = \frac{\pi x^3}{24}\varkappa^4\Phi_\varepsilon(\varkappa). \tag{54}$$

The factor $\varkappa^4$ in (54) vanishes for $\varkappa = 0$, which suppresses the large-scale components of the turbulence spectrum. If $\Phi_\varepsilon(\varkappa) \sim \varkappa^{-11/3}$, $F_\chi(\varkappa) \sim \varkappa^{1/3}$, i.e., the spectral density $F_\chi(\varkappa)$ vanishes for $\varkappa = 0$ and increases with increasing $\varkappa$. $F_\chi(\varkappa)$ has its maximum in that region where $\Phi_\varepsilon(\varkappa)$ becomes sensitive to thermal conduction, i.e., in the region corresponding to the inner scale of turbulence. Therefore in the geometrical optics approximation amplitude fluctuations are controlled by the smallest dielectric constant fluctuations, and the amplitude correlation radius is of the order of l_0.

Note the equality

$$F_\chi(0) = 0, \tag{55}$$

which is equivalent to the relation

$$\iint_{-\infty}^{\infty} B_\chi(y, z)\, dy\, dz = 0, \tag{56}$$

and for isotropic fluctuations to

$$\int_0^\infty B_\chi(\rho)\,\rho\, d\rho = 0. \tag{56a}$$

Relation (56) is a consequence of the law of energy conservation.* It shows that the correlation function B_χ must sometimes assume negative values.

Let us derive a useful relation between amplitude and phase fluctuations. Starting with expression (23)

$$D_S(\eta, \zeta) = \frac{k^2 x}{2} \int_0^\infty [D_\varepsilon(\xi, \eta, \zeta) - D_\varepsilon(\zeta, 0, 0)]\, d\xi$$

we use the $\Delta_\perp^2$ operator and obtain

$$\Delta_\perp^2 D_S(\eta, \zeta) = \frac{k^2 x}{2} \int_0^\infty \Delta_\perp^2 D_\varepsilon(\xi, \eta, \zeta)\, d\xi .$$

Comparison with (51) gives

$$B_\chi(\eta, \zeta) = -\frac{x^2}{24k^2} \Delta_\perp^2 D_S(\eta, \zeta). \tag{57}$$

Relation (57) is a consequence of the fact that in geometrical optics the amplitude of the wave is completely determined by its phase (see (44)).

Relation (57) enables us to compute $B_\chi(\eta, \zeta)$ if $D_S(\eta, \zeta)$ is known. Note that since (57) contains $\Delta_\perp^2 D_S(\eta, \zeta)$, the determination of $\langle\chi^2\rangle = B_\chi(0, 0)$ requires knowledge of the behavior of the function $D_S(\eta, \zeta)$ for small $\rho = \sqrt{\eta^2 + \zeta^2}$ satisfying the inequality $\rho \ll l_0$.

§ 41. Fluctuations in the parameters of a spherical wave

In this section we consider the fluctuations of the parameters of a spherical wave emitted from a point source in the random medium.

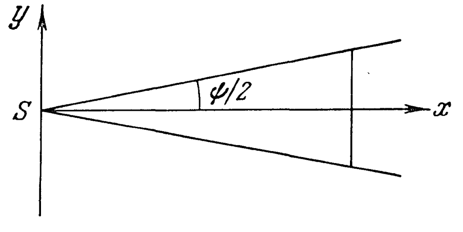

FIGURE 40. Ray geometry for the calculation of the phase-difference fluctuations for a spherical wave.

The mean square fluctuations in the total phase difference for a spherical phase are clearly equal to the corresponding fluctuations for a

* If a random variable $\varphi(x)$ ($\langle\varphi\rangle = 0$) satisfies the conservation law $\int_{-\infty}^{\infty} \varphi(x)\, dx = 0$, multiplication by $\varphi(x_0)$ and averaging give

$$\int_{-\infty}^{\infty} B(x - x_0)\, dx = \int_{-\infty}^{\infty} B(x)\, dx = 0.$$

Relation (56) is a consequence of the two-dimensional analog of this theorem.

plane wave. Consider the mean square phase difference for two rays divergent at an angle ψ from the source (Figure 40). Let the origin coincide with the source; the plane x, y passes through the source and the two observation points, and the x axis is directed from the source to the midpoint of the segment joining the observation points. We will consider only the case of observation points equidistant from the source by a distance $x/\cos\frac{\psi}{2}$. Then

$$S_1 - \langle S_1 \rangle = k\int_0^x n_1(\xi, a\xi, 0)\frac{d\xi}{\cos\frac{\psi}{2}} = \frac{k}{2\cos\frac{\psi}{2}}\int_0^x \varepsilon_1(\xi, a\xi, 0)\,d\xi, \tag{1}$$

where $= \tan\frac{\psi}{2}$. Similarly

$$S_2 - \langle S_2 \rangle = \frac{k}{2\cos\frac{\psi}{2}}\int_0^x \varepsilon_1(\xi, -a\xi, 0)\,d\xi. \tag{2}$$

Thus, for $S_1 - S_2$ we get

$$S_1 - S_2 = \frac{k}{2\cos\frac{\psi}{2}}\int_0^x [\varepsilon_1(\xi, a\xi, 0) - \varepsilon_1(\xi, -a\xi, 0)]\,d\xi, \tag{3}$$

$$\langle (S_1 - S_2)^2 \rangle = \frac{k^2}{4\cos^2\frac{\psi}{2}}\int_0^x\int_0^x \langle [\varepsilon_1(\xi, a\xi, 0) - \varepsilon_1(\xi, -a\xi, 0)] \times$$
$$\times [\varepsilon_1(\eta, a\eta, 0) - \varepsilon_1(\eta, -a\eta, 0)]\rangle\,d\xi\,d\eta. \tag{4}$$

Using the identity

$$(a - b)(c - d) = \frac{1}{2}[(a - d)^2 + (b - c)^2 - (a - c)^2 - (b - d)^2],$$

we write (4) in the form

$$\langle (S_1 - S_2)^2 \rangle = \frac{k^2}{8\cos^2\frac{\psi}{2}}\int_0^x\int_0^x [D_\varepsilon(\xi - \eta, a(\xi + \eta), 0) +$$
$$+ D_\varepsilon(\xi - \eta, -a(\xi + \eta), 0) - D_\varepsilon(\xi - \eta, a(\xi - \eta), 0) -$$
$$- D_\varepsilon(\xi - \eta, -a(\xi - \eta), 0)]\,d\xi\,d\eta. \tag{5}$$

For locally isotropic dielectric constant fluctuations

$$D_\varepsilon(\xi - \eta, a(\xi + \eta), 0) = D_\varepsilon(\xi - \eta, -a(\xi + \eta), 0) = D_\varepsilon(\sqrt{(\xi - \eta)^2 + a^2(\xi + \eta)^2}),$$

$$D_\varepsilon(\xi - \eta, a(\xi - \eta), 0) = D_\varepsilon(\xi - \eta, -a(\xi - \eta), 0) = D_\varepsilon(\sqrt{1 + a^2}\,|\xi - \eta|)$$

and (5) takes the form

$$\langle (S_1 - S_2)^2 \rangle = \frac{k^2}{4\cos^2\frac{\psi}{2}}\int_0^x\int_0^x \{D_\varepsilon(\sqrt{(\xi - \eta)^2 + a^2(\xi + \eta^2)}) -$$
$$- D_\varepsilon(\sqrt{1 + a^2}\,|\xi - \eta|)\}\,d\xi\,d\eta. \tag{6}$$

Assume the angle ψ is small, i.e., $a \ll 1$, so that $D_\varepsilon(\sqrt{1+a^2}|\xi-\eta|)$ can be expanded in a power series in a:

$$D_\varepsilon(\sqrt{1+a^2}|\xi-\eta|) = D_\varepsilon(|\xi-\eta| + \frac{a^2}{2}|\xi-\eta| + \ldots) =$$
$$= D_\varepsilon(|\xi-\eta|) + \frac{a^2}{2}|\xi-\eta| D'_\varepsilon(|\xi-\eta|) + \ldots$$

Inserting this expansion in (6), we obtain

$$\langle(S_1-S_2)^2\rangle = \frac{k^2}{4\cos^2\frac{\psi}{2}} \int_0^x\int_0^x \{D_\varepsilon(\sqrt{(\xi-\eta)^2 + a^2(\xi+\eta)^2}) -$$
$$- D_\varepsilon(|\xi-\eta|)\} d\xi\, d\eta + \frac{a^2k^2}{8\cos^2\frac{\psi}{2}} \int_0^x\int_0^x |\xi-\eta| D'_\varepsilon(|\xi-\eta|)\, d\xi\, d\eta + \ldots \quad (7)$$

The second integral in (7) is an even function of $(\xi-\eta)$ and it can be transformed using (40.7) to a one-dimensional integral. In the first integral we change over to new variables $u = \xi - \eta$, $v = \frac{1}{2}(\xi+\eta)$, i.e.,

$$\xi = v + \frac{1}{2}u, \quad \eta = v - \frac{1}{2}u, \quad \frac{D(\xi,\eta)}{D(u,v)} = 1.$$

Since the integrand in (7) is an even function of u, the integration over this variable need be carried out only for $u > 0$, provided that the result is multiplied by 2. Then,

$$\langle(S_1-S_2)^2\rangle = \frac{k^2}{2\cos^2\frac{\psi}{2}} \left\{ \int_0^{\frac{x}{2}} dv \int_0^{2v} du\, [D_\varepsilon(\sqrt{u^2+4a^2v^2}) - D_\varepsilon(u)] + \right.$$
$$\left. + \int_{\frac{x}{2}}^{x} dv \int_0^{2(x-v)} du\, [D_\varepsilon \sqrt{u^2+4a^2v^2}) - D_\varepsilon(u)] \right\} +$$
$$+ \frac{a^2k^2}{4\cos^2\frac{\psi}{2}} \int_0^x \xi(x-\xi) D'_\varepsilon(\xi)\, d\xi + \ldots \quad (8)$$

Note that integration over u in the first integral can be extended to infinity, since for $u > 2v$ the difference $D_\varepsilon(\sqrt{u^2+4a^2v^2}) - D_\varepsilon(u)$ is of the order of a^2:

$$D_\varepsilon(\sqrt{u^2+4a^2v^2}) - D_\varepsilon(u) = D_\varepsilon\left(u\sqrt{1+\frac{4a^2v^2}{u^2}}\right) - D_\varepsilon(u) =$$
$$= D_\varepsilon\left(u\left(1+\frac{2a^2v^2}{u^2}+\ldots\right)\right) - D_\varepsilon(u) = \frac{2a^2v^2}{u} D'_\varepsilon(u) + \ldots < a^2vD'_\varepsilon(u) + \ldots,$$

and its integral over u from $2v$ to ∞ converges.

The same changes can also be introduced in the second integral. Then approximately

$$\langle(S_1-S_2)^2\rangle = \frac{k^2}{2\cos^2\frac{\psi}{2}} \int_0^x dv \int_0^\infty [D_\varepsilon(\sqrt{u^2+4a^2v^2}) - D_\varepsilon(u)]\, du +$$
$$+ \frac{a^2k^2}{4\cos^2\frac{\psi}{2}} \int_0^x \xi(x-\xi) D'_\varepsilon(\xi)\, d\xi + \ldots \quad (9)$$

The inner integral in the first term in (9) is related to the structure function of plane-wave phase fluctuations $D_S(\rho)$. Indeed, multiplying (40.19) by k^2, we have (for isotropic fluctuations)

$$k^2\int_0^\infty [D_\varepsilon(\sqrt{u^2+\rho^2})-D_\varepsilon(u)]\,du=\frac{2D_0(\rho)\,k^2}{x}=\frac{2D_S(\rho)}{x}. \tag{10}$$

Putting $\rho = 2av$, we obtain

$$\langle(S_1-S_2)^2\rangle=\frac{1}{x\cos^2\frac{\psi}{2}}\int_0^x D_S(2av)\,dv+\frac{a^2k^2}{4\cos^2\frac{\psi}{2}}\int_0^x \xi(x-\xi)\,D_\varepsilon'(\xi)\,d\xi+\ldots \tag{11}$$

Let in the first integral $v = px$. Clearly $2ax = b$ is the distance between the observation points. Putting 1 for $\cos\frac{\psi}{2}$, we obtain

$$\langle(S_1-S_2)^2\rangle=\int_0^1 D_S(bp)\,dp+\frac{a^2k^2}{4\cos^2\frac{\psi}{2}}\int_0^x \xi(x-\xi)\,D_\varepsilon'(\xi)\,d\xi+\ldots \tag{12}$$

We will now estimate the various terms in this expression and establish under what conditions it is necessary to keep only the first term in (12). Let $D_\nu(\xi)=C\xi^\mu$. Then by (40.23)

$$D_S(\rho)=\frac{k^2x}{2}\int_0^\infty [C(\xi^2+\rho^2)^{\mu/2}-C\xi^\mu]\,d\xi=\frac{k^2x}{2}C\rho^{\mu+1}N,$$

where

$$N=\int_0^\infty [(1+t^2)^{\mu/2}+t^\mu]\,dt$$

and

$$\int D_S(bp)\,dp=\frac{1}{2}CNk^2x\int_0^1 (bp)^{\mu+1}\,dp=\frac{CN}{2}k^2xb^{\mu+1}\frac{1}{\mu+2}.$$

For the second integral in (12) we have

$$\int_0^x \xi(x-\xi)\,D_\varepsilon'(\xi)\,d\xi=C\mu\int_0^x (x-\xi)\,\xi^\mu\,d\xi=C\mu\frac{x^{\mu+2}}{(2+\mu)(\mu+1)}.$$

The ratio of the second to the first term in (12) is therefore (we put 1 for $\cos^2\frac{\psi}{2}$ in the denominator)

$$\delta=\frac{\mu}{2N(\mu+1)}a^2\left(\frac{x}{b}\right)^{\mu+1}.$$

But $\frac{x}{b}=\frac{1}{2a}$, so that $\delta\sim a^{1-\mu}$. Hence, for $\mu<1$ (e.g., $\mu = 2/3$), $\delta = O(a^{1-\mu})$ and the second term in (12) may be neglected. In this case we obtain a general expression relating the mean square phase-difference fluctuations (for two observation points separated by a distance b) for a spherical wave to that for a plane wave:

$$\langle (S_1 - S_2)^2_{sp} \rangle = \int_0^1 D_S(bp)\,dp. \tag{13}$$

Since $0 \leqslant D_S(bp) \leqslant D_S(b)$, the phase-difference fluctuations of a spherical wave are invariably less than those for a plane wave propagating over the same distance. This conclusion is also evident from simple qualitative arguments.

Expression (13) can also be written in the form

$$\langle (S_1 - S_2)^2_{sp} \rangle = \frac{1}{b}\int_0^b D_S(\rho)\,d\rho, \tag{13a}$$

which is readily generalized to the case when the random medium occupies only part of the space between the source and the observation points. If the distance between the two rays at one end of their path in the random medium in b_1 and the distance at the other end of the path is b_2, we write (13a) in the form

$$\langle (S_1 - S_2)^2 \rangle = \frac{1}{b_2 - b_1}\int_{b_1}^{b_2} D_S(\rho)\,d\rho. \tag{13b}$$

In (13b) one of the distances may be negative: this represents the case of two intersecting rays. If this is indeed so, the absolute value of the argument of the structure function in the integrand is taken:

$$\langle (S_1 - S_2)^2 \rangle = \frac{1}{b_2 + b_1}\int_{-b_1}^{b_2} D_S(|\rho|)\,d\rho. \tag{13c}$$

Expressions (13a)–(13c) are applied in the next section to calculate the angular correlation of the phase differences in two plane-wave beams propagating at a certain angle to one another.

Given expression (13) for $\langle (S_1 - S_2)^2_{sp} \rangle$, the relations of the previous section can be applied to find the correlation functions of the direction of propagation.

Let us compute the amplitude fluctuations. We again use the equality

$$\operatorname{grad}\chi\ \operatorname{grad}\theta + \frac{1}{2}\Delta\theta = 0 \tag{14}$$

($\chi = \ln A$) relating the amplitude to the phase (more precisely the eikonal) of the wave. The origin of the spherical system of coordinates (r, ϑ, φ) is placed at the source. θ is then expressed by

$$\theta(r, \vartheta, \varphi) = r + \frac{1}{2}\int_0^r \varepsilon_1(\rho, \vartheta, \varphi)\,d\rho. \tag{15}$$

Using the expressions for the components of grad θ in spherical coordinates,

$$\nabla_r\theta = \frac{\partial\theta}{\partial r}, \quad \nabla_\vartheta\theta = \frac{1}{r}\frac{\partial\theta}{\partial\vartheta}, \quad \nabla_\varphi\theta = \frac{1}{r\sin\vartheta}\frac{\partial\theta}{\partial\varphi},$$

we obtain

$$\nabla\chi\nabla\theta = \left(1 + \frac{1}{2}\varepsilon_1(r, \vartheta, \varphi)\right)\frac{\partial\chi}{\partial r} + (\nabla_\vartheta\chi)\frac{1}{2r}\int_0^r \frac{\partial\varepsilon_1}{\partial\vartheta}d\rho + (\nabla_\varphi\chi)\frac{1}{2r\sin\vartheta}\int_0^r \frac{\partial\varepsilon_1}{\partial\varphi}d\rho.$$

$\nabla_\vartheta\chi$ and $\nabla_\varphi\chi$ for a spherical wave are associated only with the refractive index inhomogeneities and are therefore of the order ε_1. Retaining in $\nabla\chi\nabla\theta$ only terms of first order in ε_1, we obtain

$$\nabla\chi\nabla\theta \approx \left[1 + \frac{1}{2}\varepsilon_1(r, \vartheta, \varphi)\right]\frac{\partial\chi}{\partial r}. \tag{16}$$

Unlike the plane wave case, the log amplitude χ of a spherical wave decreases as $\ln r$ due to the wave's divergence, so that $\partial\chi/\partial r$ contains a determinate component. Taking in (16) $\chi = \langle\chi\rangle + \chi'$ and ignoring terms of second order in ε_1, we obtain

$$\nabla\chi\nabla\theta \approx \frac{\partial\langle\chi\rangle}{\partial r} + \frac{1}{2}\frac{\partial\langle\chi\rangle}{\partial r}\varepsilon_1(r, \vartheta, \varphi) + \frac{\partial\chi'}{\partial r}, \tag{17}$$

so that (14) takes the form

$$\frac{\partial\chi'}{\partial r} + \frac{\partial\langle\chi\rangle}{\partial r} + \frac{1}{2}\frac{\partial\langle\chi\rangle}{\partial r}\varepsilon_1(r, \vartheta, \varphi) + \frac{1}{2}\Delta\langle\theta\rangle + \frac{1}{2}\Delta\theta' = 0. \tag{18}$$

By (15)

$$\langle\theta\rangle = r, \quad \Delta\langle\theta\rangle = \frac{1}{r^2}\frac{d}{dr}(r^2) = \frac{2}{r}.$$

Averaging (18), we get

$$\frac{\partial\langle\chi\rangle}{\partial r} = -\frac{1}{2}\Delta\langle\theta\rangle = -\frac{1}{r}. \tag{19}$$

Subtracting (19) from (18) we obtain an equation for χ':

$$\frac{\partial\chi'}{\partial r} + \frac{1}{2}\left[\Delta\theta' - \frac{\varepsilon_1}{r}\right] = 0,$$

or inserting for ε_1 an expression obtained from (15)

$$\varepsilon_1(r, \vartheta, \varphi) = 2\frac{\partial\theta'(r, \vartheta, \varphi)}{\partial r}, \tag{20}$$

we obtain finally

$$\frac{\partial\chi'(r, \vartheta, \varphi)}{\partial r} + \frac{1}{2}\left[\Delta\theta' - \frac{2}{r}\frac{\partial\theta'}{\partial r}\right] = 0. \tag{21}$$

In spherical coordinates

$$\Delta\theta' = \frac{\partial^2\theta'}{\partial r^2} + \frac{2}{r}\frac{\partial\theta'}{\partial r} + \frac{1}{r^2}\Omega(\vartheta, \varphi)\theta', \tag{22}$$

where

$$\Omega(\vartheta, \varphi) = \frac{1}{\sin\vartheta}\left[\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial}{\partial\vartheta}\right) + \frac{1}{\sin\theta}\frac{\partial^2}{\partial\varphi^2}\right]. \tag{23}$$

Thus,

$$\frac{\partial\chi'(r,\vartheta,\varphi)}{\partial r}+\frac{1}{2}\left[\frac{\partial^2\theta'}{\partial r^2}+\frac{1}{r^2}\Omega(\vartheta,\varphi)\theta'(r,\vartheta,\varphi)\right]=0. \quad (24)$$

Integrating this equation over r from 0 to R and noting that $\chi'(0,\vartheta,\varphi)=0$, $\frac{\partial\vartheta'(0,\vartheta,\varphi)}{\partial r}=0$, we obtain

$$\chi'(R,\vartheta,\varphi)=-\frac{1}{2}\left[\frac{\partial\theta'(R,\vartheta,\varphi)}{\partial r}+\int_0^R\frac{1}{r^2}\Omega(\vartheta,\varphi)\theta'(r,\vartheta,\varphi)\,dr\right]. \quad (25)$$

The first term in (25) can be ignored, as was carefully shown for a plane wave. Putting in (25)

$$\theta'(r,\vartheta,\varphi)=\frac{1}{2}\int_0^r\varepsilon_1(\rho,\vartheta,\varphi)\,d\rho,$$

we obtain

$$\chi'(R,\vartheta,\varphi)=-\frac{1}{4}\int_0^R\frac{dr}{r^2}\int_0^r d\rho\,\Omega(\vartheta,\varphi)\,\varepsilon_1(\rho,\vartheta,\varphi). \quad (26)$$

Changing the order of integration and integrating over r, we obtain

$$\chi'(R,\vartheta,\varphi)=-\frac{1}{4}\int_0^R\rho\left(1-\frac{\rho}{R}\right)\frac{1}{\rho^2}\Omega(\vartheta,\varphi)\,\varepsilon_1(\rho,\vartheta,\varphi)\,d\rho. \quad (27)$$

But as is clear from (22), (23), $\frac{1}{\rho^2}\Omega=\Delta_\perp$ is the transverse part of the Laplacian. Hence,

$$\chi'(R,\vartheta,\varphi)=-\frac{1}{4}\int_0^R\frac{\rho}{R}(R-\rho)\,\Delta_\perp\varepsilon_1(\rho,\vartheta,\varphi)\,d\rho. \quad (28)$$

Expression (28) differs from the corresponding plane-wave relation by an additional factor ρ/R in the integrand. If dielectric constant inhomogeneities are present only for $\rho\approx R$, the factor $\rho/R\approx 1$ and (28) reduces to the corresponding plane-wave expression. Since $\rho/R<1$, we see from (28) that the amplitude fluctuations for a spherical wave are invariably less than the amplitude fluctuations for a plane wave propagating over the same distance.

Let us find the mean square amplitude fluctuations of a spherical wave. By (28)

$$\langle(\chi'(R,\vartheta,\varphi))^2_{\text{sp}}\rangle=\frac{1}{16R^2}\int_0^R\int_0^R\rho_1\rho_2(R-\rho_1)(R-\rho_2)\times$$
$$\times\langle\Delta_\perp\varepsilon_1(\rho_1,\vartheta,\varphi)\,\Delta_\perp\varepsilon_1(\rho_2,\vartheta,\varphi)\rangle\,d\rho_1\,d\rho_2=$$
$$=-\frac{1}{32R^2}\int_0^R\int_0^R\rho_1\rho_2(R-\rho_1)(R-\rho_2)\,\Delta_\perp^2 D_\varepsilon(\rho_1-\rho_2,0,0)\,d\rho_1\,d\rho_2. \quad (29)$$

(We again used the relation $\Delta_{\perp}^2 B_\varepsilon = -\frac{1}{2}\Delta_{\perp}^2 D_\varepsilon$). Introducing new variables of integration

$$\rho_1 - \rho_2 = \xi, \ \frac{1}{2}(\rho_1 + \rho_2) = \eta,$$

and seeing that the integrand is an even function of ξ, we obtain

$$\langle \chi_{sp}'^2 \rangle = -\frac{1}{16R^2}\int_0^R \Delta_{\perp}^2 D_\varepsilon(\xi, 0, 0)\, d\xi \int_{\frac{\xi}{2}}^{R-\frac{\xi}{2}} \left(\eta^2 - \frac{\xi^2}{4}\right)\left[(R-\eta)^2 - \frac{\xi^2}{4}\right] d\eta. \qquad (30)$$

The function $\Delta_{\perp}^2 D_\varepsilon$ has a characteristic scale corresponding to l_0 (see previous section). In (30) η is of the order of R. Therefore, ξ can be ignored compared to η and we have

$$\int_{\frac{\xi}{2}}^{R-\frac{\xi}{2}} \left(\eta^2 - \frac{\xi^2}{4}\right)\left[(R-\eta)^2 - \frac{\xi^2}{4}\right] d\eta \approx \int_0^R \eta^2 (R-\eta)^2\, d\eta = \frac{R^5}{30}.$$

(Compare with the corresponding treatment for the plane wave case.) Hence,

$$\langle (\chi')_{sp}^2 \rangle = -\frac{R^3}{480}\int_0^R \Delta_{\perp}^2 D_\varepsilon(\xi, 0, 0)\, d\xi \approx -\frac{R^3}{480}\int_0^\infty \Delta_{\perp}^2 D_\varepsilon(\xi, 0, 0)\, d\xi. \qquad (31)$$

Comparing (31) with (40.51a), we obtain the simple relation

$$\langle (\chi')_{sp}^2 \rangle = \frac{1}{10}\langle (\chi')_{pl}^2 \rangle, \qquad (32)$$

which is applicable for any correlation function (see also Part B, §49).

Consider the expression

$$\langle \chi'(R, \vartheta_1, \varphi_1)\, \chi'(R, \vartheta_2, \varphi_2) \rangle = \frac{1}{16R^2}\int_0^R\int_0^R \rho_1\rho_2 (R-\rho_1)(R-\rho_2) \times$$
$$\times \langle \Delta_{\perp}\varepsilon_1(\rho_1, \vartheta_1, \varphi_1)\, \Delta_{\perp}\varepsilon_1(\rho_2, \vartheta_2, \varphi_2) \rangle\, d\rho_1\, d\rho_2. \qquad (33)$$

If the angles $\vartheta_1 - \vartheta_2$, $\varphi_1 - \varphi_2$ are small, the operators $\Delta_{\perp}$ acting in different directions differ only in terms of the second order of smallness and this difference may clearly be neglected. Then approximately

$$\langle \Delta_{\perp}\varepsilon_1(\rho_1, \vartheta_1, \varphi_1)\, \Delta_{\perp}\varepsilon_1(\rho_2, \vartheta_2, \varphi_2) \rangle \approx -\frac{1}{2}\Delta_{\perp}^2 D_\varepsilon(\rho_1, \rho_2; \vartheta_1, \vartheta_2; \varphi_1, \varphi_2).$$

For locally isotropic fluctuations, D_ε depends only on the distance

$$r = \sqrt{\rho_1^2 - 2\rho_1\rho_2\cos\psi + \rho_2^2},$$

where

$$\cos\psi = \cos\vartheta_1\cos\vartheta_2 + \sin\vartheta_1\sin\vartheta_2\cos(\varphi_1 - \varphi_2)$$

and ψ is the angle between the rays from the source to the observation points (this angle is assumed to be small).

We take as before $\xi = \rho_1 - \rho_2$, $\eta = \frac{1}{2}(\rho_1 + \rho_2)$. In this case

$$r = \sqrt{\xi^2 \cos^2\frac{\psi}{2} + 4\eta^2 \sin^2\frac{\psi}{2}} \approx \sqrt{\xi^2 + 4\eta^2 \sin^2\frac{\psi}{2}}.$$

The region of integration is between the straight lines $\eta = \pm\frac{1}{2}\xi$, $\eta = R \pm \frac{1}{2}\xi$. Since on changing to the variables η, ξ, the integrand in (33) is an even function of ξ, the integration over ξ need be carried out only for $\xi > 0$, multiplying the result by 2. Then

$$\langle \chi_1' \chi_2' \rangle = -\frac{1}{16R^2}\Bigg\{\int_0^{\frac{R}{2}} d\eta \int_0^{2\eta} d\xi \left(\eta^2 - \frac{\xi^2}{4}\right)\left[(R-\eta)^2 - \frac{\xi^2}{4}\right] \times$$
$$\times \Delta_\perp^2 D_\varepsilon\left(\sqrt{\xi^2 + 4\eta^2 \sin^2\frac{\psi}{2}}\right) + \int_{\frac{R}{2}}^{R} d\eta \int_0^{2(R-\eta)} d\xi \left(\eta^2 - \frac{\xi^2}{4}\right) \times$$
$$\times \left[(R-\eta)^2 - \frac{\xi^2}{4}\right] \Delta_\perp^2 D_\varepsilon\left(\sqrt{\xi^2 + 4\eta^2 \sin^2\frac{\psi}{2}}\right)\Bigg\}. \tag{34}$$

As in the derivation of the expression for $\langle (\chi')^2_{sp} \rangle$, in (34) we may approximately take

$$\left(\eta^2 - \frac{\xi^2}{4}\right)\left[(R-\eta)^2 - \frac{\xi^2}{4}\right] \approx \eta^2 (R-\eta)^2.$$

Then

$$\langle \chi_1' \chi_2' \rangle \approx -\frac{1}{16R^2}\Bigg\{\int_0^{\frac{R}{2}} \eta^2 (R-\eta)^2 d\eta \times$$
$$\times \int_0^{2\eta} d\xi \Delta_\perp^2 D_\varepsilon\left(\sqrt{\xi^2 + 4\eta^2 \sin^2\frac{\psi}{2}}\right) +$$
$$+ \int_{\frac{R}{2}}^{R} \eta^2 (R-\eta)^2 d\eta \int_0^{2(R-\eta)} d\xi \Delta_\perp^2 D_\varepsilon\left(\sqrt{\xi^2 + 4\eta^2 \sin^2\frac{\psi}{2}}\right)\Bigg\}. \tag{35}$$

We will use expression (40.51) for the plane-wave amplitude correlation function

$$B_\chi(y, z) = -\frac{x^3}{48}\int_0^\infty \Delta_\perp^2 D_\varepsilon(\xi, y, z)\, d\xi,$$

which for isotropic fluctuations can be written in the form

$$\int_0^\infty \Delta_\perp^2 D_\varepsilon\left(\sqrt{\xi^2 + b^2}\right) d\xi = -\frac{48 B_\chi(b)}{R^3}. \tag{36}$$

Integration over ξ in (35) can be extended over the entire interval $(0, \infty)$ since $\Delta_\perp^2 D_\varepsilon$ rapidly falls off for $\xi \gtrsim l_0$. Expression (35) thus takes the form

$$\langle \chi_1' \chi_2' \rangle = \frac{3}{R^5}\int_0^R \eta^2 (R-\eta)^2 B_\chi\left(2\eta \sin\frac{\psi}{2}\right) d\eta. \tag{37}$$

Let $b = 2R \sin \frac{\psi}{2}$ (b is the distance between observation points). Changing to a new variable of integration $p = \frac{\eta}{R}$ we write (37) in the form

$$\langle \chi_1' \chi_2' \rangle_{sp} = 3 \int\limits_0^1 p^2 (1-p)^2 B_\chi (bp)\, dp. \tag{38}$$

Expression (38) relates the log amplitude correlation function of a spherical wave to that of a plane wave covering the same distance in the random medium /95/. Taking in (38) $b = 0$, we obtain, after integration, expression (32).

The factor $\eta^2(R-\eta)^2$ in (37) gives the relative weight of the various sections along the path of the ray as sources of fluctuations. As is known, a lens placed directly in front of the source or near the observation point does not affect the light intensity. The factor $\eta^2(R-\eta)^2$ therefore vanishes at these points.

This effect also explains why the fluctuations of a spherical wave are smaller than the fluctuations of a plane wave: the inhomogeneities in the immediate vicinity of the source do not affect the wave intensity.

Note that expressions (13) and (38) relating phase and amplitude fluctuations of a spherical wave to those of a plane wave also show a fundamental difference. If we calculate the phase difference fluctuations for two plane waves diverging at an angle ψ, the result will coincide with (13). The result for amplitude fluctuations, however, will be different from relation (38) (see Chapter 4).

§ 42. Phase and amplitude fluctuations of a wave propagating in a locally isotropic turbulent medium

We will now compute phase and amplitude fluctuations for plane and spherical waves in a locally isotropic turbulent medium. As we have shown in the preceding, all the characteristics of amplitude and angle of arrival fluctuations in the geometrical optics approximation can be computed if the structure function of the plane-wave phase is known.

We use expression (40.22) for the eikonal structure function (the distance traveled by a wave in the random medium is denoted by L instead of x):

$$D_\theta(\rho) = 2\pi^2 L \int\limits_0^\infty [1 - J_0(\varkappa\rho)]\, \Phi_\varepsilon(\varkappa)\, \varkappa\, d\varkappa. \tag{1}$$

For the function $\Phi_\varepsilon(\varkappa)$ we use the approximation (15.9):

$$\Phi_\varepsilon(\varkappa) = 0.033 C_\varepsilon^2 \varkappa^{-11/3} \exp\left(-\frac{\varkappa^2}{\varkappa_m^2}\right), \tag{2}$$

where $\varkappa_m$ is related to the inner scale λ_0 of the dielectric-constant fluctuations by the expression*

$$\lambda_0 \varkappa_m = 5.92 \tag{3}$$

* We recall that l_0 and λ_0 coincide apart from a numerical factor.

and λ_0 is the intersection point of the asymptotic expansions of $D_T(r)$ for small and large r. Inserting (2) into (1) and using the expansion

$$1 - J_0(\varkappa\rho) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n!)^2}\left(\frac{\rho}{2}\right)^{2n} \varkappa^{2n},$$

we obtain

$$D_\theta(\rho) = -0.033\, 2\pi^2 C_\varepsilon^2 L \sum_{n=1}^{\infty} \frac{\left(-\frac{\rho^2}{4}\right)^n}{(n!)^2} \int_0^\infty \varkappa^{2n-\frac{11}{3}} \exp\left(-\frac{\varkappa^2}{\varkappa_m^2}\right) \varkappa\, d\varkappa. \tag{4}$$

Substituting $\varkappa^2 = \varkappa_m^2 t$, we obtain

$$\int_0^\infty \varkappa^{2n-\frac{11}{3}} \exp\left(-\frac{\varkappa^2}{\varkappa_m^2}\right) \varkappa\, d\varkappa = \frac{1}{2}\varkappa_m^{2n-\frac{5}{3}} \int_0^\infty t^{n-\frac{5}{6}-1} e^{-t}\, dt = \frac{1}{2}\Gamma\left(n - \frac{5}{6}\right)\varkappa_m^{2n-\frac{5}{3}}$$

and

$$D_\theta(\rho) = -0.033\, \pi^2 C_\varepsilon^2 L \sum_{n=1}^{\infty} \frac{\Gamma\left(n - \frac{5}{6}\right)}{(n!)^2}\left(-\frac{\varkappa_m^2\rho^2}{4}\right)^n \varkappa_m^{-5/3}. \tag{5}$$

Using the definition of the confluent hypergeometric function (Kummer's function)

$${}_1F_1(\alpha, \gamma, z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)} \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)}{n!\,\Gamma(n+\gamma)} z^n,$$

we write (5) in the form

$$\begin{aligned} D_\theta(\rho) &= 0.033\pi^2\Gamma\left(-\frac{5}{6}\right) C_\varepsilon^2 L \varkappa_m^{-5/3}\left[1 - {}_1F_1\left(-\frac{5}{6}, 1, -\frac{\varkappa_m^2\rho^2}{4}\right)\right] = \\ &= 2.2 C_\varepsilon^2 L \varkappa_m^{-5/3}\left[{}_1F_1\left(-\frac{5}{6}, 1, -\frac{\varkappa_m^2\rho^2}{4}\right) - 1\right]. \end{aligned} \tag{6}$$

Consider the behavior of $D_\theta(\rho)$ in the region of large and small ρ. For $\varkappa_m\rho \gg 1$, i.e., $\rho \gg \lambda_0$, we use the asymptotic expansion of ${}_1F_1$ for large negative values of the argument:

$${}_1F_1(\alpha, \gamma, z) \approx \frac{\Gamma(\gamma)}{\Gamma(\gamma - \alpha)}(-z)^{-\alpha},$$

whence

$${}_1F_1\left(-\frac{5}{6}, 1, -\frac{\varkappa_m^2\rho^2}{4}\right) \approx \frac{1}{\Gamma\left(\frac{11}{6}\right)} \frac{\varkappa_m^{5/3}\rho^{5/3}}{2^{5/3}}$$

and by /96, 97/

$$D_\theta(\rho) \approx 0.73 C_\varepsilon^2 L \rho^{5/3} \quad (\rho \gg \lambda_0). \tag{7}$$

The asymptotic expression of $D_\theta(\rho)$ for small ρ can be found by taking the first two terms of the series for ${}_1F_1$:

$${}_1F_1\left(-\frac{5}{6}, 1, -\frac{\varkappa_m^2\rho^2}{4}\right) = 1 + \frac{5}{6}\frac{\varkappa_m^2\rho^2}{4} + \dots$$

Moreover, putting $\varkappa_m = 5.92/\lambda_0$, we obtain

$$D_0(\rho) \approx 0.82 C_\varepsilon^2 L \lambda_0^{-1/3} \rho^2 \quad (\rho \ll \lambda_0). \tag{8}$$

We can now compute the correlation functions of the angles of arrival. The transverse correlation function $B_\beta(\rho)$ is related to $D_0(\rho)$ by (40.38):

$$B_\beta(\rho) = \frac{1}{2} \frac{D_0'(\rho)}{\rho}. \tag{9}$$

Inserting (6) and using the equality

$$\frac{d}{dz}\, {}_1F_1(\alpha, \gamma, z) = \frac{\alpha}{\gamma}\, {}_1F_1(\alpha+1, \gamma+1, z),$$

we obtain

$$B_\beta(\rho) = 0.46 C_\varepsilon^2 L \varkappa_m^{1/3}\, {}_1F_1\left(\frac{1}{6}, 2, -\frac{\varkappa_m^2 \rho^2}{4}\right). \tag{10}$$

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Taking $\rho = 0$, we obtain the mean square fluctuations of the angle of arrival

$$\langle \beta^2 \rangle = 0.46 C_\varepsilon^2 L \varkappa_m^{1/3} = 0.82 C_\varepsilon^2 L \lambda_0^{-1/3}. \tag{11}$$

Dividing (10) by (11), we obtain the normalized transverse correlation function for the angle of arrival

$$b_\beta(\rho) = {}_1F_1\left(\frac{1}{6}, 2, -\frac{\varkappa_m^2 \rho^2}{4}\right). \tag{12}$$

The argument of $b_\beta(\rho)$ is the dimensionless variable $\varkappa_m^2 \rho^2/4$, therefore it follows that the correlation radius for angle of arrival fluctuations is of the order $\varkappa_m^{-1} \sim \lambda_0$, i.e., comparable with the inner scale of turbulence.

For the longitudinal autocorrelation function of the angle of arrival we have from (40.35)

$$B_\alpha(\rho) = \frac{1}{2} D_0''(\rho).$$

Differentiating, we obtain

$$B_\alpha(\rho) = 0.46 C_\varepsilon^2 L \varkappa_m^{1/3} \left[{}_1F_1\left(\frac{1}{6}, 2, -\frac{\varkappa_m^2 \rho^2}{4}\right) - \frac{\varkappa_m^2 \rho^2}{24}\, {}_1F_1\left(\frac{7}{6}, 3, -\frac{\varkappa_m^2 \rho^2}{4}\right) \right]. \tag{13}$$

Putting $\rho = 0$, we obtain the expected equality $\langle \alpha^2 \rangle = \langle \beta^2 \rangle$, and for the normalized longitudinal correlation function we get

$$b_\alpha(\rho) = {}_1F_1\left(\frac{1}{6}, 2, -\frac{\varkappa_m^2 \rho^2}{4}\right) - \frac{\varkappa_m^2 \rho^2}{24}\, {}_1F_1\left(\frac{7}{6}, 3, -\frac{\varkappa_m^2 \rho^2}{4}\right). \tag{14}$$

The correlation scale for α is also λ_0. The functions (12) and (14) are plotted in Figures 41 and 42.

From (10) and (13) we conclude that the magnitude and the correlation radius of angle of arrival fluctuations are determined by the inner scale of turbulence, λ_0. However, if the angle of arrival of a wave is measured by an antenna (an objective) with a characteristic dimension d, averaging over the aperture will suppress the effects of small inhomogeneities and the magnitude of the fluctuations will be determined by the parameter d (see below).

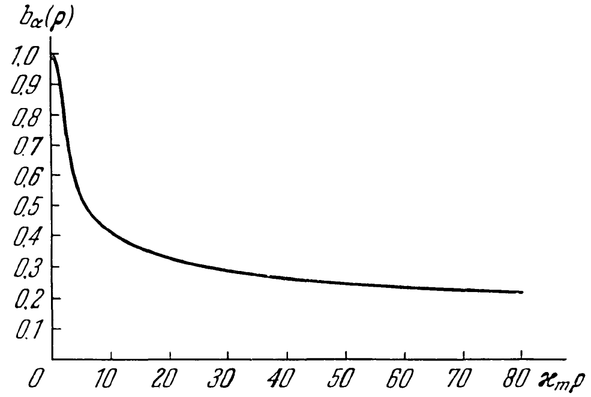


FIGURE 41. The correlation function b_α for angle-of-arrival fluctuations in the plane through both rays (see Figure 39) in the geometrical optics approximation.

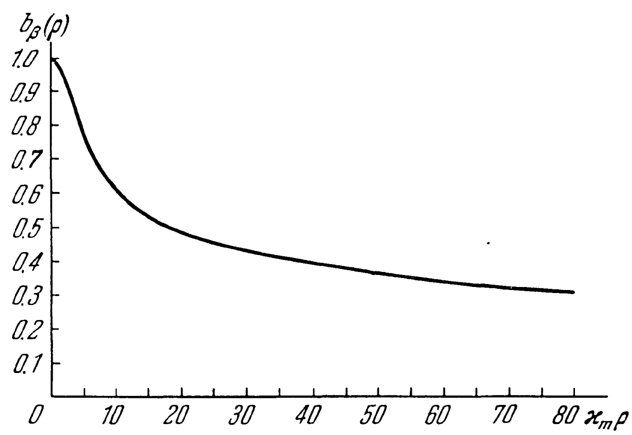


FIGURE 42. The correlation function b_β for angle-of-arrival fluctuations in planes perpendicular to the plane through the two rays (see Figure 39) in the geometrical optics approximation.

Consider the amplitude correlation function. It is related to $D_0(\rho)$ by (40.57):

$$B_\chi(\rho) = -\frac{L^2}{24}\Delta_\perp^2 D_0(\rho) = -\frac{L^2}{24}\frac{1}{\rho}\frac{d}{d\rho}\left\{\rho\frac{d}{d\rho}\left[\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dD_0(\rho)}{d\rho}\right)\right]\right\}.$$

Inserting (6) and differentiating, we obtain

$$B_\chi(\rho) = 0.0126C_\varepsilon^2 L^3 \varkappa_m^{7/3}\left[{}_1F_1\left(\frac{7}{6}, 3, -\frac{\varkappa_m^2\rho^2}{4}\right) - \right.$$
$$\left. -\frac{7}{36}\varkappa_m^2\rho^2{}_1F_1\left(\frac{13}{6}, 4 - \frac{\varkappa_m^2\rho^2}{4}\right) + \frac{91}{13824}\varkappa_m^4\rho^4{}_1F_1\left(\frac{19}{6}, 5 - \frac{\rho^2\varkappa_m^2}{4}\right)\right]. \quad (15)$$

Taking $\rho = 0$, we obtain

$$\langle \chi^2 \rangle = 0.0126 C_\varepsilon^2 L^3 \varkappa_m^{7/3} = 0.80 C_\varepsilon^2 L^3 \lambda_0^{-7/3},$$

$$b_\chi(\rho) = {}_1F_1\left(\frac{7}{6}, 3, -\frac{\varkappa_m^2\rho^2}{4}\right) - \frac{7}{36}\varkappa_m^2\rho^2 {}_1F_1\left(\frac{13}{6}, 4, -\frac{\varkappa_m^2\rho^2}{4}\right) + \\ + \frac{91}{13824}\varkappa_m^4\rho^4 {}_1F_1\left(\frac{19}{6}, 5, -\frac{\varkappa_m^2\rho^2}{4}\right). \qquad (16)$$

Successive application of the equalities

$$z\,{}_1F_1(\alpha, \gamma, z) = (\gamma - 1)\left[{}_1F_1(\alpha, \gamma - 1, z) - {}_1F_1(\alpha - 1, \gamma - 1, z)\right],$$

and

$${}_1F_1(\alpha, \gamma, z) = \frac{\alpha - \gamma}{\alpha - 1}\,{}_1F_1(\alpha - 1, \gamma, z) + \frac{\gamma - 1}{\alpha - 1}\,{}_1F_1(\alpha - 1, \gamma - 1, z),$$

gives

$$b_\chi(\rho) = {}_1F_1\left(\frac{7}{6}, 1, -\frac{\varkappa_m^2\rho^2}{4}\right). \qquad (17)$$

From (16) we see that the mean square of the log amplitude fluctuations increases as the inner scale of turbulence λ_0 decreases. From (17) it also follows that the amplitude correlation radius in the geometrical optics approximation is equal to the inner scale of turbulence.

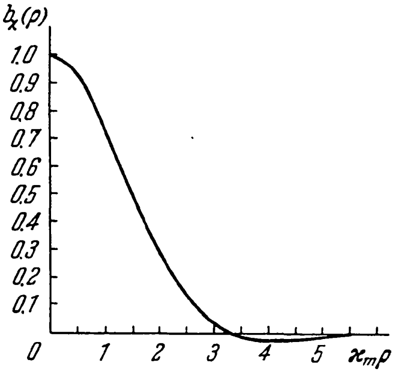


FIGURE 43. The correlation function of log-amplitude fluctuations in the geometrical optics approximation.

The function (17) is plotted in Figure 43. For large values of the argument

$$b_\chi(\rho) \approx -\frac{1}{6\Gamma\left(\frac{5}{6}\right)}\left(\frac{\varkappa_m\rho}{2}\right)^{-7/3}.$$

Let us now find the characteristics of the spherical wave fluctuations. The structure function of the fluctuations of the eikonal for a spherical wave $D_{\theta,\,\mathrm{sp}}(\rho)$ is related to the structure function of the eikonal for a plane wave $D_\theta(\rho)$ by (41.13):

$$D_{\theta,\ \mathrm{sp}}(\rho) = \int\limits_0^1 D_\theta(\rho t)\,dt. \tag{18}$$

We do not insert expression (6) in its general form, but will consider the cases $\rho \ll \lambda_0$ and $\rho \gg \lambda_0$ separately. In the first case, we have from (8)

$$D_\theta(\rho) = 0.82 C_\varepsilon^2 L \lambda_0^{-1/3} \rho^2,$$

so that

$$D_{\theta,\ \mathrm{sp}}(\rho) = \frac{1}{3} D_\theta(\rho) \quad (\rho \ll \lambda_0). \tag{19}$$

If, using this expression, we calculate the mean square angle-of-arrival fluctuations for a spherical wave, we obtain the simple relation

$$\langle \alpha_{\mathrm{sp}}^2 \rangle = \frac{1}{3} \langle \alpha^2 \rangle. \tag{20}$$

For $\rho \gg \lambda_0$, the region where $\rho t \ll \lambda_0$ makes a negligible contribution to the integral in (18) and we may use relation (7). In this case

$$D_{\theta,\ \mathrm{sp}}(\rho) = \frac{3}{8} D_\theta(\rho) = 0.27 C_\varepsilon^2 L \rho^{5/3} \qquad (\rho \gg \lambda_0). \tag{21}$$

The phase structure functions for a spherical wave thus differ from the corresponding plane wave functions only by numerical coefficients which are associated with the effective "average" separation between the two paths.

With regard to amplitude fluctuations, we have already noted that the mean square log amplitude fluctuations of a spherical wave is 1/10 of the corresponding plane wave fluctuations.

In concluding this section, it remains to calculate the phase difference correlation function for two plane waves diverging at a certain angle γ. This function will be needed in the following when we interpret the phenomenon of "dancing" of star images in telescopes.

Let two plane waves (Figure 44) propagating at an angle γ to each other be observed at points A and B. The random phase difference between the two points is $(S_1 - S_2)$ for one wave and $(S_3 - S_4)$ for the other. The subscripts correspond to the ray numbers in Figure 44. Consider the product

$$\langle (S_1 - S_2)(S_3 - S_4) \rangle = \frac{1}{2} [\langle (S_1 - S_4)^2 \rangle + \langle (S_2 - S_3)^2 \rangle - \\ - \langle (S_1 - S_3)^2 \rangle - \langle (S_2 - S_4)^2 \rangle].$$

To the individual terms in this expression we can apply the relations (13a)–(13c) of the previous section, using for the plane-wave phase structure function

$$D_S(\rho) = 0.73 C_\varepsilon^2 k^2 L \rho^{5/3} \tag{22}$$

(this expression is obtained from (7) by multiplying by k^2). We should distinguish between two particular cases. If $L\tan\left(\frac{\gamma}{2}\right) < \frac{b}{2}$ and rays 2, 3 do not intersect, we should use (13b) for $\langle (S_2 - S_3)^2 \rangle$. Otherwise, this term is computed from (13c).

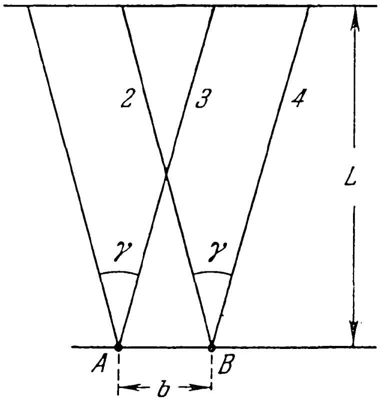


FIGURE 44. Ray geometry in calculating the phase-difference fluctuations for two plane waves propagating at a certain angle to each other.

Elementary manipulations lead to the following expression for the correlation coefficient (normalized to unity for $\gamma = 0$):

$$b_{\Delta S}(\gamma) = \begin{cases} \dfrac{3}{16}\,\dfrac{(1+x)^{8/3} - (1-x)^{8/3} - 2x^{8/3}}{x} & \text{for } x \equiv \dfrac{2L\tan\frac{\gamma}{2}}{b} < 1, \\[2ex] \dfrac{3}{16}\,\dfrac{(x+1)^{8/3} + (x-1)^{8/3} - 2x^{8/3}}{x} & \text{for } x = \dfrac{2L\tan\frac{\gamma}{2}}{b} > 1. \end{cases} \tag{23}$$

§ 43. Limits of applicability of the first geometrical optics approximation

The geometrical optics method used in the preceding section called for the use of two different expansions. The first expansion was in powers of k^{-1}, i.e., actually in powers of the ratio λ/λ_0, where λ_0 is the inner scale of turbulence. This expansion gave the eikonal equation and an equation relating the amplitude and phase of the wave. The eikonal equation can be solved exactly for wave propagation in a layered medium. In this case the applicability of the geometrical optics approximation is determined by higher terms in the expansion in powers of λ/λ_0. However, for wave propagation in a random medium, the eikonal equation is solved approximately after expanding in powers of the small parameter $\varepsilon_1 = \varepsilon - \langle \varepsilon \rangle$. In this case the applicability of the method is again limited by nonlinear effects, associated with terms of the order ε_1^2. In considering the applicability of the geometrical optics method as a whole, we should start with an analysis of the second part of the problem.

The expansion of $\theta(\boldsymbol{r})$ in powers of $\nu = \sqrt{\langle\varepsilon_1^2\rangle}$ was derived in §39. Combining (39.19) and (39.21), we write for θ to terms of the order of ε_1^3:

$$\theta(\boldsymbol{l}_0 s) = \theta(\boldsymbol{l}_0 s + \delta\boldsymbol{x}(s)) - \delta\boldsymbol{x}(s)\nabla\theta(\boldsymbol{l}_0 s + \delta\boldsymbol{x}(s)) =$$
$$= s + \frac{1}{2}\int_0^s \varepsilon_1(\boldsymbol{l}_0 s')ds' + \frac{1}{2}\int_0^s \delta\boldsymbol{x}(s')\nabla\varepsilon_1(\boldsymbol{l}_0 s')\,ds' - \delta\boldsymbol{x}(s)\,\delta\boldsymbol{l}(s) + \cdots \quad (1)$$

Inserting (39.12) and (39.13), we obtain after simple manipulations

$$\theta(\boldsymbol{l}_0 s) = s + \frac{1}{2}\int_0^s \varepsilon_1(\boldsymbol{l}_0 s)\,ds' - \frac{1}{2}\int_0^s ds'\int_0^{s'} ds'' \times$$
$$\times (s - s')\left[\nabla\varepsilon_1(\boldsymbol{l}_0 s')\nabla\varepsilon_1(\boldsymbol{l}_0 s'') - (\boldsymbol{l}_0\nabla\varepsilon_1(\boldsymbol{l}_0 s'))(\boldsymbol{l}_0\nabla\varepsilon_1(\boldsymbol{l}_0 s''))\right] + \cdots \quad (2)$$

Let us calculate the mean correction to θ due to nonlinear effects:

$$\langle\delta\theta(\boldsymbol{l}_0 s)\rangle = -\frac{1}{2}\int_0^s (s - s')\,ds'\int_0^{s'}\langle\nabla\varepsilon_1(\boldsymbol{l}_0 s')\nabla\varepsilon_1(\boldsymbol{l}_0 s'') - (\boldsymbol{l}_0\nabla\varepsilon_1(\boldsymbol{l}_0 s'))(\boldsymbol{l}_0\nabla\varepsilon_1(\boldsymbol{l}_0 s''))\rangle\,ds''. \quad (3)$$

To find the mean in the integrand, we differentiate the equality

$$B_\varepsilon(\boldsymbol{r}_1 - \boldsymbol{r}_2) = \langle\varepsilon_1(\boldsymbol{r}_1)\,\varepsilon_1(\boldsymbol{r}_2)\rangle,$$

which gives

$$\left\langle\frac{\partial\varepsilon_1(\boldsymbol{r}_1)}{\partial x_i^1}\frac{\partial\varepsilon_1(\boldsymbol{r}_2)}{\partial x_k^2}\right\rangle = -\frac{\partial^2 B_\varepsilon(\boldsymbol{\rho})}{\partial\xi_i\,\partial\xi_k}, \quad (4)$$

where $\boldsymbol{\rho} = \boldsymbol{r}_1 - \boldsymbol{r}_2$, $\boldsymbol{\rho} = \{\xi_1, \xi_2, \xi_3\}$. For locally isotropic turbulence, $B_\varepsilon(\boldsymbol{\rho}) = B_\varepsilon(\rho)$ and

$$\frac{\partial B_\varepsilon(\rho)}{\partial\xi_i} = B_\varepsilon'(\rho)\frac{\xi_i}{\rho}, \quad \frac{\partial^2 B_\varepsilon(\rho)}{\partial\xi_i\,\partial\xi_k} = B_\varepsilon''(\rho)\frac{\xi_i\xi_k}{\rho^2} + \frac{B_\varepsilon'(\rho)}{\rho^3}(\rho^2\delta_{ik} - \xi_i\xi_k). \quad (5)$$

Hence,

$$\langle\nabla\varepsilon_1(\boldsymbol{l}_0 s')\nabla\varepsilon_1(\boldsymbol{l}_0 s'') - (\boldsymbol{l}_0\nabla\varepsilon_1(\boldsymbol{l}_0 s'))(\boldsymbol{l}_0\nabla\varepsilon_1(\boldsymbol{l}_0 s''))\rangle = -(\delta_{ik} - l_i^0 l_k^0)\frac{\partial^2 B_\varepsilon(\rho)}{\partial\xi_i\,\partial\xi_k}. \quad (6)$$

In our case $\boldsymbol{\rho} = \boldsymbol{l}_0 s' - \boldsymbol{l}_0 s'' = \boldsymbol{l}_0(s' - s'')$, $\rho = s' - s''$, $\boldsymbol{\rho}/\rho = \boldsymbol{l}_0$ and $\xi_i/\rho = l_i^0$. Therefore,

$$\langle\nabla\varepsilon_1'\nabla\varepsilon_1'' - (\boldsymbol{l}_0\nabla\varepsilon_1')(\boldsymbol{l}_0\nabla\varepsilon_1'')\rangle =$$
$$= -(\delta_{ik} - l_i^0 l_k^0)\left[B_\varepsilon''(\rho)\,l_i^0 l_k^0 + \frac{B_\varepsilon'(\rho)}{\rho}(\delta_{ik} - l_i^0 l_k^0)\right] = -\frac{2B_\varepsilon'(\rho)}{\rho}. \quad (7)$$

(We used the equality $\delta_{ii} = 3$.) Differentiating the equality $D_\varepsilon(\rho) = 2B_\varepsilon(0) - 2B_\varepsilon(\rho)$ we obtain $-2B_\varepsilon'(\rho) = D_\varepsilon'(\rho)$, so that (7) takes the form

$$\langle\nabla\varepsilon_1'\nabla\varepsilon_1'' - (\boldsymbol{l}_0\nabla\varepsilon_1')(\boldsymbol{l}_0\nabla\varepsilon_1'')\rangle = \frac{D_\varepsilon'(\rho)}{\rho} = \frac{D_\varepsilon'(s' - s'')}{s' - s''}. \quad (8)$$

Insertion of (8) in (3) gives

$$\langle\delta\theta(l_0 s)\rangle = -\frac{1}{2}\int_0^s (s-s')\,ds' \int_0^{s'} \frac{D'_\varepsilon(s'-s'')}{s'-s''}\,ds''. \qquad (9)$$

Substitution of the variables $s'' = s' - \rho$, followed by changing the order of integration over s' and ρ and then integrating over s' gives

$$\langle\delta\theta(l_0 s)\rangle = -\frac{1}{4}\int_0^s \frac{(s-\rho)^2}{\rho} D'_\varepsilon(\rho)\,d\rho. \qquad (10)$$

The function $D'_\varepsilon(\rho)/\rho$ has a sharp maximum near $\rho = 0$ (if $D_\varepsilon(\rho) = C_\varepsilon^2\rho^{2/3}$, we have $\frac{D'_\varepsilon(\rho)}{\rho} \to \infty$ for $\rho \to 0$, but viscosity and thermal conduction effects reduce this peak to a finite value). The main contribution in (10) thus comes from the region where ρ is small, and we may take $(s-\rho)^2 \approx s^2$, extending the limit of integration to infinity. Then

$$\langle\delta\theta\rangle = -\frac{s^2}{4}\int_0^\infty \frac{D'_\varepsilon(\rho)}{\rho}\,d\rho. \qquad (11)$$

Comparison with expression (40.32) for the mean square angle-of-arrival fluctuations (the distance is denoted by L instead of x)

$$\langle\alpha^2\rangle = \frac{L}{4}\int_0^\infty \frac{D'_\varepsilon(\rho)}{\rho}\,d\rho,$$

gives the relation

$$\langle\delta\theta\rangle = -L\langle\alpha^2\rangle = -L\sigma_\alpha^2(L). \qquad (12)$$

Let us now establish the conditions when the correction $\langle\delta\theta\rangle$ may be neglected. This nonlinear correction should be compared with the characteristic value of fluctuations in the optical path difference over distances comparable with the correlation radius of angle-of-arrival fluctuations (over these distances the wave front is rotated as a whole). The corresponding optical path difference is $\lambda_0\sigma_\alpha$. Hence, the nonlinear correction to θ can be ignored if $L\sigma_\alpha^2 \ll \lambda_0\sigma_\alpha$ or

$$\sigma_\alpha \ll \frac{\lambda_0}{L}. \qquad (13)$$

Condition (13) can be written in a different form. Inserting in (13)

$$\sigma_\alpha^2 = 0.82 C_\varepsilon^2 L \lambda_0^{-1/3},$$

we obtain

$$\frac{L^2\sigma_\alpha^2}{\lambda_0^2} = 0.82 C_\varepsilon^2 L^3 \lambda_0^{-7/3} \ll 1.$$

The product $C_\varepsilon^2 L^3 \lambda_0^{-7/3}$ is proportional to the mean square amplitude fluctuations. Condition (13) can thus be written in the form

$$\langle \chi^2 \rangle \ll 1, \tag{14}$$

i.e., the amplitude fluctuations must be small. It is necessary to consider another limiting condition, namely that in deriving expression (39.19) for θ, we used a Taylor series expansion in ε_1:

$$\varepsilon_1 (\boldsymbol{l}_0 s + \delta \boldsymbol{x}(s)) = \varepsilon_1 (\boldsymbol{l}_0 s) + \nabla \varepsilon_1 (\boldsymbol{l}_0 s)\, \delta \boldsymbol{x}(s) + \ldots$$

This expansion is clearly justified only if $\delta \boldsymbol{x}$ is small compared to the inner scale of turbulence, since only for $|\delta \boldsymbol{x}^2| \ll \lambda_0^2$ are the increments of ε_1 linear in $\delta \boldsymbol{x}$. Therefore, in addition to (14), it is necessary to require that $\langle \delta \boldsymbol{x}^2 \rangle \ll \lambda_0^2$. To compute $\langle \delta \boldsymbol{x}^2 \rangle$, we use (39.13):

$$\langle \delta \boldsymbol{x}^2 \rangle = \frac{1}{4} \int_0^s \int_0^s (s - s')(s - s'') \langle \nabla \varepsilon_1 (\boldsymbol{l}_0 s') \nabla \varepsilon_1 (\boldsymbol{l}_0 s'') - (\boldsymbol{l}_0 \nabla \varepsilon_1')(\boldsymbol{l}_0 \nabla \varepsilon_1'') \rangle \, ds' ds''. \tag{15}$$

Inserting (8), we obtain

$$\langle \delta \boldsymbol{x}^2 \rangle = \frac{1}{4} \int_0^s ds' \int_0^s ds'' (s - s')(s - s'') \frac{D_\varepsilon' (s' - s'')}{s' - s''}. \tag{16}$$

Substitution of variables $s' - s'' = \rho$, $s - s' = \xi$ gives

$$\langle \delta \boldsymbol{x}^2 \rangle = \frac{1}{4} \int_0^s \xi \, d\xi \int_{-\xi}^{s-\xi} (\xi + \rho) \frac{D_\varepsilon' (\rho)}{\rho} \, d\rho. \tag{17}$$

In (17), as in (10), the main contribution to the integral is from the region of small ρ; we may thus replace $\xi + \rho$ by ξ and extend the integration over ρ from minus infinity to plus infinity. This gives

$$\langle \delta \boldsymbol{x}^2 \rangle = \frac{s^3}{6} \int_0^\infty \frac{D_\varepsilon' (\rho)}{\rho} \, d\rho, \tag{18}$$

where we made use of $D_\varepsilon' (\rho)/\rho$ being an even function. Expressing the integral in (18) in terms of $\sigma_\alpha^2 (L)$ and substituting L for s, we obtain

$$\langle \delta \boldsymbol{x}^2 \rangle = \frac{2}{3} L^2 \sigma_\alpha^2 (L). \tag{19}$$

The condition $\langle \delta \boldsymbol{x}^2 \rangle \ll \lambda_0^2$ thus takes the form

$$\frac{L^2 \sigma_\alpha^2}{\lambda_0^2} \ll 1,$$

and inserting the expression for σ_α^2 we again end up with condition (14). That $\langle \delta \boldsymbol{x}^2 \rangle \ll \lambda_0^2$ coincides with the condition $\langle \chi^2 \rangle \ll 1$ is easily understood by a simple argument. Indeed, amplitude fluctuations correspond to changes in the cross section of the ray tube, and whenever this change is considerable, both $\langle \delta \boldsymbol{x}^2 \rangle$ and $\langle \chi^2 \rangle$ are large.

The condition $\langle \chi^2 \rangle \ll 1$ arose in connection with the approximate solution of the eikonal equation. If this equation is solved exactly (e.g., for layered media), condition (14) is not necessary.

Let us now consider the conditions when the higher geometrical optics approximations can be dropped.* The wave field $\Psi(\boldsymbol{r})$ was sought in the form of a series

$$\Psi(\boldsymbol{r}) = \left[F_0 + \frac{1}{k} F_1 + \frac{1}{k^2} F_2 + \cdots\right] \exp(ik\theta),$$

and we determined only the two parameters θ and F_0. The equation for F_1 has the form (see (38.7))

$$2\nabla F_1 \nabla\theta + F_1 \Delta\theta = i\Delta F_0. \qquad (20)$$

We insert in (20) the above expressions for θ, $F_0 = A$, and

$$\nabla\theta = n\boldsymbol{l} = \left(1 + \frac{1}{2}\varepsilon_1\right)(\boldsymbol{l}_0 + \delta\boldsymbol{l} + \cdots) = \boldsymbol{l}_0 + \left(\frac{1}{2}\boldsymbol{l}_0\varepsilon_1 + \delta\boldsymbol{l}\right) + \cdots$$

For a plane wave

$$\Delta\theta = \operatorname{div}\left(\frac{1}{2}\boldsymbol{l}_0\varepsilon_1 + \delta\boldsymbol{l}\right) + \cdots$$

For $\chi = \ln(A/A_0)$ we have (40.46), where as has been established only the integral term need be retained:

$$\chi(x, y, z) = -\frac{1}{4}\int_0^x (x-\xi)\,\Delta_\perp \varepsilon_1(\xi, y, z)\,d\xi. \qquad (21)$$

Differentiating $A = A_0 \exp(\chi)$, we obtain

$$\Delta A = A_0 \exp(\chi)\,[\Delta\chi + (\nabla\chi)^2].$$

Since by assumption (14) is satisfied, we may take $|\chi| \ll 1$. In this case $\Delta A \approx A_0 \Delta\chi$. Differentiating (21), we get

$$\Delta\chi(x, y, z) = -\frac{1}{4}\int_0^x (x-\xi)\,\Delta_\perp^2 \varepsilon_1(\xi, y, z)\,d\xi - \frac{1}{4}\Delta_\perp \varepsilon_1(x, y, z).$$

In this expression we may also retain only the integral term. Then

$$\Delta A(x, y, z) \approx -\frac{A_0}{4}\int_0^x (x-\xi)\,\Delta_\perp^2 \varepsilon_1(\xi, y, z)\,d\xi. \qquad (22)$$

We now have all the expressions to be inserted in equation (20). Since the right-hand side of (20) is linear in ε_1 and $\nabla\theta$ includes the term $\boldsymbol{l}_0$, F_1 clearly also contains a term linear in ε_1. If we retain only this term in the expression for F_1, we may take $\nabla\theta = \boldsymbol{l}_0$ and $\Delta\theta = 0$, since all the other terms in equation (20) give a contribution to F_1 of higher order in ε_1. Equation (20) thus takes the form

$$2\frac{\partial F_1}{\partial x} = -\frac{iA_0}{4}\int_0^x (x-\xi)\,\Delta_\perp^2 \varepsilon_1(\xi, y, z)\,d\xi, \qquad (23)$$

where we made use of the fact that $\boldsymbol{l}_0 = \{1, 0, 0\}$.

* On this subject also see /94/.

Integrating (23) with the initial condition $F_1(0, y, z) = 0$, we obtain

$$F_1(x, y, z) = -\frac{iA_0}{8}\int_0^x dx'\int_0^{x'}(x'-\xi)\,\Delta_\perp^2\varepsilon_1(\xi, y, z)\,d\xi. \tag{24}$$

Changing the order of integration, and then integrating over x':

$$F_1(x, y, z) = -\frac{iA_0}{16}\int_0^x (x-\xi)^2\,\Delta_\perp^2\,\varepsilon_1(\xi, y, z)\,d\xi. \tag{25}$$

Let us now compute the function $Q(x, y, z) = \langle |F_1(x, y, z)|^2\rangle$:

$$Q(L, y, z) = \left(\frac{A_0}{16}\right)^2\int_0^L\int_0^L (L-\xi_1)^2(L-\xi_2)^2\langle\Delta_\perp^2\varepsilon_1(\xi_1, y, z)\Delta_\perp^2\varepsilon_1(\xi_2, y, z)\rangle d\xi_1 d\xi_2. \tag{26}$$

For homogeneous turbulence,

$$\langle\Delta_\perp^2\varepsilon_1(\xi_1, y_1, z_1)\,\Delta_\perp^2\varepsilon_1(\xi_2, y_2, z_2)\rangle = \Delta_\perp^4 B_\varepsilon(\xi_1-\xi_2, y_1-y_2, z_1-z_2) =$$
$$= -\frac{1}{2}\Delta_\perp^4 D_\varepsilon(\xi_1-\xi_2, y_1-y_2, z_1-z_2). \tag{27}$$

Insertion of (27) in (26) gives

$$Q(L, y, z) = -\frac{1}{2}\left(\frac{A_0}{16}\right)^2\int_0^L\int_0^L (L-\xi_1)^2(L-\xi_2)^2\,\Delta_\perp^4 D_\varepsilon(\xi_1-\xi_2, 0, 0)\,d\xi_1\,d\xi_2. \tag{28}$$

We change over to new variables $\xi_1 - \xi_2 = \xi$, $\xi_1 + \xi_2 = 2x$. Since the function $\Delta_\perp^4 D_\varepsilon(\xi, 0, 0)$ falls off rapidly, the integral over ξ can be taken from minus infinity to plus infinity. Then

$$Q(L, y, z) = -\frac{1}{2}\left(\frac{A_0}{16}\right)^2\int_{-\infty}^{\infty}\Delta_\perp^4 D_\varepsilon(\xi, 0, 0)\,d\xi\int_0^L\left(L-x-\frac{\xi}{2}\right)^2\left(L-x+\frac{\xi}{2}\right)^2 dx \tag{29}$$

Since the main contribution to the integral over ξ comes from the region where $|\xi| \lesssim \lambda_0$, in the inner integral ξ can be ignored in comparison to $L - x$. Integrating over x, we obtain

$$Q(L, y, z) = -\frac{L^5}{10}\left(\frac{A_0}{16}\right)^2\int_{-\infty}^{\infty}\Delta_\perp^4 D_\varepsilon(\xi, 0, 0)\,d\xi. \tag{30}$$

Q can be expressed in terms of the spectrum of the dielectric constant fluctuations. Differentiating

$$D_\varepsilon(\xi, y, z) = 2\iiint_{-\infty}^{\infty}\left[1 - e^{i(\varkappa_1\xi+\varkappa_2 y+\varkappa_3 z)}\right]\Phi_\varepsilon(\varkappa_1, \varkappa_2, \varkappa_3)\,d^3\varkappa$$

with respect to y and z, we obtain

$$\Delta_\perp^4 D_\varepsilon(\xi, y, z) = -2\iiint_{-\infty}^{\infty}(\varkappa_2^2+\varkappa_3^2)^4 e^{i(\varkappa_1\xi+\varkappa_2 y+\varkappa_3 z)}\Phi_\varepsilon(\varkappa_1, \varkappa_2, \varkappa_3)\,d^3\varkappa.$$

Taking $y = z = 0$ and integrating over ξ, we obtain

$$\int_{-\infty}^{\infty} \Delta_{\perp}^{4} D_{\varepsilon}(\xi, 0, 0)\, d\xi = -2 \iiint_{-\infty}^{\infty} \Phi_{\varepsilon}(\varkappa_1, \varkappa_2, \varkappa_3)(\varkappa_2^2 + \varkappa_3^2)^4\, d^3\varkappa \int_{-\infty}^{\infty} e^{i\varkappa_1\xi}\, d\xi =$$

$$= -4\pi \iiint_{-\infty}^{\infty} \Phi_{\varepsilon}(\varkappa_1, \varkappa_2, \varkappa_3)(\varkappa_2^2 + \varkappa_3^2)^4\, \delta(\varkappa_1)\, d^3\varkappa =$$

$$= -4\pi \iint_{-\infty}^{\infty} (\varkappa_2^2 + \varkappa_3^2)^4\, \Phi_{\varepsilon}(0, \varkappa_2, \varkappa_3)\, d\varkappa_2\, d\varkappa_3.$$

For locally isotropic turbulence, when

$$\Phi_{\varepsilon}(0, \varkappa_2, \varkappa_3) = \Phi_{\varepsilon}\left(\sqrt{\varkappa_2^2 + \varkappa_3^2}\right) = \Phi_{\varepsilon}(\varkappa),$$

we can change over to polar coordinates in the last integral and integrate over the angular variable:

$$\int_{-\infty}^{\infty} \Delta_{\perp}^{4} D_{\varepsilon}(\xi, 0, 0)\, d\xi = -8\pi^2 \int_{0}^{\infty} \varkappa^9 \Phi_{\varepsilon}(\varkappa)\, d\varkappa.$$

Hence, (30) takes the form

$$Q(L, y, z) = \frac{4\pi^2 L^5}{5}\left(\frac{A_0}{16}\right)^2 \int_{0}^{\infty} \varkappa^9 \Phi_{\varepsilon}(\varkappa)\, d\varkappa. \tag{31}$$

Inserting in (31) $\Phi_{\varepsilon}(\varkappa) = 0.033 C_{\varepsilon}^2 \varkappa^{-11/3} \exp(-\varkappa^2/\varkappa_m^2)$ and integrating, we obtain

$$\frac{1}{k^2} Q = \frac{L^5}{5}\left(\frac{A_0}{16}\right)^2 \frac{0.033\,\Gamma\left(\frac{19}{6}\right)}{2} C_{\varepsilon}^2 \varkappa_m^{19/3} \lambda^2 = 3.4 \cdot 10^{-5} A_0^2 L^5 C_{\varepsilon}^2 \varkappa_m^{19/3} \lambda^2 = 2.6 A_0^2 L^5 C_{\varepsilon}^2 \lambda_0^{-19/3} \lambda^2. \tag{32}$$

To find the limits of applicability of the first geometrical optics approximation we should compare

$$Qk^{-2} \text{ with } \langle (A - A_0)^2 \rangle \approx A_0^2 \langle \chi^2 \rangle$$

requiring that the ratio of the two quantities be small. Using the relation

$$\langle \chi^2 \rangle = 0.80 C_{\varepsilon}^2 L^3 \lambda_0^{-7/3},$$

we obtain the required condition in the form

$$\frac{2.6 A_0^2 L^5 C_{\varepsilon}^2 \lambda^2 \lambda_0^{-19/3}}{0.80 C_{\varepsilon}^2 A_0^2 L^3 \lambda_0^{-7/3}} = 3.2 \frac{L^2 \lambda^2}{\lambda_0^4} \ll 1. \tag{33}$$

Thus, in addition to the constraint $\langle \chi^2 \rangle \ll 1$, condition (33) must also be satisfied, according to which the radius of the first Fresnel zone is at most equal to the inner scale of turbulence λ_0 /98, 99/. Note that since F_1 is a purely imaginary quantity, condition (33) essentially guarantees that neglect of the higher approximations in the expansion

$$\Psi = \left(F_0 + \frac{1}{k} F_1 + \cdots\right) \exp(ik\theta)$$

introduces no phase distortion.

§ 44. Fluctuations of sound wave parameters

In concluding this section, we consider sound wave fluctuations in the geometrical optics approximation. In Chapter 2, Part B we derived equation (34.15) for the nondimensional acoustic pressure of the wave $\Pi \equiv \frac{p_a}{\rho_0 c_0^2}$:

$$\Delta\Pi + k^2\Pi = -\frac{\partial}{\partial x_i}\left(\frac{T'}{T_0}\frac{\partial \Pi}{\partial x_i}\right) - \frac{2}{ik}\frac{\partial^2}{\partial x_i \partial x_k}\left(\frac{u_i}{c_0}\frac{\partial \Pi}{\partial x_k}\right), \tag{1}$$

where T' is temperature fluctuation, $\boldsymbol{u}$ is the wind velocity, and ω has been replaced with kc_0. Dividing equation (1) by Π we use the following relations, which are readily verified by direct calculation:

$$\frac{\Delta\Pi}{\Pi} = \Delta \ln \Pi + (\nabla \ln \Pi)^2, \tag{2}$$

$$\frac{1}{\Pi}\frac{\partial}{\partial x_i}\left(\frac{T'}{T_0}\frac{\partial \Pi}{\partial x_i}\right) = \operatorname{div}\left[\frac{T'}{T_0}(\nabla \ln \Pi)\right] + \frac{T'}{T_0}(\nabla \ln \Pi)^2, \tag{3}$$

$$\frac{1}{\Pi}\frac{\partial^2}{\partial x_i \partial x_j}\left(\frac{u_i}{c_0}\frac{\partial \Pi}{\partial x_j}\right) = \frac{\partial^2}{\partial x_i \partial x_j}\left(\frac{u_i}{c_0}\frac{\partial \ln \Pi}{\partial x_j}\right) + \frac{u_i}{c_0}\frac{\partial}{\partial x_i}(\nabla \ln \Pi)^2 + \\ + \frac{\partial \ln \Pi}{\partial x_i}\left[\frac{\partial}{\partial x_j}\left(\frac{u_i}{c_0}\frac{\partial \ln \Pi}{\partial x_j}\right) + \frac{u_i}{c_0}(\nabla \ln \Pi)^2\right]. \tag{4}$$

In (2) through (4) we put $\ln \Pi = \chi + ik\theta$, where $\chi = \ln A$ and θ is the eikonal. Inserting these expressions in (1) and equating to zero the terms of this equation with the coefficient k^2, we obtain the equation

$$(\nabla\theta)^2\left[1 + \frac{T'}{T_0} + 2\frac{u_i}{c_0}\frac{\partial\theta}{\partial x_i}\right] = 1. \tag{5}$$

Since equation (1) is accurate only up to terms linear in T' and $\boldsymbol{u}$, equation (5) can be written with the same accuracy in the form

$$(\nabla\theta)^2 = 1 - \frac{T'}{T_0} - 2\frac{u_i}{c_0}\frac{\partial\theta}{\partial x_i}. \tag{6}$$

For arbitrary $T'(\boldsymbol{r})$ and $\boldsymbol{u}(\boldsymbol{r})$, equation (6) can be solved approximately by taking

$$\nabla\theta = n\boldsymbol{l} = (1 + n_1)(\boldsymbol{l}_0 + \delta\boldsymbol{l} + \cdots) = \boldsymbol{l}_0 + n_1\boldsymbol{l}_0 + \delta\boldsymbol{l} + \cdots,$$

where n_1 is the deviation of the refractive index from unity, $\boldsymbol{l}_0$ is the unit vector tangent to the ray, $|\boldsymbol{l}_0| = 1$, $\boldsymbol{l}_0\delta\boldsymbol{l} = 0$. Inserting this expansion in (6) and retaining only the terms linear in n_1, T', and $\boldsymbol{u}$, we obtain

$$n_1 = -\frac{T'}{2T_0} - \frac{\boldsymbol{u}\boldsymbol{l}_0}{c_0}. \tag{7}$$

The deviation of the index of refraction for sound from unity is thus composed of two terms. The first of these terms is associated with fluctuations in the local velocity of sound and the second with "wind drift." Using (7), we can write equation (6) in the usual form of the eikonal equation

$$(\nabla\theta)^2 = (1 + n_1)^2 = n^2. \tag{8}$$

It should be remembered, however, that on passing from (5) to (8) we repeatedly made use of the assumption that the parameter n_1 was small (in the case of electromagnetic waves the eikonal equation was obtained only using the expansion in powers of $1/k$).

Substituting (2) through (4) in (1) and collecting all the terms to the first power in k, we obtain an equation relating χ to θ. This equation contains $(\nabla\theta)^2$. If we use expression (6) for $(\nabla\theta)^2$ retaining only terms linear in T'/T_0 and $\boldsymbol{u}/c_0$, this equation takes the form

$$2\nabla\chi\left[\nabla\theta+\frac{T'}{T_0}\nabla\theta+\frac{\boldsymbol{u}}{c_0}+2\left(\frac{\boldsymbol{u}}{c_0}\nabla\theta\right)\nabla\theta\right]=$$
$$=-\Delta\theta-\operatorname{div}\left(\frac{T'}{T_0}\nabla\theta\right)-2\frac{\partial\theta}{\partial x_i}\frac{\partial}{\partial x_j}\left(\frac{u_i}{c_0}\frac{\partial\theta}{\partial x_j}\right). \tag{9}$$

The right-hand side of (9) is a small quantity, which is a linear function of T' and $\boldsymbol{u}$. Therefore, $\nabla\chi$ is clearly of the order of smallness of T' and u. Hence, the coefficient of $\nabla\chi$ needs to be taken only in the zero approximation, i.e., it equals $2\boldsymbol{l}_0$. We thus have

$$2\nabla\chi\boldsymbol{l}_0=2\frac{d\chi}{ds}. \tag{10}$$

In the right-hand side of (9) we may put $\nabla\theta=\boldsymbol{l}_0$ in the second and third term, as they already include the small factors T'/T_0 and $\boldsymbol{u}/c$. We thus obtain the equation

$$2\frac{d\chi}{ds}=-\Delta\theta-\operatorname{div}\left(\boldsymbol{l}_0\frac{T'}{T_0}\right)-2l_i^0\frac{\partial}{\partial x_j}\left(\frac{u_i}{c_0}l_j^0\right). \tag{11}$$

In the following, we consider for simplicity the case of a plane incident wave, when $\boldsymbol{l}_0=\text{const}$. In this case

$$\operatorname{div}\left(\boldsymbol{l}_0\frac{T'}{T_0}\right)=\boldsymbol{l}_0\nabla\left(\frac{T'}{T_0}\right)=\frac{d}{ds}\left(\frac{T'}{T_0}\right)$$

and

$$l_i^0\frac{\partial}{\partial x_j}\left(\frac{u_i}{c_0}l_j^0\right)=l_j^0\frac{\partial}{\partial x_j}\left(\frac{u_i l_i^0}{c_0}\right)=\frac{d}{ds}\left(\frac{u_s}{c_0}\right),$$

where $u_s=l_i^0u_i$ is the projection of the velocity vector on the ray. Then (11) takes the form

$$2\frac{d\chi}{ds}=-\Delta\theta-\frac{d}{ds}\left(\frac{T'}{T_0}\right)-2\frac{d}{ds}\left(\frac{u_s}{c_0}\right)=-\Delta\theta+2\frac{dn_1}{ds}. \tag{12}$$

Integration of (12) along the path of the ray gives

$$2\chi=2n_1-\int\Delta\theta\,ds. \tag{13}$$

As we have established before, the integral term in (13) is much greater than the local term n_1 and therefore only the former need be retained in the calculation of amplitude fluctuations. In this case the relation between the amplitude and the eikonal of the sound wave is expressed by the same formula as for the electromagnetic field.

All the previous calculations of amplitude, phase, and other fluctuations can now be extended to the case of sound waves, provided that the parameter ε_1 entering the electromagnetic wave relations is replaced by its acoustic analog

$$\varepsilon_1' = 2n_1 = -\frac{T'}{T_0} - \frac{2ul_0}{c_0}. \qquad (14)$$

The structure function of ε_1' can be expressed in terms of the structure functions $D_T(\boldsymbol{r})$ and $D_{ij}(\boldsymbol{r})$ on the T' and $\boldsymbol{u}$ fields. If T' and $\boldsymbol{u}$ are assumed to be uncorrelated (as is always true for locally isotropic turbulence), we easily find

$$D_{\varepsilon'}(\boldsymbol{r}) = \frac{D_T(\boldsymbol{r})}{T_0^2} + \frac{4l_i^0 l_j^0}{c_0^2} D_{ij}(\boldsymbol{r}). \qquad (15)$$

In contrast to the electromagnetic case, $D_{\varepsilon'}(\boldsymbol{r})$ is not an isotropic function; it depends on the relative orientation of the vectors $\boldsymbol{l}_0$ and $\boldsymbol{r}$.

In locally isotropic turbulence, we have for $D_{ij}(\boldsymbol{r})$

$$D_{ij}(\boldsymbol{r}) = D_{tt}(\boldsymbol{r})\,\delta_{ij} + (D_{rr} - D_{tt})\, n_i n_j, \qquad (16)$$

where $\boldsymbol{n} = \boldsymbol{r}/r$. Using (16), we get

$$l_i^0 l_j^0 D_{ij}(\boldsymbol{r}) = D_{tt} + (D_{rr} - D_{tt})(\boldsymbol{l}_0\boldsymbol{r})^2 = D_{tt}\sin^2\varphi + D_{rr}\cos^2\varphi, \qquad (17)$$

where φ is the angle between $\boldsymbol{l}_0$ and $\boldsymbol{n}$. Thus,

$$D_{\varepsilon'}(\boldsymbol{r}) = \frac{D_T(r)}{T_0^2} + \frac{4}{c_0^2}\left[D_{tt}(r)\sin^2\varphi + D_{rr}(r)\cos^2\varphi\right]. \qquad (18)$$

Consider the phase structure function of a plane wave. We derived for this function an expression in the form of a double integral over two parallel rays, where the integrand contained the structure function of ε whose argument is the vector $\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$ joining the two integration points on the two rays. We are dealing with a case where the path length L is much greater than the distance ρ between the two rays. In this case, the angle φ is small in most of the integration region. The condition $\sin^2\varphi \ll 1$ is not satisfied only in a small part of the integration region, and this small region only makes a small contribution to the integral. In calculating sound wave fluctuations, we may therefore take $\varphi = 0$ in (18), i.e., put

$$D_{\varepsilon'}(r) \approx \frac{D_T(r)}{T_0^2} + \frac{4D_{rr}(r)}{c_0^2}. \qquad (19)$$

The functions $D_T(r)$ and $D_{rr}(r)$ in general have somewhat different inner scales, but since the Prandtl number for air is close to unity, these scales can be taken as equal with fair accuracy. All the preceding expressions thus remain without change,* provided that C_ε is taken in the form

$$C_\varepsilon^2 = \frac{C_T^2}{T_0^2} + \frac{4C^2\varepsilon^{2/3}}{c_0^2}, \qquad (20)$$

* More detailed consideration of the anisotropy of $D_{\varepsilon'}(r)$ somewhat alters the numerical coefficients in the expressions for the mean square fluctuations.

where C is the numerical constant entering the "2/3 law" of the longitudinal velocity component, and C_T^2 is the structure characteristic entering the "2/3 law" of the temperature field.

B. THE METHOD OF SMOOTH PERTURBATIONS

The geometrical optics method considered in Part A is adequate for the treatment of short wavelengths. The limits of its applicability are set by the conditions $\lambda \ll \lambda_0$, $\lambda^2 L^2 \ll \lambda_0^4$, $\langle\chi^2\rangle \ll 1$, where λ is the wavelength, λ_0 is the inner scale of turbulence, L is the path length of the waves in the random medium, and $\langle\chi^2\rangle$ is the mean square fluctuation of the log amplitude of the wave. The first two constraints are associated with the approximate methoc of solution of the wave equation, and the third constraint is traceable to the approximate solution of the eikonal equation.

It is possible to improve on the geometrical optics method. In particular, the second restriction can be largely eliminated if diffraction effects are taken into account.

The most convenient method for the construction of improved solutions is the method of smooth perturbations advanced by S. M. Rytov /100/ and applied to the problem of wave propagation in a random medium by Obukhov /101/, Chernov /92, 102–104/, and other authors /105–112/.

This method, like the geometrical optics approximation, is adapted to the case of short waves ($\lambda \ll \lambda_0$). However, the constraint $\lambda^2 L^2 \ll \lambda_0^4$ limiting the applicability of the geometrical optics approximation, does not apply here. Moreover, the method of smooth perturbations is applicable even when the first and higher order geometrical optics approximations are not valid (see §45).

§45. Derivation of the fundamental equations of the method of smooth perturbations

As in the geometrical optics approximation, we start with the scalar equatior since for $\lambda \ll \lambda_0$ the polarization corrections are small. We consider a monochromatic wave with a time factor exp $(-i\omega t)$. The wave equation thus takes the form

$$\Delta \Psi + k^2 (1 + \varepsilon_1) \Psi = 0 \tag{1}$$

Dividing equation (1) by Ψ and noting that $\frac{\Delta \Psi}{\Psi} = \Delta \ln \Psi + (\nabla \ln \Psi)^2$, we obtain

$$\Delta \Phi + (\nabla \Phi)^2 + k^2 + k^2 \varepsilon_1 = 0, \tag{2}$$

where $\Phi = \ln \Psi$ is the complex phase. Equation (2) should be considered in conjunction with the radiation condition.

Equation (2), like the eikonal equation, is nonlinear. The main advantages of the geometrical optics method, which is responsible for its wide popularity, is that in layered media the nonlinear eikonal equation can be solved exactly (a similar situation exists in the quasi-classical approximation to quantum mechanics).

However, in our case, when $\varepsilon_1(\boldsymbol{r})$ is a random function from some quite arbitrary statistical ensemble, an exact solution of the eikonal equation cannot be obtained and the equation must be solved by successive approximations. Therefore a better procedure is to solve not the eikonal equation, which itself is an approximation, but rather equation (2), which is an exact consequence of (1).

We introduce the small parameter $\nu = \sqrt{\langle \varepsilon_1^2 \rangle}$ and take $\varepsilon_1(\boldsymbol{r}) = \nu\alpha(\boldsymbol{r})$. The function $\Phi(\boldsymbol{r})$ is sought as a series in powers of ν:

$$\Phi(\boldsymbol{r}) = \Phi_0(\boldsymbol{r}) + \nu\,\Phi^{(1)}(\boldsymbol{r}) + \nu^2\,\Phi^{(2)}(\boldsymbol{r}) + \ldots$$

Inserting this expansion in our equation

$$\Delta\Phi + (\nabla\Phi)^2 + k^2 + k^2\nu\alpha(\boldsymbol{r}) = 0$$

and equating to zero groups of terms corresponding to equal powers of ν, we obtain the set of equations

$$\Delta\Phi_0 + (\nabla\Phi_0)^2 + k^2 = 0, \tag{3}$$

$$\Delta\Phi^{(1)} + 2\nabla\Phi_0\nabla\Phi^{(1)} = -k^2\alpha(\boldsymbol{r}), \tag{4}$$

$$\Delta\Phi^{(2)} + 2\nabla\Phi_0\nabla\Phi^{(2)} = -(\nabla\Phi^{(1)})^2, \tag{5}$$

$$\Delta\Phi^{(3)} + 2\nabla\Phi_0\nabla\Phi^{(3)} = -2\nabla\Phi^{(1)}\nabla\Phi^{(2)}, \tag{6}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

The nonlinear equation (3) can be solved exactly, since it corresponds to wave propagation in a medium without fluctuations. All the remaining equations for $\Phi^{(i)}$ $(i \geqslant 1)$ have the same form

$$\Delta u + 2\nabla\Phi_0\nabla u = -f(\boldsymbol{r}), \tag{7}$$

where the right-hand side of each equation is known if all the preceding equations have been solved. Equation (7) is linear. The substitution

$$u = e^{-\Phi_0} w \tag{8}$$

reduces it to the equation

$$\Delta w + k^2 w = -e^{\Phi_0} f. \tag{9}$$

The solution of equation (9) in free space satisfying the radiation condition is given by the well-known expression

$$w(\boldsymbol{r}) = \frac{1}{4\pi}\int \frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|} e^{\Phi_0(\boldsymbol{r}')} f(\boldsymbol{r}')\, d^3r'. \tag{10}$$

Therefore, the solution of equation (7) is

$$u(\boldsymbol{r}) = \frac{1}{4\pi}\int \frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|} e^{\Phi_0(\boldsymbol{r}') - \Phi_0(\boldsymbol{r})} f(\boldsymbol{r}')\, d^3r'. \tag{11}$$

Instead of $\Phi_0(\boldsymbol{r})$ in (11) it is better to replace it by a function $\Psi_0(\boldsymbol{r}) = \exp(\Phi_0(\boldsymbol{r}))$, which is the solution of the equation

$$\Delta\Psi_0 + k^2\Psi_0 = 0$$

representing an unperturbed wave. Thus, (11) takes the form

$$u(\boldsymbol{r}) = \frac{1}{4\pi}\int \frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|}\frac{\Psi_0(\boldsymbol{r}')}{\Psi_0(\boldsymbol{r})} f(\boldsymbol{r}')\, d^3r'. \tag{12}$$

Using for $f(\boldsymbol{r})$ the right-hand sides of equations (4), (5), ..., we obtain $\Phi^{(1)}, \Phi^{(2)}, \ldots$. Writing $\Phi_1 = \nu\Phi^{(1)}$, $\Phi_2 = \nu^2\Phi^{(2)}$ and returning to the function $\varepsilon_1(\boldsymbol{r})$, we obtain

$$\Phi(\boldsymbol{r}) = \Phi_0(\boldsymbol{r}) + \Phi_1(\boldsymbol{r}) + \Phi_2(\boldsymbol{r}) + \ldots, \tag{13}$$

$$\Phi_1(\boldsymbol{r}) = \frac{k^2}{4\pi}\int \frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|}\frac{\Psi_0(\boldsymbol{r}')}{\Psi_0(\boldsymbol{r})}\varepsilon_1(\boldsymbol{r}')\, d^3r', \tag{14}$$

$$\Phi_2(\boldsymbol{r}) = \frac{1}{4\pi}\int \frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|}\frac{\Psi_0(\boldsymbol{r}')}{\Psi_0(\boldsymbol{r})}[\nabla\Phi_1(\boldsymbol{r}')]^2\, d^3r'. \tag{15}$$

Expansion (13) is a series in powers of the small quantity ε_1. However, as in geometrical optics, each term of this series is equivalent to the sum of an infinite subseries of the perturbation series for the equation

$$\Delta\Psi + k^2\Psi = -k^2\varepsilon_1\Psi.$$

Indeed, $\Psi(\boldsymbol{r})$ is expressed in terms of Φ by $\Psi(\boldsymbol{r}) = e^{\Phi} = \Psi_0 e^{\Phi_1} e^{\Phi_2}\ldots$ Expanding $\exp\Phi_1$ in a series we obtain

$$\Psi(\boldsymbol{r}) = \Psi_0(\boldsymbol{r})\left[1 + \Phi_1 + \frac{1}{2!}\Phi_1^2 + \ldots\right]e^{\Phi_2}\ldots \tag{16}$$

Inserting the expression for Φ_1, we readily see that even if $\Phi_2, \Phi_3, \ldots$ are ignored, expression (16) is still a series describing multiple scattering.

Let us consider in more detail the case of an incident plane wave $\Psi_0(\boldsymbol{r})$. Taking the x axis along the propagation vector of this wave, we obtain

$$\Psi_0(\boldsymbol{r}) = A_0 \exp(ikx). \tag{17}$$

Inserting (17) in (14) and (15), we find

$$\Phi_1(x, y, z) = \frac{k^2}{4\pi}\int \frac{\exp\{ik[\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}-(x-x')]\}}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} \times \varepsilon_1(x', y', z')\, dx'dy'dz', \tag{18}$$

$$\Phi_2(x, y, z) = \frac{1}{4\pi}\int \frac{\exp\{ik[\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}-(x-x')]\}}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} \times [\nabla\Phi_1(x', y', z')]^2\, dx'\,dy'\,dz'. \tag{19}$$

So far in deriving the expression for Φ we have not made use of the fact that short wavelengths are being considered. We will now use the condition $\lambda \ll \lambda_0$ to achieve further simplification of (18), (19). The general technique for tackling expressions of this type for $k \to \infty$ is the method of stationary phase (see, e.g., /94/). If, however, we wish to avoid the restriction $\sqrt{\lambda L} \ll \lambda_0$, this method is not applicable in its standard form. To make this point, consider in more detail the phase factor $\exp(iS)$, where

$$S = k[\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}-(x-x')]. \tag{20}$$

Let us find the equations of the surfaces on which $S = S_m = \pi m$. After simple manipulations, we obtain this equation in the form

$$\rho_m^2 = (y-y')^2 + (z-z')^2 = m\lambda(x-x') + \frac{m^2\lambda^2}{4}. \qquad (21)$$

Equation (21) defines a family of paraboloids of revolution with their apexes at the points $x'_m = x + m\lambda/4$. For the distance $\rho_{m+1} - \rho_m = \Delta\rho_m$ between two adjacent surfaces we obtain

$$\Delta\rho_m = \sqrt{\rho_m^2 + \lambda(x-x') + \frac{m\lambda^2}{2} + \frac{\lambda^2}{4}} - \rho_m. \qquad (22)$$

For $m \gg 1$ and $(x-x') \gg m\lambda$, expression (22) takes the form

$$\Delta\rho_m \approx \frac{1}{2}\sqrt{\frac{\lambda(x-x')}{m}}. \qquad (23)$$

If $\sqrt{\lambda(x-x')} \gg \lambda_0$, the phase factor $\exp(iS)$ varies with ρ much more slowly than the function $\varepsilon_1(x', y', z')$, which reverses its sign over distances of the order of a few λ_0.

Therefore in (18) near the x axis (i.e., for moderate values of m) the factor $\exp(iS)$ cannot be regarded as a rapidly oscillating function, so that the method of stationary phase is not applicable. This approach is legitimate only if $\sqrt{\lambda(x-x')} \ll \lambda_0$, i.e., when the first geometrical optics approximation is valid. Farther away from the x' axis, i.e., for large m, $\Delta\rho_m$ may become less than λ_0. The corresponding value of m can be found from the inequality

$$\frac{1}{2}\sqrt{\frac{\lambda(x-x')}{m}} \ll \lambda_0,$$

which gives

$$m \gg \frac{\lambda(x-x')}{4\lambda_0^2} > \frac{\lambda L}{4\lambda_0^2},$$

where L is the distance to the observation point traversed by the wave in the random medium. Inserting the last expression in (21), we obtain the corresponding value of ρ:

$$\rho \gg \frac{\lambda L}{\lambda_0}.$$

Note that the part of the space defined by the reverse inequality is the region from which the major part of the received field is scattered. Indeed, the maximum scattering angle is of the order of λ/λ_0, which corresponds to a transverse dimension of the order of $\lambda L/\lambda_0$.

In the region $x' < x$, for $\rho \gg \frac{\lambda L}{\lambda_0}$ the function $\exp(iS)$ oscillates rapidly compared to $\varepsilon_1(\mathbf{r}')$. In the region $x' > x$, the distance between the successive surfaces $S = \pi m$ and $S = \pi(m+1)$ is at most $\lambda/2$ (for $x' = x$, $\Delta\rho_m = \lambda/2$), i.e., for short wavelengths $\lambda \ll \lambda_0$, $\exp(iS)$ always varies more rapidly than $\varepsilon_1(\mathbf{r}')$.

We divide the integration region in (18) into two by the $x' = x$ plane and seek the contribution to Φ_1 from the subregion $x' > x$. Here, as we have already shown, $\exp(iS)$ oscillates rapidly compared to $\varepsilon_1(\mathbf{r}')$ and the integral can therefore be evaluated using the technique of stationary phase.

Applying this method to the integrals over y', z', we expand the exponential in a series near the stationary point $y' = y$, $z' = z$ and take the value of the slowly varying factor at that point,

$$\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}-(x-x') =$$
$$= |x-x'|\sqrt{1+\frac{(y-y')^2+(z-z')^2}{|x-x'|^2}}-(x-x') =$$
$$= |x-x'|-(x-x')+\frac{(y-y')^2+(z-z')^2}{2|x-x'|}+\ldots =$$
$$= 2(x'-x)-\frac{(y-y')^2+(z-z')^2}{2(x-x')}+\ldots$$

Let Φ_{11} be the contribution to $\Phi_1(x, y, z)$ from $x' > x$. Then

$$\Phi_{11}(x, y, z) \approx \frac{k^2}{4\pi}\int_x^\infty \frac{e^{-2ik(x-x')}}{(x'-x)}\varepsilon_1(x', y, z)\,dx' \times$$
$$\times \iint_{-\infty}^{\infty} \exp\left\{\frac{ik}{2(x'-x)}[(y-y')^2+(z-z')^2]\right\} dy'\,dz' =$$
$$= \frac{k^2}{4\pi}\int_x^\infty \frac{e^{2ik(x'-x)}}{(x'-x)}\varepsilon_1(x', y, z)\frac{2\pi(x'-x)}{ik}\,dx' = \frac{k}{2i}\int_0^\infty e^{2ikx'}\varepsilon_1(x+x', y, z)\,dx'. \tag{24}$$

Since Φ_{11} is a random variable, we can estimate its mean square fluctuations:

$$\langle(\Phi_{11}(x, y, z))^2\rangle = -\frac{k^2}{4}\iint_{00}^{\infty\infty} e^{2ik(x'+x'')}B_\varepsilon(x'-x'')\,dx'\,dx''. \tag{25}$$

Changing to new variables $\xi = x' - x''$ and $2\eta = x' + x''$, we obtain

$$\langle\Phi_{11}^2(x, y, z)\rangle = -\frac{k^2}{2}\int_0^\infty B_\varepsilon(\xi)\,d\xi\int_{\xi/2}^\infty e^{4ik\eta}\,d\eta = -\frac{ik}{8}\int_0^\infty e^{2ik\xi}B_\varepsilon(\xi)\,d\xi. \tag{26}$$

The integral in (26) can be approximately evaluated if the slowly varying function $B_\varepsilon(\xi)$ is taken outside the integrand. Assuming an absorbing medium ($\mathrm{Im}\,k > 0$), we obtain

$$\langle\Phi_{11}^2(x, y, z)\rangle \approx \frac{B_\varepsilon(0)}{16} = \frac{\sigma_\varepsilon^2}{16}\,. \tag{27}$$

Thus the error resulting when the integral over the region $x' > x$, which lies beyond the observation point, is omitted from (18) is of the order $\frac{1}{4}\sigma_\varepsilon \ll 1$ for $k\lambda_0 \gg 1$. As we shall see from the following, the integrals over the region $x' < x$ invariably give contributions proportional to $L^\alpha k^\beta$, where $\alpha, \beta > 0$; in this case we can always neglect σ_ε compared to these "integral" terms which increase with increasing L.

Let us now consider the part of the integral in (18) taken over the region $x' < x$, $\rho \gg \frac{\lambda(x-x')}{\lambda_0}$. This integral is clearly not greater in its order of magnitude than Φ_{11}. Therefore, the integral in (18) need be taken only over the region $x' < x$, $\rho < \frac{\lambda(x-x')}{\lambda_0}$, which introduces an error of the order of σ_ε (similar estimates clearly apply to the integral in (19) too). For $\rho < \frac{\lambda(x-x')}{\lambda_0}$ the integrand can be markedly simplified if we use the inequality $\lambda \ll \lambda_0$. In this case $\rho \ll (x-x')$ and we may use the expansion

$$k\left[\sqrt{(x-x')^2+\rho^2}-(x-x')\right] = k\left[|x-x'|-(x-x')+\right.$$
$$\left.+\frac{\rho^2}{2|x-x'|}+\ldots\right] = \frac{k\rho^2}{2(x-x')}+O\left(\frac{k\rho^4}{(x-x')^3}\right). \tag{28}$$

The main contribution to the integral in (18) comes from the region where $\rho < \frac{\lambda(x-x')}{\lambda_0}$. Suppose that in this region it is only necessary to retain the first term in (28), i.e., suppose that

$$\frac{k\rho^4}{(x-x')^3} \ll 2\pi,$$

or, since $\rho < \frac{\lambda(x-x')}{\lambda_0}$, we have the condition

$$\frac{\lambda^3(x-x')}{\lambda_0^4} < \frac{\lambda^3 L}{\lambda_0^4} \ll 1. \tag{29}$$

Condition (29) is far from being as rigid a constraint on L as the condition required to apply the first geometrical optics approximation $\frac{\lambda L}{\lambda_0^2} \ll 1$, since (29) can be written in the form

$$\frac{\lambda L}{\lambda_0^2}\,\frac{\lambda^2}{\lambda_0^2} \ll 1 \quad \text{or} \quad \frac{\lambda L}{\lambda_0^2} \ll \frac{\lambda_0^2}{\lambda^2}.$$

When condition (29) is met, we may use the first term of expansion (28) not only in the region $\rho < \lambda_0^{-1}\lambda(x-x')$ but for any value of ρ, since the subregion where constraint (29) is broken gives only a small contribution to the integral in (18) (it is of the order of σ_ε). Actually the integral over the region $\rho > \lambda_0^{-1}\lambda(x-x')$ using the approximate kernel (28) does not coincide with the integral taken over the same region with the exact kernel. However, the two kernels are both rapidly oscillating functions in this region, and estimate (27) is applicable to both.

Thus, when condition (29) is satisfied and σ_ε is small compared to the integral over $x' < x$ (the second estimate will be improved in what follows), expressions (18) and (19) can be approximately written in the form

$$\Phi_1(L,y,z) = \frac{k^2}{4\pi}\int_0^L dx' \iint_{-\infty}^{\infty} dy'\,dz' \frac{\exp\left\{ik\frac{(y-y')^2+(z-z')^2}{2(L-x')}\right\}}{L-x'}\varepsilon_1(x',y',z'), \tag{30}$$

$$\Phi_2(L,y,z) = \frac{1}{4\pi}\int_0^L dx' \iint_{-\infty}^{\infty} dy'\,dz' \frac{\exp\left\{ik\frac{(y-y')^2+(z-z')^2}{2(L-x')}\right\}}{L-x'}[\nabla\Phi_1(x',y',z')]^2, \tag{31}$$

where the origin is placed at the boundary of the random medium. Expression (30) is the exact solution of the differential equation

$$\frac{\partial^2\Phi_1}{\partial y^2}+\frac{\partial^2\Phi_1}{\partial z^2}+2ik\frac{\partial\Phi_1}{\partial x} = -k^2\varepsilon_1(\boldsymbol{r}), \tag{32}$$

which is obtained from equation (4) by substituting $\nabla\Phi_0 = i\boldsymbol{k} = (ik, 0, 0)$, and neglecting the term $\frac{\partial^2\Phi_1}{\partial x^2}$. This can be verified by differentiating (30) and substituting the derivatives in the left-hand side of (32). When (30) is differentiated with respect to the upper limit, the following fairly obvious equality is used:

$$\lim_{x'\to L} \frac{k}{2\pi i (L-x')} \exp\left\{\frac{ik\,[(y-y')^2+(z-z')^2]}{2(L-x')}\right\} = \delta(y-y')\,\delta(z-z'),$$

this being a generalization of the well known equality in the theory of heat conduction (diffusion) to the case of an imaginary diffusion coefficient. Equation (32) itself is analogous to the equation of diffusion in the y, z plane perpendicular to the propagation direction; the coordinate x here is the analog of time and the coefficient of diffusion is $D = \frac{i}{2k}$. Expression (30) is thus the distribution of the "concentration" of the diffusing substance set up by a "nonstationary" source $\varepsilon_1(x', y', z')$.

In concluding this section, let us consider the relationship between the method of smooth perturbations and geometrical optics. Suppose that $\sqrt{\lambda L} \lesssim \lambda_0$, so that in (30) $\varepsilon_1(\boldsymbol{r}')$ may be regarded as a smoothly varying function (we do not impose the more rigid constraint $\sqrt{\lambda L} \ll \lambda_0$). Then the integral

$$\Phi_1(x, y, z) = \frac{k^2}{4\pi}\int_0^x \frac{dx'}{x-x'} \iint_{-\infty}^{\infty} \exp\left\{-\frac{k(\eta^2+\zeta^2)}{2i(x-x')}\right\} \varepsilon_1(x', y+\eta, z+\zeta)\, d\eta\, d\zeta \qquad (33)$$

can be treated by the method of stationary phase, which in this case amounts to expansion of the function $\varepsilon_1(x', y+\eta, z+\zeta)$ in powers of η, ζ. Since the exponential is an even function of η, ζ, only terms which are even with respect to η, ζ need be retained in this expansion:

$$\begin{aligned}\Phi_1(x, y, z) = \frac{k^2}{4\pi}\int_0^x \frac{dx'}{x-x'} \iint_{-\infty}^{\infty} \exp\left\{-\frac{k(\eta^2+\zeta^2)}{2i(x-x')}\right\}\Big[\varepsilon_1(x', y, z) + \\ + \frac{1}{2}\frac{\partial^2\varepsilon_1(x', y, z)}{\partial y^2}\eta^2 + \frac{1}{2}\frac{\partial^2\varepsilon_1(x', y, z)}{\partial z^2}\zeta^2 + \frac{1}{24}\frac{\partial^4\varepsilon_1(x', y, z)}{\partial y^4}\eta^4 + \\ + \frac{1}{4}\frac{\partial^4\varepsilon_1(x', y, z,)}{\partial y^2\partial z^2}\eta^2\zeta^2 + \frac{1}{24}\frac{\partial^4\varepsilon_1(x', y, z)}{\partial z^4}\zeta^4 + \ldots\Big]\, d\eta\, d\zeta.\end{aligned} \qquad (34)$$

Integration over η, ζ gives after simple manipulations

$$\begin{aligned}\Phi_1(x, y, z) = \frac{k}{4\pi}\int_0^x \Big\{2\pi i\varepsilon_1(x', y, z) - \frac{\pi}{k}(x-x')\,\Delta_\perp\varepsilon_1(x', y, z) + \\ + \frac{\pi i^3(x-x')^2}{4k^2}\left[\frac{\partial^4\varepsilon_1}{\partial y^4} + 2\frac{\partial^4\varepsilon_1}{\partial y^2\partial z^2} + \frac{\partial^4\varepsilon_1}{\partial z^4}\right] + \ldots\Big\}\, dx' = \\ = \frac{ik}{2}\int_0^x \varepsilon_1(\xi, y, z)\, d\xi - \frac{1}{4}\int_0^x (x-\xi)\,\Delta_\perp\varepsilon_1(\xi, y, z)\, d\xi - \\ - \frac{i}{16k}\int_0^x (x-\xi)^2\,\Delta_\perp^2\varepsilon_1(\xi, y, z)\, d\xi + \ldots\end{aligned} \qquad (35)$$

Let us compare expression (35), which is a power series in $1/k$, with the solution obtained by geometrical optics (in the first approximation to

ε_1). In Part A we derived expression (40.2) for the eikonal θ, which gave for the phase correction due to the inhomogeneities

$$\frac{k}{2}\int_0^x \varepsilon_1(\xi, y, z)\, d\xi.$$

This expression coincides with the first term in (35). The expression for the log amplitude of a plane wave derived using geometrical optics has the form (40.46):

$$\chi(x, y, z) = -\frac{1}{4}\int_0^x (x-\xi)\,\Delta_\perp \varepsilon_1(\xi, y, z)\, d\xi$$

(here the first term of (40.46) is omitted, since it is shown in §40 to be negligibly small compared to the integral term).

This expression coincides with the second term in (35). Finally, in considering the limits of applicability of the first geometrical optics approximation, we obtained a correction of second order in $1/k$ which is given by (43.25):

$$\frac{F_1(x, y, z)}{kA_0} = -\frac{i}{16k}\int_0^x (x-\xi)^2\,\Delta_\perp^2 \varepsilon_1(\xi, y, z)\, d\xi.$$

This expression coincides with the third term in (35).

Expansion of the integral in (30) in powers of k^{-1} using the method of stationary phase thus gives the corresponding expansion derived using geometrical optics, including its higher approximations. As we have seen, expansion (35) can be obtained if $\sqrt{\lambda L} \ll \lambda_0$, when the stationary phase method is applicable to the integral in (30). If, however, this condition is not satisfied, the expansion is meaningless, as is clear from the structure of (30). Thus, when the condition $\sqrt{\lambda L} \ll \lambda_0$ is broken, the entire geometrical optics method falls through, since together with the first approximation in k^{-1}, all the higher terms of the expansion in powers of k^{-1} become meaningless. On the other hand expression (30) for $\Phi_1(\boldsymbol{r})$ remains valid as long as the weaker constraint (29) is not broken.

The method of smooth perturbations thus has a much wider region of application than the geometrical optics method, even if the latter is improved by introduction of a finite number of higher order terms. In a certain sense, the method of smooth perturbations corresponds to a summation over an infinite number of higher approximations given by geometrical optics.

§ 46. Phase and amplitude fluctuations of a plane wave

We will now apply the method of smooth perturbations to calculate phase and amplitude fluctuations of a plane wave propagating in a turbulent atmosphere. It is more convenient to start with equation (45.32):

$$\frac{\partial^2\Phi_1}{\partial y^2} + \frac{\partial^2\Phi_1}{\partial z^2} + 2ik\frac{\partial\Phi_1}{\partial x} = -k^2\varepsilon_1(x, y, z). \qquad (1)$$

We solve (1) by the method of spectral expansions: $\varepsilon_1(x, y, z)$ and $\Phi_1(x, y, z)$ are represented as two-dimensional stochastic Fourier—Stieltjes integrals (5.19):

$$\varepsilon_1(x, y, z) = \varepsilon_1(x, 0, 0) + \iint\limits_{-\infty}^{\infty} [e^{i(\varkappa_2 y + \varkappa_3 z)} - 1]\, u_\varepsilon(d\varkappa_2, d\varkappa_3, x), \tag{2}$$

$$\Phi_1(x, y, z) = \Phi_1(x, 0, 0) + \iint\limits_{-\infty}^{\infty} [e^{i(\varkappa_2 y + \varkappa_3 z)} - 1]\, u_\Phi(d\varkappa_2, d\varkappa_3, x). \tag{3}$$

Inserting expansions (2), (3) in equation (1), we obtain

$$\begin{gathered} -\iint\limits_{-\infty}^{\infty} e^{i(\varkappa_2 y + \varkappa_3 z)} (\varkappa_2^2 + \varkappa_3^2)\, u_\Phi(d\varkappa_2, d\varkappa_3, x) + 2ik \frac{d\Phi_1(x, 0, 0)}{dx} + \\ + 2ik \iint\limits_{-\infty}^{\infty} [e^{i(\varkappa_2 y + \varkappa_3 z)} - 1] \frac{\partial u_\Phi(d\varkappa_2, d\varkappa_3, x)}{\partial x} = -k^2 \varepsilon_1(x, 0, 0) - \\ - k^2 \iint\limits_{-\infty}^{\infty} [e^{i(\varkappa_2 y + \varkappa_3 z)} - 1]\, u_\varepsilon(d\varkappa_2, d\varkappa_3, x). \end{gathered} \tag{4}$$

Putting $y = z = 0$ in (4), we find

$$-\iint\limits_{-\infty}^{\infty} (\varkappa_2^2 + \varkappa_3^2)\, u_\Phi(d\varkappa_2, d\varkappa_3, x) + 2ik \frac{d\Phi_1(x, 0, 0)}{dx} = -k^2 \varepsilon_1(x, 0, 0). \tag{5}$$

Subtraction of (5) from (4) gives

$$\begin{gathered} -\iint\limits_{-\infty}^{\infty} [e^{i(\varkappa_2 y + \varkappa_3 z)} - 1]\, (\varkappa_2^2 + \varkappa_3^2)\, u_\Phi(d\varkappa_2, d\varkappa_3, x) + \\ + 2ik \iint\limits_{-\infty}^{\infty} [e^{i(\varkappa_2 y + \varkappa_3 z)} - 1] \frac{\partial u_\Phi(d\varkappa_2, d\varkappa_3, x)}{\partial x} = \\ = -k^2 \iint\limits_{-\infty}^{\infty} [e^{i(\varkappa_2 y + \varkappa_3 z)} - 1]\, u_\varepsilon(d\varkappa_2, d\varkappa_3, x). \end{gathered} \tag{6}$$

The function $u_\Phi(d\varkappa_2, d\varkappa_3, x)$ thus satisfies the equation

$$2ik \frac{\partial u_\Phi}{\partial x} - (\varkappa_2^2 + \varkappa_3^2)\, u_\Phi = -k^2 u_\varepsilon(d\varkappa_2, d\varkappa_3, x). \tag{7}$$

In the following we will write $\varkappa_2^2 + \varkappa_3^2 = \varkappa^2$. The solution of equation (7) with the boundary condition $u_\Phi(d\varkappa_2, d\varkappa_3, 0) = 0$ (this boundary condition follows from expression (45.30) for Φ_1) has the form

$$u_\Phi(d\varkappa_2, d\varkappa_3, x) = \frac{ik}{2} \int\limits_0^x \exp\left\{-\frac{i\varkappa^2(x - x')}{2k}\right\} u_\varepsilon(d\varkappa_2, d\varkappa_3, x')\, dx'. \tag{8}$$

Expression (8) is a two-dimensional Fourier transform of (45.30).

We can now proceed to calculate the structure functions of phase and amplitude. Let $\Phi = \ln A + iS$, where A is the amplitude and S is the phase of the wave. Then $\Phi_0 = \ln A_0 + iS_0$, where A_0 and S_0 are the amplitude and the phase of the incident wave,

$$\Phi_1 = \Phi - \Phi_0 = \ln \frac{A}{A_0} + i(S - S_0),$$

i.e.,

$$\Phi_1 = \chi + iS_1. \tag{9}$$

where $\chi = \ln(A/A_0)$, $S_1 = S - S_0$. We are interested in the structure (correlation) functions of amplitude and phase and in their cross-correlation. The three functions can be obtained from the following two functions /113/:

$$\langle(\Phi_1(\boldsymbol{r}_1) - \Phi_1(\boldsymbol{r}_2))(\Phi_1^*(\boldsymbol{r}_1) - \Phi_1^*(\boldsymbol{r}_2))\rangle = D_1(\boldsymbol{r}_1, \boldsymbol{r}_2), \tag{10}$$

$$\langle[\Phi_1(\boldsymbol{r}_1) - \Phi_1(\boldsymbol{r}_2)]^2\rangle = D_2(\boldsymbol{r}_1, \boldsymbol{r}_2). \tag{11}$$

Indeed, putting $\Phi_1 = \chi + iS_1$, we obtain

$$D_1(\boldsymbol{r}_1, \boldsymbol{r}_2) = D_\chi(\boldsymbol{r}_1, \boldsymbol{r}_2) + D_S(\boldsymbol{r}_1, \boldsymbol{r}_2), \tag{12}$$

$$D_2(\boldsymbol{r}_1, \boldsymbol{r}_2) = D_\chi(\boldsymbol{r}_1, \boldsymbol{r}_2) - D_S(\boldsymbol{r}_1, \boldsymbol{r}_2) + 2iD_{\chi S}(\boldsymbol{r}_1, \boldsymbol{r}_2), \tag{13}$$

where

$$D_\chi(\boldsymbol{r}_1, \boldsymbol{r}_2) = \langle[\chi(\boldsymbol{r}_1) - \chi(\boldsymbol{r}_2)]^2\rangle,$$

$$D_S(\boldsymbol{r}_1, \boldsymbol{r}_2) = \langle[S_1(\boldsymbol{r}_1) - S_1(\boldsymbol{r}_2)]^2\rangle,$$

$$D_{\chi S}(\boldsymbol{r}_1, \boldsymbol{r}_2) = \langle[\chi(\boldsymbol{r}_1) - \chi(\boldsymbol{r}_2)][S_1(\boldsymbol{r}_1) - S_1(\boldsymbol{r}_2)]\rangle.$$

Solving equations (12), (13), we obtain

$$D_\chi = \frac{1}{2}[D_1 + \operatorname{Re} D_2], \tag{14}$$

$$D_S = \frac{1}{2}[D_1 - \operatorname{Re} D_2], \tag{15}$$

$$D_{\chi S} = \frac{1}{2}\operatorname{Im} D_2. \tag{16}$$

The structure functions D_1 and D_2 can be expressed in terms of the spectral density u_Φ. Consider the function $D_1(\boldsymbol{r}_1, \boldsymbol{r}_2)$ for $x_1 = x_2$. In this case, inserting (3) in (10), we obtain

$$D_1(x, y, z; x, y', z') = \iiiint_{-\infty}^{\infty} [e^{i(\varkappa_2 y + \varkappa_3 z)} - e^{i(\varkappa_2 y' + \varkappa_3 z')}] \times$$

$$\times [e^{-i(\varkappa_2' y + \varkappa_3' z)} - e^{-i(\varkappa_2' y' + \varkappa_3' z')}] \langle u_\Phi(d\varkappa_2, d\varkappa_3, x) u_\Phi^*(d\varkappa_2', d\varkappa_3', x)\rangle, \tag{17}$$

$$D_2(x, y, z; x, y', z') = \iiiint_{-\infty}^{\infty} [e^{i(\varkappa_2 y + \varkappa_3 z)} - e^{i(\varkappa_2 y' + \varkappa_3 z')}] \times$$

$$\times [e^{i(\varkappa_2' y + \varkappa_3' z)} - e^{i(\varkappa_2' y' + \varkappa_3' z')}] \langle u_\Phi(d\varkappa_2, d\varkappa_3, x) u_\Phi(d\varkappa_2', d\varkappa_3', x)\rangle. \tag{18}$$

We use (8) to calculate the average values:

$$\langle u_\Phi(d\varkappa_2, d\varkappa_3, x) u_\Phi^*(d\varkappa_2', d\varkappa_3', x)\rangle =$$

$$= \frac{k^2}{4} \int_0^x \int_0^x \exp\left\{-\frac{i}{2k}[\varkappa^2(x - x') - \varkappa'^2(x - x'')]\right\} \times$$

$$\times \langle u_\varepsilon(d\varkappa_2, d\varkappa_3, x') u_\varepsilon^*(d\varkappa_2', d\varkappa_3', x'')\rangle\, dx'\, dx''. \tag{19}$$

Using the relation

$$\langle u_\varepsilon(d\varkappa_2, d\varkappa_3, x')\, u^*_\varepsilon(d\varkappa'_2, d\varkappa'_3, x'')\rangle =$$
$$= \delta(\varkappa_2-\varkappa'_2)\,\delta(\varkappa_3-\varkappa'_3)\,F_\varepsilon(\varkappa_2, \varkappa_3, x'-x'')\,d\varkappa_2\,d\varkappa_3\,d\varkappa'_2\,d\varkappa'_3, \tag{20}$$

where F_ε is the two-dimensional spectral density of the dielectric constant fluctuations, and inserting (20) in (19), we get

$$\langle u_\Phi(d\varkappa_2, d\varkappa_3, x)\, u^*_\Phi(d\varkappa'_2, d\varkappa'_3, x)\rangle = \delta(\varkappa_2-\varkappa'_2)\,\delta(\varkappa_3-\varkappa'_3)\,d\varkappa_2 d\varkappa_3 d\varkappa'_2 d\varkappa'_3 \times$$
$$\times \frac{k^2}{4}\int_0^x\!\!\int_0^x \exp\left\{\frac{i\varkappa^2(x'-x'')}{2k}\right\} F_\varepsilon(\varkappa_2, \varkappa_3, x'-x'')\,dx'dx''. \tag{21}$$

Let $F_1(\varkappa_2, \varkappa_3, x)$ denote

$$F_1(\varkappa_2, \varkappa_3, x) = \frac{k^2}{4}\int_0^x\!\!\int_0^x \exp\left\{\frac{i\varkappa^2(x'-x'')}{2k}\right\} F_\varepsilon(\varkappa_2, \varkappa_3, x'-x'')\,dx'dx''. \tag{22}$$

Insertion of (21) in (17) and integration over $\varkappa'$ give

$$D_1(x, y, z; x, y', z') = \iint_{-\infty}^{\infty} [e^{i(\varkappa_2 y+\varkappa_3 z)} - e^{i(\varkappa_2 y'+\varkappa_3 z')}] \times$$
$$\times [e^{-i(\varkappa_2 y+\varkappa_3 z)} - e^{-i(\varkappa_2 y'+\varkappa_3 z')}]\,F_1(\varkappa_2, \varkappa_3, x)\,d\varkappa_2\,d\varkappa_3 =$$
$$= 2\iint_{-\infty}^{\infty} \{1-\cos[\varkappa_2(y-y')+\varkappa_3(z-z')]\}\,F_1(\varkappa_2, \varkappa_3, x)\,d\varkappa_2 d\varkappa_3. \tag{23}$$

Let us now compute the function

$$\langle u_\Phi(d\varkappa_2, d\varkappa_3, x)\, u_\Phi(d\varkappa'_2, d\varkappa'_3, x)\rangle =$$
$$= -\frac{k^2}{4}\int_0^x\!\!\int_0^x \exp\left\{-\frac{i}{2k}[\varkappa^2(x-x')+\varkappa'^2(x-x'')]\right\} \times$$
$$\times \langle u_\varepsilon(d\varkappa_2, d\varkappa_3, x')\, u_\varepsilon(d\varkappa'_2, d\varkappa'_3, x'')\rangle\,dx'\,dx''. \tag{24}$$

Since ε_1 is real, putting $\varepsilon_1^* = \varepsilon_1$ in expansion (2), we obtain

$$\iint_{-\infty}^{\infty} [e^{i(\varkappa_2 y+\varkappa_3 z)} - 1]\,u_\varepsilon(d\varkappa_2,\ d\varkappa_3, x) = \iint_{-\infty}^{\infty} [e^{-i(\varkappa_2 y+\varkappa_3 z)} - 1]\,u^*_\varepsilon(d\varkappa_2, d\varkappa_3, x).$$

Substitution of the variables $\varkappa_2 \to -\varkappa_2$, $\varkappa_3 \to -\varkappa_3$ in the integral on the right-hand side gives

$$\iint_{-\infty}^{\infty} [e^{i(\varkappa_2 y+\varkappa_3 z)} - 1]\,u_\varepsilon(d\varkappa_2, d\varkappa_3, x) = \iint_{-\infty}^{\infty} [e^{i(\varkappa_2 y+\varkappa_3 z)} - 1]\,u^*_\varepsilon(-d\varkappa_2, -d\varkappa_3, x),$$

from which follows the relation

$$u_\varepsilon(d\varkappa_2, d\varkappa_3, x) = u^*_\varepsilon(-d\varkappa_2, -d\varkappa_3, x)$$

or, taking $d\varkappa_2 = -d\varkappa''_2$, $d\varkappa_3 = -d\varkappa''_3$,

$$u^*_\varepsilon(d\varkappa''_2, d\varkappa''_3, x) = u_\varepsilon(-d\varkappa''_2, -d\varkappa''_3, x).$$

Inserting this expression in (20) we obtain

$$\langle u_\varepsilon(d\varkappa_2, d\varkappa_3, x')\, u_\varepsilon(-d\varkappa_2'', -d\varkappa_3'', x'')\rangle =$$
$$= \delta(\varkappa_2 - \varkappa_2'')\,\delta(\varkappa_3 - \varkappa_3'')\, F_\varepsilon(\varkappa_2, \varkappa_3, x' - x'')\, d\varkappa_2\, d\varkappa_3\, d\varkappa_2''\, d\varkappa_3''.$$

Putting $\varkappa_2'' = -\varkappa_2'$, $\varkappa_3'' = -\varkappa_3'$, we finally get

$$\langle u_\varepsilon(d\varkappa_2, d\varkappa_3, x')\, u_\varepsilon(d\varkappa_2', d\varkappa_3', x'')\rangle =$$
$$= \delta(\varkappa_2 + \varkappa_2')\,\delta(\varkappa_3 + \varkappa_3')\, F_\varepsilon(\varkappa_2, \varkappa_3, x' - x'')\, d\varkappa_2\, d\varkappa_3\, d\varkappa_2'\, d\varkappa_3'. \tag{20a}$$

Inserting (20a) in (24) we obtain

$$\langle u_\Phi(d\varkappa_2, d\varkappa_3, x)\, u_\Phi(d\varkappa_2', d\varkappa_3', x)\rangle =$$
$$= \delta(\varkappa_2 + \varkappa_2')\,\delta(\varkappa_3 + \varkappa_3')\, d\varkappa_2\, d\varkappa_2'\, d\varkappa_3\, d\varkappa_3'\, F_2(\varkappa_2, \varkappa_3, x), \tag{25}$$

where

$$F_2(\varkappa_2, \varkappa_3, x) = -\frac{k^2}{4}\int_0^x\int_0^x \exp\left\{-\frac{i\varkappa^2}{2k}(2x - x' - x'')\right\} F_\varepsilon(\varkappa_2, \varkappa_3, x' - x'')\, dx'\, dx''. \tag{26}$$

Inserting (25) in (18) and integrating over $\varkappa_2'$, $\varkappa_3'$, we find

$$D_2(x, y, z; x, y', z') = 2\iint_{-\infty}^{\infty} \{1 - \cos[\varkappa_2(y - y') + \varkappa_3(z - z')]\} \times$$
$$\times F_2(\varkappa_2, \varkappa_3, x)\, d\varkappa_2\, d\varkappa_3. \tag{27}$$

The functions F_1 and F_2 defined by (22), (26) thus give the spectral expansions of D_1, D_2.

Let us compute these functions. We will have to make use of the fact, mentioned in Chapter 1, that the function $F_\varepsilon(\varkappa_2, \varkappa_3, \xi)$ is markedly different from zero only for $\varkappa\xi \lesssim 1$. Therefore the relevant region of integration in (22) is only for $|\varkappa(x' - x'')| \lesssim 1$. In this region, however,

$$\left|\frac{\varkappa^2(x' - x'')}{2k}\right| \lesssim \frac{\varkappa}{2k}.$$

The maximum values of $\varkappa$ in which we are interested are of the order $\varkappa_m \sim \frac{1}{\lambda_0}$. Therefore

$$\left|\frac{\varkappa^2(x' - x'')}{2k}\right| \lesssim \frac{\varkappa_m}{2k} \sim \frac{\lambda}{2\lambda_0} \ll 1$$

since the wavelength is small compared to the inner scale of turbulence. Thus, in the main region of integration the argument of the exponential function is small and we may take

$$\exp\left\{\frac{i\varkappa^2(x' - x'')}{2k}\right\} \approx 1.$$

Consequently,

$$F_1(\varkappa_2, \varkappa_3, x) \approx \frac{k^2}{4}\int_0^x\int_0^x F_\varepsilon(\varkappa_2, \varkappa_3, x' - x'')\, dx'\, dx''. \tag{28}$$

$F_\varepsilon(\varkappa_2, \varkappa_3, x' - x'')$ is an even function of $(x' - x'')$. Using the relation

$$\int_0^x\int_0^x f(x' - x'')\, dx'\, dx'' = 2\int_0^x (x - \xi) f(\xi)\, d\xi \tag{29}$$

from Part A, we obtain

$$F_1(\varkappa_2, \varkappa_3, x) = \frac{k^2}{2}\int\limits_0^x (x-\xi)\, F_\varepsilon(\varkappa_2, \varkappa_3, \xi)\, d\xi. \tag{30}$$

We are concerned with the values of the structure functions D_1, D_2 in the region $\sqrt{(y-y')^2+(z-z')^2} \ll x$ only. Therefore we need consider only those $\varkappa$ which satisfy the inequality $\varkappa x \gg 1$. On the other hand, in (30) the significant ξ are only those in the region $\varkappa\xi \lesssim 1$, i.e., $\xi \ll x$. Therefore, in (30) we may take $x - \xi \approx x$, extending the integration over ξ to infinity:

$$F_1(\varkappa_2, \varkappa_3, L) = \frac{k^2 L}{2}\int\limits_0^\infty F_\varepsilon(\varkappa_2, \varkappa_3, \xi) d\xi. \tag{31}$$

Using the relation

$$\int\limits_0^\infty F_\varepsilon(\varkappa_2, \varkappa_3, \xi)\, d\xi = \pi\Phi_\varepsilon(0, \varkappa_2, \varkappa_3), \tag{32}$$

between two-dimensional and three-dimensional spectral densities (see (5.28)), we finally obtain

$$F_1(\varkappa_2, \varkappa_3, L) = \frac{\pi}{2} k^2 L\Phi_\varepsilon(0, \varkappa_2, \varkappa_3)\,. \tag{33}$$

Let us now compute F_2. In (26) we introduce new variables of integration $\xi = x' - x''$, $\eta = \frac{1}{2}(x' + x'')$. The region of integration over the variables ξ, η is a rhombus whose limits are the straight lines $\eta = \frac{1}{2}\xi$, $\eta = -\frac{1}{2}\xi$, $\eta = x - \frac{1}{2}\xi$, $\eta = x + \frac{1}{2}\xi$. As we have already noted, the main contribution to the integral comes from the region of small ξ: $|\xi| \ll x$, since outside this region $F_\varepsilon(\varkappa_2, \varkappa_3, \xi)$ becomes exceedingly small. The integration of ξ therefore can be extended from the interior of the rhombus to an infinite strip between the straight lines $\eta = 0$, $\eta = x$. Thus,

$$F_2(\varkappa_2, \varkappa_3, x) \approx -\frac{k^2}{4}\int\limits_0^x \exp\left\{-\frac{i\varkappa^2(x-\eta)}{k}\right\} d\eta \int\limits_{-\infty}^\infty F_\varepsilon(\varkappa_2, \varkappa_3, \xi)\, d\xi. \tag{34}$$

Making use of (32), we find

$$F_2(\varkappa_2, \varkappa_3, x) = -\frac{\pi k^2}{2}\Phi_\varepsilon(0, \varkappa_2, \varkappa_3)\int\limits_0^x \exp\left\{-\frac{i\varkappa^2(x-\eta)}{k}\right\} d\eta.$$

Integration over η finally gives

$$F_2(\varkappa_2, \varkappa_3, L) = \frac{i\pi}{2}\frac{k^3}{\varkappa^2}\left(1 - \exp\left(-\frac{i\varkappa^2 L}{k}\right)\right)\Phi_\varepsilon(0, \varkappa_2, \varkappa_3). \tag{35}$$

From (14)–(16) we obtain analogous relations for the spectral functions:

$$F_\chi = \frac{1}{2}[F_1 + \operatorname{Re} F_2], \tag{14a}$$

$$F_S = \frac{1}{2}[F_1 - \operatorname{Re} F_2], \tag{15a}$$

$$F_{\chi S} = \frac{1}{2}\,\mathrm{Im}\,F_2. \tag{16a}$$

Inserting the above expressions for F_1, F_2, we obtain

$$F_\chi(\varkappa_2, \varkappa_3, L) = \frac{\pi k^2 L}{4}\left(1 - \frac{k}{\varkappa^2 L}\sin\frac{\varkappa^2 L}{k}\right)\Phi_\varepsilon(0, \varkappa_2, \varkappa_3), \tag{36}$$

$$F_S(\varkappa_2, \varkappa_3, L) = \frac{\pi k^2 L}{4}\left(1 + \frac{k}{\varkappa^2 L}\sin\frac{\varkappa^2 L}{k}\right)\Phi_\varepsilon(0, \varkappa_2, \varkappa_3), \tag{37}$$

$$F_{\chi S}(\varkappa_2, \varkappa_3, L) = \frac{\pi k^3}{4\varkappa^2}\left(1 - \cos\frac{\varkappa^2 L}{k}\right)\Phi_\varepsilon(0, \varkappa_2, \varkappa_3). \tag{38}$$

For locally isotropic turbulence,

$$\Phi_\varepsilon(0, \varkappa_2, \varkappa_3) = \Phi_\varepsilon\left(\sqrt{\varkappa_2^2 + \varkappa_3^2}\right) = \Phi_\varepsilon(\varkappa)$$

and the functions F_χ, F_S, $F_{\chi S}$ depend only on $\varkappa$:

$$F_\chi(\varkappa, L) = \frac{\pi k^2 L}{4}\left(1 - \frac{k}{\varkappa^2 L}\sin\frac{\varkappa^2 L}{k}\right)\Phi_\varepsilon(\varkappa), \tag{36a}$$

$$F_S(\varkappa, L) = \frac{\pi k^2 L}{4}\left(1 + \frac{k}{\varkappa^2 L}\sin\frac{\varkappa^2 L}{k}\right)\Phi_\varepsilon(\varkappa), \tag{37a}$$

$$F_{\chi S}(\varkappa, L) = \frac{\pi k^3}{2\varkappa^2}\sin^2\frac{\varkappa^2 L}{2k}\,\Phi_\varepsilon(\varkappa). \tag{38a}$$

One property of expressions (36)—(38) is worth noting. If $\frac{\varkappa_m^2 L}{k} \ll 1$, i.e., $\lambda L \ll \lambda_0^2$, expressions (36)—(38) can be written approximately as

$$F_\chi(\varkappa, L) \approx \frac{\pi}{24} L^3 \varkappa^4 \Phi_\varepsilon(\varkappa), \tag{39}$$

$$F_S(\varkappa, L) \approx \frac{\pi k^2 L}{2}\Phi_\varepsilon(\varkappa), \tag{40}$$

$$F_{\chi S}(\varkappa, L) \approx \frac{\pi}{8} k L^2 \varkappa^2 \Phi_\varepsilon(\varkappa). \tag{41}$$

When $\lambda L \ll \lambda_0^2$, as we have seen before, the method of smooth perturbation reduces to the method of geometrical optics. Therefore, expressions (39) and (40) coincide with (40.54) and (40.21) derived by geometrical optics method ((40) differs from (40.21) by a factor k^2, since the above expression is written for the phase and not for the eikonal).

If the condition $\lambda L \ll \lambda_0^2$ does not apply, relations (39)—(41) are valid for small $\varkappa$, such that $\varkappa \ll \sqrt{k/L}$. Geometrical optics is thus always applicable to the large-scale part of the turbulence spectrum.

Even if $\Phi_\varepsilon(\varkappa) \sim \varkappa^{-11/3}$, i.e., the spectrum is infinite for $\varkappa \to 0$, we have the relation $F_\chi(0, L) = 0$, whence we see that the integral over $F_\chi(\varkappa, L)$ converges. This means that in addition to the structure function of the log-amplitude, its correlation function

$$B_\chi(\eta, \xi) = \iint_{-\infty}^{\infty} \cos(\varkappa_2\eta + \varkappa_3\zeta)\, F_\chi(\varkappa_2, \varkappa_3, x)\, d\varkappa_2\, d\varkappa_3 \tag{42}$$

also exists, satisfying the equality

$$\iint_{-\infty}^{\infty} B_\chi(\eta, \zeta)\, d\zeta\, d\eta = 0. \tag{43}$$

Similarly, in the expression for $F_{\chi S}$ the factor $\frac{1}{\varkappa^2}\left(1-\cos\frac{\varkappa^2 L}{k}\right)$ is proportional to $\varkappa^2$ for $\varkappa \to 0$, so that even if $\Phi_\varepsilon(\varkappa) \sim \varkappa^{-11/3}$, the integral of $F_{\chi S}$ over $\varkappa_2, \varkappa_3$ converges. This implies the existence of a cross-correlation function

$$B_{\chi S}(\boldsymbol{r}_1, \boldsymbol{r}_2) = \frac{\langle \chi(\boldsymbol{r}_1) S_1(\boldsymbol{r}_2) + \chi(\boldsymbol{r}_2) S_1(\boldsymbol{r}_1)\rangle}{2} =$$
$$= \iint_{-\infty}^{\infty} \cos(\varkappa_2\eta + \varkappa_3\zeta) F_{\chi S}(\varkappa_2, \varkappa_3, L)\, d\varkappa_2\, d\varkappa_3, \tag{44}$$

related to $D_{\chi S}$ by the equality

$$D_{\chi S}(\rho) = 2B_{\chi S}(0) - 2B_{\chi S}(\rho). \tag{45}$$

In particular, for $\boldsymbol{r}_1 = \boldsymbol{r}_2$, we have

$$\langle \chi(L) S_1(L)\rangle = \iint_{-\infty}^{\infty} F_{\chi S}(\varkappa_2, \varkappa_3, L)\, d\varkappa_2,\, d\varkappa_3. \tag{46}$$

§ 47. Phase and amplitude structure functions in a locally isotropic turbulent flow

Let us find the structure functions D_1, D_2 for locally isotropic turbulence, when

$$\Phi_\varepsilon(\varkappa) = 0.033 C_\varepsilon^2 \varkappa^{-11/3} e^{-\varkappa^2/\varkappa^2_m}. \tag{1}$$

If $F_{1,2}$ depend on $\varkappa = \sqrt{\varkappa_2^2 + \varkappa_3^2}$, relations (46.23), (46.27) take the form

$$D_{1,2}(\rho) = 4\pi \int_0^\infty [1 - J_0(\varkappa\rho)]\, F_{1,2}(\varkappa, L)\, \varkappa\, d\varkappa. \tag{2}$$

Inserting (46.33) in (1), we obtain

$$D_1(\rho) = \frac{0.033\ 4\pi^2}{2} k^2 L C_\varepsilon^2 \int_0^\infty [1 - J_0(\varkappa\rho)]\, \varkappa^{-8/3} \exp\left(-\frac{\varkappa^2}{\varkappa_m^2}\right) d\varkappa. \tag{3}$$

Substitution of the variable $\varkappa = \varkappa_m \sqrt{t}$ gives

$$D_1(\rho) = 0.033\pi^2 k^2 L C_\varepsilon^2 \varkappa_m^{-5/3} \int_0^\infty [1 - J_0(\varkappa_m \rho \sqrt{t})]\, t^{-11/6} e^{-t}\, dt. \tag{4}$$

The integral in (4) is evaluated by expanding J_0 in a series. The calculations proceed along the same lines as in Part A when the geometrical optics approximation was used. After integrating we finally have

$$D_1(\rho) = 0.033\pi^2 \frac{6}{5} \Gamma\left(\frac{1}{6}\right) k^2 L C_\varepsilon^2 \varkappa_m^{-5/3} \left[{}_1F_1\left(-\frac{5}{6}, 1, -\frac{\varkappa_m^2 \rho^2}{4}\right) - 1\right]. \tag{5}$$

Let us now compute $D_2(\rho)$. Inserting (46.35) and (1) in (2), we obtain

$$D_2(\rho) = 0.033\, 2\pi^2 i k^3 C_\varepsilon^2 \int_0^\infty [1 - J_0(\varkappa\rho)] \times$$
$$\times \left(1 - \exp\left(-\frac{i\varkappa^2 L}{k}\right)\right) \varkappa^{-14/3} \exp\left(-\frac{\varkappa^2}{\varkappa_m^2}\right) d\varkappa. \quad (6)$$

For $\varkappa \to 0$,

$$[1 - J_0(\varkappa\rho)]\left(1 - \exp\left(-\frac{i\varkappa^2 L}{k}\right)\right) \sim \varkappa^4,$$

and the integral in (6) converges at the origin.

Let

$$A(p) = \int_0^\infty [1 - J_0(\varkappa\rho)]\left(1 - \exp\left(-\frac{i\varkappa^2 L}{k}\right)\right) \varkappa^{-p} \exp\left(-\frac{\varkappa^2}{\varkappa_m^2}\right) d\varkappa. \quad (7)$$

We are interested in the value of $A(14/3)$. The integral in (7) converges for all $p < 5$. We split it into two terms:

$$B(p) = \int_0^\infty [1 - J_0(\varkappa\rho)]\, \varkappa^{-p} \exp\left(-\frac{\varkappa^2}{\varkappa_m^2}\right) d\varkappa, \quad (8)$$

$$C(p) = \int_0^\infty [1 - J_0(\varkappa\rho)]\, \varkappa^{-p} \exp\left\{-\varkappa^2\left(\frac{1}{\varkappa_m^2} + \frac{iL}{k}\right)\right\} dx. \quad (9)$$

The integrals in (8) and (9) converge for $p < 3$. At $p = 3$ the functions $B(p)$ and $C(p)$ have poles. However, the singularities at $p = 3$ mutually cancel when the difference $B(p) - C(p)$ is formed and the function $A(p)$ for $3 < p < 5$ can be defined as the analytical continuation in p of the difference $B(p) - C(p)$. The integrals in (8) and (9) have the same form as (3) and are evaluated by the same method:

$$B(p) = -\frac{1}{2}\Gamma\left(-\frac{p-1}{2}\right)\varkappa_m^{-(p-1)}\left[{}_1F_1\left(-\frac{p-1}{2}, 1, -\frac{\varkappa_m^2\rho^2}{4}\right) - 1\right]. \quad (10)$$

$C(p)$ may be obtained from $B(p)$ if we substitute $1/\varkappa_m^2 + iL/k$ for $1/\varkappa_m^2$, i.e.,

$$\varkappa_m^2 \to \frac{1}{\frac{1}{\varkappa_m^2} + \frac{iL}{k}}.$$

Thus

$$C(p) = -\frac{1}{2}\Gamma\left(-\frac{p-1}{2}\right)\left(\frac{1}{\varkappa_m^2} + \frac{iL}{k}\right)^{(p-1)/2} \times$$
$$\times \left[{}_1F_1\left(-\frac{p-1}{2},\ 1,\ -\frac{\varkappa_m^2\rho^2}{4}\frac{1}{1 + \frac{i\varkappa_m^2 L}{k}}\right) - 1\right]. \quad (11)$$

Expressions (10) and (11) have poles at $p = 3$ (the factor $F\left(-\frac{p-1}{2}\right)$ is infinite). We form the difference $A(p) = B(p) - C(p)$:

$$A(p) = -\frac{1}{2}\Gamma\left(-\frac{p-1}{2}\right)\varkappa_m^{-(p-1)}\left\{{}_1F_1\left(-\frac{p-1}{2}, 1, -\frac{\varkappa_m^2\rho^2}{4}\right) - 1 - \right.$$
$$\left. -\left(1+\frac{i\varkappa_m^2 L}{k}\right)^{(p-1)/2}\left[{}_1F_1\left(-\frac{p-1}{2}, 1, -\frac{\varkappa_m^2\rho^2}{4}\left(1+\frac{i\varkappa_m^2 L}{k}\right)^{-1}\right) - 1\right]\right\}. \quad (12)$$

We will now show that the expression in curly brackets vanishes for $p = 3$. Indeed, for $p = 3$ we have ${}_1F_1(-1, 1, z) = 1 - z$, i.e.,

$$ {}_1F_1\left(-\frac{p-1}{2}, 1, -\frac{\varkappa_m^2\rho^2}{4}\right) - 1 \to \frac{\varkappa_m^2\rho^2}{4},$$
$$ {}_1F_1\left(-\frac{p-1}{2}, 1, -\frac{\varkappa_m^2\rho^2}{4}\left(1+\frac{i\varkappa_m^2 L}{k}\right)^{-1}\right) - 1 \to \frac{\varkappa_m^2\rho^2}{4}\left(1+\frac{i\varkappa_m^2 L}{k}\right)^{-1}$$

and the expression in curly brackets is zero, thus offsetting the pole of the gamma-function $\Gamma\left(-\frac{p-1}{2}\right)$. We can now legitimately consider expression (12) for $p < 5$ (at $p = 5$ the next pole of the gamma-function appears). Setting $p = 14/3$ in (12) and inserting (12) in (6), we obtain

$$D_2(\rho) = -0.033\pi^2\frac{36}{55}\Gamma\left(\frac{1}{6}\right)ik^3C_\varepsilon^2\varkappa_m^{-11/3}\left\{{}_1F_1\left(-\frac{11}{6}, 1, -\frac{\varkappa_m^2\rho^2}{4}\right) - 1 - \right.$$
$$\left. -\left(1+\frac{i\varkappa_m^2 L}{k}\right)^{11/6}\left[{}_1F_1\left(-\frac{11}{6}, 1, -\frac{\varkappa_m^2\rho^2}{4}\left(1+\frac{i\varkappa_m^2 L}{k}\right)^{-1}\right) - 1\right]\right\}. \quad (13)$$

$D_2(\rho)$ is a function of two dimensionless parameters:

$$g = \frac{\varkappa_m^2\rho^2}{4} = 8.8\frac{\rho^2}{\lambda_0^2}, \qquad D = \frac{\varkappa_m^2 L}{k} = 5.6\frac{\lambda L}{\lambda_0^2}$$

(D is called the wave parameter). Expression (13) is conveniently written as

$$D_2(\rho) = -0.033\, i\pi^2\frac{36}{55}\Gamma\left(\frac{1}{6}\right)k^2C_\varepsilon^2\,\varkappa_m^{-5/3}L\frac{1}{D}\times$$
$$\times\left\{{}_1F_1\left(-\frac{11}{6}, 1, -g\right) - 1 - (1+iD)^{11/6}\left[{}_1F_1\left(-\frac{11}{6}, 1, -\frac{g}{1+iD}\right) - 1\right]\right\}. \quad (13a)$$

Consider the limiting cases of large and small D. If $D \to 0$, we expand the right-hand side of (13a) in a series to terms of the order D^2 (this expansion is obtained with somewhat less difficulty directly from (6), by expanding $\exp(-i\varkappa^2 L/k)$ in a series):

$$D_2(\rho) = -0.033\pi^2\Gamma\left(\frac{1}{6}\right)k^2C_\varepsilon^2\,\varkappa_m^{-5/3}L\left\{\frac{6}{5}\left[{}_1F_1\left(-\frac{5}{6}, 1, -g\right) - 1\right] + \right.$$
$$\left. + \frac{iD}{2}\left[{}_1F_1\left(\frac{1}{6}, 1, -g\right) - 1\right] + \frac{D^2}{36}\left[{}_1F_1\left(\frac{7}{6}, 1, -g\right) - 1\right] + \ldots\right\}. \quad (14)$$

Using (14) we can find the functions D_S, D_χ, and $D_{\chi S}$ for the case $D \ll 1$, where geometrical optics is applicable. Inserting (5) and (14) in (46.14)–(46.16), we obtain

$$D_\chi(\rho) = \frac{0.033\,\pi^2\Gamma\left(\frac{1}{6}\right)}{72}C_\varepsilon^2 L^3\varkappa_m^{7/3}\left[1 - {}_1F_1\left(\frac{7}{6}, 1, -g\right)\right]. \quad (15)$$

For $\rho \to \infty$, $F\left(\frac{7}{6}, 1, -g\right) \to 0$, so that

$$D_\chi(\infty) = 2B_\chi(0) = \frac{0.033\,\pi^2\,\Gamma\left(\frac{1}{6}\right)}{72}\,C_\varepsilon^2 L^3 \varkappa_m^{7/3}.$$

Hence we obtain for $\langle\chi^2\rangle = B_\chi(0)$ the previous expression (42.16), originally derived using geometrical optics. If from $D_\chi(\rho)$ we change over to $B_\chi(\rho) = \frac{1}{2}[D_\chi(\infty) - D_\chi(\rho)]$, we obtain expression (42.17).

In computing $D_S(\rho)$, it suffices to retain only the first term in (14):

$$D_S(\rho) = 0.033\pi^2\,\frac{6}{5}\,\Gamma\left(\frac{1}{6}\right) k^2 L C_\varepsilon^2 \varkappa_m^{-5/3}\left[{}_1F_1\left(-\frac{5}{6}, 1, -g\right) - 1\right]. \qquad (16)$$

This expression coincides with (42.6) derived using the geometrical optics method ((42.6) does not contain the factor k^2, as it is written for the eikonal, while (16) is written for the phase).

Finally,

$$D_{\chi S}(\rho) = \frac{0.033\pi^2\,\Gamma\left(\frac{1}{6}\right)}{4}\,C_\varepsilon^2 k L^2 \varkappa_m^{1/3}\left[1 - {}_1F_1\left(\frac{1}{6}, 1, -g\right)\right]. \qquad (17)$$

Consider the function

$$R_{\chi S}^2 = \frac{[D_{\chi S}(\rho)]^2}{D_\chi(\rho)\,D_S(\rho)} = \frac{\langle[\chi(\boldsymbol{r}_1) - \chi(\boldsymbol{r}_2)]\,[S_1(\boldsymbol{r}_1) - S_1(\boldsymbol{r}_2)]\rangle^2}{\langle[\chi(\boldsymbol{r}_1) - \chi(\boldsymbol{r}_2)]^2\rangle\,\langle[S_1(\boldsymbol{r}_1) - S_1(\boldsymbol{r}_2)]^2\rangle},$$

which is the square of the correlation coefficient between phase and log amplitude fluctuations at two points $\boldsymbol{r}_1$ and $\boldsymbol{r}_2 = \boldsymbol{r}_1 + \boldsymbol{\rho}$. Inserting (15), (16), and (17) in the expression for $R_{\chi S}^2$, we obtain

$$R_{\chi S}^2 = \frac{15\left[1 - {}_1F_1\left(\frac{1}{6}, 1, -g\right)\right]^2}{4\left[1 - {}_1F_1\left(\frac{7}{6}, 1, -g\right)\right]\left[{}_1F_1\left(-\frac{5}{6}, 1, -g\right) - 1\right]}. \qquad (18)$$

For $\rho = 0$, $R_{\chi S} = \sqrt{3/28} \approx 0.33$; for $\rho \to \infty$, $R_{\chi S} \to 0$. For small ρ the correlation coefficient is markedly less than 1 because the same spectral components of the turbulence have different influences on amplitude and phase fluctuations. For $\rho \to \infty$ the maximum contribution to phase fluctuations is from inhomogeneities of size comparable with ρ, whereas the amplitude fluctuations (for $D \ll 1$) are determined by inhomogeneities of the order λ_0. This difference is responsible for the small $R_{\chi S}$.

Let us now consider the case when the wave parameter D is large. In this case diffraction effects are appreciable. Therefore,

$$(1 + iD)^{11/6} \approx D^{11/6}\exp\left(\frac{11}{12}i\pi\right), \quad -\frac{g}{1 + iD} \approx \frac{ig}{D}$$

and

$$D_2(\rho) = -0.033\;i\pi^2\,\frac{36}{55}\,\Gamma\left(\frac{1}{6}\right) k^2 C_\varepsilon^2 L \varkappa_m^{-5/3}\,\frac{1}{D}\left\{{}_1F_1\left(-\frac{11}{6}, 1, -g\right) - 1 - \right.$$
$$\left. - \exp\left(\frac{11}{12}i\pi\right) D^{11/6}\left[{}_1F_1\left(-\frac{11}{6}, 1, \frac{ig}{D}\right) - 1\right]\right\}. \qquad (19)$$

For $D \gg 1$, the function $D_2(\rho)$ depends on the two dimensionless parameters, g and $\frac{g}{D} = \frac{k\rho^2}{4L} = \frac{\pi\rho^2}{2\lambda L}$. The parameter g/D is proportional to the square of the ratio of ρ to the radius of the first Fresnel zone.

First let us consider the case $g \ll 1$. Then $g/D \ll 1$ also and we may take the first terms in the series expansion of ${}_1F_1$. This gives

$$D_1(\rho) \approx \frac{0.033\,\pi^2\Gamma\left(\frac{1}{6}\right)}{4} C_\varepsilon^2 k^2 L \varkappa_m^{1/3} \rho^2 + \ldots, \tag{20}$$

$$D_2(\rho) \approx \exp\left(\frac{11}{12}\pi i\right) \frac{0.033\,\pi^2 \frac{6}{5}\Gamma\left(\frac{1}{6}\right)}{4} C_\varepsilon^2 k^{13/6} L^{5/6} \rho^2 + \ldots \tag{21}$$

Thus, for $g \ll 1$, i.e., $\rho \ll \lambda_0$, D_1 and D_2 are quadratic functions of ρ. This also applies to $D_\chi, D_S, D_{\chi S}$:

$$D_\chi(\rho) = \frac{0.033\,\pi^2\Gamma\left(\frac{1}{6}\right)}{8} C_\varepsilon^2 k^2 L \varkappa_m^{1/3} \left[1 - \frac{6}{5}\cos\frac{\pi}{12}\left(\frac{k}{\varkappa_m^2 L}\right)^{1/6}\right]\rho^2 + \ldots =$$

$$= 0.41\, C_\varepsilon^2 k^2 L \lambda_0^{-1/3}\left[1 - 0.87\left(\frac{\lambda_0^2}{\lambda L}\right)^{1/6}\right]\rho^2 + \ldots, \tag{22}$$

$$D_S(\rho) = 0.41\, C_\varepsilon^2 k^2 L \lambda_0^{-1/3}\left[1 + 0.87\left(\frac{\lambda_0^2}{\lambda L}\right)^{1/6}\right]\rho^2 + \ldots \tag{23}$$

We see that for $D \gg 1$ and $g \ll 1$, the functions $D_\chi(\rho)$ and $D_S(\rho)$ coincide (apart from a small correction of the order $D^{-1/6}$).

For $D_{\chi S}(\rho)$ we have

$$D_{\chi S}(\rho) = 0.071\, C_\varepsilon^2 k^{13/6} L^{5/6} \rho^2 + \ldots, \tag{24}$$

and the correlation coefficient is

$$R_{\chi S} = \frac{D_{\chi S}}{\sqrt{D_S D_\chi}} = 0.23\left(\frac{\lambda L}{\lambda_0^2}\right)^{-1/6}. \tag{25}$$

For $D \gg 1$, $R_{\chi S}(0) \ll 1$, whereas for small wave parameters we had $R_{\chi S}(0) \approx 0.33$. The relatively large value of $R_{\chi S}$ for $D \ll 1$ is attributable to the fact that in the geometrical optics approximation amplitude fluctuations (the ray tube cross section changes) are determined by phase fluctuations.

Let us now consider the most important case $D \gg 1$, $g \gg 1$. The parameter $\frac{g}{D} = \frac{k\rho^2}{4L}$ is quite arbitrary in this case. Using the asymptotic expression for ${}_1F_1$ for large g, we obtain

$$D_1(\rho) = \frac{0.033\,\pi^2\left(\frac{6}{5}\right)\Gamma\left(\frac{1}{6}\right)}{\Gamma\left(\frac{11}{6}\right)2^{5/3}} C_\varepsilon^2 k^2 L \rho^{5/3} = 0.73\, C_\varepsilon^2 k^2 L \rho^{5/3}, \tag{26}$$

$$D_2(\rho) = -i\,\frac{0.033\,\pi^2\frac{36}{55}\Gamma\left(\frac{1}{6}\right)}{\Gamma\left(\frac{17}{6}\right)2^{\frac{11}{3}}} C_\varepsilon^2 k^3 \rho^{11/3}\left\{1 - \Gamma\left(\frac{17}{6}\right)\left(\frac{iD}{g}\right)^{11/6}\left[{}_1F_1\left(-\frac{11}{6},\, 1, -\frac{g}{iD}\right) - 1\right]\right\}. \tag{27}$$

Using (26) and (27), we write the expression for $D_\chi(\rho)$:

$$D_\chi(\rho)=\frac{1}{2}\frac{0.033\,\pi^2\left(\frac{6}{5}\right)\Gamma\left(\frac{1}{6}\right)}{\Gamma\left(\frac{11}{6}\right)2^{\frac{5}{3}}}\left\{C_\varepsilon^2k^2L\rho^{5/3}+\frac{1}{4}\left(\frac{6}{11}\right)^2C_\varepsilon^2k^3\rho^{11/3}\Gamma\left(\frac{17}{6}\right)\times\right.$$
$$\left.\times\left(\frac{D}{g}\right)^{11/6}\mathrm{Re}\left[i^{17/6}{}_1F_1\left(-\frac{11}{6},1,-\frac{g}{iD}\right)-1\right]\right\}. \qquad (28)$$

First let us find $D_\chi(\infty)$ since this quantity determines the mean square fluctuations. To this end we have to establish the asymptotic behavior of the function ${}_1F_1$ entering (28). The asymptotic series for ${}_1F_1(\alpha,\gamma,z)$ is

$${}_1F_1(\alpha,\gamma,z)=\frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha)}(-z)^{-\alpha}G(\alpha,\alpha+1-\gamma,-z)+\frac{\Gamma(\gamma)}{\Gamma(\alpha)}e^zz^{\alpha-\gamma}G(\gamma-\alpha,1-\alpha,z), \qquad (29)$$

where

$$G(\alpha,\beta,z)=1+\frac{\alpha\beta}{1!z}+\frac{\alpha(\alpha+1)\beta(\beta+1)}{2!z^2}+\dots$$

Consider the case $\alpha=-11/6$, $\gamma=1$. Since z is a purely imaginary number, $|e^z|=1$. The difference $\alpha-\gamma$ is negative, therefore it suffices to consider the first term in (29). In the expression for G we take the first two terms, since all the rest only introduce a small correction:

$${}_1F_1\left(-\frac{11}{6},1,-\frac{g}{iD}\right)=\frac{1}{\Gamma\left(\frac{17}{6}\right)}\left(\frac{g}{iD}\right)^{11/6}\left[1+\left(\frac{11}{6}\right)^2\frac{iD}{g}+\dots\right].$$

We insert this expansion in (28). In taking the real part of $i^{17/6}{}_1F_1(-11/6, 1, -g/iD)$, the leading term of the expansion drops out, since after multiplication by $i^{17/6}$ it becomes a purely imaginary quantity. Thus

$$\lim_{\rho\to\infty}D_\chi(\rho)=\lim_{\rho\to\infty}\frac{1}{2}0.73\left\{C_\varepsilon^2k^2L\rho^{5/3}+\frac{1}{4}\left(\frac{6}{11}\right)^2\Gamma\left(\frac{17}{6}\right)C_\varepsilon^2\times\right.$$
$$\left.\times k^3\rho^{11/3}\left(\frac{D}{g}\right)^{11/6}\left[-\frac{\left(\frac{11}{6}\right)^2}{\Gamma\left(\frac{17}{6}\right)}\left(\frac{g}{D}\right)^{5/6}-\cos\left(\frac{17}{12}\pi\right)\right]\right\}.$$

Inserting $\frac{g}{D}=\frac{k\rho^2}{4L}$, we see that terms with $\rho^{5/3}$ mutually cancel and the desired limit is

$$D_\chi(\infty)=\frac{0.73}{2}\,2^{5/3}\frac{6}{11}\Gamma\left(\frac{11}{6}\right)\left(-\cos\frac{17\pi}{12}\right)C_\varepsilon^2k^{7/6}L^{11/6}=0.154\,C_\varepsilon^2k^{7/6}L^{11/6}. \qquad (30)$$

Hence we obtain an expression for the mean square fluctuations $\langle\chi\rangle^2=\frac{1}{2}D_\chi(\infty)$:

$$\langle\chi^2\rangle=0.077\,C_\varepsilon^2k^{7/6}L^{11/6}. \qquad (31)$$

Let us derive the expression for the normalized amplitude correlation function

$$b_\chi(\rho)=\frac{B_\chi(\rho)}{B_\chi(0)}=\frac{\frac{1}{2}[D_\chi(\infty)-D_\chi(\rho)]}{\frac{1}{2}D_\chi(\infty)}=1-\frac{D_\chi(\rho)}{D_\chi(\infty)}. \qquad (32)$$

Inserting (28) and (31), we obtain after simple manipulations

$$b_\chi(\rho) = \mathrm{Re}\,{}_1F_1\left(-\frac{11}{6},\ 1,\ \frac{ik\rho^2}{4L}\right) - \mathrm{ctg}\,\frac{\pi}{12}\,\mathrm{Im}\,{}_1F_1\left(-\frac{11}{6},\ 1,\ \frac{ik\rho^2}{4L}\right) - \frac{1}{\frac{6}{11}\Gamma\left(\frac{11}{6}\right)\sin\frac{\pi}{12}}\left(\frac{k\rho^2}{4L}\right)^{5/6} \quad (\rho \gg \lambda_0). \tag{33}$$

From (33) it follows that for $D \gg 1$ in the region $\rho \gg \lambda_0$ the function $b_\chi(\rho)$ depends only on the dimensionless argument $\frac{k\rho^2}{4L} = \frac{\pi\rho^2}{2\lambda L}$, i.e., on the ratio of the distance between the observation points to the radius of the first Fresnel zone. Formally, we may consider (33) for $\rho \to 0$ too (we have $b_\chi(0) = 1$), but in this case the condition $g \gg 1$, i.e., $\rho \gg \lambda_0$, assumed in the derivation of this expression, is broken. Expression (33) is applicable for $g \gg 1$, i.e., $\frac{g}{D} \gg \frac{1}{D}$. For $\frac{g}{D} = \frac{k\rho^2}{4L} \lesssim \frac{1}{D} \ll 1$, expression (33) is no longer valid and should be replaced by a new expression which follows from (22), (31), and (32):

$$b_\chi(\rho) = 1 - 12.3\frac{\rho^2}{(\lambda L)^{5/6}\lambda_0^{1/3}} + \ldots \quad (\rho \ll \lambda_0). \tag{33a}$$

For $\lambda_0 \ll \rho \ll \sqrt{\lambda L}$ the function $b_\chi(\rho)$ is determined by expression (33) with only the first term of the series for ${}_1F_1$ retained:

$$b_\chi(\rho) = 1 - \frac{1}{\frac{6}{11}\Gamma\left(\frac{11}{6}\right)\sin\frac{\pi}{12}}\left(\frac{k\rho^2}{4L}\right)^{5/6} + \frac{11}{6}\,\mathrm{ctg}\,\frac{\pi}{12}\,\frac{k\rho^2}{4L} - \frac{55}{144}\left(\frac{k\rho^2}{4L}\right)^2 + \ldots =$$
$$= 1 - 2.36\left(\frac{k\rho^2}{L}\right)^{5/6} + 1{,}71\,\frac{k\rho^2}{L} - 0.024\left(\frac{k\rho^2}{L}\right)^2 + \ldots \tag{34}$$

To obtain an asymptotic expression for $b_\chi(\rho)$ for $\rho \to \infty$, we should take the first four terms in expression (29) for $G\left(-\frac{11}{6}, -\frac{11}{6}, -\frac{ik\rho^2}{4L}\right)$. The result gives

$$b_\chi(\rho) = -A\left(\frac{k\rho^2}{4L}\right)^{-7/6} \qquad \left(\frac{k\rho^2}{4L} \gg 1\right), \tag{35}$$

where

$$A = \frac{11\cdot 25}{6^6\,\Gamma\left(\frac{11}{6}\right)\sin\frac{\pi}{12}} = 0.0242.$$

The function (33) is plotted in Figure 45.

Consider the phase structure function D_S. It is best obtained from the equality

$$D_\chi(\rho) + D_S(\rho) = D_1(\rho).$$

Putting

$$D_\chi(\rho) = 2B_\chi(0) - 2B_\chi(\rho) = 2\langle\chi^2\rangle\,[1 - b_\chi(\rho)]$$

and using expression (26) for $D_1(\rho)$ for $D \gg 1$, $g \gg 1$, we obtain

$$D_S(\rho) = 0{,}73\,C_\varepsilon^2 k^2 L\rho^{5/3} - 0.154\,C_\varepsilon^2 k^{7/6} L^{11/6}\,[1 - b_\chi(\rho)]. \tag{36}$$

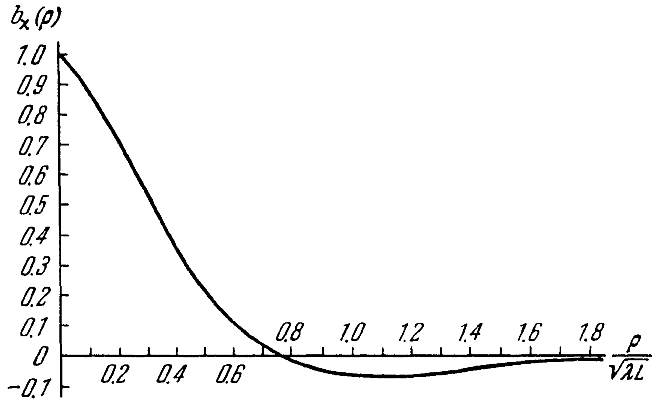


FIGURE 45. The log amplitude correlation function for $\sqrt{\lambda L} \gg l_0$.

For $\rho \gg \sqrt{\lambda L}$ the second term in (36) approaches a constant value, whereas the first term increases, so that in the expression for D_S it is only necessary to retain the first term:

$$D_S(\rho) = 0.73\, C_\varepsilon^2 k^2 L \rho^{5/3} \quad (\rho \gg \sqrt{\lambda L}). \tag{37}$$

For $\rho \ll \sqrt{\lambda L}$ we use the first two terms in the expansion (34) of $b_\chi(\rho)$:

$$1 - b_\chi(\rho) \approx 2.36\, k^{5/6} L^{-5/6} \rho^{5/3}.$$

Inserting in (36), we obtain

$$D_S(\rho) = \frac{1}{2} 0.73\, C_\varepsilon^2 k^2 L \rho^{5/3} \quad (\lambda_0 \ll \rho \ll \sqrt{\lambda L}). \tag{36a}$$

Thus the expressions for $D_S(\rho)$ for $\rho \gg \sqrt{\lambda L}$ and $\rho \ll \sqrt{\lambda L}$ differ only by a coefficient 1/2. For $\rho \ll \lambda_0$ the function $D_S(\rho)$ has the quadratic form in (23).

The expression for $D_{\chi S}$ is readily found from (27) taking the imaginary part of $D_2(\rho)$. We give here only the expression for $\frac{1}{2} D_{\chi S}(\infty) = B_{\chi S}(0)$. Using the asymptotic expansion of ${}_1F_1$, we obtain

$$B_{\chi S}(0) = \langle \chi S_1 \rangle = \mathrm{ctg}\frac{\pi}{12} \langle \chi^2 \rangle = 0.28\, C_\varepsilon^2 k^{7/6} L^{11/6}. \tag{38}$$

Hence it is obvious that $\langle \chi S_1 \rangle \sim \langle \chi^2 \rangle$.

The expression for the angle-of-arrival correlation functions can be derived from $D_S(\rho)$ using relations analogous to (40.35) and (40.38) (an additional factor k^{-2} should be introduced in the right-hand sides of these relations). The mean square fluctuations of the propagation direction for $D \gg 1$ is by (23)

$$\langle \alpha^2 \rangle = 0.41\, C_\varepsilon^2 L \lambda_0^{-1/3}.$$

This relation differs from the corresponding expression (42.11) obtained by the geometrical optics method only by a factor of 1/2. The angle-of-arrival correlation radius, as in the case $D \ll 1$, is of the order of λ_0.

In concluding this section, let us consider a qualitative interpretation of these results. The basic relations for the phase structure function and the expressions for the mean square amplitude fluctuations can be derived, apart from numerical coefficients, from simple qualitative considerations.

First consider the case of geometrical optics. Let two parallel rays of length L be a distant ρ from each other, so that $\rho \gg l_0$. These rays are divided into segments each of length ρ. The number N of these segments along each ray is clearly $N = L/\rho$. The phase difference which arises when the wave traverses a given pair of segments is of the order $\Delta S \sim k\rho\Delta n(\rho)$. Δn is zero on the average. The mean square phase difference after traversing one pair of segments is $\langle \Delta S^2 \rangle_1 \sim k^2\rho^2 \langle \Delta n^2(\rho) \rangle$. The total mean square phase difference after traversing all of the N segments is

$$\langle \Delta S^2 \rangle = N \langle \Delta S^2 \rangle_1 = k^2\rho^2 \langle \Delta n^2(\rho) \rangle \frac{L}{\rho}.$$

Putting

$$\langle \Delta n^2(\rho) \rangle = C_n^2 \rho^{2/3} \quad \left(C_n^2 = \frac{1}{4} C_\varepsilon^2 \right),$$

we obtain

$$\langle \Delta S^2 \rangle \sim C_n^2 k^2 L \rho^{5/3},$$

and this is the expression for the phase structure function for $\rho \gg l_0$. If now $\rho \ll l_0$ the segments should be made of length l_0, and then within each segment n will be approximately constant. Thus, $\Delta S \sim k l_0 \Delta n(\rho)$, $N = \frac{L}{l_0}$ and

$$\langle \Delta S^2 \rangle_1 \sim k^2 l_0^2 \langle \Delta n^2(\rho) \rangle,$$
$$\langle \Delta S^2 \rangle \sim k^2 l_0 L \langle \Delta n^2(\rho) \rangle.$$

In this case (for $\rho \ll l_0$)

$$\langle \Delta n^2(\rho) \rangle \sim C_n^2 l_0^{2/3} \left(\frac{\rho}{l_0} \right)^2,$$

so that

$$\langle \Delta S^2 \rangle \sim C_n^2 k^2 L l_0^{-1/3} \rho^2.$$

Let us now consider amplitude fluctuations. Let a plane wave be incident on a collecting lens with a focal length F and radius R. Draw a plane perpendicular to the optical axis of the lens and at a distance x from the lens, i.e., at a distance $F - x$ from its focus. The cross section of the beam at this plane is found from the equality $R/F = r/(F - x)$, i.e., $R/r = F/(F - x)$. If the wave amplitude before the lens is A_0 and in the beam at x it is A, we clearly have $A_0^2 R^2 = A^2 r^2$, therefore $A = A_0 R/r$ and

$$\delta A = A - A_0 = A_0 \left(\frac{R}{r} - 1 \right) = \frac{A_0 x}{F - x}.$$

Hence the relative change in amplitude

$$\frac{\delta A}{A_0} = \frac{x}{F - x}.$$

Let $x \ll F$. Then $\delta A/A_0 \sim x/F$. Hence it follows that given a selection of lenses with various focal lengths, the largest effect will be caused by lenses with the smallest focal length. The focal length of a lens is given by the well known expression $F = R/(n-1)$, where R is the radius of curvature of the lens. If turbulent inhomogeneities act as lenses, we have $n-1 \sim C_n R^{1/3}$ and $F \sim R^{2/3}/C_n$. Then for the smallest lenses $R \sim l_0$ and we have $F \sim l_0^{2/3}/C_n$ and $\frac{\delta A}{A} \sim \frac{C_n x}{l_0^{2/3}}$. Let the ray be intercepted by a great many lenses with random values of F. The number of such lenses is $N = \frac{L}{l_0}$, where L is the path length. The parameter x is different for different lenses, but it is always of the order of L. The resultant effect is then approximately given by

$$\left\langle \left(\frac{\delta A}{A}\right)^2 \right\rangle \sim \left(\frac{C_n L}{l_0^{2/3}}\right)^2 \frac{L}{l_0} = C_n^2 l_0^{-7/3} L^3.$$

We have obtained an expression for the mean square amplitude fluctuations in the geometrical optics approximation.

Let us now consider the diffraction effects. They become pronounced in small-angle scattering, with angles of the order λ/l, where l is the scale size of the inhomogeneity. Diffraction effects do not strongly modify the mechanism producing phase fluctuations. The main changes are observed in connection with amplitude fluctuations.

In geometrical optics the beam radius in the plane x was equal to r, whereas diffraction effects change the beam radius by an amount of the order $\frac{\lambda x}{R}$. If $\frac{\lambda x}{R} \ll r \approx R$ (for weak fluctuations $r \approx R$), i.e., $\lambda x \ll R^2$, this effect is insignificant. If, however, $\frac{\lambda}{R} x \gg R$, i.e., $\lambda x \gg R^2$, the diffraction pattern entirely determines the intensity distribution in section x and the focusing action of the lens is unimportant (the diffraction pattern from a weak collecting lens is roughly the same as that of a weak dispersing lens). Therefore lenses with $\lambda L \gg R^2$ do not focus and have little effect on amplitude fluctuations. If the turbulent medium contains lenses of various sizes, some of which satisfy the condition $\lambda L \ll R^2$ and some not, amplitude fluctuations are mainly caused by the smallest of these inhomogeneities which still retain the focusing properties, i.e., by inhomogeneities for which $\lambda L \sim R^2$. Hence it follows that for $\lambda L \gg l_0^2$, the parameter l_0 in the expression for $\langle(\delta A/A)^2\rangle$ should be replaced by $\sqrt{\lambda L}$. Then we have

$$\left\langle \left(\frac{\delta A}{A}\right)^2 \right\rangle \sim C_n^2 k^{7/6} L^{11/6},$$

and this is the expression obtained by the method of smooth perturbations for the case $\sqrt{\lambda L} \gg l_0$. It follows from the preceding argument that this expression can be interpreted as the result of the application of geometrical optics to a medium from which all small scale inhomogeneities measuring less than $\sqrt{\lambda L}$ have been removed. This interpretation is sometimes highly useful (see Chapter 5, Part B).

Note that the above expressions are valid only for $L \ll F$. If this condition is not met and, instead, if $F \ll L$, the situation changes considerably. If we consider a lens with a short focal length located at a large distance from the observation point, then irrespective of the "sign" of the lens, its very presence will broaden the beam (the rays will diverge from a real or an imaginary focus, the distance $2F$ between these foci being small compared to the distance from the lens to the observation point). In this case an increase in the number and the "strength" of lenses far from the observation point has virtually no effect on the amplitude. The amplitude is changed appreciably only by those lenses whose foci are close to the observation point. In this case, the "effective" section of the path is always close to the observation point, so that any further increase in path length hardly affects the intensity of fluctuations: saturation of amplitude fluctuations is observed.

Quantitative calculations of this effect are fairly involved and have not been carried out to completion. We will give two different methods, in the last section of this chapter and in Chapter 5, Part B.

§ 48. Phase and amplitude fluctuations in a locally homogeneous turbulent medium with smoothly varying mean characteristics

We have so far considered the case where the intensity of the dielectric constant fluctuations is constant in space.

The previous results can be readily generalized to the case when C_ε^2 is a function of position /108/. Such a model of turbulence was considered in Chapter 1. The corresponding structure function $D_\varepsilon(\boldsymbol{r}_1, \boldsymbol{r}_2) = \langle[\varepsilon_1(\boldsymbol{r}_1) - \varepsilon_1(\boldsymbol{r}_2)]^2\rangle$ has the form

$$D_\varepsilon(\boldsymbol{r}_1, \boldsymbol{r}_2) = C_\varepsilon^2\left(\frac{\boldsymbol{r}_1+\boldsymbol{r}_2}{2}\right) D_\varepsilon^{(0)}(\boldsymbol{r}_1 - \boldsymbol{r}_2). \qquad (1)$$

The spectral densities are also similarly expressed:

$$\Phi_\varepsilon = C_\varepsilon^2\left(\frac{\boldsymbol{r}_1+\boldsymbol{r}_2}{2}\right)\Phi_\varepsilon^{(0)}(\varkappa) = \Phi_\varepsilon\left(\varkappa, \frac{\boldsymbol{r}_1+\boldsymbol{r}_2}{2}\right), \qquad (2)$$

$$F_\varepsilon = C_\varepsilon^2\left(\frac{\boldsymbol{r}_1+\boldsymbol{r}_2}{2}\right) F_\varepsilon^{(0)}(\varkappa_2, \varkappa_3, x_1 - x_2) = F_\varepsilon\left(\varkappa_2, \varkappa_3, x_1 - x_2, \frac{\boldsymbol{r}_1+\boldsymbol{r}_2}{2}\right). \qquad (3)$$

Expressions (46.22) and (46.26) for the two-dimensional spectral densities F_1 and F_2 remain valid, since they were derived without actually using the assumption that C_ε^2 was constant:

$$F_1(\varkappa_2, \varkappa_3, x) = \frac{k^2}{4}\int_0^x\int_0^x \exp\left\{\frac{i\varkappa^2(x'-x'')}{2k}\right\} F_\varepsilon\left(\varkappa_2, \varkappa_3, x'-x'', \frac{x'+x''}{2}\right) dx'\,dx'', \qquad (4)$$

$$F_2(\varkappa_2, \varkappa_3, x) = -\frac{k^2}{4}\int_0^x\int_0^x \exp\left\{-\frac{i\varkappa^2}{2k}(2x-x'-x'')\right\} F_\varepsilon\left(\varkappa_2, \varkappa_3, x'-x'', \frac{x'+x''}{2}\right) dx'\,dx''. \qquad (5)$$

Here we consider only the dependence of the function F_ε on the longitudinal coordinate $(x' + x'')/2$, since the characteristic distance L_0 over which the function $C_\varepsilon(r)$ changes appreciably is large compared to the relevant transverse dimensions (the radius of the amplitude correlation function $\sqrt{\lambda L}$ or the argument ρ of $D_s(\rho)$). Inserting for F_ε in (4) its expression from (3), we obtain

$$F_1(\varkappa_1, \varkappa_2, x) = \frac{k^2}{4}\int_0^x\int_0^x C_\varepsilon^2\left(\frac{x'+x''}{2}\right)\exp\left\{\frac{i\varkappa^2(x'-x'')}{2k}\right\}F_\varepsilon^{(0)}(\varkappa_2, \varkappa_3, x'-x'')\,dx'dx''. \quad (6)$$

We introduce new variables $\xi = x' - x''$, $\eta = \frac{1}{2}(x' + x'')$. Integration over ε may be extended, as before, from minus infinity to plus infinity, taking the exponential equal to unity. Using the relation

$$\int_{-\infty}^{\infty} F_\varepsilon^{(0)}(\varkappa_2, \varkappa_3, \xi)\,d\xi = 2\pi\Phi_\varepsilon^{(0)}(0, \varkappa_2, \varkappa_3), \quad (7)$$

we obtain

$$F_1(\varkappa_1, \varkappa_2, x) = \frac{1}{2}\pi k^2\Phi_\varepsilon^{(0)}(0, \varkappa_2, \varkappa_3)\int_0^x C_\varepsilon^2(\eta)\,d\eta$$

or, using the notation (2),

$$F_1(\varkappa_1, \varkappa_2, L) = \frac{1}{2}\pi k^2\int_0^L \Phi_\varepsilon(0, \varkappa_2, \varkappa_3, x)\,dx. \quad (8)$$

The function F_2 is similarly expressed:

$$F_2(\varkappa_2, \varkappa_3, x) = -\frac{k^2}{4}\int_0^x C_\varepsilon^2(\eta)\exp\left\{-\frac{i\varkappa^2}{k}(x-\eta)\right\}d\eta\int_{-\infty}^{\infty} F_\varepsilon^{(0)}(\varkappa_2, \varkappa_3, \xi)\,d\xi =$$

$$= -\frac{\pi k^2}{2}\Phi_\varepsilon^{(0)}(0, \varkappa_2, \varkappa_3)\int_0^x C_\varepsilon^2(\eta)\exp\left\{-\frac{i\varkappa^2(x-\eta)}{k}\right\}d\eta$$

or

$$F_2(\varkappa_2, \varkappa_3, L) = -\frac{\pi k^2}{2}\int_0^L \exp\left\{-\frac{i\varkappa^2(L-x)}{k}\right\}\Phi_\varepsilon(0, \varkappa_2, \varkappa_3, x)\,dx. \quad (9)$$

Expressions (8) and (9) include integrals over the variable spectral density. In F_1 we have the function Φ_ε simply averaged over the ray, whereas in F_2 the averaging is carried out with a weighting factor $\exp\left\{-\frac{i\varkappa^2(L-x)}{k}\right\}$.

Let us calculate the functions D_1 and D_2 for the case when $\Phi_\varepsilon^{(0)}$ is given by

$$\Phi_\varepsilon^{(0)}(\varkappa) = 0.033\,\varkappa^{-11/3}\exp\left(-\frac{\varkappa^2}{\varkappa_m^2}\right). \quad (10)$$

Inserting (10) in (8), we obtain

$$D_1(\rho) = 4\pi\int_0^\infty [1 - J_0(\varkappa\rho)]\,F_1(\varkappa, L)\,\varkappa\,d\varkappa =$$

$$2\pi^2\cdot 0.033\,k^2\int_0^L C_\varepsilon^2(x)\,dx\int_0^\infty [1 - J_0(\varkappa\rho)]\varkappa^{-8/3}\exp\left(-\frac{\varkappa^2}{\varkappa_m^2}\right)d\varkappa. \quad (11)$$

This expression differs from (47.3) only in that the factor LC_ε^2 is replaced by

$$LC_\varepsilon^2 \to \int_0^L C_\varepsilon^2(x)\,dx. \tag{12}$$

Hence, the expression for $D_1(\rho)$ can be obtained from the corresponding expressions with $C_\varepsilon^2 = \text{const}$ using the substitution (12):

$$D_1(\rho) = 0.033\,\pi^2 \frac{6}{5} \Gamma\left(\frac{1}{6}\right) k^2 \varkappa_m^{-5/3} \int_0^L C_\varepsilon^2(x)\,dx \left[{}_1F_1\left(-\frac{5}{6},\ 1, -\frac{\varkappa_m^2 \rho^2}{4}\right) - 1\right]. \tag{13}$$

Let us find an expression for $D_2(\rho)$ using the spectrum (10):

$$D_2(\rho) = -2\pi^2\ 0.033\, k^2 \int_0^L C_\varepsilon^2(x)\,dx \int_0^\infty [1 - J_0(\varkappa\rho)]\,\varkappa^{-8/3} \times$$
$$\times \exp\left\{-\varkappa^2\left[\frac{1}{\varkappa_m^2} + \frac{i(L-x)}{k}\right]\right\} d\varkappa.$$

Using (47.9) and (47.12) with L replaced by $(L-x)$ and taking $p = 8/3$, we obtain

$$D_2(\rho) = -0.033\,\pi^2 \frac{6}{5} \Gamma\left(\frac{1}{6}\right) k^2 \varkappa_m^{-5/3} \int_0^L C_\varepsilon^2(x) \left[1 + \frac{i\varkappa_m^2 (L-x)}{k}\right]^{5/6} \times$$
$$\times \left[{}_1F_1\left(-\frac{5}{6},\ 1, -\frac{g}{1 + \frac{i\varkappa_m^2 (L-x)}{k}}\right) - 1\right] dx, \tag{14}$$

where as before $g = \frac{\varkappa_m^2 \rho^2}{4}$.

As in the case $C_\varepsilon^2 = \text{const}$, we consider the cases $D \ll 1$ and $D \gg 1$ separately.

If $D \ll 1$, we have $1 + \frac{i\varkappa_m^2 (L-x)}{k} \approx 1$ and $D_2(\rho) \approx -D_1(\rho)$. Therefore $D_S = \frac{1}{2}[D_1 - \operatorname{Re} D_2] \approx D_1$, i.e.,

$$D_S(\rho) = 0.033\,\pi^2 \frac{6}{5} \Gamma\left(\frac{1}{6}\right) k^2 \varkappa_m^{-5/3} \times \left[{}_1F_1\left(-\frac{5}{6},\ 1, -\frac{\varkappa_m^2 \rho^2}{4}\right) - 1\right] \int_0^L C_\varepsilon^2(x)\,dx. \tag{15}$$

The phase fluctuations are thus determined by the integral of $C_\varepsilon^2(x)$ along the ray.

Let us find $D_\chi(\rho)$. We expand (14) in a series in $D = \frac{\varkappa_m^2 L}{k}$ up to terms of second order (this expansion is more easily obtained using the original integral which led to (14)):

$$D_2(\rho) = -D_1(\rho) - 0.033\,\pi^2 ik\Gamma\left(\frac{1}{6}\right) \varkappa_m^{1/3} \times$$
$$\times \left[{}_1F_1\left(\frac{1}{6},\ 1, -g\right) - 1\right] \int_0^L (L-x)\, C_\varepsilon^2(x)\,dx - \frac{1}{2}\pi^2\, 0.033\, \Gamma\left(\frac{7}{6}\right) \varkappa_m^{7/3} \times$$
$$\times \left[{}_1F_1\left(\frac{7}{6},\ 1, -g\right) - 1\right] \int_0^L (L-x)^2\, C_\varepsilon^2(x)\,dx + \ldots \tag{16}$$

We can now calculate $D_\chi(\rho) = \frac{1}{2}[D_1(\rho) + \operatorname{Re} D_2(\rho)]$:

$$D_\chi(\rho) = \frac{\pi^2}{24} 0.033\, \Gamma\left(\frac{1}{6}\right) \varkappa_m^{7/3} \left[1 - {}_1F_1\left(\frac{7}{6}, 1, -g\right)\right] \int_0^L (L-x)^2 C_\varepsilon^2(x)\, dx. \tag{17}$$

Comparison of this expression with (47.15) for $C_\varepsilon^2 = \text{const}$ shows that the factor $\frac{1}{3} L^3 C_\varepsilon^2$ in (17) is replaced by the integral

$$\frac{1}{3} L^3 C_\varepsilon^2 \to \int_0^L (L-x)^2 C_\varepsilon^2(x)\, dx. \tag{18}$$

Therefore all the expressions for amplitude fluctuations for $D \ll 1$ can be obtained from the corresponding expressions for the case $C_\varepsilon^2 = \text{const}$ by the simple substitution in (18). For instance,

$$\langle \chi^2 \rangle = 2.4\, \lambda_0^{-7/6} \int_0^L C_\varepsilon^2(x) (L-x)^2\, dx \quad (D \ll 1). \tag{19}$$

Now consider the case $D \gg 1$. Here, over almost all of the integration region in (14), we have $\varkappa_m^2 (L-x)/k \gg 1$, so that

$$\left[1 + \frac{i\varkappa_m^2 (L-x)}{k}\right]^{5/6} \approx \left[\frac{i\varkappa_m^2 (L-x)}{k}\right]^{5/6}.$$

Similarly

$${}_1F_1\left(-\frac{5}{6}, 1, -\frac{g}{1 + \frac{i\varkappa_m^2}{k}(L-x)}\right) \approx {}_1F_1\left(-\frac{5}{6}, 1, \frac{ik\rho^2}{4(L-x)}\right),$$

so that

$$D_2(\rho) = -i^{5/6}\, 0.033\, \pi^2 \frac{6}{5} \Gamma\left(\frac{1}{6}\right) k^{7/6} \int_0^L C_\varepsilon^2(x) (L-x)^{5/6} \times$$
$$\times \left[{}_1F_1\left(-\frac{5}{6}, 1, \frac{ik\rho^2}{4(L-x)}\right) - 1\right] dx. \tag{20}$$

We see from this expression that the function $D_2(\rho)$ is in fact determined by the form of the function $C_\varepsilon^2(x)$. Since $C_\varepsilon^2(\varkappa)$ is an arbitrary function, we cannot find the particular form of $D_2(\rho)$ without specifying the spatial behavior of the fluctuations. For $\langle \chi^2 \rangle$, however, we can find the general expression. Since

$$D_\chi(\rho) = \frac{1}{2}[D_1(\rho) + \operatorname{Re} D_2(\rho)]$$

and $\langle \chi^2 \rangle = \frac{1}{2} D_\chi(\infty)$, we have to find $D_1(\rho)$ and $D_2(\rho)$ for $\rho \to \infty$. From (13) it is easy to show

$$D_1(\rho) \approx \frac{0.033\, \pi^2 \frac{6}{5} \Gamma\left(\frac{1}{6}\right)}{2^{5/3} \Gamma\left(\frac{11}{6}\right)} k^2 \rho^{5/3} \int_0^L C_\varepsilon^2(x)\, dx + O\left(\frac{1}{\rho}\right). \tag{21}$$

Inserting the asymptotic expansion

$$ {}_1F_1\left(-\frac{5}{6},\ 1,\ \frac{ik\rho^2}{4(L-x)}\right)=\frac{1}{\Gamma\left(\frac{11}{6}\right)}\left(\frac{k\rho^2}{4i(L-x)}\right)^{5/6}+O\left(\frac{1}{\rho}\right) $$

in (20), we find

$$ \operatorname{Re} D_2(\rho)=-\frac{0.033\,\pi^2\,\frac{6}{5}\,\Gamma\left(\frac{1}{6}\right)k^{7/6}}{2^{5/3}\,\Gamma\left(\frac{11}{6}\right)}\,k^{5/6}\rho^{5/3}\int_0^L C_\varepsilon^2(x)\,dx+ $$

$$ +\cos\left(\frac{5}{12}\pi\right)0.033\,\pi^2\,\frac{6}{5}\,\Gamma\left(\frac{1}{6}\right)k^{7/6}\int_0^L C_\varepsilon^2(x)(L-x)^{5/6}\,dx. \qquad (22) $$

Adding (21) and (22), we find

$$ D_\chi(\rho)=0.033\,\pi^2\cos\left(\frac{5}{12}\pi\right)\frac{3}{5}\,\Gamma\left(\frac{1}{6}\right)k^{7/6}\int_0^L C_\varepsilon^2(x)(L-x)^{5/6}\,dx+O\left(\frac{1}{\rho}\right), \qquad (23) $$

from which we obtain the final expression

$$ \langle\chi\rangle^2=\frac{3\cdot 0.033\,\pi^2}{10}\cos\left(\frac{5}{12}\pi\right)\Gamma\left(\frac{1}{6}\right)k^{7/6}\int_0^L C_\varepsilon^2(x)(L-x)^{5/6}\,dx= $$

$$ =0.141\,k^{7/6}\int_0^L C_\varepsilon^2(x)(L-x)^{5/6}\,dx \quad (D\gg 1). \qquad (24) $$

If we set $C_\varepsilon^2(x)=\text{const}$ in (24), we obtain the previous expression (47.31).

The function $D_S(\rho)$ can be found from the equality $D_S+D_\chi=D_1$. For $\rho\gg\sqrt{\lambda L}$, when $D_\chi(\rho)$ is constant and small compared to D_1, we have $D_S(\rho)\approx D_1(\rho)$. Therefore,

$$ D_S(\rho)=0.73\,k^2\rho^{5/3}\int_0^L C_\varepsilon^2(x)\,dx \quad (\rho\gg\sqrt{\lambda L}). \qquad (25) $$

For $\rho\lesssim\sqrt{\lambda L}$, the function $D_S(\rho)$ strongly depends on $D_\chi(\rho)$, i.e., on the actual form of $C_\varepsilon^2(x)$, and it is no longer a universal function.

It follows from the preceding that the contribution of the various inhomogeneities to phase fluctuations is independent of their position (e.g., the phase shift introduced by a plane-parallel plate is independent of the exact position of the plate along the beam). On the other hand, the effect of dielectric constant inhomogeneities on amplitude fluctuations is more pronounced the farther the inhomogeneity is located from the observation point. This is consistent with the well known fact that a lens situated immediately in front of the observation point does not affect the light intensity at that point.*

* At a first glance it would appear that expressions (19), (24) contradict the reciprocity principle, since the values of the integrals depend on whether the origin is at the source or at the observation point. Interchanging the position of the source and the observation point, we obtain from (19), (24) different values of $\langle\chi^2\rangle$. This apparent contradiction is resolved if we remember that expressions (19), (24) apply to a plane wave. Application of the reciprocity principle to a plane wave implies an infinitely removed point source situated behind an inhomogeneous layer in free space. Therefore, in terms of plane waves, the reciprocity principle maintains that if a point source is placed at the observation point, and the observation point is moved to infinity (into the free space beyond the inhomogeneous layer), the level of fluctuations will not change. This transposition, however, is by no means equivalent to simply interchanging the plane wave source and the observation point. Note that for a spherical wave the weighting function before $C_\varepsilon^2(x)$ in the integral expression for $\langle\chi^2\rangle$ is symmetric in relation to x and $L-x$, so that $\langle\chi^2\rangle$ is the same for waves propagating in opposite directions.

§ 49. Amplitude fluctuations of a spherical wave

To calculate the fluctuations of a spherical wave, we use expression (45.14) which gives $\Phi_1(\boldsymbol{r})$ for any function $\Psi_0(\boldsymbol{r})$:

$$\Phi_1(\boldsymbol{r}) = \frac{k^2}{4\pi}\int \frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|}\frac{\Psi_0(\boldsymbol{r}')}{\Psi_0(\boldsymbol{r})}\varepsilon_1(\boldsymbol{r}')\,d^3r'. \tag{1}$$

If $\Psi_0(\boldsymbol{r})$ is a spherical wave propagating from the origin,

$$\frac{\Psi_0(\boldsymbol{r}')}{\Psi_0(\boldsymbol{r})} = \frac{r}{r'}e^{ik(r'-r)} \tag{2}$$

and (1) takes the form

$$\Phi_1(\boldsymbol{r}) = \frac{k^2 r}{4\pi}\int \frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|+ik(r'-r)}}{r'|\boldsymbol{r}-\boldsymbol{r}'|}\varepsilon_1(\boldsymbol{r}')\,d^3r'. \tag{3}$$

For $\lambda \ll \lambda_0$, as in the case of a plane wave, the important region in (3) is only that between the source and the observation point adjoining the straight line between the two points. Let the x axis be directed to the observation point. Then $\boldsymbol{r} = \{x, 0, 0\}$ and in the relevant integration region $|x - x'| \gg |y'|, |z'|$. Therefore, as for a plane wave, we may take in the exponential term

$$|\boldsymbol{r}-\boldsymbol{r}'| = x - x' + \frac{y'^2+z'^2}{2(x-x')} + \ldots, \tag{4}$$

and in the denominator of (3) simply $|\boldsymbol{r}-\boldsymbol{r}'| = x - x' + \ldots$. The integration over x' is carried out for $0 < x' < x$, and the integrals over y' and z' are taken from minus infinity to plus infinity. The variable r' in the exponential function may also be written in the form $r' = x' + \frac{y'^2+z'^2}{2x'} + \ldots$, and in the denominator we simply replace it by x'. We thus obtain for Φ_1

$$\Phi_1(x, 0, 0) = \frac{k^2 x}{4\pi}\int_0^x \frac{dx'}{(x-x')x'}\iint_{-\infty}^{\infty} dy'\,dz'\exp\left\{ik\frac{x(y'^2+z'^2)}{2x'(x-x')}\right\}\varepsilon_1(x', y', z'). \tag{5}$$

Let us calculate $\langle\Phi_1\Phi_1^*\rangle$ and $\langle\Phi_1^2\rangle$, which give the mean square fluctuations of log amplitude and phase. At first we assume homogeneous turbulence. For this case

$$\langle\Phi_1\Phi_1^*\rangle = \frac{k^4x^2}{16\pi^2}\int_0^x\int_0^x \frac{dx'\,dx''}{x'(x-x')(x-x'')x''}\iiiint_{-\infty}^{\infty}\exp\left\{\frac{ikx}{2}\times\right.$$
$$\left.\times\left[\frac{y'^2+z'^2}{x'(x-x')} - \frac{y''^2+z''^2}{x''(x-x'')}\right]\right\}B_\varepsilon(x'-x'', y'-y'', z'-z'')\,dy'\,dz\,dy''\,dz''. \tag{6}$$

Putting $y' - y'' = \eta$, $z' - z'' = \zeta$, $y'' = y$, $z'' = z$, we obtain

$$\langle\Phi_1\Phi_1^*\rangle = \frac{k^4x^2}{16\pi^2}\int_0^x\int_0^x \frac{dx'\,dx''}{x'(x-x')(x-x'')x''}\iint_{-\infty}^{\infty}B_\varepsilon(x'-x'', \eta, \zeta)\,d\eta\,d\zeta\,\times$$
$$\times\iint_{-\infty}^{\infty}\exp\left\{\frac{ikx}{2}\left[-\frac{y^2+z^2}{x''(x-x'')} + \frac{y^2+2y\eta+\eta^2+z^2+2z\zeta+\zeta^2}{x'(x-x')}\right]\right\}dy\,dx. \tag{7}$$

The inner integral over y and z is readily evaluated; it is equal to

$$\frac{2\pi i x' x'' (x-x')(x-x'')}{kx(x'-x'')(x'+x''-x)} \exp\left\{-\frac{ikx(\eta^2+\zeta^2)}{2(x'-x'')(x'+x''-x)}\right\}.$$

Inserting in (7) gives

$$\langle\Phi_1\Phi_1^*\rangle = \frac{ik^3x}{8\pi}\int_0^x\int_0^x \frac{dx'\,dx''}{(x'-x'')(x'+x''-x)} \times$$
$$\iint_{-\infty}^{\infty} B_\varepsilon(x'-x'', \eta, \zeta)\exp\left\{\frac{ikx(\eta^2+\zeta^2)}{2(x'-x'')(x-x'-x'')}\right\}d\eta\,d\zeta. \quad (8)$$

We will use the two-dimensional spectral expansion of $B_\varepsilon(x'-x'', \eta, \zeta)$, (5.31):

$$B_\varepsilon(x'-x'', \eta, \zeta) = \iint_{-\infty}^{\infty} e^{i(\varkappa_2\eta+\varkappa_3\zeta)} F_\varepsilon(\varkappa_2, \varkappa_3, x'-x'')\,d\varkappa_2\,d\varkappa_3. \quad (9)$$

Inserting (9) in (8), we obtain

$$\langle\Phi_1\Phi_1^*\rangle = -\frac{ik^3x}{8\pi}\int_0^x\int_0^x \frac{dx'\,dx''}{(x'-x'')(x-x'-x'')}\iint_{-\infty}^{\infty} F_\varepsilon(\varkappa_2, \varkappa_3, x'-x'')d\varkappa_2 d\varkappa_3 \times$$
$$\times \iint_{-\infty}^{\infty} \exp\left\{i(\varkappa_2\eta+\varkappa_3\zeta) - \frac{kx(\eta^2+\zeta^2)}{2i(x'-x'')(x-x'-x'')}\right\}d\eta\,d\zeta. \quad (10)$$

Integration over η, ζ readily gives

$$\langle\Phi_1\Phi_1^*\rangle = \frac{k^2}{4}\int_0^x\int_0^x dx'\,dx'' \iint_{-\infty}^{\infty} F_\varepsilon(\varkappa_2, \varkappa_3, x'-x'') \times \exp\left\{\frac{i\varkappa^2(x'-x'')(x-x'-x'')}{2kx}\right\}d\varkappa_2\,d\varkappa_3, \quad (11)$$

where $\varkappa^2 = \varkappa_2^2 + \varkappa_3^2$. In the main region of integration over $\varkappa$ we have $|\varkappa(x'-x'')| \lesssim 1$, and thus

$$\left|\frac{\varkappa^2(x'-x'')(x-x'-x'')}{2kx}\right| \lesssim \left|\frac{\varkappa(x-x'-x'')}{2kx}\right| \sim \frac{\varkappa}{k} \ll 1,$$

since $\varkappa/k \lesssim \lambda/\lambda_0 \ll 1$. The exponential factor in (11) may therefore be taken equal to unity:

$$\langle\Phi_1\Phi_1^*\rangle = \frac{k^2}{4}\int_0^x\int_0^x dx'\,dx'' \iint_{-\infty}^{\infty} F_\varepsilon(\varkappa_2, \varkappa_3, x'-x'')\,d\varkappa_2\,d\varkappa_3 =$$
$$= \frac{k^2}{4}\iint_{-\infty}^{\infty} d\varkappa_2\,d\varkappa_3 \int_0^x\int_0^x F_\varepsilon(\varkappa_2, \varkappa_3, x'-x'')\,dx'\,dx''. \quad (12)$$

The integrand in the inner integral is an even function of $(x'-x'')$. Using the relation

$$\int_0^x\int_0^x F_\varepsilon(\varkappa_2, \varkappa_3, x'-x'')\,dx'\,dx'' = 2\int_0^x (x-\xi)F_\varepsilon(\varkappa_2, \varkappa_3, \xi)\,d\xi,$$

we obtain

$$\langle\Phi_1\Phi_1^*\rangle = \frac{k^2}{2}\iint_{-\infty}^{\infty} d\varkappa_2\,d\varkappa_3 \int_0^x (x-\xi)F_\varepsilon(\varkappa_2, \varkappa_3, \xi)\,d\xi. \quad (13)$$

In calculating amplitude fluctuations, we have already established that the main contribution is due to wavenumbers of the order $1/\sqrt{\lambda x}$. Hence it follows that the only significant integration region in (13) is $\xi \lesssim \sqrt{\lambda x} \ll x$. We may therefore take $x - \xi \approx x$ and extend the integration over x to infinity. Using the relation

$$\int_0^\infty F_\varepsilon(\varkappa_2, \varkappa_3, \xi)\, d\xi = \pi\Phi_\varepsilon(0, \varkappa_2, \varkappa_3),$$

we obtain

$$\langle\Phi_1\Phi_1^*\rangle = \frac{\pi k^2 x}{2} \iint_{-\infty}^{\infty} \Phi_\varepsilon(0, \varkappa_2, \varkappa_3)\, d\varkappa_2\, d\varkappa_3. \tag{14}$$

Let us now calculate $\langle\Phi_1^2\rangle$:

$$\langle\Phi_1^2\rangle = \frac{k^4 x^2}{16\pi^2} \int_0^x\int_0^x \frac{dx'\, dx''}{x'x''(x-x')(x-x'')} \iiiint_{-\infty}^{\infty} \exp\left\{\frac{ikx}{2}\left[\frac{y'^2+z'^2}{x'(x-x')} + \right.\right.$$
$$\left.\left. + \frac{y''^2+z''^2}{x''(x-x'')}\right]\right\} B_\varepsilon(x'-x'', y'-y'', z'-z'')\, dy'\, dz'\, dy''\, dz''. \tag{15}$$

As with integral (6), we introduce new variables $y' - y'' = \eta$, $z' - z'' = \zeta$, $y'' = y$, $z'' = z$. Integration over y, z readily gives an expression analogous to (8):

$$\langle\Phi_1^2\rangle = \frac{ik^3 x}{8\pi} \int_0^x\int_0^x \frac{dx'\, dx''}{x'(x-x') + x''(x-x'')} \iint_{-\infty}^{\infty} B_\varepsilon(x'-x'', \eta, \zeta) \times$$
$$\times \exp\left\{\frac{ikx(\eta^2+\zeta^2)}{2[x'(x-x') + x''(x-x'')]}\right\} d\eta\, d\zeta. \tag{16}$$

Using again the spectral expansion (9) and integrating over η, ζ, we obtain

$$\langle\Phi_1^2\rangle = -\frac{k^2}{4} \int_0^x\int_0^x dx'\, dx'' \iint_{-\infty}^{\infty} d\varkappa_2\, d\varkappa_3\, F_\varepsilon(\varkappa_2, \varkappa_3, x'-x'') \times$$
$$\times \exp\left\{-\frac{i\varkappa^2[x'(x-x') + x''(x-x'')]}{2kx}\right\}. \tag{17}$$

Changing to new variables $u = x' - x''$, $v = \frac{1}{2}(x' + x'')$, we write the argument of the exponential function in the form

$$-\frac{i\varkappa^2[x'(x-x') + x''(x-x'')]}{2kx} = -\frac{i\varkappa^2}{2kx}\left[2vx - 2v^2 - \frac{u^2}{2}\right]. \tag{18}$$

In the important region of the integration $\varkappa u \lesssim 1$, so that

$$\left|\frac{\varkappa^2 u^2}{4kx}\right| \lesssim \frac{1}{4kx} \ll 1$$

and consequently the term $u^2/2$ in (18) can be dropped. In view of the rapid decrease of the function $F_\varepsilon(\varkappa_2, \varkappa_3, u)$ the integration over u can be extended to infinite limits. We thus obtain

$$\langle\Phi_1^2\rangle = -\frac{\pi k^2}{2} \iint_{-\infty}^{\infty} \Phi_\varepsilon(0, \varkappa_2, \varkappa_3)\, d\varkappa_2\, d\varkappa_3 \int_0^x \exp\left\{-\frac{i\varkappa^2 v(x-v)}{kx}\right\} dv. \tag{19}$$

In what follows we will be interested in Re $\langle\Phi_1^2\rangle$, in terms of which the phase and amplitude fluctuations are expressed. Indeed, using the relation $\Phi_1 = \chi + iS_1$, we obtain

$$\langle\Phi_1\Phi_1^*\rangle = \langle\chi^2\rangle + \langle S^2\rangle, \quad \langle\Phi_1^2\rangle = \langle\chi^2\rangle - \langle S_1^2\rangle + 2i\langle\chi S_1\rangle,$$

from which

$$\langle\chi^2\rangle = \frac{1}{2}\left[\langle\Phi_1\Phi_1^*\rangle + \mathrm{Re}\,\langle\Phi_1^2\rangle\right].$$

Taking the real part of (19), we obtain

$$\mathrm{Re}\,\langle\Phi_1^2\rangle = -\frac{\pi k^2}{2}\iint\limits_{-\infty}^{\infty}\Phi_\varepsilon(0, \varkappa_2, \varkappa_3)\,d\varkappa_2\,d\varkappa_3\int\limits_0^x\cos\left\{\frac{\varkappa^2 v(x-v)}{kx}\right\}dv. \tag{20}$$

Consider the integral

$$I = \int\limits_0^x\cos\frac{\varkappa^2 v(x-v)}{kx}\,dv. \tag{21}$$

The substitution $\xi = -\frac{x}{2} + v$ gives

$$I = \int\limits_{-x/2}^{x/2}\cos\frac{\varkappa^2\left(\frac{x^2}{4} - \xi^2\right)}{kx}\,d\xi = 2\int\limits_0^{x/2}\cos\frac{\varkappa^2\left(\frac{x^2}{4} - \xi^2\right)}{kx}\,d\xi =$$
$$= 2\left[\cos\frac{\varkappa^2 x}{4k}\int\limits_0^{x/2}\cos\frac{\varkappa^2\xi^2}{kx}\,d\xi + \sin\frac{\varkappa^2 x}{4k}\int\limits_0^{x/2}\sin\frac{\varkappa^2\xi^2}{kx}\,d\xi\right].$$

The integrals entering this expression are expressed in terms of Fresnel integrals

$$C(z) = \int\limits_0^z\cos\frac{\pi t^2}{2}\,dt, \quad S(z) = \int\limits_0^z\sin\frac{\pi t^2}{2}\,dt,$$

and we obtain

$$I = x\sqrt{\frac{2\pi k}{\varkappa^2 x}}\left[\cos\frac{\varkappa^2 x}{4k}C\left(\sqrt{\frac{\varkappa^2 x}{2\pi k}}\right) + \sin\frac{\varkappa^2 x}{4k}S\left(\sqrt{\frac{\varkappa^2 x}{2\pi k}}\right)\right]. \tag{22}$$

Inserting (22) in (20), we find

$$\mathrm{Re}\,\langle\Phi_1^2\rangle = -\frac{\pi k^2 x}{2}\iint\limits_{-\infty}^{\infty}\sqrt{\frac{2\pi k}{\varkappa^2 x}}\left[\cos\frac{\varkappa^2 x}{4k}C\left(\sqrt{\frac{\varkappa^2 x}{2\pi k}}\right) + \right.$$
$$\left. + \sin\frac{\varkappa^2 x}{4k}S\left(\sqrt{\frac{\varkappa^2 x}{2\pi k}}\right)\right]\Phi_\varepsilon(0, \varkappa_2, \varkappa_3)\,d\varkappa_2\,d\varkappa_3. \tag{23}$$

Inserting (14) and (23) in the equality

$$\langle\chi^2\rangle = \frac{1}{2}\left[\langle\Phi_1\Phi_1^*\rangle + \mathrm{Re}\,\langle\Phi_1^2\rangle\right],$$

we obtain

$$\langle\chi^2\rangle = \frac{\pi k^2 x}{4} \iint\limits_{-\infty}^{\infty} \Phi_\varepsilon(0, \varkappa_2, \varkappa_3)\left\{1 - \sqrt{\frac{2\pi k}{\varkappa^2 x}} \times \right.$$
$$\left. \times \left[\cos\frac{\varkappa^2 x}{4k} C\left(\sqrt{\frac{\varkappa^2 x}{2\pi k}}\right) + \sin\frac{\varkappa^2 x}{4k} S\left(\sqrt{\frac{\varkappa^2 x}{2\pi k}}\right)\right]\right\} d\varkappa_2\, d\varkappa_3. \tag{24}$$

In the case of isotropic turbulence, when $\Phi_\varepsilon(0, \varkappa_2, \varkappa_3) = \Phi_\varepsilon(\varkappa)$, we may change to polar coordinates for $\varkappa_2, \varkappa_3$ in (24) and integrate over the angular variable:

$$\langle\chi^2\rangle = \frac{1}{2}\pi^2 k^2 x \int\limits_0^\infty \Phi_\varepsilon(\varkappa)\left\{1 - \sqrt{\frac{2\pi k}{\varkappa^2 x}} \times \right.$$
$$\left. \times \left[\cos\frac{\varkappa^2 x}{4k} C\left(\sqrt{\frac{\varkappa^2 x}{2\pi k}}\right) + \sin\frac{\varkappa^2 x}{4k} S\left(\sqrt{\frac{\varkappa^2 x}{2\pi k}}\right)\right]\right\} \varkappa\, d\varkappa. \tag{25}$$

For $\varkappa \to 0$ the expression in braces in (25) approaches zero as $\varkappa^4$. The integral in (25) therefore converges even if $\Phi_\varepsilon(\varkappa)$ has a singularity for $\varkappa \to 0$ (e.g., if $\Phi_\varepsilon(\varkappa) \sim \varkappa^{-11/3}$). Physically, this means that amplitude fluctuations are not strongly affected by the large-scale components of the turbulence spectrum (as is the case with plane waves). We can thus extend expression (25) to the case of locally isotropic turbulence, although originally it was derived assuming homogeneous turbulence. The expression for the mean square phase fluctuations of a spherical wave differs from (25) only by the sign of the first term in braces. Because of this small difference, however, the expression for $\langle S_1^2\rangle$ in the case of locally isotropic turbulence turns out to be infinite (as for the plane wave case).

In the limiting case $\sqrt{\lambda x} \ll \lambda_0$, we have $\frac{\varkappa^2 x}{2\pi k} \ll 1$ over the important region of integration. Thus, expanding the integrand in a series, we obtain

$$\langle\chi^2\rangle = \frac{\pi^2}{120} x^3 \int\limits_0^\infty \Phi_\varepsilon(\varkappa)\varkappa^5 d\varkappa \quad (\sqrt{\lambda x} \ll \lambda_0). \tag{26}$$

Comparing this expression with the corresponding result derived using geometrical optics, we conclude that they are identical. In particular, for $\sqrt{\lambda x} \ll \lambda_0$, the mean square fluctuations of the log amplitude for a spherical wave is 1/10 of that for a plane wave /65, 95, 112/.

Let us calculate the amplitude fluctuations using the spectrum

$$\Phi_\varepsilon(\varkappa) = 0.033 C_\varepsilon^2 \varkappa^{-11/3} \exp\left(-\frac{\varkappa^2}{\varkappa_m^2}\right). \tag{27}$$

The calculations are best carried out starting with the intermediate expression (19). If (27) is inserted in (19), we obtain a divergent integral. However, following the procedure used with plane-wave fluctuations, we first evaluate these integrals with the spectrum

$$\Phi_\varepsilon(\varkappa) = 0.033 C_\varepsilon^2 \varkappa^{-p} \exp\left(-\frac{\varkappa^2}{\varkappa_m^2}\right) \tag{28}$$

for such values of p where the integrals converge, and then after forming the necessary combination, we take $p = 11/3$.

Inserting (28) in (19), we obtain

$$\langle\Phi_1^2\rangle = -\pi^2k^2\int_0^\infty \Phi_\varepsilon(\varkappa)\varkappa\,d\varkappa\int_0^x \exp\left\{-\frac{i\varkappa^2 v(x-v)}{kx}\right\}dv = -0.033\pi^2C_\varepsilon^2k^2\int_0^x dv\int_0^\infty \varkappa^{1-p}\exp\left\{-\varkappa^2\left[\frac{1}{\varkappa_m^2}+\frac{iv(x-v)}{kx}\right]\right\}d\varkappa. \quad (29)$$

The inner integral is evaluated and is equal to

$$\frac{1}{2}\Gamma\left(1-\frac{p}{2}\right)\left[\frac{1}{\varkappa_m^2}+\frac{iv(x-v)}{kx}\right]^{\frac{p}{2}-1}.$$

Consequently,

$$\langle\Phi_1^2\rangle = -\frac{1}{2}\,0.033\pi^2\Gamma\left(1-\frac{p}{2}\right)C_\varepsilon^2k^2\int_0^x\left[\frac{1}{\varkappa_m^2}+\frac{iv(x-v)}{kx}\right]^{\frac{p}{2}-1}dv. \quad (30)$$

Substituting $v = \frac{x}{2}+\xi$ gives

$$\langle\Phi_1^2\rangle = -0.033\pi^2\Gamma\left(1-\frac{p}{2}\right)C_\varepsilon^2k^2\int_0^{x/2}\left[\frac{1}{\varkappa_m^2}+\frac{ix}{4k}-\frac{i\xi^2}{kx}\right]^{\frac{p}{2}-1}d\xi. \quad (31)$$

In the following we are interested in the case $\sqrt{\lambda x}\gg\lambda_0$ (the opposite case was considered by the geometrical optics method). Then in (31) $1/\varkappa_m^2 \ll |ix/4k|$ and $1/\varkappa_m^2$ can be omitted:

$$\langle\Phi_1^2\rangle \approx -0.033\pi^2\Gamma\left(1-\frac{p}{2}\right)C_\varepsilon^2k^2\left(\frac{ix}{4k}\right)^{\frac{p}{2}-1}\int_0^{x/2}\left[1-\frac{4\xi^2}{x^2}\right]^{(p-2)/2}d\xi = -\left(i^{\frac{p}{2}-1}\right)0.033\pi^2\Gamma\left(1-\frac{p}{2}\right)2^{1-p}C_\varepsilon^2k^{3-\frac{p}{2}}x^{p/2}\int_0^1(1-t^2)^{\frac{p}{2}-1}dt. \quad (32)$$

The integral in (32) may be expressed in terms of the gamma function:

$$\int_0^1(1-t^2)^{\frac{p}{2}-1}dt = \frac{1}{2}\,\mathrm{B}\left(\frac{1}{2},\frac{p}{2}\right) = \frac{1}{2}\,\frac{\sqrt{\pi}\,\Gamma\left(\frac{p}{2}\right)}{\Gamma\left(\frac{p+1}{2}\right)}.$$

Inserting this expression in (32), using the equality

$$\Gamma\left(\frac{p}{2}\right)\Gamma\left(1-\frac{p}{2}\right) = \frac{\pi}{\sin\frac{\pi p}{2}}$$

and taking the real part of (32), we obtain

$$\mathrm{Re}\,\langle\Phi_1^2\rangle = -\frac{0.033\pi^3\sqrt{\pi}}{2^{p+1}\Gamma\left(\frac{p+1}{2}\right)\cos\frac{\pi p}{4}}C_\varepsilon^2k^{3-\frac{p}{2}}x^{p/2}. \quad (33)$$

Let us now compute $\langle \Phi_1 \Phi_1^* \rangle$. Using (14), we get

$$\langle \Phi_1 \Phi_1^* \rangle = \pi^2 k^2 x \int_0^\infty \Phi_\varepsilon(\varkappa)\, \varkappa\, d\varkappa = 0.033\pi^2 C_\varepsilon^2 k^2 x \int_0^\infty \varkappa^{1-p} \exp\left(-\frac{\varkappa^2}{\varkappa_m^2}\right) d\varkappa =$$
$$= \frac{1}{2}\, 0.033\pi^2 \Gamma\left(1-\frac{p}{2}\right) C_\varepsilon^2 k^2 \varkappa_m^{2-p} x. \qquad (34)$$

Both expressions (33) and (34), were obtained for $p < 2$, when the corresponding integrals converge. However, as we have shown in connection with (25), the expression for $\langle \chi^2 \rangle$, which is equal to one-half the sum of (33) and (34), is finite for $p > 2$ also, and in particular for $p = 11/3$. Therefore in the expression

$$\langle \chi^2 \rangle = \frac{1}{2}\left\{\frac{1}{2}\, 0.033\pi^2 \Gamma\left(1-\frac{p}{2}\right) C_\varepsilon^2 k^2 \varkappa_m^{2-p} x - \frac{0.033\pi^3 \sqrt{\pi}}{2^{p+1}\Gamma\left(\frac{p+1}{2}\right)\cos\frac{\pi p}{4}} C_\varepsilon^2 k^{3-\frac{p}{2}} x^{p/2}\right\} \qquad (35)$$

we may take $p = 11/3$.* Note that the ratio of the first to the second term in (35) is of the order

$$\left(\frac{\lambda_0^2}{\lambda x}\right)^{5/6} \ll 1,$$

since $\sqrt{\lambda x}/\lambda_0$ is assumed to be large. Therefore in (35) it is only necessary to retain the second term, which gives

$$\langle \chi^2 \rangle = \frac{0.033\pi^3 \sqrt{\pi}}{2^{17/6}\Gamma\left(\frac{14}{6}\right)\left(-\cos\frac{11}{12}\pi\right)} C_\varepsilon^2 k^{7/6} x^{11/6} = 0.031 C_\varepsilon^2 k^{7/6} x^{11/6}. \qquad (36)$$

Expression (36) for $\langle \chi^2 \rangle$ differs from the corresponding plane-wave expression only by its numerical coefficient, 0.031 instead of 0.077. Thus, for $\sqrt{\lambda x} \gg \lambda_0$, the mean square log amplitude fluctuations of a spherical wave are approximately 2.5 times smaller than the corresponding plane-wave fluctuations.

The difference between plane-wave and spherical-wave amplitude fluctuations gradually diminishes as the distance of propagation x into the random medium increases. For $x \ll \lambda_0^2/\lambda$ the ratio of these amplitude fluctuations is 10, for $x \gg \frac{\lambda_0^2}{\lambda}$ it drops to 2.5, and for $x \gg \frac{L_0^2}{\lambda}$, where L_0 is the outer scale of turbulence, the fluctuations are of comparable magnitude. The latter follows from (25), which shows that for very large x the second term in braces may be neglected, as in the analogous plane-wave formula, so that identical equations are obtained in both cases.

§ 50. The limits for application of the first approximation of the method of smooth perturbations

The method of smooth perturbations is based on an approximate solution of the equation for the logarithm of the field Φ using an expansion in powers

* It is easily seen that for $p = 2$ expression (35) remains finite, i.e., this expression is an analytical function of p for $p < 4$ and we may safely use it for $p = \frac{11}{3} < 4$.

of the small parameter ε_1. Moreover, as in geometrical optics, we also use an expansion in powers of the small parameter λ/λ_0. This expansion, however, imposes much less rigid constraints on the solution than in the case of geometrical optics. As we have shown in §45, the change from the exact kernel of the diffraction problem $\exp(ikR)/R$ to the kernel $\exp\left(ik\frac{\rho^2}{2x}\right)\Big/x$

and at the same time omitting the integral over the region beyond the observation point (which is equivalent to replacing the complete Laplacian $\Delta\Phi$ by the transverse operator $\Delta_\perp\Phi$) is permissible if $\langle\Phi_{11}^2\rangle = \frac{1}{16}\sigma_\varepsilon^2$ is small compared to $\langle\Phi_1^2\rangle$. Instead of the last function we may use $\langle\chi^2\rangle$ (since $\langle S_1^2\rangle$ is always greater than $\langle\chi^2\rangle$). Using expression (47.31)

$$\langle\chi^2\rangle = 0.077C_\varepsilon^2k^{7/6}L^{11/6},$$

we can write this condition in the form

$$\frac{1}{16}\sigma_\varepsilon^2 \ll 0.077C_\varepsilon^2k^{7/6}L^{11/6}. \tag{1}$$

Condition (1) contains the parameter σ_ε^2 which is determined by large scale components in the turbulence. It can be estimated using the approximate relation $\sigma_\varepsilon^2 \approx 2C_\varepsilon^2L_0^{2/3}$ where L_0 is the outer scale of turbulence, near which the structure function $D_\varepsilon(r)$ saturates. Using this relation, we write (1) in the form

$$k^{7/6}L^{11/6}L_0^{-2/3} \gg 1 \tag{2}$$

or

$$\frac{L}{L_0} \gg \left(\frac{\lambda}{L}\right)^{7/4}. \tag{2a}$$

We see from (2a) that for $\lambda \ll L$ and $L > L_0$ (which is always true in practice) this condition does not impose important restrictions on the relevant parameters. A more significant constraint is (45.29)

$$L \ll \frac{\lambda_0^4}{\lambda^3}, \tag{3}$$

used in changing over from the exact kernel of the diffraction problem to the Fresnel kernel.

Let us now derive some estimates associated with the approximate solution of the nonlinear equation for Φ /94, 114–116/. The solution for Φ was obtained in the form of a series $\Phi = \Phi_1 + \Phi_2 + \ldots$, where only the first term Φ_1 was used. For Φ_2 we have by (45.31)

$$\Phi_2(L, y, z) = \frac{1}{4\pi}\int_0^L dx' \iint_{-\infty}^{\infty} dy'\,dz' \frac{\exp\left\{ik\frac{(y-y')^2+(z-z')^2}{2(L-x')}\right\}}{L-x'} [\nabla\Phi_1(x', y', z')]^2. \tag{4}$$

Let us first calculate the average value of the correction

$$\langle\Phi_2\rangle = \frac{1}{4\pi}\int_0^L dx' \iint_{-\infty}^{\infty} dy'\,dz' \frac{\exp\left\{ik\frac{(y-y')^2+(z-z')^2}{2(L-x')}\right\}}{L-x'} \langle[\nabla\Phi_1(x', y', z')]^2\rangle. \tag{5}$$

To obtain this estimate, it is necessary to find $\langle[\nabla\Phi_1]^2\rangle$. For this purpose, we consider the correlation function of the gradient Φ_1 in the plane $x = L$:

$$\langle\nabla\Phi_1(\boldsymbol{r}_1)\nabla\Phi_1(\boldsymbol{r}_2)\rangle = B_{\nabla\Phi}(\boldsymbol{r}_1-\boldsymbol{r}_2). \tag{6}$$

In computing $\nabla\Phi_1$ it suffices to take only its "transverse" part $\nabla_\perp$ (as in the substitution of $\Delta_\perp$ for Δ). Hence,

$$B_{\nabla\Phi}(\boldsymbol{r}_1-\boldsymbol{r}_2) \approx \langle\nabla_\perp\Phi_1(\boldsymbol{r}_1)\nabla_\perp\Phi_1(\boldsymbol{r}_2)\rangle = \nabla_\perp(\boldsymbol{r}_1)\nabla_\perp(\boldsymbol{r}_2)\langle\Phi_1(\boldsymbol{r}_1)\Phi_2(\boldsymbol{r}_2)\rangle.$$

But $\langle\Phi_1(\boldsymbol{r}_1)\Phi_1(\boldsymbol{r}_2)\rangle$ is a function of $\boldsymbol{r}_1-\boldsymbol{r}_2$ only, and we may therefore replace $\nabla_\perp(\boldsymbol{r}_2)$ by $-\nabla_\perp(\boldsymbol{r}_1)$:

$$B_{\nabla\Phi}(\boldsymbol{r}_1-\boldsymbol{r}_2) = -\nabla_\perp^2\langle\Phi_1(\boldsymbol{r}_1)\Phi_1(\boldsymbol{r}_2)\rangle. \tag{7}$$

We apply the operator $\nabla_\perp^2 = \Delta_\perp$ to (46.11):

$$D_2(\boldsymbol{r}_1-\boldsymbol{r}_2) \equiv \langle[\Phi_1(\boldsymbol{r}_1)-\Phi_1(\boldsymbol{r}_2)]^2\rangle = 2\langle\Phi_1^2\rangle - 2\langle\Phi_1(\boldsymbol{r}_1)\Phi_1(\boldsymbol{r}_2)\rangle.$$

This gives

$$-\Delta_\perp\langle\Phi_1(\boldsymbol{r}_1)\Phi_1(\boldsymbol{r}_2)\rangle = \frac{1}{2}\Delta_\perp D_2(\boldsymbol{r}_1-\boldsymbol{r}_2),$$

since $\langle\Phi_1^2\rangle$ depends only on the longitudinal coordinates.

Inserting this expression in (7), we find

$$B_{\nabla\Phi}(\boldsymbol{r}_1-\boldsymbol{r}_2) = \frac{1}{2}\Delta_\perp D_2(\boldsymbol{r}_1-\boldsymbol{r}_2). \tag{8}$$

We put $\boldsymbol{r}_1 = \boldsymbol{r}_2$:

$$\langle[\nabla\Phi_1]^2\rangle = B_{\nabla\Phi}(0) = \frac{1}{2}\Delta_\perp D_2(0). \tag{9}$$

To obtain the right-hand side of (9), we use the spectral expansion of $D_2(\rho)$:

$$D_2(\boldsymbol{\rho}) = 2\iint\limits_{-\infty}^{\infty}[1-e^{i(\varkappa_2\eta+\varkappa_3\zeta)}]F_2(\varkappa_2, \varkappa_3, x')\,d\varkappa_2\,d\varkappa_3, \tag{10}$$

where (see (46.35))

$$F_2(\varkappa_2, \varkappa_3, x') = \frac{i\pi}{2}\frac{k^3}{\varkappa^2}\left[1-\exp\left(-\frac{i\varkappa^2x'}{k}\right)\right]\Phi_\varepsilon(0, \varkappa_2, \varkappa_3). \tag{11}$$

Acting on (10) with $\Delta_\perp$, then putting $\eta = \zeta = 0$, and inserting (11), we obtain

$$\langle[\nabla\Phi_1(x', y', z')]^2\rangle = \frac{i\pi k^3}{2}\iint\limits_{-\infty}^{\infty}\left[1-\exp\left(-\frac{i\varkappa^2x'}{k}\right)\right]\Phi_\varepsilon(0, \varkappa_2, \varkappa_3)\,d\varkappa_2\,d\varkappa_3. \tag{12}$$

Expression (12) is independent of y', z'. In (5) we may therefore integrate over these variables, which gives

$$\langle\Phi_2(L, y, z)\rangle = \frac{i}{2k}\int\limits_0^L\langle[\nabla\Phi_1(x', y', z')]^2\rangle\,dx'. \tag{13}$$

Inserting (12) and integrating over x', we find

$$\langle \Phi_2(L, y, z)\rangle = -\frac{\pi k^2}{4}\iint\limits_{-\infty}^{\infty}\left\{L+\frac{ik}{\varkappa^2}\left[1-\exp\left(-\frac{i\varkappa^2 L}{k}\right)\right]\right\}\Phi_\varepsilon(0, \varkappa_2, \varkappa_3)\,d\varkappa_2\,d\varkappa_3. \quad (14)$$

Consider separately the real and the imaginary part of $\langle\Phi_2\rangle$:

$$\operatorname{Re}\langle \Phi_2(L, y, z)\rangle = -\frac{\pi k^2 L}{4}\iint\limits_{-\infty}^{\infty}\left[1-\frac{k}{\varkappa^2 L}\sin\frac{\varkappa^2 L}{k}\right]\Phi_\varepsilon(0, \varkappa_2, \varkappa_3)\,d\varkappa_2\,d\varkappa_3, \quad (14a)$$

$$\operatorname{Im}\langle \Phi_2(L, y, z)\rangle = -\frac{\pi k^3}{2}\iint\limits_{-\infty}^{\infty}\frac{\sin^2\frac{\varkappa^2 L}{2k}}{\varkappa^2}\Phi_\varepsilon(0, \varkappa_2, \varkappa_3)\,d\varkappa_2\,d\varkappa_3. \quad (15)$$

In §46 we derived expressions (46.36)—(46.38) for the spectral densities of the fluctuations F_χ and $F_{\chi S}$. Comparing (14) with (46.36) we see that

$$\operatorname{Re}\langle \Phi_2(L, y, z)\rangle = -\iint\limits_{-\infty}^{\infty}F_\chi(L, \varkappa_2, \varkappa_3)\,d\varkappa_2\,d\varkappa_3 = -\langle\chi^2(L)\rangle, \quad (16)$$

$$\operatorname{Im}\langle \Phi_2(L, y, z)\rangle = -\iint\limits_{-\infty}^{\infty}F_{\chi S}(L, \varkappa_2, \varkappa_3)\,d\varkappa_2\,d\varkappa_3 = -\langle\chi(L)\,S_1(L)\rangle. \quad (17)$$

In §47 (see (47.38)) we established that $\langle\chi S_1\rangle$ and $\langle\chi^2\rangle$ are proportional to each other (differing only by a numerical coefficient). Therefore both (16) and (17) essentially lead to the same condition for the application of the first approximation of the method of smooth perturbations, namely

$$\langle\chi^2(L)\rangle \ll 1. \quad (18)$$

When this condition is satisfied both the log amplitude and the phase factor of the incident wave do not differ appreciably from the unperturbed values $\ln A_0$, $\exp(ikx)$ used in the calculations.

Condition (18) is connected with the law of conservation of energy. The wave field Ψ is related to Φ by $\Psi = \exp\Phi$. The energy flux density (ignoring the insignificant fluctuations in the direction of propagation, which introduce corrections of a higher order of smallness) is proportional to

$$\Psi\Psi^* = \exp(\Phi+\Phi^*) = A_0^2\exp(\Phi_1+\Phi_1^*+\Phi_2+\Phi_2^*+\ldots) = A_0^2\exp\{2\chi+2\operatorname{Re}\Phi_2+\ldots\}.$$

Writing $\Phi_2 = \langle\Phi_2\rangle + [\Phi_2 - \langle\Phi_2\rangle]$, we obtain

$$\Psi\Psi^* = A_0^2\exp\{2\operatorname{Re}\langle\Phi_2\rangle\}\exp\{2\chi+2\operatorname{Re}[\Phi_2-\langle\Phi_2\rangle]+\ldots\}.$$

Let us average this expression. When a plane wave propagates in an infinite space in which there is no absorption, the energy flux density is clearly conserved, i.e., we should have $\langle\Psi\Psi^*\rangle = \text{const}$. Since the random variables χ and $\operatorname{Re}[\Phi_2-\langle\Phi_2\rangle]$ are in the argument of the exponential function, averaging is impossible unless we know the probability distribution of these variables. We have established in the preceding that χ is expressed by an integral of the random variable ε_1. If the distance L is much greater than the correlation radius L_0 of dielectric constant fluctuations, the probability distribution of χ approaches the normal distribution by virtue of the central limit theorem of probability theory.*

Similar arguments can be applied to $\Phi_2 - \langle\Phi_2\rangle$, but in any event we ignore this term in averaging, as it only contributes terms of higher order of smallness.

For any random variable ξ (with normal probability distribution) we have

$$\langle \exp \xi\rangle = \exp\left\{\frac{1}{2}\langle\xi^2\rangle\right\} \qquad (\langle\xi\rangle = 0).$$

Using this equality, we obtain

$$\langle\Psi\Psi^*\rangle = A_0^2 \exp\{2\,\mathrm{Re}\,\langle\Phi_2\rangle\}\exp\{2\langle\chi^2\rangle\} = A_0^2\exp\{2\,[\langle\chi^2\rangle + \mathrm{Re}\,\langle\Phi_2\rangle]\}. \tag{19}$$

Inserting (16) gives

$$\langle\Psi\Psi^*\rangle = A_0^2 = \mathrm{const}.$$

Thus, in order to avoid a contradiction with the principle of energy conservation, the second approximation of the method of smooth perturbations must be taken into consideration in the expression for the mean field. If we retain only the first term Φ_1, we should impose the additional condition (18), so as to make the fractional energy transferred from the regular incident wave to the fluctuations small.

It would appear at first glance that the introduction of the attenuation of the mean field described by (16) and (17) should markedly expand the region of validity of the method of smooth perturbations. In this case, the incident wave is written in the form

$$\Phi_0 = \ln A_0 + ikr - \langle\chi^2(\boldsymbol{r})\rangle - i\langle\chi(\boldsymbol{r})S_1(\boldsymbol{r})\rangle, \tag{20}$$

where $\langle\chi^2\rangle$ and $\langle\chi S_1\rangle$ are computed in the first approximation. Proceeding from (20), we can also improve the value of the fluctuating component of the field Φ_1. In this case the expressions for the mean square fluctuations of the log amplitude and the phase coincide with the preceding expressions, but the fluctuations are now considered as deviations from the mean field (20), and not from the unperturbed wave. More detailed analysis, which is unfortunately too tedious for the present treatment, leads to the conclusion, however, that in this case the next order correction in the expression for $\langle\Phi\rangle$ is proportional to $\langle\chi^2\rangle^2$. The expression for $\langle\Phi\rangle$ obtained up to terms of fourth order using the "2/3 law" model is

$$\langle\Phi\rangle = \ln A_0 + ikr - \langle\chi^2(\boldsymbol{r})\rangle - i\langle\chi(\boldsymbol{r})S_1(\boldsymbol{r})\rangle - 2{,}9\,(1 + i\sqrt{3})\,\langle\chi^2(\boldsymbol{r})\rangle^2. \tag{21}$$

Therefore, expression (20) describing the attenuation of the mean field is also valid only for small $\langle\chi\rangle^2$.

This conclusion at first seems to contradict expression (19), which shows that for Φ_0 defined by (20) the law of energy conservation is not

* This is so if certain special conditions imposed on the moments of ε_1 are satisfied; we will not go into these details here.

violated. Actually, however, this simply means that for $\langle\chi^2\rangle \sim 1$ all terms in a perturbation theoretical series are equally significant both in the expression for the mean square logarithm of the field and in the expression for the mean square log amplitude.

Since the method of smooth perturbations is also limited by the condition of small amplitude fluctuations (18), one naturally tends to ask whether there is any point in using this method at all. Perhaps it would be simpler to apply the perturbation technique to the original wave equation only? If in the method of smooth perturbations the log amplitude is replaced by the first term in its series expansion, the resulting expression can in fact be directly obtained by the ordinary perturbation technique.

Formally, the advantage of the method of smooth perturbations is that the condition of smallness is imposed not on the field fluctuations but on its logarithm field; this clearly is a much less rigid constraint. There is, however, still another advantage. In the usual perturbation theory, the scattered field is a random complex variable with a normal distribution (by virtue of the central limit theorem). Hence it follows that the amplitude probability distribution is in general a displaced Rayleigh distribution.* For this probability distribution, however, the ratio $\langle[A-\langle A\rangle]^2\rangle/\langle A\rangle^2$ is at most $\frac{4-\pi}{\pi}=0.27$. At the same time the method of smooth perturbations leads to a log normal distribution for the amplitude fluctuations. This distribution allows any value of the ratio $\langle[A-\langle A\rangle]^2\rangle/\langle A\rangle^2$. Experimental values of $\langle[A-\langle A\rangle]^2\rangle/\langle A\rangle^2$ are often found to be greater than 0.27 (see Chapter 4) which cannot be explained by the ordinary perturbation theory. On the other hand, the log-normal distribution shows excellent agreement with experiment.

Thus, despite the substantial restriction (18), the method of smooth perturbations has a wider region of application than the conventional perturbation technique, which describes single scattering only.

Experimental studies of amplitude fluctuations of light propagating in the ground layer of the atmosphere established that the method of smooth perturbations is experimentally justified when

$$\sqrt{\langle\chi^2\rangle} = 0.28 C_\varepsilon k^{7/12} L^{11/12} < 0.8.$$

In this case, the ratio $\langle[A-\langle A\rangle]^2\rangle/\langle A\rangle^2$ is equal to 0.90. There is thus a wide range of conditions when the method of smooth perturbations is still applicable, while the ordinary perturbation technique is invalid.

If, however, the calculated value of

$$\sqrt{\langle\chi^2\rangle} = 0.28 C_\varepsilon k^{7/12} L^{11/12}$$

is greater than 0.8, a marked divergence is observed between theory and experiment (see Figure 60 on p. 300). This is due to the fact that the first approximation in the method of smooth perturbations is not applicable.

In the next section, we will present a computational technique that can be used for large fluctuations of the log amplitude, when condition (18) is broken.

§ 51. [This section has been omitted in the translation at the author's request.]

* [Rice-Nakagami distribution.]

Chapter 4

EXPERIMENTAL DATA ON THE PROPAGATION OF LIGHT, RADIO WAVES, AND SOUND IN A TURBULENT ATMOSPHERE AND THEIR INTERPRETATION

We will now consider some applications of the theory developed in the previous chapter to particular problems connected with the propagation of light, radio waves, and sound in a turbulent atmosphere (Part A). Comparison of theoretical calculations with experimental data is carried out in Part B.

A. SOME APPLICATIONS OF THE THEORY OF WAVE PROPAGATION IN A TURBULENT MEDIUM TO ATMOSPHERIC OPTICS, ACOUSTICS, AND RADIO METEOROLOGY

Time rather than space characteristics of the received signal are generally easier to measure in experimental research concerned with the effect of atmospheric turbulence on the propagation of electromagnetic and sound waves. We will therefore concentrate on the frequency spectrum of phase and amplitude fluctuations of the received signal. We will also consider the averaging effect of the receiving aperture on phase and amplitude fluctuations. This averaging is sometimes quite substantial, and comprehensive interpretation of the experimental data is impossible if it is ignored.

§ 52. Frequency spectra of phase and amplitude fluctuations

In Chapter 3, Part B we derived expressions for the spatial correlation and structure functions of wave amplitude and phase. Using these expressions, we will now proceed to obtain the time correlation functions.

In the first approximation, the space-to-time conversion can be accomplished assuming that the entire random ensemble of inhomogeneities is transported as a whole, ignoring any fluctuations in the transport velocity and evolution of the inhomogeneities while they are moving (Taylor's hypothesis of frozen turbulence). It is only at the next stage that we introduce fluctuations in the transport velocity and assess the effect of evolutionary trends.

First, note that the main contribution comes from the transverse motion of inhomogeneities, i. e., motion perpendicular to the direction of wave propagation. Indeed the wave amplitude at the observation point is determined by an integral taken over a parabolic region which is highly elongated along the ray; its transverse dimension is $\sqrt{\lambda L}$, whereas the longitudinal dimension is $L \gg \sqrt{\lambda L}$. Let us resolve the velocity $\boldsymbol{v}$ of the displacement of the inhomogeneities into two components $\boldsymbol{v}_{\parallel}$ and $\boldsymbol{v}_{\perp}$ along the propagation vector and at right angles to it, respectively.

The longitudinal and the transverse displacements of the inhomogeneities in time τ are $v_{\parallel}\tau$ and $v_{\perp}\tau$, respectively, Now, if $v_{\perp}\tau \sim \sqrt{\lambda L}$, the amplitude at the observation point will have changed substantially, as all the original inhomogeneities affecting the field amplitude will have been displaced by new inhomogeneities. The corresponding longitudinal displacement, on the other hand, is of the order

$$v_{\parallel} \frac{\sqrt{\lambda L}}{v_{\perp}} \ll L$$

and it introduces but a small change in the inhomogeneities at the two end-points of the overall path. As we are observing an integrated effect, the contribution from this factor is obviously insignificant. Thus, if

$$\frac{v_{\parallel}}{v_{\perp}} \ll \sqrt{\frac{L}{\lambda}},$$

we may consider only the transverse motion of the inhomogeneities.

A more important constraint is imposed on the ratio $v_{\parallel}/v_{\perp}$ by flcutuations in the direction of the vector $\boldsymbol{v}$. The direction fluctuations of the velocity vector are generally of the order of 0.1. Therefore, if $v_{\perp} \sim 0.1 v_{\parallel}$, the transverse component of motion of the inhomogeneities acquires a random component comparable with the mean value of $v_{\perp}$. In this case, the fluctuating component of $v_{\perp}$ can no longer be ignored.

First consider the case when the inhomogeneities move at constant velocity $v_{\perp}$ across the ray path. In this case, $\chi(\boldsymbol{r}, t)$ and $\chi(\boldsymbol{r}, t+\tau)$ are related by the obvious equality

$$\chi(\boldsymbol{r}, t+\tau) = \chi(\boldsymbol{r} - v_{\perp}\tau, t). \tag{1}$$

Consider the time correlation function

$$R_{\chi}(\tau) = \langle \chi(\boldsymbol{r}, t+\tau)\, \chi(\boldsymbol{r}, t) \rangle. \tag{2}$$

Inserting (1), we obtain the relation

$$R_{\chi}(\tau) = B_{\chi}(v_{\perp}\tau), \tag{3}$$

where $B_{\chi}(\rho)$ is the spatial correlation function found in the previous chapter.

Let us now consider the frequency spectrum of the fluctuations, $w_\chi(f)$. The functions $w_\chi(f)$ and $R_\chi(\tau)$ are related by

$$R_\chi(\tau) = \int\limits_0^\infty \cos(2\pi f\tau) w_\chi(f)\, df \tag{4}$$

(we use the expansion in positive frequencies $f = \frac{\omega}{2\pi}$ so as to facilitate direct comparison of the theoretical findings with experiment). As $R_\chi(\tau)$ and $w_\chi(f)$ are even functions, we invert (4) to obtain

$$w_\chi(f) = 4\int\limits_0^\infty \cos(2\pi f\tau) R_\chi(\tau)\, d\tau = 4\int\limits_0^\infty \cos(2\pi\tau f) B_\chi(v_\perp\tau)\, d\tau. \tag{5}$$

It is easier to calculate the spectrum $w_\chi(f)$ if $B_\chi(v_\perp\tau)$ is represented as a special expansion:

$$R_\chi(\tau) = B_\chi(v_\perp\tau) = 2\pi\int\limits_0^\infty F_\chi(\varkappa, L) J_0(\varkappa v_\perp\tau)\varkappa\, d\varkappa. \tag{6}$$

Inserting (6) in (5), changing the order of integration, and seeing that

$$\int\limits_0^\infty J_0(\varkappa v_\perp\tau)\cos 2\pi f\tau\, d\tau = \begin{cases} (\varkappa^2 v_\perp^2 - 4\pi^2 f^2)^{-1/2} & \text{for } \varkappa^2 v_\perp^2 > 4\pi^2 f^2 \\ 0 & \text{for } \varkappa^2 v_\perp^2 < 4\pi^2 f^2, \end{cases}$$

we obtain

$$w_\chi(f) = 8\pi\int\limits_{\frac{2\pi f}{v_\perp}}^\infty F_\chi(\varkappa, L)\frac{\varkappa\, d\varkappa}{\sqrt{\varkappa^2 v_\perp^2 - 4\pi^2 f^2}} = \frac{8\pi}{v_\perp}\int\limits_0^\infty F_\chi\left(\sqrt{\varkappa^2 + \frac{4\pi^2 f^2}{v_\perp^2}}, L\right) d\varkappa. \tag{7}$$

(The last equality is obtained by the substitution $\sqrt{\varkappa^2 v_\perp^2 - 4\pi^2 f^2} = \varkappa' v_\perp$.) Expression (7) relates the frequency spectrum of the fluctuations in χ with its two-dimensional spatial spectrum F_χ. We now use expression (46.36a).

$$F_\chi(\varkappa, L) = \frac{\pi k^2 L}{4}\left[1 - \frac{k}{\varkappa^2 L}\sin\frac{\varkappa^2 L}{k}\right]\Phi_\varepsilon(\varkappa), \tag{8}$$

in which we insert

$$\Phi_\varepsilon(\varkappa) = AC_\varepsilon^2\varkappa^{-11/3}\exp\left(-\frac{\varkappa^2}{\varkappa_m^2}\right), \quad A = 0.033.$$

This gives

$$w_\chi(f) = \frac{2\pi^2 Ak^2 LC_\varepsilon^2}{v_\perp}\int\limits_0^\infty\left[1 - \frac{\sin\frac{L}{k}\left(\varkappa^2 + \frac{4\pi^2 f^2}{v_\perp^2}\right)}{\frac{L}{k}\left(\varkappa^2 + \frac{4\pi^2 f^2}{v_\perp^2}\right)}\right]\times$$

$$\times\left(\varkappa^2 + \frac{4\pi^2 f^2}{v_\perp^2}\right)^{-11/6}\exp\left(-\frac{\varkappa^2 + \frac{4\pi^2 f^2}{v_\perp^2}}{\varkappa_m^2}\right) d\varkappa. \tag{9}$$

We define f_0 by

$$f_0^2 = \frac{kv_\perp^2}{4\pi^2 L} = \frac{v_\perp^2}{2\pi\lambda L}, \qquad f_0 = \frac{v_\perp}{\sqrt{2\pi\lambda L}},$$

and change over to a new variable of integration $z = \frac{\varkappa^2 L}{k}$. The dimensionless ratio

$$\sqrt{\frac{4\pi^2 L f^2}{kv_\perp^2}} = \frac{f}{f_0}$$

is designated by Ω. Thus (9) takes the form

$$w_\chi(f) = \frac{\pi}{2} A C_\varepsilon^2 k^{7/6} L^{11/6} \frac{1}{f_0} \int_0^\infty \left[1 - \frac{\sin(z+\Omega^2)}{z+\Omega^2}\right] \times$$
$$\times (z+\Omega^2)^{-11/6} \exp\left(-\frac{z+\Omega^2}{D}\right) z^{-1/2} dz, \tag{10}$$

where $D = \frac{\varkappa_m^2 L}{k}$ is the wave parameter. The coefficient $AC_\varepsilon^2 k^{7/6} L^{11/6}$ in (10) is proportional to the mean square of the fluctuations $\langle \chi^2(L) \rangle$. Consider the case of large $D \gg 1$. Then we may omit the factor $\exp\left(-\frac{z+\Omega^2}{D}\right)$ in (10) and after substituting $z = \Omega^2 x$ we have

$$w_\chi(f) = \frac{\pi}{2} A C_\varepsilon^2 k^{7/6} L^{11/6} \frac{\Omega^{-8/3}}{f_0} \times \int_0^\infty \left[1 - \frac{\sin \Omega^2(x+1)}{\Omega^2(x+1)}\right] (x+1)^{-11/6} x^{-1/2} dx. \tag{11}$$

Noting that according to the definition of the beta function (Euler's integral of the first kind)

$$\int_0^\infty \frac{x^{1/2-1}}{(1+x)^{1/2+4/3}} dx = B\left(\frac{1}{2}, \frac{4}{3}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\frac{11}{6}\right)} = N = 1.69,$$

we obtain

$$w_\chi(f) = \frac{\pi}{2} A C_\varepsilon^2 k^{7/6} L^{11/6} \frac{\Omega^{-8/3}}{f_0} \left[N - \frac{1}{\Omega^2} \int_0^\infty \sin\left[\Omega^2(x+1)\right] (x+1)^{-17/6} x^{-1/2} dx\right]. \tag{12}$$

Consider the integral

$$f(\Omega) = \int_0^\infty \sin\left[\Omega^2(x+1)\right] (x+1)^{-17/6} x^{-1/2} dx =$$
$$= \operatorname{Im} \int_0^\infty e^{i\Omega^2(x+1)} (x+1)^{-17/6} x^{-1/2} dx. \tag{13}$$

We have the known equality (see, e.g., /117/)

$$\int_0^\infty e^{-zx}x^{\alpha-1}(1+x)^{\gamma-\alpha-1}dx = \Gamma(\alpha)\left[\frac{\Gamma(1-\gamma)\,{}_1F_1(\alpha,\gamma,z)}{\Gamma(\alpha-\gamma+1)} + \right.$$
$$\left. + \frac{\Gamma(\gamma-1)}{\Gamma(\alpha)} z^{1-\gamma}\,{}_1F_1(\alpha+1-\gamma, 2-\gamma, z)\right], \quad (\mathrm{Re}\,\alpha > 0). \tag{14}$$

Putting $z = -i\Omega^2$, $\alpha = \frac{1}{2}$, $\gamma = -\frac{4}{3}$, we obtain

$$f(\Omega) = \mathrm{Im}\, e^{i\Omega^2}\sqrt{\pi}\left[\frac{\Gamma\left(\frac{7}{3}\right)}{\Gamma\left(\frac{17}{6}\right)}\,{}_1F_1\left(\frac{1}{2}, -\frac{4}{3}, -i\Omega^2\right) + \right.$$
$$\left. + \frac{\Gamma\left(-\frac{7}{3}\right)}{\sqrt{\pi}}(-i\Omega^2)^{7/3}\,{}_1F_1\left(\frac{17}{6}, \frac{10}{3}, -i\Omega^2\right)\right].$$

Thus,

$$w_\chi(f) = \frac{\pi}{2} A C_\varepsilon^2 k^{7/6} L^{11/6} \frac{\Omega^{-8/3}}{f_0}\left\{N - \mathrm{Im}\frac{\sqrt{\pi}\, e^{i\Omega^2}}{\Omega^2}\left[\frac{\Gamma\left(\frac{7}{3}\right)}{\Gamma\left(\frac{17}{6}\right)} \times \right.\right.$$
$$\left.\left. \times\,{}_1F_1\left(\frac{1}{2}, -\frac{4}{3}, -i\Omega^2\right) + (-i\Omega^2)^{7/3}\frac{\Gamma\left(-\frac{7}{3}\right)}{\sqrt{\pi}}\,{}_1F_1\left(\frac{17}{6}, \frac{10}{3}, -i\Omega^2\right)\right]\right\}. \tag{15}$$

For small frequencies $\Omega \ll 1$, using the first four terms in the expansion of ${}_1F_1(1/2, -4/3, -i\Omega^2)$ and the first term in the expansion of ${}_1F_1(17/6, 10/3, -i\Omega^2) \approx 1$ we obtain the expression

$$w_\chi(f) = \frac{\pi}{2} A C_\varepsilon^2 k^{7/6} L^{11/6} \frac{\Omega^{-8/3}}{f_0}\left[\frac{5}{48}N\Omega^4 - \frac{1}{2}\Gamma\left(-\frac{7}{3}\right)\Omega^{8/3} + \ldots\right] =$$
$$= \frac{\pi}{2}\cdot\frac{1}{2}\frac{27}{56}\Gamma\left(\frac{5}{3}\right) A C_\varepsilon^2 k^{7/6} L^{11/6}\frac{1}{f_0}\left[1 + \frac{5\cdot 56}{24\cdot 27\Gamma\left(\frac{5}{3}\right)}\Omega^{4/3} + \ldots\right]. \tag{16}$$

Inserting the numerical coefficients and substituting

$$0.077 C_\varepsilon^2 k^{7/6} L^{11/6} = \langle\chi^2\rangle,$$

we obtain

$$w_\chi(f) \approx 0.15\,\langle\chi^2\rangle\frac{1}{f_0}[1 + 0.48\Omega^{4/3} + \ldots], \quad (\Omega \ll 1). \tag{16a}$$

For $\Omega \gg 1$ we use again the asymptotic expansion

$${}_1F_1(\alpha, \gamma, z) = \frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha)}(-z)^{-\alpha}G(\alpha, \alpha-\gamma+1, -z) +$$
$$+ \frac{\Gamma(\gamma)}{\Gamma(\alpha)}e^z z^{\alpha-\gamma}G(\gamma-\alpha, 1-\alpha, z), \tag{17}$$

where

$$G(\alpha, \beta, z) = 1 + \frac{\alpha\cdot\beta}{1!z} + \frac{\alpha(\alpha+1)\beta(\beta+1)}{2!z^2} + \ldots$$

In our case, the main contribution to both the hypergeometric functions in (15) comes from the second term of the asymptotic expression, but these terms, being proportional to $\Omega^{11/6}$, mutually cancel on substitution in (15). Therefore, to obtain the asymptotic form of (15) for $\Omega \gg 1$, we should take the first terms in the asymptotic expansion (17) and the expression in square brackets in (15) is found to be proportional to Ω^{-1}. Therefore, for $\Omega \gg 1$, $w_\chi(f)$ has the form

$$w_\chi(f) = \frac{\pi}{2} NAC_\varepsilon^2 k^{7/6} L^{11/6} \frac{\Omega^{-8/3}}{f_0} = 1.14\langle\chi^2\rangle \frac{\Omega^{-8/3}}{f_0}. \tag{18}$$

Therefore, the function $w_\chi(f)$, which is equal to $w_\chi(0) = 0.15\frac{\langle\chi^2\rangle}{f_0}$ for $f = 0$, gradually increases with increasing f, reaches a maximum near $f \sim f_0$, and then falls off as $f^{-8/3}$. The dimensionless function $F(\Omega) = \frac{f_0 w_\chi(f)}{\langle\chi^2\rangle}$ depends on the dimensionless parameter $\Omega = \frac{f}{f_0}$. The function $F(\Omega)$ plotted using (15) is shown in Figure 48.

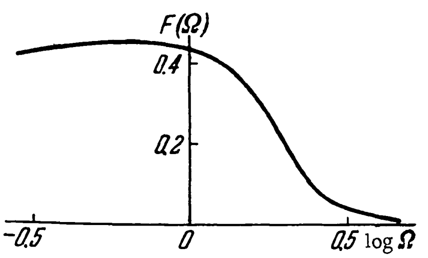


FIGURE 48. The frequency spectrum of the log-amplitude fluctuations assuming the inhomogeneities have constant velocity.

The function $w_\chi(f)$ was computed assuming the inhomogeneities had constant velocity. We have shown, however, that the transverse velocity component $v_\perp$ is subject to fluctuations, which differ for different points of the ray path. As the result, the amplitude distribution pattern in the plane $x = L$ no longer moves as a whole with a constant transport velocity. For observations at a fixed point, this effect is manifested in the form of fluctuations in the transport velocity of the diffraction pattern (at that point). Since the velocity of the diffraction pattern is determined by the resultant action of all the inhomogeneities along the ray path, the velocity fluctuations can be regarded as having a normal probability distribution.

Let us estimate the magnitude of these fluctuations. The mean square fluctuation in $v_\perp$, for which we use the symbol $\sigma_\perp^2$, is equal to $\sigma_\perp^2 = {}^2/_3\,\sigma^2$, where σ^2 is the mean square fluctuations in the wind speed (assuming that the mean square fluctuations of the three velocity components of the wind are equal). On the other hand, the mean value of $v_\perp$ is $\langle v\rangle \sin\alpha$, where α is the angle between the wind velocity and the wave propagation vector. Therefore, the ratio

$$\frac{\sigma_\perp}{\langle v_\perp\rangle} = \frac{0.8\sigma}{\langle v\rangle \sin\alpha}$$

is largely dependent on the angle α. Generally $\frac{\sigma}{\langle v\rangle}\sim 0.1$, so that for $\alpha=\frac{\pi}{2}$ we have $\frac{\sigma_{\perp}}{\langle v\rangle}\sim 8-10\%$, but this ratio may markedly increase for smaller α.

An expression for the frequency spectrum $w_\chi(f)$, averaged over the fluctuations in $v_\perp$, can be obtained from (15) by multiplying this expression by the probability density for $v_\perp$ and integrating. If the relation between $\langle v_\perp\rangle$ and σ is quite arbitrary, this calculation is difficult to carry out analytically. Relatively simple expressions, however, can be obtained if $\langle v_\perp\rangle = 0$, i.e., $\alpha=0$. In this case it is convenient to start with expression (6), which we multiply by $\frac{1}{\sqrt{2\pi}\sigma}e^{\frac{-v_\perp^2}{2\sigma}}$ and then integrate over $v_\perp$. Using the equality

$$\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}\sigma}e^{\frac{-v_\perp^2}{2\sigma^2}}J_0(\varkappa\tau v_\perp)\,dv_\perp = e^{-\frac{\sigma^2\tau^2\varkappa^2}{4}}I_0\left(\frac{\sigma^2\varkappa^2\tau^2}{4}\right),$$

where I_0 is a modified Bessel function, we obtain

$$\langle R_\chi(\tau)\rangle = 2\pi\int_0^\infty F_\chi(\varkappa, L)\,e^{\frac{-\sigma^2\tau^2\varkappa^2}{4}}I_0\left(\frac{\sigma^2\tau^2\varkappa^2}{4}\right)\varkappa\,d\varkappa. \tag{19}$$

Inserting (19) in (5) gives

$$\langle w_\chi(f)\rangle = 8\pi\int_0^\infty F_\chi(\varkappa,L)\varkappa\,d\varkappa\int_0^\infty \cos(2\pi f\tau)\,e^{-\frac{\sigma^2\tau^2\varkappa^2}{4}}I_0\left(\frac{\varkappa^2\sigma^2\tau^2}{4}\right)d\tau. \tag{20}$$

Noting that

$$\int_0^\infty \cos(2\pi f\tau)\,e^{-\frac{\sigma^2\tau^2\varkappa^2}{4}}I_0\left(\frac{\sigma^2\varkappa^2\tau^2}{4}\right)d\tau = \frac{1}{\sqrt{2\pi}\sigma\varkappa}e^{-\frac{\pi^2f^2}{\sigma^2\varkappa^2}}K_0\left(\frac{\pi^2f^2}{\sigma^2\varkappa^2}\right)$$

(see, e.g., /117/), where K_0 is Macdonald's function (modified Bessel function) we find

$$\langle w_\chi(f)\rangle = \frac{8\pi}{\sqrt{2\pi}\sigma}\int_0^\infty F_\chi(\varkappa, L)\,e^{-\frac{\pi^2f^2}{\sigma^2\varkappa^2}}K_0\left(\frac{\pi^2f^2}{\sigma^2\varkappa^2}\right)d\varkappa. \tag{21}$$

Inserting in (21) expression (8) for $F_\chi(\varkappa, L)$ and ignoring, as before, the factor $\exp(-\varkappa^2/\varkappa_m^2)$ (this is legitimate for $\frac{\varkappa_m^2 L}{k}\gg 1$), we obtain

$$\langle w_\chi(f)\rangle = \frac{\pi\sqrt{2\pi}AC_\varepsilon^2k^2L}{\sigma}\int_0^\infty \exp\left(-\frac{\pi^2f^2}{\sigma^2\varkappa^2}\right)K_0\left(\frac{\pi^2f^2}{\sigma^2\varkappa^2}\right)\varkappa^{-11/3}\left[1-\frac{k}{\varkappa^2L}\sin\frac{\varkappa^2L}{k}\right]d\varkappa. \tag{22}$$

We define a characteristic frequency

$$f_1 = \frac{\sigma}{\pi}\left(\frac{k}{L}\right)^{1/2},$$

which is proportional to the ratio of the characteristic velocity to the radius of a Fresnel zone, and a dimensionless ratio

$$\Omega_1 = \frac{\pi f}{\sigma}\sqrt{\frac{L}{k}} = \frac{f}{f_1}.$$

Thus, substituting in the integral in (22) $\frac{\pi f}{\sigma\varkappa}=x$, we obtain

$$\langle w_{\chi}(f)\rangle=\frac{\pi\sqrt{2\pi}Ak^2LC_{\varepsilon}^2}{\sigma}\left(\frac{\pi f}{\sigma}\right)^{-8/3}\int_0^{\infty}\left[1-\frac{x^2}{\Omega_1^2}\sin\frac{\Omega_1^2}{x^2}\right]e^{-x^2}K_0(x^2)\,x^{5/3}dx. \qquad (23)$$

For $\Omega_1\gg 1$, i.e., $f\gg f_1$, the second term in square brackets in the integrand may be ignored. The remaining integral is evaluated using the tables in /117/, and we obtain

$$\langle w_{\chi}(f)\rangle=\frac{\pi\sqrt{2\pi}Ak^2LC_{\varepsilon}^2}{\sigma}\left(\frac{\pi f}{\sigma}\right)^{-8/3}\frac{\sqrt{\pi}}{2^{/3}}\frac{\left[\Gamma\left(\frac{4}{3}\right)\right]^2}{\Gamma\left(\frac{11}{6}\right)}=0.32\frac{\langle\chi^2\rangle}{f_1}(\Omega_1)^{-8/3}. \qquad (24)$$

Thus, in this case the high-frequency part of the spectrum varies with frequency as $f^{-8/3}$. Comparison of this expression with (18) shows that when the mean velocity is zero the numerical coefficient in (24) is less than that in (18). Since in both cases

$$\int_0^{\infty}\langle w_{\chi}(f)\rangle\,df=\int_0^{\infty}w_{\chi}(f)\,df=\langle\chi^2\rangle,$$

the decrease of the function $\langle w_{\chi}(f)\rangle$ for $f\gg f_1$ should be compensated by its increase in the region $f\ll f_1$ compared to that in (16). Indeed, it follows from (22) that $\langle w_{\chi}(f)\rangle\sim\ln(f_1/f)$ for $f\ll f_1$ (since $K_0(z)\approx\ln{}^1/z$ for $|z|\ll 1$). Thus, in this case $w_{\chi}(0)=\infty$. This result is consistent with the fact that for zero mean velocity, zero velocities have the maximum probability. In the general case of an arbitrary $\langle v_{\perp}\rangle/\sigma$ ratio, the calculation of $\langle w_{\chi}(f)\rangle$ becomes extremely difficult. However the qualitative situation is fairly clear in this case also. For $\sigma\ll\langle v_{\perp}\rangle$ the spectrum (15) is only slightly "smeared." For $\sigma\sim\langle v_{\perp}\rangle$ the peak of the spectral curve begins to shift toward zero, and for $\sigma\gg\langle v_{\perp}\rangle$ this maximum begins to increase markedly.

We will now investigate when it is possible to neglect the nonuniform motion of the diffraction pattern. Amplitude fluctuations are mainly attributable to inhomogeneities in the medium whose scale is of the order $l=\sqrt{\lambda L}$ and moving with a velocity $\langle v_{\perp}\rangle$. Besides the mean-wind motion, the inhomogeneities also exhibit intrinsic random velocity components of the order $v_l\sim(\varepsilon l)^{1/3}$, where ε is the rate of dissipation of the turbulent energy. The time during which these velocities remain constant (their "lifetime") depends only on ε and l and is of the order $\tau\sim\varepsilon^{-1/3}l^{2/3}$ /13/. On the other hand, the time required for an inhomogeneity moving with velocity $\langle v_{\perp}\rangle$ to cover a distance $l=\sqrt{\lambda L}$ is of the order $T\sim l/\langle v_{\perp}\rangle$. Clearly, the variation of the transport velocity connected with the evolution may be ignored if $\tau\gg T$, i.e., $\varepsilon^{-1/3}l^{2/3}\gg l/\langle v_{\perp}\rangle$.

This leads to the condition $\langle v_{\perp}\rangle\gg(\varepsilon l)^{1/3}$, i.e., the velocity difference over the distance l should be small compared to the mean transport velocity. If, however, $\langle v_{\perp}\rangle=0$, $\langle v_{\perp}\rangle$ in the last constraint should be replaced by $\sigma_{\perp}$, namely the characteristic velocity for the transport of the inhomogeneities of scale l by large eddies with dimensions comparable with the outer scale of turbulence, L_0. For $\sigma_{\perp}$ we have the estimate $\sigma_{\perp}\sim(\varepsilon L_0)^{1/3}$.

Thus, the time variation of the transport velocity may be ignored if

$$L_0 \gg \sqrt{\lambda L}. \qquad (25)$$

Let us now proceed to find the spectrum of the phase fluctuations. At first we adopt Taylor's hypothesis of "frozen" turbulence, when by analogy with (1) we write

$$S_1(\boldsymbol{r},\ t+\tau) = S_\perp(\boldsymbol{r} - \boldsymbol{v}_\perp\tau,\ t). \qquad (26)$$

Consider the time structure function of the phase:

$$H_S(\tau) = \langle [S_1(\boldsymbol{r}, t+\tau) - S_1(\boldsymbol{r}, t)]^2\rangle = \langle [S_1(\boldsymbol{r} - \boldsymbol{v}_\perp\tau, t) - S_1(\boldsymbol{r}, t)]^2\rangle = D_S(\boldsymbol{v}_\perp\tau). \qquad (27)$$

It can be represented as a Fourier integral

$$H_S(\tau) = 2\int_0^\infty [1 - \cos 2\pi f\tau]\, w_S(f)\, df, \qquad (28)$$

while the function $D_S(\boldsymbol{v}_\perp\tau) = D_S(v_\perp\tau)$ is expressible in the form

$$D_S(v_\perp\tau) = 4\pi\int_0^\infty [1 - J_0(\varkappa v_\perp\tau)]\, F_S(\varkappa, L)\, \varkappa\, d\varkappa.$$

Inserting these expansions in (27) and differentiating the resulting equality with respect to τ, we have

$$4\pi\int_0^\infty \sin(2\pi f'\tau)\, f' w_S(f')\,df' = 4\pi v_\perp\int_0^\infty J_1(\varkappa v_\perp\tau)\, F_S(\varkappa, L)\, \varkappa^2 d\varkappa. \qquad (29)$$

Multiplying (29) by $e^{2\pi i f\tau}$, where $f > 0$, we integrate over τ from minus infinity to plus infinity. Seeing that

$$\int_{-\infty}^\infty \sin(2\pi f'\tau)\, e^{2\pi i f\tau} d\tau = \frac{\pi}{i}[\delta(2\pi f + 2\pi f') - \delta(2\pi f - 2\pi f')] =$$

$$= \frac{1}{2i}[\delta(f + f') - \delta(f - f')],$$

we obtain on the left-hand side

$$\frac{2\pi}{i}\int_0^\infty [\delta(f+f') - \delta(f - f')]\, f' w_S(f')\, df' = -\frac{2\pi}{i} f w_S(f), \qquad (30)$$

where we made use of the fact that $\delta(f + f') = 0$ for $f > 0$ and $0 < f' < \infty$

For the right-hand side of equality (29) we obtain

$$4\pi v_\perp\int_0^\infty F_S(\varkappa, L)\, \varkappa^2\, d\varkappa \int_{-\infty}^\infty e^{2\pi i f\tau} J_1(\varkappa v_\perp\tau) d\tau. \qquad (31)$$

Consider the integral

$$\int_{-\infty}^{\infty} e^{2\pi i f\tau} J_1(\varkappa v_\perp \tau)\, d\tau = i \int_{-\infty}^{\infty} \sin(2\pi f\tau) J_1(\varkappa v_\perp \tau)\, d\tau =$$

$$= 2i \int_0^{\infty} \sin(2\pi f\tau) J_1(\varkappa v_\perp \tau)\, d\tau = \begin{cases} 2i \dfrac{2\pi f}{\varkappa v_\perp \sqrt{\varkappa^2 v_\perp^2 - 4\pi^2 f^2}} & (2\pi f < \varkappa v_\perp), \\ 0 & (2\pi f > \varkappa v_\perp) \end{cases}$$

(see, e.g., /117/). Consequently, (31) takes the form

$$16\pi^2 i f \int_{\frac{2\pi f}{v_\perp}}^{\infty} \frac{F_S(\varkappa, L)\, \varkappa\, d\varkappa}{\sqrt{\varkappa^2 v_\perp^2 - 4\pi^2 f^2}}. \tag{32}$$

Equating (30) and (32), we obtain

$$w_S(f) = 8\pi \int_{\frac{2\pi f}{v_\perp}}^{\infty} \frac{F_S(\varkappa, L)\, \varkappa\, d\varkappa}{\sqrt{\varkappa^2 v_\perp^2 - 4\pi^2 f^2}} = \frac{8\pi}{v_\perp} \int_0^{\infty} F_S\left(\sqrt{\varkappa^2 + \frac{4\pi^2 f^2}{v_\perp^2}},\, L\right) d\varkappa. \tag{33}$$

Expression (33) has the same form as expression (7) for the spectral density of the amplitude fluctuations. Inserting in (33)

$$F_S(\varkappa, L) = \frac{\pi k^2 L}{4}\left(1 + \frac{k}{\varkappa^2 L} \sin \frac{\varkappa^2 L}{k}\right) \Phi_\varepsilon(\varkappa),$$

we see that the expression for $w_S(f)$ differs from expression (15) for $w_\chi(f)$ only by the sign of the second term in the curly brackets:

$$w_S(f) = \frac{\pi}{2} A C_\varepsilon^2 k^{7/6} L^{11/6} \frac{\Omega^{-8/3}}{f_0} \left\{ N + \operatorname{Im} \frac{\sqrt{\pi}\, e^{i\Omega^2}}{\Omega^2} \left[\frac{\Gamma\left(\frac{7}{3}\right)}{\Gamma\left(\frac{17}{6}\right)} \times \right.\right.$$

$$\left.\left. \times\, {}_1F_1\left(\frac{1}{2}, -\frac{4}{3}, -i\Omega^2\right) + (-i\Omega^2)^{7/3} \frac{\Gamma\left(-\frac{7}{3}\right)}{\sqrt{\pi}}\, {}_1F_1\left(\frac{17}{6}, \frac{10}{3}, -i\Omega^2\right) \right] \right\}. \tag{34}$$

Let us establish the asymptotic behavior of $w_S(f)$ for small and large Ω. For small $\Omega \ll 1$, as we have seen in connection with $w_\chi(f)$, the second term in the curly brackets is equal to $N + O(\Omega^{2/3})$ (therefore in the expression for $w_\chi(f)$, which differs from (34) in its sign, N cancels out for $\Omega \ll 1$). Thus for $\Omega \ll 1$

$$w_S(f) = N\pi A C_\varepsilon^2 k^{7/6} L^{11/6} \frac{\Omega^{-8/3}}{f_0} = 8.2 \cdot 10^{-3} C_\varepsilon^2 k^2 L v_\perp^{5/3} f^{-8/3}. \tag{35}$$

For $\Omega \gg 1$ the second term in brackets in (34) may be neglected, and we find

$$w_S(f) = 4.1 \cdot 10^{-3} C_\varepsilon^2 k^2 L v_\perp^{5/3} f^{-8/3}. \tag{36}$$

We see that these asymptotic expansions differ from one another only by a factor of $^1/_2$ (as we have noted above, the large-scale region of the phase fluctuations are described by the geometrical optics approximation, where the expression for the phase fluctuations differs by a factor of 2 from the corresponding expression for the case $D=\frac{\varkappa^2 L}{k}\gg 1$. Also note that the functions $w_S(f)$ and $w_\chi(f)$ coincide for $\Omega\gg 1$ (see (18)).

Thus, with the exception of a relatively narrow transition region $f\sim f_0$, the spectrum of phase fluctuations is proportional to $f^{-8/3}$. Naturally, this conclusion is not applicable in the region of very low frequencies of the order of $v_\perp/L_0$ (L_0 is the outer scale of turbulence), where the "2/3 law" for the refractive index fluctuations is not observed.

Let us also consider the frequency spectrum of the spatial phase difference $\delta_\rho S$ at two points in the plane $x=L$, separated by a distance ρ,

$$\delta_\rho S(t)=S_1(\boldsymbol{r},t)-S_1(\boldsymbol{r}+\boldsymbol{\rho},t),\ \delta_\rho S(t+\tau)=S_1(\boldsymbol{r}-\boldsymbol{v}_\perp\tau,t)-S_1(\boldsymbol{r}+\boldsymbol{\rho}-\boldsymbol{v}_\perp\tau,t),$$
$$R_{\delta S}(\tau)=\langle[S_1(\boldsymbol{r},t)-S_1(\boldsymbol{r}+\boldsymbol{\rho},t)]\times[S_1(\boldsymbol{r}-\boldsymbol{v}_\perp\tau,t)-S_1(\boldsymbol{r}+\boldsymbol{\rho}-\boldsymbol{v}_\perp\tau,t)]\rangle. \qquad (37)$$

Using the identity $(a-b)(c-d)=\frac{1}{2}[(a-d)^2+(b-c)^2-(a-c)^2-(b-d)^2]$, we find

$$R_{\delta S}(\tau)=\frac{1}{2}\{D_S(\boldsymbol{\rho}-\boldsymbol{v}_\perp\tau)+D_S(\boldsymbol{\rho}+\boldsymbol{v}_\perp\tau)-2D_S(\boldsymbol{v}_\perp\tau)\}. \qquad (38)$$

Expression (38) holds true for any relative orientation of the vectors $\boldsymbol{\rho}$ and $\boldsymbol{v}_\perp$. Consider in more detail the particular case of $\boldsymbol{\rho}$ parallel to $\boldsymbol{v}_\perp$. Then $\boldsymbol{\rho}=\frac{\rho}{v_\perp}\boldsymbol{v}_\perp$ and

$$D_S(\boldsymbol{\rho}-\boldsymbol{v}_\perp\tau)=D_S\left(\boldsymbol{v}_\perp\left(\frac{\rho}{v_\perp}-\tau\right)\right).$$

Using (27), we obtain

$$D_S(\boldsymbol{\rho}-\boldsymbol{v}_\perp\tau)=H\left(\frac{\rho}{v_\perp}-\tau\right)=2\int_0^\infty\left\{1-\cos\left[2\pi f\left(\frac{\rho}{v_\perp}-\tau\right)\right]\right\}w_S(f)\,df.$$

Similarly

$$D_S(\boldsymbol{\rho}+\boldsymbol{v}_\perp\tau)=H\left(\frac{\rho}{v_\perp}+\tau\right)=2\int_0^\infty\left\{1-\cos\left[2\pi f\left(\frac{\rho}{v_\perp}+\tau\right)\right]\right\}w_S(f)\,df.$$

Inserting these expressions and (28) in (38), we obtain after simple aminpulations

$$R_{\delta S}(\tau)=\int_0^\infty\cos(2\pi f\tau)\,2\left[1-\cos\frac{2\pi\rho f}{v_\perp}\right]w_S(f)\,df. \qquad (39)$$

It follows from this equality that for $\boldsymbol{\rho}\parallel\boldsymbol{v}_\perp$ the frequency spectrum of the spatial phase difference $w_{\delta S}(f)$ is related to $w_S(f)$ by the equality

$$w_{\delta S}(f)=2\left[1-\cos\frac{2\pi f\rho}{v_\perp}\right]w_S(f)=4\sin^2\left(\frac{\pi\rho f}{v_\perp}\right)w_S(f). \qquad (40)$$

At low frequencies for $f \ll \frac{v_\perp}{\rho}$, we can replace the sine term in (40) by its argument, so that $w_{\delta S}(f) \sim f^2 w_S(f)$. If ρ is not greater than the outer scale of turbulence L_0, then in this region we have $w_S \sim f^{-8/3}$, so that $w_{\delta S} \sim f^{-2/3}$. Although in this case $w_{\delta S}(0) = \infty$, the singularity is integrable, so that

$$\int_0^f w_{\delta S}(f')\, df' \sim f^{1/3},$$

i. e.,

$$\lim_{f \to 0} \int_0^f w_{\delta S}(f')\, df' = 0.$$

This means that the region near $f = 0$ contains only a small fraction of the "energy" of the fluctuations.

At frequencies high compared to $\frac{v_\perp}{\rho}$, the frequency spectrum $w_{\delta S}(f)$ is zero for $f_n = \frac{n v_\perp}{\rho}$. This result is readily understood. The field $S_1(x, y, z)$ in the plane $x = L$ can be represented as a superposition of sine waves having different "wavelengths" Λ, which move in this plane with a velocity $v_\perp$. A sine wave with period Λ causes phase fluctuations with a frequency $f = \frac{v_\perp}{\Lambda}$. Let the two observation points be a distance ρ from each other. Then if $\rho = n\Lambda_n$ (i. e., an integer number of wavelengths of period Λ_n will fit between the observation points), the phase difference caused by the corresponding sine wave is zero, since at both observation points the sine wave has the same value. Therefore the corresponding frequency

$$f_n = \frac{v_\perp}{\Lambda_n} = \frac{n v_\perp}{\rho}$$

drops out from the spectrum of phase difference fluctuations.

In practice $v_\perp$ is not strictly constant but fluctuates. As is evident from (40), this will lead to a certain displacement of the spectrum zeros and in the final analysis they are "filled up". For $f \gg \frac{v_\perp}{\rho}$ even small fluctuations in $v_\perp$ obliterate all the zeros. Indeed, let $v_\perp = v_0 + \delta v$, where $v_0 = \langle v_\perp \rangle$, $v_\perp^{-1} \approx v_0^{-1} - \frac{\delta v}{v_0^2}$.

Then

$$\frac{\pi\rho f}{v_\perp} = \frac{\pi\rho f}{v_0} - \frac{\pi\rho f\,\delta v}{v_0^2}.$$

If $\frac{\pi\rho f\,\delta v}{v_0^2}$ is comparable with $\frac{\pi}{2}$, the minimum of the sine curve replaces its maximum, and the spectrum has no oscillations. The corresponding δv is of the order

$$\delta v \sim \frac{v_0^2}{2\rho f},$$

or, inserting for δv its rms value σ_v, we obtain

$$\frac{\sigma_v}{v_0} \sim \frac{v_0}{2\rho f}.$$

Thus the "cutoff" frequency f_m above which the oscillations in the spectrum are smeared is of the order

$$f_m \sim \frac{1}{2}\,\frac{v_0}{\rho}\,\frac{v_0}{\sigma_v}. \qquad (41)$$

As the ratio v_0/σ_v increases, the valleys in the spectrum $w_{\delta S}$ become more pronounced and remain prominent over a wider range. For frequencies $f > f_m$, on the other hand, the factor $\sin^2 \frac{\pi\rho f}{v_\perp}$ can be replaced by its mean value of $^1/_2$. Then

$$w_{\delta S}(f) = 2w_S(f) \quad (f > f_m). \qquad (42)$$

Thus, $w_{\delta S} \sim (f)^{-8/3}$ for $f > f_m$. Note that in view of the rapid decrease of $w_S(f)$ for large f, high frequencies do not make a substantial contribution to the integral $\int\limits_0^\infty w_{\delta S}(f)\,df$. We have already seen that the contribution from very low frequencies to this integral is also insignificant. The value of this integral is thus mainly determined by frequencies of the order $v_\perp/\rho$.

If $\frac{v_\perp}{\rho} \ll \frac{v_\perp}{\sqrt{\lambda L}}$, i.e., $\rho \gg \sqrt{\lambda L}$, then in (40) $w_S(f)$ can be replaced by its asymptotic expression for $f \ll \frac{v_\perp}{\sqrt{\lambda L}}$, i.e., $\Omega \ll 1$. In this case

$$w_{\delta S}(f) = 0.033\,C_\varepsilon^2 k^2 L v_\perp^{5/3} \sin^2\left(\frac{\pi\rho f}{v_\perp}\right) f^{-8/3} \quad (\rho \gg \sqrt{\lambda L}). \qquad (43)$$

In the opposite case, when $\rho \ll \sqrt{\lambda L}$, we may use the asymptotic expression (36), which gives

$$w_{\delta S}(f) = 0.016 C_\varepsilon^2 k^2 L v_\perp^{5/3} \sin^2\left(\frac{\pi\rho f}{v_\perp}\right) f^{-8/3} \quad (\rho \ll \sqrt{\lambda L}). \qquad (44)$$

For $\rho \gg \sqrt{\lambda L}$, $\langle \delta S^2 \rangle = D_S(\rho) = 0.73 C_\varepsilon^2 k^2 L \rho^{5/3}$. Inserting this expression in (43), we obtain

$$w_{\delta S}(f) = 0.045\,\frac{\langle \delta S^2 \rangle}{f_S} \sin^2\left(\pi \frac{f}{f_S}\right) \cdot \left(\frac{f}{f_S}\right)^{-8/3}, \qquad (45)$$

where $f_S = \frac{v_\perp}{\rho}$ is the characteristic frequency of the phase difference fluctuations. For $\rho \ll \sqrt{\lambda L}$, $D_S(\rho) = \langle \delta S^2 \rangle$ has an additional factor of $^1/_2$, as in (44), and expression (45) is therefore also applicable for $\rho \ll \sqrt{\lambda L}$. This expression breaks down only for $\rho \approx \sqrt{\lambda L}$.

§ 53. The effect of receiver aperture averaging on amplitude fluctuations

We have computed the amplitude correlation function in a plane perpendicular to the direction of propagation and found that for $\frac{\varkappa_m^2 L}{k} \gg 1$ the amplitude correlation distance is of the order $\sqrt{\lambda L}$. Now suppose that the signal is received by a receiving device with a finite aperture. We will refer to a telescope objective, although all our arguments can be extended to antenna reflectors or sonic transducers. If the diameter of the objective is much greater than the amplitude correlation radius $\sqrt{\lambda L}$, the objective will contain wave front sections with fluctuations of opposite sign, so that the overall light flux through a large objective will fluctuate relatively weakly compared to the flux through a small (compared to $\sqrt{\lambda L}$) objective.

Let $I(y, z)$ be the light intensity. Then the total light flux P through the objective is

$$P = \iint_{\Sigma} I(y, z)\, dy\, dz, \qquad (1)$$

where Σ is the aperture area. The fluctuations in P, defined as $P' = P - \langle P \rangle$, are expressed in the form

$$P' = \iint_{\Sigma} I'(y, z)\, dy\, dz, \qquad (2)$$

where $I' = I - \langle I \rangle$. For the mean square fluctuations we have

$$\langle P'^2 \rangle = \iint_{\Sigma}\iint_{\Sigma} \langle I'(y_1, z_1)\, I'(y_2, z_2) \rangle\, dy_1\, dz_1\, dy_2\, dz_2. \qquad (3)$$

The function $B_I = \langle I'(y_1, z_1)\, I'(y_2, z_2) \rangle$ depends only on the distance between points 1 and 2, i.e.,

$$B_I = B_I(y_1 - y_2,\ z_1 - z_2) = B_I(\rho).$$

We introduce an auxiliary function $F(y, z)$, which is zero outside the aperture and 1 on its surface. Relation (3) is then written in the form

$$\langle P'^2 \rangle = \iiiint_{-\infty}^{\infty} B_I(y_1 - y_2, z_1 - z_2)\, F(y_1, z_1)\, F(y_2, z_2)\, dy_1\, dz_1\, dy_2\, dz_2. \qquad (4)$$

Changing over to new variables $y = y_1 - y_2$, $z = z_1 - z_2$, we write (4) as

$$\langle P'^2 \rangle = \iint_{-\infty}^{\infty} B_I(y, z)\, dy\, dz \iint_{-\infty}^{\infty} F(y_1, z_1)\, F(y_1 - y, z_1 - z)\, dy_1\, dz_1. \qquad (5)$$

Let

$$K(y, z) = \iint_{-\infty}^{\infty} F(y_1, z_1)\, F(y_1 - y, z_1 - z)\, dy_1\, dz_1. \qquad (6)$$

The function K depends only on the geometry of the aperture. In practice, the aperture is generally a circle of radius R, and it is for this case that the function K is evaluated. $F(y_1, z_1) = 1$ for $y_1^2 + z_1^2 \leqslant R^2$; $F(y - y_1, z - z_1) = 1$ for $(y - y_1)^2 + (z - z_1)^2 \leqslant R^2$.

The first region is a circle of radius R centered at the origin, and the second is the same circle centered at the point (y_1, z_1). $K(y, z)$ is the area formed by the intersection of the two circles. Clearly $K(y, z) = K(\sqrt{y^2 + z^2})$. An elementary calculation gives for this area*

$$K(y, z) = K(\rho) = \begin{cases} 2R^2\left[\arccos\left(\frac{\rho}{2R}\right) - \frac{\rho}{2R}\sqrt{1 - \frac{\rho^2}{4R^2}}\right], & \rho < 2R, \\ 0, & \rho > 2R. \end{cases} \tag{7}$$

Thus, for a circular aperture (5) takes the form

$$\begin{aligned} \langle P'^2 \rangle &= 2R^2 \iint\limits_{\rho < 2R} B_I(\rho)\left[\arccos\left(\frac{\rho}{2R}\right) - \frac{\rho}{2R}\sqrt{1 - \frac{\rho^2}{4R^2}}\right]\rho\, d\rho\, d\varphi = \\ &= 4\pi R^2 \int\limits_0^{2R} B_I(\rho)\left[\arccos\left(\frac{\rho}{2R}\right) - \frac{\rho}{2R}\sqrt{1 - \frac{\rho^2}{4R^2}}\right]\rho\, d\rho. \end{aligned} \tag{8}$$

The function in square brackets in (8) is equal to $\pi/2$ for $\rho = 0$ and falls off to zero for $\rho = 2R$. If $\sqrt{\lambda L} \gg R$, the function $B_I(\rho)$ is approximately equal to $B_I(0)$ for $0 \leqslant \rho \leqslant 2R$ and it can be taken outside the integral. Therefore, noting that

$$\int\limits_0^1 (\arccos x - x\sqrt{1 - x^2})\, x\, dx = \frac{\pi}{16},$$

we find

$$\langle P'^2 \rangle = (\pi R^2)^2 B_I(0).$$

This means that for $R \ll \sqrt{\lambda L}$ the total light flux fluctuates just like the light intensity.

Instead of $\langle P'^2 \rangle$ it is convenient to study the ratio

$$Q(R) = \frac{\langle P'^2 \rangle}{\langle P \rangle^2} = \frac{4}{\pi R^2} \int\limits_0^{2R} \frac{B_I(\rho)}{\langle I \rangle^2}\left[\arccos\left(\frac{\rho}{2R}\right) - \frac{\rho}{2R}\sqrt{1 - \frac{\rho^2}{4R^2}}\right]\rho\, d\rho. \tag{9}$$

For a point aperture we have

$$Q(0) = \frac{B_I(0)}{\langle I \rangle^2}.$$

The averaging action of the aperture is conveniently characterized by the ratio $G(R) = \frac{Q(R)}{Q(0)}$ which shows by what factor the fluctuations of the total light flux through an aperture of radius R are less than those through a point aperture:

$$G(R) = \frac{4}{\pi R^2} \int\limits_0^{2R} b_I(\rho)\left[\arccos\left(\frac{\rho}{2R}\right) - \frac{\rho}{2R}\sqrt{1 - \frac{\rho^2}{4R^2}}\right]\rho\, d\rho, \tag{10}$$

* [arccos ≡ $\cos^{-1}$.]

where $b_I(\rho) = \frac{B_I(\rho)}{B_I(0)}$ is the normalized correlation coefficient of the intensity fluctuations. The function $G(R)$ is unity for $R = 0$ and is monotonically decreasing as R increases.

To obtain the final expression for $G(R)$, we still require the function $b_I(\rho)$. The intensity I is proportional to the square of the amplitude:

$$I = \alpha A^2 = \alpha A_0^2 e^{2\chi}.$$

We have noted before that χ is normally distributed. Therefore, it follows that

$$\langle I \rangle = \alpha A_0^2 \langle e^{2\chi} \rangle = \alpha A_0^2 e^{2\langle \chi^2 \rangle}.$$

Consider the expression

$$\langle I(\boldsymbol{r}_1) I(\boldsymbol{r}_2) \rangle = \alpha^2 A_0^4 \langle e^{2\chi(\boldsymbol{r}_1)+2\chi(\boldsymbol{r}_2)} \rangle = \alpha^2 A_0^4 \exp\{4 [\langle \chi^2 \rangle + B_\chi(\boldsymbol{r}_1 - \boldsymbol{r}_2)]\}.$$

By definition

$$b_I(\boldsymbol{r}_1 - \boldsymbol{r}_2) = \frac{\langle I(\boldsymbol{r}_1) I(\boldsymbol{r}_2) \rangle - \langle I \rangle^2}{\langle I^2 \rangle - \langle I \rangle^2}.$$

Inserting the above expressions, we obtain

$$b_I(\rho) = \frac{e^{4B_\chi(\rho)} - 1}{e^{4\langle \chi^2 \rangle} - 1}. \qquad (11)$$

For $\langle \chi^2 \rangle \ll 1$, we have from (11)

$$b_I(\rho) = \frac{B_\chi(\rho)}{\langle \chi^2 \rangle} = b_\chi(\rho),$$

i. e., the intensity correlation radius is again of the order $\sqrt{\lambda L}$.

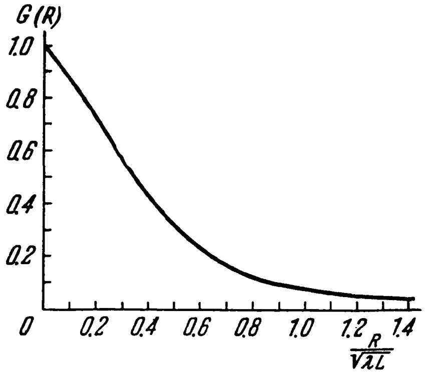


FIGURE 49. The relative reduction in the fluctuations of the total light flux through an aperture of radius R (for $l_0 \ll \sqrt{\lambda L} \ll L_0$).

Figure 49 plots the curve of $G(R) = \frac{Q(R)}{Q(0)}$ for small values of $\langle\chi^2\rangle$. The function G virtually depends only on the ratio $R/\sqrt{\lambda L}$, which is plotted on the abscissa. For $R \gg \sqrt{\lambda L}$, $G(R) \sim \left(\frac{R}{\sqrt{\lambda L}}\right)^{-7/3}$. At first glance it would seem that $G(R)$ for $R \to \infty$ should decrease in inverse proportion to the number of independent inhomogeneities contained inside the aperture, and this is inconsistent with the above asymptotic behavior. This is not so, however, since for large R, the function $\left[\arccos\left(\frac{\rho}{2R}\right) - \frac{\rho}{2R}\sqrt{1-\frac{\rho}{4R^2}}\right]$ in (10) can be replaced by its value at the origin ($\pi/2$). Therefore, the integral will approach zero, and not a finite constant value, since as has been shown earlier

$$\int_0^\infty b_\chi(\rho)\,\rho\,d\rho = 0.$$

This actually explains the rapid decrease of $G(R)$.

Let us now consider the frequency spectra of fluctuations in the light flux. By analogy with (52.1), we may write

$$P'(t) = \iint_\Sigma I'(y, z)\,dy\,dz,$$
$$P'(t+\tau) = \iint_\Sigma I'(y' - v_y\tau, z' - v_z\tau)\,dy'\,dz'. \tag{12}$$

Multiplying these expressions, averaging, and introducing the functions $F(y, z), F(y'\ z')$, we obtain

$$R_P(\tau) \equiv \langle P'(t)\,P'(t+\tau)\rangle =$$
$$= \iiiint_{-\infty}^{\infty} F(y, z)\,F(y', z')\,B_I(y - y' + v_y\tau, z - z' + v_z\tau)\,dy\,dz\,dy'\,dz'. \tag{13}$$

$B_I(y, z)$ can be represented as a two-dimensional spatial spectral expansion

$$B_I(y, z) = \iint_{-\infty}^{\infty} F_I(\varkappa_2, \varkappa_3, L)\,e^{i(\varkappa_2 y + \varkappa_3 z)}\,d\varkappa_2\,d\varkappa_3. \tag{14}$$

Inserting (14) in (13), we get

$$R_P(\tau) = \iint_{-\infty}^{\infty} F_I(\varkappa_2, \varkappa_3, L)\,e^{i(\varkappa_2 v_y + \varkappa_3 v_z)\tau}\,d\varkappa_2\,d\varkappa_3 \times$$
$$\times \iint_{-\infty}^{\infty} F(y, z)\,e^{i(\varkappa_2 y + \varkappa_3 z)}\,dy\,dz \iint_{-\infty}^{\infty} F(y', z')\,e^{-i(\varkappa_2 y' + \varkappa_3 z')}\,dy'\,dz'. \tag{15}$$

The function

$$V_\Sigma(\varkappa_2, \varkappa_3) = \iint_{-\infty}^{\infty} F(y, z)\,e^{i(\varkappa_2 y + \varkappa_3 z)}\,dy\,dz = \iint_\Sigma e^{i(\varkappa_2 y + \varkappa_3 z)}\,dy\,dz \tag{16}$$

describes the Fraunhoffer diffraction by the aperture Σ. Using (16), we have

$$R_P(\tau) = \iint_{-\infty}^{\infty} F_I(\varkappa_2, \varkappa_3, L)\,|V_\Sigma(\varkappa_2, \varkappa_3)|^2 e^{i(\varkappa_2 v_y + \varkappa_3 v_z)\tau} d\varkappa_2\, d\varkappa_3. \quad (16a)$$

We can now write for the frequency spectrum

$$w_P(f) = 2\int_{-\infty}^{\infty} \cos(2\pi f\tau) R_P(\tau)\, d\tau = 2\int_{-\infty}^{\infty} e^{-2\pi i f\tau} R_P(\tau)\, d\tau =$$

$$= 2\iint_{-\infty}^{\infty} F_I(\varkappa_2, \varkappa_3, L)\,|V_\Sigma(\varkappa_2, \varkappa_3)|^2 d\varkappa_2\, d\varkappa_3 \int_{-\infty}^{\infty} e^{i[\varkappa_2 v_y + \varkappa_3 v_z - 2\pi f]\tau} d\tau =$$

$$= 4\pi \iint_{-\infty}^{\infty} F_I(\varkappa_2, \varkappa_3, L)\,|V_\Sigma(\varkappa_2, \varkappa_3)|^2 \delta(\varkappa_2 v_y + \varkappa_3 v_z - 2\pi f)\, d\varkappa_2\, d\varkappa_3. \quad (17)$$

Consider a circular aperture, as before. In this case,

$$V_\Sigma(\varkappa_2, \varkappa_3) = \int_0^R \rho\, d\rho \int_0^{2\pi} d\varphi e^{i\varkappa\rho\cos\varphi} = \pi R^2 \frac{2J_1(\varkappa R)}{\varkappa R}. \quad (18)$$

The function $F_1(\varkappa_2, \varkappa_3, L)$ for locally isotropic fluctuations also depends only on $\varkappa = \sqrt{\varkappa_2^2 + \varkappa_3^2}$. In (17) we introduce polar coordinates. Then $\varkappa_2 v_y + \varkappa_3 v_z = \varkappa v_\perp \cos\varphi$ and

$$w_P(f) = 4\pi \int_0^{\infty} F_I(\varkappa, L)\left(\pi R^2 \frac{2J_1(\varkappa R)}{\varkappa R}\right)^2 \varkappa\, d\varkappa \int_0^{2\pi} \delta(\varkappa v_\perp \cos\varphi - 2\pi f)\, d\varphi. \quad (19)$$

The integral over the δ-function is readily evaluated:

$$\int_0^{2\pi} \delta(\varkappa v_\perp \cos\varphi - 2\pi f)\, d\varphi = \begin{cases} \dfrac{2}{\sqrt{\varkappa^2 v_\perp^2 - 4\pi^2 f^2}} & \text{for } 4\pi^2 f^2 < \varkappa^2 v_\perp^2, \\ 0 & \text{for } 4\pi^2 f^2 > \varkappa^2 v_\perp^2. \end{cases}$$

Inserting in (19), we get

$$w_P(f) = 8\pi(\pi R^2)^2 \int_{2\pi f/v_\perp}^{\infty} F_I(\varkappa, L)\left[\frac{2J_1(\varkappa R)}{\varkappa R}\right]^2 \frac{\varkappa\, d\varkappa}{\sqrt{\varkappa^2 v_\perp^2 - 4\pi^2 f^2}} =$$

$$= \frac{8\pi(\pi R^2)^2}{v_\perp} \int_0^{\infty} F_I\left(\sqrt{\varkappa^2 + \frac{4\pi^2 f^2}{v_\perp^2}}, L\right)\left[\frac{2J_1\left(R\sqrt{\varkappa^2 + \frac{4\pi^2 f^2}{v_\perp^2}}\right)}{R\sqrt{\varkappa^2 + \frac{4\pi^2 f^2}{v_\perp^2}}}\right]^2 d\varkappa. \quad (20)$$

The spectrum of P' is sometimes replaced by the spectrum of the normalized fluctuations of the ratio $w_{P/\langle P\rangle}(f)$. The latter is obtained if (20) is divided by $\langle P\rangle^2 = (\pi R^2 \langle I\rangle)^2$:

$$w_{\frac{P}{\langle P\rangle}} = \frac{8\pi}{v_\perp} \int_0^{\infty} \frac{F_I\left(\sqrt{\varkappa^2 + \frac{4\pi^2 f^2}{v_\perp^2}}, L\right)}{\langle I\rangle^2}\left[\frac{2J_1\left(R\sqrt{\varkappa^2 + \frac{4\pi^2 f^2}{v_\perp^2}}\right)}{R\sqrt{\varkappa^2 + \frac{4\pi^2 f^2}{v_\perp^2}}}\right]^2 d\varkappa. \quad (21)$$

Expression (21) differs from the corresponding expression (52.7) by the presence of an additional factor

$$\left[\frac{2J_1\left(R\sqrt{\varkappa^2+\frac{4\pi^2f^2}{v_\perp^2}}\right)}{R\sqrt{\varkappa^2+\frac{4\pi^2f^2}{v_\perp^2}}}\right]^2 \tag{22}$$

in the integrand. For $R=0$ this factor is unity, and we obtain the previous expression. For $R\neq 0$ this factor is less than unity, which corresponds to partial suppression of the normalized fluctuations due to aperture averaging. If $\frac{2\pi fR}{v_\perp}\gg 1$, i.e., $f\gg\frac{v_\perp}{2\pi R}$, the factor (22) is small for all $\varkappa$. This means that for high frequencies the averaging action of the aperture weakens the effect of inhomogeneities of all scales. If, however, $f\ll\frac{v_\perp}{2\pi R}$, the factor (22) is close to unity for $\varkappa\ll\frac{1}{R}$ and becomes small for $\varkappa\gg\frac{1}{R}$. The contribution of large-scale inhomogeneities in the low-frequency part of the spectrum is thus virtually unaffected, whereas all scales which are small compared to the aperture radius are suppressed.

For weak fluctuations

$$\frac{F_I(\varkappa, L)}{\langle I\rangle^2}=F_\chi(\varkappa, L)=\frac{\pi k^2L}{4}\left[1-\frac{k}{\varkappa^2L}\sin\frac{\varkappa^2L}{k}\right]AC_\varepsilon^2\varkappa^{-11/3}, \tag{23}$$

where $A=0.033$. Inserting (23) in (21), putting

$$f_0^2=\frac{kv_\perp}{4\pi^2L},\quad \Omega^2=\left(\frac{f}{f_0}\right)^2,\quad \rho^2=\frac{kR^2}{L}=\frac{2\pi R^2}{\lambda L},$$

and changing variables from $\varkappa$ to $z=\frac{L\varkappa^2}{k}$ in the integral, we obtain

$$w_{\frac{P}{\langle P\rangle}}(f)=0.67\frac{\langle\chi^2\rangle}{f_0}\int_0^\infty\left[\frac{2J_1(\rho\sqrt{z+\Omega^2})}{\rho\sqrt{z+\Omega^2}}\right]^2\left[1-\frac{\sin(z+\Omega^2)}{z+\Omega^2}\right]\times$$
$$\times(z+\Omega^2)^{-11/6}z^{-1/2}\,dz. \tag{24}$$

The dimensionless function $F(\Omega,\ \rho)=f_0w_{P/\langle P\rangle}(f)/\langle\chi^2\rangle$ thus depends on two dimensionless parameters: ρ, the ratio of the aperture radius to the radius of the first Fresnel zone, and Ω, the ratio of frequency to the characteristic frequency

$$f_0\sim\frac{v_\perp}{\sqrt{\lambda L}}.$$

The function $F(\Omega,\rho)$ for various ρ is shown in Figure 50. For $\rho=0$ it coincides with $F(\Omega)$ (Figure 48).

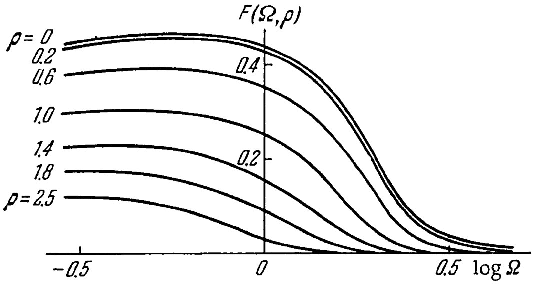


FIGURE 50. The frequency spectrum of fluctuations in the light flux vs. aperture diameter.

§ 54. Scintillation of sources of finite angular size

It is a well-known fact that planets scintillate less than stars at the same zenith distance. This effect has much in common with the reduction of scintillation due to aperture averaging. Let γ be the angular size of the planet. Then the beam reaching the observation point is a bundle of plane waves originating at various points of the planetary disk with angular width γ. At the boundary of the refracting atmosphere the distance between the rays drawn to the extreme points of the planetary disk is γL, where L is the thickness of the atmosphere in the direction of the planet. If $\gamma L > \sqrt{\lambda L}$, the fluctuations of different rays are uncorrelated and the scintillation is attenuated. It follows from the above that for $\gamma \gg \gamma_0 = \sqrt{\lambda/L}$ the scintillation is definitely attenuated, whereas for $\gamma \ll \gamma_0$ this effect is insignificant. The angle γ_0 may be called the correlation angle.

Let us now proceed with a quantitative treatment /118/. Let $i(\theta, \varphi)$ be the intensity of the light emitted by the planet which reaches the observation point from the direction θ, φ (the polar axis of the coordinate system is directed to the center of the planetary disk). Different points of the disk appear as incoherent light sources and therefore the total intensity is equal to the integral of i over the entire disk:

$$I = \int_0^{2\pi} d\varphi \int_0^{\gamma} \sin\theta \, d\theta \, i(\theta, \varphi). \qquad (1)$$

A similar relation is obtained for the intensity fluctuations:

$$I' = \int_0^{2\pi} d\varphi \int_0^{\gamma} \sin\theta \, d\theta \, i'(\theta, \varphi). \qquad (2)$$

In what follows we will consider relative fluctuations $Y' = \frac{I'}{\langle I \rangle}$. From (1) it follows that

$$\langle I \rangle = 4\pi \sin^2 \frac{\gamma}{2} \langle i \rangle,$$

so that

$$Y' = \frac{1}{4\pi \sin^2 \frac{\gamma}{2}} \int_0^{2\pi} d\varphi \int_0^{\gamma} \sin\theta\, d\theta \frac{i(\theta,\varphi)}{\langle i \rangle} \approx \frac{1}{\pi\gamma^2} \int_0^{2\pi} d\varphi \int_0^{\gamma} \sin\theta d\theta \frac{i(\theta,\varphi)}{\langle i \rangle}. \qquad (3)$$

For the mean square of the relative fluctuations we have (when $\gamma \ll 1$ we may take $\sin\theta \approx \theta$)

$$\langle Y'^2 \rangle = \sigma_\gamma^2 = \frac{1}{(\pi\gamma^2)^2} \int_0^{2\pi} d\varphi_1 \int_0^{2\pi} d\varphi_2 \int_0^{\gamma} \theta_1\, d\theta_1 \int_0^{\gamma} \theta_2\, d\theta_2 \frac{\langle i'(\theta_1,\varphi_1)\, i'(\theta_2,\varphi_2)\rangle}{\langle i \rangle^2}. \qquad (4)$$

The product $\langle i'(\theta_1, \varphi_1)\; i'(\theta_2, \varphi_2)\rangle$ can be related to the log-amplitude correlation function of two plane waves propagating at a certain angle ψ with respect to each other. Using the equality $i = A_0^2 \exp(2\chi(\theta, \varphi))$ and remembering that χ is normally distributed, we readily arrive at an expression analogous to (53.11):

$$\frac{\langle i'(\theta_1,\varphi_1)\, i'(\theta_2,\varphi_2)\rangle}{\langle i \rangle^2} = \frac{\langle i(\theta_1,\varphi_1)\, i(\theta_2,\varphi_2)\rangle - \langle i \rangle^2}{\langle i \rangle^2} = e^{4B_\chi(\theta_1,\varphi_1,\theta_2,\varphi_2)} - 1, \qquad (5)$$

where $B_\chi(\theta_1, \varphi_1, \theta_2, \varphi_2) = \langle \chi(\theta_1, \varphi_1)\chi(\theta_2, \varphi_2)\rangle$.

Thus, to compute σ_γ^2, we should first find $B_\chi(\theta_1, \varphi_1, \theta_2, \varphi_2)$. Let us calculate this function. For $\chi(x, y, z)$ we have from (45.30)

$$\chi(x,y,z) = \frac{k^2}{4\pi} \int_0^x d\xi \iint_{-\infty}^{\infty} d\eta\, d\zeta\, \varepsilon_1(\xi,\eta,\zeta) \frac{\cos\left\{k \frac{(y-\eta)^2 + (z-\zeta)^2}{2(x-\xi)}\right\}}{x-\xi}. \qquad (6)$$

This expression is written in a special system of coordinates, with the x axis pointing along the vector $\boldsymbol{k}$. To rewrite this expression in covariant form, let $\boldsymbol{k} = k\boldsymbol{n}$, where $\boldsymbol{n}$ is the unit vector in the direction of propagation. Clearly,

$$x - \xi = \boldsymbol{n}(\boldsymbol{r} - \boldsymbol{\rho}),$$

where $\boldsymbol{r} = (x, y, z)$, $\boldsymbol{\rho} = (\xi, \eta, \zeta)$,

$$(y-\eta)^2 + (z-\zeta)^2 = (\boldsymbol{r}-\boldsymbol{\rho})^2 - (\boldsymbol{n}(\boldsymbol{r}-\boldsymbol{\rho}))^2.$$

The integral is taken over an infinite layer where $\xi > 0$, i.e., $\boldsymbol{n\rho} > 0$, and $(x - \xi) > 0$, i.e., $\boldsymbol{n}(\boldsymbol{r} - \boldsymbol{\rho}) > 0$. If

$$\theta(x) = \begin{cases} 1 & \text{for } x > 0, \\ 0 & \text{for } x < 0, \end{cases}$$

the inequalities $\boldsymbol{n\rho} > 0$, $\boldsymbol{n}(\boldsymbol{r} - \boldsymbol{\rho}) > 0$ can be satisfied by introducing in the integrand an additional factor $\theta(\boldsymbol{n\rho})\theta(\boldsymbol{n}(\mathbf{r} - \boldsymbol{\rho}))$. Then (6) may be rewritten in the form

$$\chi(\boldsymbol{r}) = \frac{k^2}{4\pi} \iiint_{-\infty}^{\infty} \theta(\boldsymbol{n\rho})\, \theta(\boldsymbol{n}(\boldsymbol{r}-\boldsymbol{\rho}))\, \varepsilon_1(\boldsymbol{\rho}) \frac{\cos\left\{k \frac{(\boldsymbol{r}-\boldsymbol{\rho})^2 - (\boldsymbol{n}(\boldsymbol{r}-\boldsymbol{\rho}))^2}{2\boldsymbol{n}(\boldsymbol{r}-\boldsymbol{\rho})}\right\}}{\boldsymbol{n}(\boldsymbol{r}-\boldsymbol{\rho})} d^3\rho. \qquad (7)$$

Expression (7) is now applicable to any direction of the vector n. Note that in (7) for $n \neq \{1, 0, 0\}$ the integral is taken over a region whose boundaries are not perpendicular to the x-axis. If the boundaries of the random layer are in fact perpendicular to the x-axis, a certain error creeps into the result, although it is of no particular significance in practice.

Let $n = \{\cos\psi, 0, \sin\psi\}$, $r = (L, 0, 0)$. Then

$$n(r-\rho) = \cos\psi \cdot (L-\xi) - \sin\psi \cdot \zeta,$$

$$(r-\rho)^2 - (n(r-\rho))^2 = [(L-\xi)\sin\psi + \zeta\cos\psi]^2 + \eta^2,$$

$$\chi(r) = \frac{k^2}{4\pi}\iiint\limits_{-\infty}^{\infty}\theta(\xi\cos\psi + \zeta\sin\psi)\,\theta((L-\xi)\cos\psi - \zeta\sin\psi)\times$$

$$\times\,\varepsilon_1(\xi, \eta, \zeta)\cos\left\{k\frac{[(L-\xi)\sin\psi + \zeta\cos\psi]^2 + \eta^2}{2[(L-\xi)\cos\psi - \zeta\sin\psi]}\right\}\frac{d\xi\,d\eta\,d\zeta}{(L-\xi)\cos\psi - \zeta\sin\psi}. \qquad (7a)$$

Let the angle ψ be small, so that

$$(L-\xi)\cos\psi - \zeta\sin\psi \approx (L-\xi) - \zeta\psi.$$

In the argument of the cosine

$$[(L-\xi)\psi + \zeta]^2 + \eta^2 = \rho_\perp^2$$

is the square of the distance of the point ρ from the beam axis, which passes through the observation point. In the main region of integration its order of magnitude does not exceed $\rho_\perp \sim \frac{\lambda L}{\lambda_0}$ (here λ_0 is the inner scale of turbulence, λ/λ_0 is the diffraction angle). Expanding the argument of the cosine in a series

$$\frac{k\rho_\perp^2}{2[(L-\xi)\cos\psi - \zeta\sin\psi]} \approx \frac{k\rho_\perp^2}{2(L-\xi-\zeta\psi)} = \frac{k\rho_\perp^2}{2(L-\xi)}\left[1 + \frac{\zeta\psi}{L-\xi} + \ldots\right]$$

we will try to find out when only the first term needs to be retained. Here ζ is of the order $L\psi$. Therefore the necessary condition takes the form

$$\frac{k\rho_\perp^2}{2(L-\xi)}\frac{\zeta\psi}{L-\xi} \sim \frac{1}{\lambda L}\left(\frac{\lambda L}{\lambda_0}\right)^2\frac{L\psi^2}{L} = \frac{\lambda L}{\lambda_0^2}\psi^2 \ll 1,$$

$$\psi^2 \ll \frac{\lambda_0^2}{\lambda L} = \psi_0^2. \qquad (8)$$

The condition given in (8) is not difficult to satisfy since the correlation angle is $\gamma_0 \approx \sqrt{\frac{\lambda}{L}}$ and $\frac{\gamma_0}{\psi_0} \sim \frac{\lambda}{\lambda_0} \ll 1$. Consequently, ψ may assume values which are much greater than γ_0, and this is quite sufficient for our purposes. Using constraint (8), we can write the cosine in (7) in the form

$$\cos\left[k\frac{[(L-\xi)\psi + \zeta]^2 + \eta^2}{2(L-\xi)}\right].$$

Moreover, the denominator $(L-\xi)\cos\psi - \zeta\sin\psi$ in (7a) can be replaced by $L-\xi$. Since $\psi \ll 1$, the limits of integration over ξ in (7a) can be taken as 0 and L. For the log-amplitude of the wave propagating at an angle ψ to the x-axis we thus obtain

$$\chi(\psi) = \frac{k^2}{4\pi}\int_0^L \frac{d\xi}{L-\xi}\iint_{-\infty}^{\infty} d\eta\, d\zeta\, \varepsilon_1(\xi, \eta, \zeta)\cos\left[k\,\frac{[(L-\xi)\psi+\zeta]^2+\eta^2}{2(L-\xi)}\right]. \tag{9}$$

For a wave propagating to the observation point along the x-axis (its log-amplitude is denoted $\chi(0)$) we have expression (6) with $x = L$, $y = z = 0$. Multiplying the two expressions and averaging, we obtain

$$B_\chi(\psi) = \left(\frac{k^2}{4\pi}\right)^2\int_0^L \frac{d\xi}{L-\xi}\int_0^L \frac{d\xi'}{L-\xi'}\iiiint_{-\infty}^{\infty} B_\varepsilon(\xi-\xi', \eta-\eta', \zeta-\zeta')\times$$
$$\times\cos\left[k\,\frac{\eta'^2+\zeta'^2}{2(L-\xi')}\right]\cos\left\{k\,\frac{[(L-\xi)\psi+\zeta]^2+\eta^2}{2(L-\xi)}\right\}d\eta\, d\zeta\, d\eta'\, d\zeta'. \tag{10}$$

Further treatment is simplified if we take

$$B_\varepsilon(\xi-\xi', \eta-\eta', \zeta-\zeta') = \iint_{-\infty}^{\infty} F_\varepsilon(\varkappa_2, \varkappa_3, \xi-\xi')\, e^{i[\varkappa_2(\eta-\eta')+\varkappa_3(\zeta-\zeta')]}\, d\varkappa_2\, d\varkappa_3.$$

After this substitution, the integrals over η, η', ζ, ζ' are readily computed by reducing them to Poisson's integral. The integral over $(\xi+\xi')/2$ can be solved exactly, and the integral over $(\xi-\xi')$ is evaluated approximately by remembering that the function $F_\varepsilon(\varkappa, \xi)$ falls off rapidly for $\xi \ll \frac{2\pi}{\varkappa}$ (this is not the first time we have used computations of this kind, and therefore the details are omitted in this case). For $B_\chi(\psi)$ we thus obtain the expression

$$B_\chi(\psi) = \frac{\pi k^2}{8}\iint_{-\infty}^{\infty}\Phi_\varepsilon(0, \varkappa_2, \varkappa_3)\left\{\frac{\exp\left[-i\left(\frac{\varkappa^2 L}{k}-\varkappa_2 L\psi\right)\right]-1}{i\left(\frac{\varkappa^2}{k}-\psi\varkappa_2\right)}+\right.$$
$$\left.+\frac{2i\left(1-e^{i\varkappa_2 L\psi}\right)}{\varkappa_2\psi}-\frac{\exp\left[i\left(\frac{\varkappa^2 L}{k}+\varkappa_2 L\psi\right)\right]-1}{i\left(\frac{\varkappa^2}{k}+\psi\varkappa_2\right)}\right\}d\varkappa_2\, d\varkappa_3. \tag{11}$$

Changing to polar coordinates $\varkappa_2 = \varkappa\cos\varphi$, $\varkappa_3 = \varkappa\sin\varphi$, we use the equality

$$\int_0^{2\pi}\frac{e^{i\varkappa L\psi\cos\varphi}-1}{i\varkappa L\psi\cos\varphi}\,d\varphi = \int_0^{2\pi}d\varphi\int_0^1 e^{itL\varkappa\psi\cos\varphi}\,dt$$

and similar expressions for the other terms in (11). Integration over φ then leads to a Bessel function. Inserting

$$\Phi_\varepsilon(\varkappa) = AC_\varepsilon^2\varkappa^{-11/3} \quad (A = 0.033),$$

we obtain for $B_\chi(\psi)$

$$B_\chi(\psi)=\frac{\pi^2 A}{4}C_\varepsilon^2 k^2 L\int_0^1 dt\int_0^\infty\left[1-\cos\frac{\varkappa^2 Lt}{k}\right]J_0(\varkappa t L\psi)\,\varkappa^{-8/3}\,d\varkappa. \tag{12}$$

The integral in (12) can be evaluated by expanding the function J_0 in a series. For $b_\chi(\psi)=B_\chi(\psi)/B_\chi(0)$ we thus obtain

$$b_\chi(\psi)=\frac{11}{6\Gamma\left(-\frac{5}{6}\right)\cos\frac{5\pi}{12}}\sum_{k=0}^{\infty}\frac{\Gamma\left(k-\frac{5}{6}\right)\cos\left(\frac{\pi}{2}\left(k+\frac{5}{6}\right)\right)}{(k!)^2\left(k+\frac{11}{6}\right)}\left(\frac{\psi}{\gamma_0}\right)^{2k}-$$
$$-\frac{11}{16\Gamma\left(\frac{11}{6}\right)\cos\frac{5\pi}{12}}\left(\frac{\psi}{\gamma_0}\right)^{5/3}, \tag{13}$$

where

$$\gamma_0=\frac{2}{\sqrt{kL}}=\sqrt{\frac{2\lambda}{\pi L}}$$

(this definition of γ_0 differs by a numerical factor from the one used in the preceding qualitative considerations). As could have been expected, the function $b_\chi(\psi)$ depends only on the ratio ψ/γ_0. The function $b_\chi(\psi)$ computed using the first five terms of the series in (13) is shown in Figure 51.

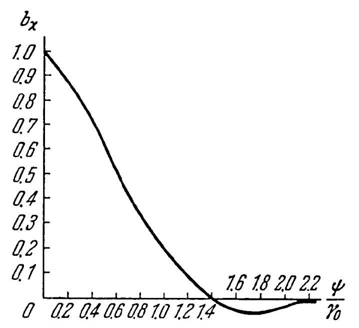


FIGURE 51. The log-amplitude correlation function for two plane waves propagating at an angle ψ with respect to each other.

Let us now return to the calculation of σ_γ^2 from (4). Expression (4) can be written in a somewhat more convenient form. For $\gamma \ll 1$ we may treat (θ_1, φ_1) and (θ_2, φ_2) as cylindrical coordinates of the points of a circle of radius γ; $\sin\theta\, d\theta\, d\varphi \approx \theta\, d\theta\, d\varphi = d\Sigma$ is an element of area of this circle, $\pi\gamma^2=\Sigma$ is its area. Let $\rho_1=(\theta_1, \varphi_1)$ and $\rho_2=(\theta_2,\varphi_2)$ be the radius-vectors of the corresponding points. According to (5) and (13), the integrand in (4) depends only on $\psi=|\rho_1-\rho_2|$. Therefore expression (4) may be written in the form

$$\sigma_\gamma^2=\frac{1}{\Sigma^2}\iint_\Sigma\iint_\Sigma\left[e^{4\langle\varkappa^2\rangle b_\chi(|\rho_1-\rho_2|)}-1\right]d\Sigma_1\,d\Sigma_2. \tag{14}$$

We again introduce a function $F_\gamma(\rho)$ which is zero everywhere outside the circle of radius γ and 1 inside this circle. Then

$$\sigma_\gamma^2 = \frac{1}{\Sigma^2}\iiiint\limits_{-\infty}^{\infty} F(\rho_1) F(\rho_2)\left[e^{4\langle\chi^2\rangle b_\chi(|\varrho_1-\varrho_2|)} - 1\right] d\Sigma_1\, d\Sigma_2 =$$
$$= \frac{1}{\Sigma^2}\iint\limits_{-\infty}^{\infty}\left[e^{4\langle\chi^2\rangle b_\chi(\rho)} - 1\right] d\Sigma_\rho \iint\limits_{-\infty}^{\infty} F(\rho_1) F(\rho_1 - \rho)\, d\Sigma_1. \tag{15}$$

Using the definition of the function $K(\rho)$ (see (53.6)) and its explicit expression (53.7) for a circle, we find

$$\sigma_\gamma^2(\gamma) = \frac{1}{\Sigma^2}\iint\limits_{-\infty}^{\infty}\left[e^{4\langle\chi^2\rangle b_\chi(\rho)} - 1\right] K(\rho)\, d\Sigma_\rho =$$
$$= \frac{1}{(\pi\gamma^2)^2}\int_0^{2\pi} d\varphi \int_0^{2\gamma} \rho\, d\rho \left[e^{4\langle\chi^2\rangle b_\chi(\rho)} - 1\right] 2\gamma^2\left[\arccos\left(\frac{\rho}{2\gamma}\right) - \right.$$
$$\left. - \frac{\rho}{2\gamma}\sqrt{1 - \frac{\rho^2}{4\gamma^2}}\right] = \frac{16}{\pi}\int_0^1 \left[e^{4\langle\chi^2\rangle b_\chi(2\gamma x)} - 1\right] \times$$
$$\times \left[\arccos x - x\sqrt{1 - x^2}\right] x\, dx. \tag{16}$$

For $\gamma = 0$, expression (16) gives

$$\sigma_\gamma^2(0) = \left[e^{4\langle\chi^2\rangle} - 1\right]\frac{16}{\pi}\int_0^1 \left[\arccos x - x\sqrt{1 - x^2}\right] x\, dx = e^{4\langle\chi^2\rangle} - 1.$$

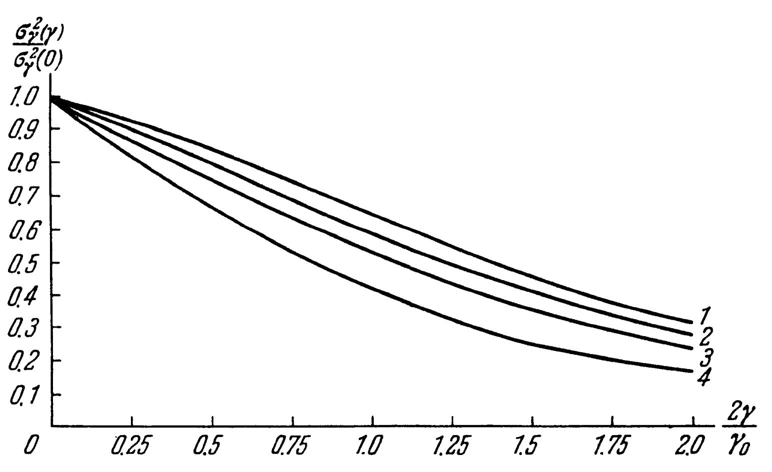


FIGURE 52. Relative reduction of intensity fluctuations for an extended source as a function of its diameter γ for various values of the parameter $4\langle\chi^2\rangle$:

1) $4\langle\chi^2\rangle \to 0$; 2) $4\langle\chi^2\rangle = 0.5$; 3) $4\langle\chi^2\rangle = 1$; 4) $4\langle\chi^2\rangle = 2.5$.

Consider the function

$$\frac{\sigma_\gamma^2(\gamma)}{\sigma_\gamma^2(0)} = \frac{16\int\limits_0^1 [e^{4\langle\chi^2\rangle b_\chi(2\gamma x)} - 1]\,[\arccos x - x\sqrt{1-x^2}]\,x\,dx}{\pi\,[e^{4\langle\chi^2\rangle} - 1]}, \tag{17}$$

which is equal to the ratio of the mean square intensity fluctuations of an extended ($\gamma \neq 0$) and a point ($\gamma = 0$) source. Since $b_\chi(\psi)$ depends on the ratio ψ/γ_0, $\sigma_\gamma^2(\gamma)/\sigma_\gamma^2(0)$ depends on the ratio $2\gamma/\gamma_0$ and also on the parameter $4\langle\chi^2\rangle$.

§ 55. "Quivering" of the image in the focal plane of a telescope

Phase fluctuations are responsible for the well-known astronomical effect of image "quivering." To a first approximation, this effect can be described in the following manner. Consider an interferometer (or a phasemeter) with baseline b receiving a wave which has traversed a turbulent atmospheric layer; and suppose that the mean position of the wave front is parallel to the baseline. A random deflection of the wave front through an angle α causes a phase difference $\Delta S = kb \sin\alpha \approx kb\alpha$, i. e., α may be expressed in terms of ΔS as $\alpha = \frac{\Delta S}{kb}$. The mean square fluctuation in α is

$$\langle\alpha^2\rangle = \sigma_\alpha^2 = \frac{\langle\Delta S^2\rangle}{k^2b^2} = \frac{D_S(b)}{k^2b^2}.$$

If we are observing through an aperture of radius R, this expression can be applied approximately by letting $b = 2R$. It is clear, however, that apertures and telescope objectives constitute a somewhat more complicated case. Phase inhomogeneities whose scale is smaller than R do not displace the position of the image: they only cause a certain blurring of the image, i. e., deterioration in the quality of the image. Large-scale inhomogeneities, on the other hand, cause an actual change in the position of the image. Let us consider this problem in more detail. Suppose a perturbed wave is incident on a lens with focal distance F. A certain intensity distribution is formed in the focal plane of the lens. The effective arrival angle can be determined from the "center of gravity" of this intensity distribution, whose coordinates are random variables. The mean square fluctuations in the center of gravity coordinates can always be converted into the mean square fluctuations of the effective angle of arrival.

This problem can also be formulated in simpler terms. Consider a lensless aperture; the incident wave is diffracted and a certain diffraction pattern (the angular distribution of the field intensity) is established far from the aperture (in the Fraunhofer diffraction zone). We know that if the intensity distribution in the focal plane of a lens is converted to angular units, the resulting angular distribution will coincide with the Fraunhofer diffraction pattern of a lenseless aperture. Since we are interested in angle of arrival fluctuations in a telescope, we should concentrate on the angular distribution of the diffracted field in the Fraunhofer approximation.

Thus, let a wave Ψ be incident on an aperture Σ. For $x = 0$ (in the plane of Σ) the wave function is given by

$$\Psi_0(y, z) = A_0 \exp(\chi(y, z) + iS(y, z)). \tag{1}$$

The field behind the aperture can be found from the wave equation

$$\Delta\Psi + k^2\Psi = 0$$

with boundary condition (1) for $x = 0$ and the radiation condition for $x = \infty$. The standard solution of this problem is

$$\Psi(x, y, z) = \iint\limits_{-\infty}^{\infty} \Psi_0(\eta, \zeta)\, w(x, y-\eta, z-\zeta)\, d\eta\, d\zeta, \tag{2}$$

where

$$w(x, y-\eta, z-\zeta) = -\frac{1}{2\pi}\frac{\partial}{\partial x}\left(\frac{e^{ikr}}{r}\right) = \frac{k}{2\pi i}\frac{e^{ikr}}{r}\frac{x}{r}\left(1-\frac{1}{ikr}\right), \tag{3}$$

$r^2 = x^2 + (y-\eta)^2 + (z-\zeta)^2$. Let also $R_0^2 = x^2 + y^2 + z^2$.

In our case the integral in (2) is taken only over the aperture Σ. The distance

$$r = \sqrt{x^2 + (y-\eta)^2 + (z-\zeta)^2} = \sqrt{R_0^2 - 2y\eta - 2z\zeta + \eta^2 + \zeta^2}$$

can be expanded in powers of η, ζ:

$$r = R_0\sqrt{1 - \frac{2(y\eta + z\zeta)}{R_0^2} + \frac{\eta^2+\zeta^2}{R_0^2}} = R_0 - \frac{y\eta + z\zeta}{R_0} + \frac{O(\eta^2, \zeta^2)}{R_0}. \tag{4}$$

The origin is placed at the center of the aperture. The last term in the expansion in (4) is then of the order R^2/R_0, where R is the size of the aperture. In the exponential function in (3) we can omit this term of the expansion if $\frac{kR^2}{R_0} \ll 1$, i.e., if

$$R^2 \ll \lambda R_0.$$

This is precisely the condition that the point is situated in the Fraunhofer diffraction region. Let

$$\frac{y}{R_0} = \sin\alpha, \quad \frac{z}{R_0} = \sin\beta. \tag{5}$$

α and β are the angles between the normal to the aperture and the direction to the observation point x, y, z measured in the planes $(x.y)$ and (x,z), respectively. Clearly,

$$\frac{x}{R_0} = \sqrt{1 - \sin^2\alpha - \sin^2\beta}.$$

Replacing r in the exponential in (3) by $R_0 - (\eta \sin\alpha + \zeta \sin\beta)$ and in the denominator simply by R_0, dropping $1/ikR_0$, and taking α and β to be small angles (this is true for $\lambda \ll \lambda_0$), we obtain

$$\Psi(x, y, z) = \frac{k}{2\pi i}\frac{e^{ikR_0}}{R_0}\iint_{\Sigma}\Psi_0(\eta, \zeta)\, e^{-ik(\alpha\eta+\beta\zeta)}\, d\eta\, d\zeta. \tag{6}$$

The angular distribution of intensity is given by the product $\Psi\Psi^*$:

$$I(\alpha, \beta) = \Psi\Psi^* = \frac{k^2}{4\pi^2 R_0^2}\iiiint_{\Sigma\,\Sigma}\Psi_0(\eta, \zeta)\Psi_0^*(\eta', \zeta')\, e^{-ik[\alpha(\eta-\eta')+\beta(\zeta-\zeta')]}\, d\eta\, d\zeta\, d\eta'\, d\zeta'. \tag{7}$$

Let us now find the angular coordinates of the center of gravity of this distribution. Denoting these coordinates by α_0, β_0, we have by definition

$$\alpha_0 = \frac{\iint_{-\infty}^{\infty} I(\alpha, \beta)\,\alpha\, d\alpha\, d\beta}{\iint_{-\infty}^{\infty} I(\alpha, \beta)\, d\alpha\, d\beta}, \quad \beta_0 = \frac{\iint_{-\infty}^{\infty} I(\alpha, \beta)\,\beta\, d\alpha\, d\beta}{\iint_{-\infty}^{\infty} I(\alpha, \beta)\, d\alpha\, d\beta}. \tag{8}$$

Let us first evaluate the integral in the denominator:

$$\iint_{-\infty}^{\infty} I(\alpha, \beta)\, d\alpha\, d\beta = \frac{k^2}{4\pi^2 R_0^2}\iiiint_{\Sigma\,\Sigma}\Psi_0(\eta, \zeta)\,\psi_0^*(\eta', \zeta')\, d\eta\, d\zeta\, d\eta'\, d\zeta' \times$$
$$\times \iint_{-\infty}^{\infty} e^{-ik[\alpha(\eta-\eta')+\beta(\zeta-\zeta')]}\, d\alpha\, d\beta = \frac{k^2}{4\pi^2 R_0^2}4\pi^2\iiiint_{\Sigma\,\Sigma}\Psi_0(\eta, \zeta)\,\Psi_0^*(\eta', \zeta') \times$$
$$\times\,\delta(k(\eta - \eta'))\,\delta(k(\zeta - \zeta'))\, d\eta\, d\zeta\, d\eta'\, d\zeta'.$$

Noting that

$$\delta(k(\eta - \eta')) = \frac{1}{k}\,\delta(\eta - \eta'),$$

we obtain

$$\iint_{-\infty}^{\infty} I(\alpha, \beta)\, d\alpha\, d\beta = \frac{1}{R_0^2}\iint_{\Sigma}|\Psi_0(\eta, \zeta)|^2\, d\eta\, d\zeta. \tag{9}$$

We can now proceed with the evaluation of the numerators:

$$\iint_{-\infty}^{\infty} \alpha I(\alpha, \beta)\, d\alpha\, d\beta = \frac{k^2}{4\pi^2 R_0^2}\iiiint_{\Sigma\,\Sigma}\Psi_0(\eta, \zeta)\,\Psi_0^*(\eta', \zeta')\, d\eta\, d\zeta\, d\eta'\, d\zeta' \times$$
$$\times \int_{-\infty}^{\infty} e^{-ik(\zeta-\zeta')\beta}\, d\beta \int_{-\infty}^{\infty} e^{-ik(\eta-\eta')\alpha}\,\alpha\, d\alpha. \tag{10}$$

The last integral in (10) is found by the following technique: consider the equality

$$\int_{-\infty}^{\infty} e^{-iu\alpha}\, d\alpha = 2\pi\delta(u).$$

Differentiation with respect to u gives

$$\int_{-\infty}^{\infty} \alpha e^{-iu\alpha}\, d\alpha = 2\pi\delta'(u)\, i.$$

Again differentiating the function $\delta(ku) = \frac{1}{k}\,\delta(u)$ with respect to u, we find $\delta'(ku)\,k = \frac{1}{k}\,\delta'(u)$, from which

$$\delta'(ku) = \frac{1}{k^2}\,\delta'(u).$$

Using these equalities, we get

$$\iint_{-\infty}^{\infty} \alpha I(\alpha, \beta)\, d\alpha\, d\beta = \frac{k^2}{R_0^2} \iiiint_{\Sigma\ \Sigma} \Psi_0(\eta, \zeta)\Psi_0^*(\eta', \zeta') \times$$

$$\times\, \delta(k(\zeta - \zeta'))\, i\delta'(k(\eta - \eta'))\, d\eta\, d\eta'\, d\zeta\, d\zeta' = -\frac{i}{kR_0^2} \iint_{\Sigma} \Psi_0(\eta, \zeta) \frac{\partial \Psi_0^*(\eta, \zeta)}{\partial \eta}\, d\eta\, d\zeta. \qquad (11)$$

In the last equality we made use of the relation

$$\int f(x)\delta'(x - x_0)\, dx = -f'(x_0).$$

Dividing (11) by (9), we obtain

$$\alpha_0 = \frac{1}{ik} \frac{\iint_{\Sigma} \Psi_0(\eta, \zeta) \frac{\partial \Psi_0^*(\eta, \zeta)}{\partial \eta}\, d\eta\, d\zeta}{\iint_{\Sigma} |\Psi_0(\eta, \zeta)|^2\, d\eta\, d\zeta}. \qquad (12)$$

An identical expression can be written for β_0. Expression (12), however, is better represented in a somewhat different form. Integrating by parts and seeing that $\Psi_0 = 0$ on the boundary of Σ, we find

$$\alpha_0 = -\frac{1}{ik} \frac{\iint_{\Sigma} \frac{\partial \Psi_0(\eta, \zeta)}{\partial \eta} \Psi_0^*(\eta, \zeta)\, d\eta\, d\zeta}{\iint_{\Sigma} |\Psi(\eta, \zeta)|^2\, d\eta\, d\zeta}. \qquad (12a)$$

Taking half the sum of these two expressions, we obtain

$$\alpha_0 = \frac{1}{k} \frac{\iint_{\Sigma} \frac{1}{2i}\left[\Psi_0(\eta, \zeta) \frac{\partial \Psi_0^*(\eta, \zeta)}{\partial \eta} - \Psi_0^*(\eta, \zeta) \frac{\partial \Psi_0(\eta, \zeta)}{\partial \eta}\right] d\eta\, d\zeta}{\iint_{\Sigma} |\Psi_0(\eta, \zeta)|^2\, d\eta\, d\zeta} =$$

$$= \frac{1}{k} \frac{\iint_{\Sigma} \mathrm{Im}\left[\Psi_0(\eta, \zeta) \frac{\partial \Psi_0^*(\eta, \zeta)}{\partial \eta}\right] d\eta\, d\zeta}{\iint_{\Sigma} |\Psi_0(\eta, \zeta)|^2\, d\eta\, d\zeta}. \qquad (13)$$

Insertion of $\Psi_0 = A_0 e^{\chi + iS_1}$ gives

$$\alpha_0 = -\frac{1}{k}\frac{\iint\limits_{\Sigma} e^{2\chi(\eta,\zeta)}\frac{\partial S_1(\eta,\zeta)}{\partial\eta}\,d\eta\,d\zeta}{\iint\limits_{\Sigma} e^{2\chi(\eta,\zeta)}\,d\zeta\,d\eta}. \tag{14}$$

This expression shows that the main contribution in (14) is from phase fluctuations: for $S_1 = \text{const}$, $\alpha_0 = 0$. Amplitude fluctuations introduce only a second-order correction, so that in the first approximation they may be ignored and we may put $\chi = 0$. In this case

$$\alpha_0 = -\frac{1}{k}\frac{1}{\Sigma}\iint\limits_{\Sigma}\frac{\partial S_1(\eta,\zeta)}{\partial\eta}\,d\eta\,d\zeta. \tag{15}$$

Let us now find the mean square σ_α^2 of fluctuations in α_0:

$$\sigma_\alpha^2 = \frac{1}{k^2\Sigma^2}\iiiint\limits_{\Sigma\ \Sigma}\left\langle\frac{\partial S_1(\eta,\zeta)}{\partial\eta}\,\frac{\partial S_1(\eta',\zeta')}{\partial\eta'}\right\rangle d\eta\,d\zeta\,d\eta'\,d\zeta'. \tag{16}$$

The integrand can be transformed as follows:

$$\left\langle\frac{\partial S_1(\eta,\zeta)}{\partial\eta}\,\frac{\partial S_1(\eta',\zeta')}{\partial\eta'}\right\rangle = \frac{\partial^2}{\partial\eta\,\partial\eta'}\langle S_1(\eta,\zeta)\,S_1(\eta',\zeta')\rangle =$$
$$= \frac{\partial^2}{\partial\eta\,\partial\eta'}B_S(\eta-\eta',\zeta-\zeta') = -\frac{\partial}{\partial\eta^2}B_S(\eta-\eta',\zeta-\zeta') =$$
$$= \frac{1}{2}\frac{\partial^2}{\partial\eta^2}D_S(\eta-\eta',\zeta-\zeta').$$

Again introducing the function

$$F(\eta,\zeta) = \begin{cases}1 \text{ inside } \Sigma,\\ 0 \text{ outside } \Sigma,\end{cases}$$

we obtain

$$\sigma_\alpha^2 = \frac{1}{2k^2\Sigma^2}\iiiint\limits_{-\infty}^{\infty}F(\eta,\zeta)\,F(\eta',\zeta')\frac{\partial^2 D_S(\eta-\eta',\zeta-\zeta')}{\partial\eta^2}\,d\eta\,d\zeta\,d\eta'\,d\zeta' =$$
$$= \frac{1}{2k^2\Sigma^2}\iint\limits_{-\infty}^{\infty}\frac{\partial^2 D_S(\eta,\zeta)}{\partial\eta^2}K(\eta,\zeta)\,d\eta\,d\zeta, \tag{17}$$

where we proceeded along the same lines as in passing from (53.4) to (53.5).
If the fluctuations are isotropic, we can write

$$D_S(\eta,\zeta) = D_S(\sqrt{\eta^2+\zeta^2}) = D_S(\rho),$$
$$\frac{\partial^2 D_S(\rho)}{\partial\eta^2} = \frac{\eta^2}{\rho^2}D_S''(\rho) + \frac{\zeta^2}{\rho^2}\frac{D_S'(\rho)}{\rho}. \tag{18}$$

We will only consider the case of a circular aperture of radius R. Introducing in (17) polar coordinates $\eta = \rho\cos\varphi$, $\zeta = \rho\sin\varphi$ and inserting (53.7) for K, we obtain

$$\sigma_\alpha^2 = \frac{1}{2k^2\pi^2 R^4} \int_0^{2R} \rho\, d\rho \int_0^{2\pi} d\varphi \left[\cos^2\varphi\, D_S''(\rho) + \sin^2\varphi \frac{D_S'(\rho)}{\rho} \right] \times$$
$$\times\, 2R^2 \left[\arccos\left(\frac{\rho}{2R}\right) - \frac{\rho}{2R} \sqrt{1 - \frac{\rho^2}{4R^2}} \right] =$$
$$= \frac{1}{\pi k^2 R^2} \int_0^{2R} \left[D_S''(\rho) + \frac{D_S'(\rho)}{\rho} \right] \left[\arccos\left(\frac{\rho}{2R}\right) - \frac{\rho}{2R} \sqrt{1 - \frac{\rho^2}{4R^2}} \right] \rho\, d\rho. \qquad (19)$$

We can now substitute an explicit expression for the structure function $D_S(\rho)$. We distinguish between two cases. If $2R \ll \sqrt{\lambda L}$, we have

$$D_S(\rho) = \frac{1}{2} \cdot 0.73 C_\varepsilon^2 L \rho^{5/3} \quad (\rho \ll \sqrt{\lambda L}) \qquad (20)$$

(the small quadratic region of $D_S(\rho)$ for $\rho \ll \lambda_0$ can be neglected, since the integral in (19), with $D_S(\rho)$ expressed by (20), converges at the origin). If now $2R \gg \sqrt{\lambda L}$, over most of the region of integration we have

$$D_S(\rho) = 0.73 C_\varepsilon^2 k^2 L \rho^{5/3} \quad (\rho \gg \sqrt{\lambda L}), \qquad (21)$$

and we will use this expression over the entire region.

We will carry out the integration using (20) in (19). (For $\sqrt{\lambda L} \ll 2R$ the final result only differs by a factor of 2 in the numerical coefficient). Introducing a new variable of integration $x = \frac{\rho}{2R}$, we obtain

$$\sigma_\alpha^2 = \frac{0.73}{2} C_\varepsilon^2 L (2R)^{-1/3} \frac{400}{9\pi} \int_0^1 x^{2/3} [\arccos x - x\sqrt{1 - x^2}]\, dx. \qquad (22)$$

The integral I on the right-hand side of (22) is readily evaluated. The second term is reduced to a beta function (Euler's integral of the first kind) by substituting $x^2 = t$, and the first term is reduced to the same function by integration by parts.

Thus,

$$I = \frac{9\sqrt{\pi}}{55} \frac{\Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\frac{11}{6}\right)}$$

and the expression for σ_α^2 takes the form

$$\sigma_\alpha^2 = 0.97 \frac{0.73}{2} C_\varepsilon^2 L (2R)^{-1/3} \quad (\lambda_0 \ll 2R \ll \sqrt{\lambda L}). \qquad (23)$$

It differs from the result obtained from (1) for $b = 2R$ only in the numerical coefficient 0.97. For $2R \gg \sqrt{\lambda L}$ we have

$$\sigma_\alpha^2 = 0.71 C_\varepsilon^2 L (2R)^{-1/3} \quad (2R \gg \sqrt{\lambda L}). \qquad (23a)$$

Let us briefly consider the second aspect of this problem, namely image blurring. Expressions (23) and (23a) give the mean square of the fluctuations of the center of gravity of the image, but the internal structure of a moving image also undergoes fluctuations. The corresponding calculations can be carried out if we consider the mean intensity in a coordinate system which moves together with the center of gravity (expression (15)). We will not go into these highly tedious calculations and it suffices to point out that on changing over to a moving system of coordinates the effect of large-scale inhomogeneities is eliminated and the internal structure of the moving image is found to be determined by small-scale fluctuations only. The angular size of the dancing spot is determined by the mean square fluctuations of the arrival angle

$$\left.\frac{D_S(\rho)}{k^2\rho^2}\right|_{\rho=0}$$

(it was calculated in the previous chapter). Note that calculations of the field structure behind a lens are also given in /113, 119-123/.

B. EXPERIMENTAL DATA ON TROPOSPHERIC PROPAGATION OF LIGHT, SOUND, AND RADIO WAVES

§ 56. Propagation of light in the atmospheric layer close to the ground

Experiments conducted in the atmosphere close to the ground are highly attractive, since in conjunction with light scintillation measurements one can measure the fluctuations in the index of refraction (i. e., the parameter C_ε) and carry out tests for various accurately known values of L. Surface experiments thus provide experimental data which can be readily compared with the theory /124 — 127, 170/.

The results published in /124 — 126, 170/ were obtained at the Tsimlyanskaya Station of the Institute of Atmospheric Physics (USSR Academy of Sciences), situated on a flat stretch of steppe. The local topography ensured homogeneous turbulence along the entire ray path (the light propagated in the horizontal direction approximately at a constant height above the ground). The distances from the source to the receiver were equal to 250, 500, 1000, and 2000 m. At distances less than 250 m the scintillation effect became comparable with instrumental noise and no measurements were made. Distances over 2000 m were not practical due to topographic irregularities. The mean height of the ray above the ground was about 2 m.

The measurements in /124 — 126/ were carried out at nighttime, when the refractive index fluctuations were relatively small. In /170/ daytime measurements were made in hot summer weather, when the refractive index fluctuated widely.

Let us first consider the findings of /124 — 126/.

The source was a 30-watt incandescent lamp. The light was focused by a high-power lens on a 0.5 mm dia. aperture. Then came a rotating shutter with 150 slits turning at 100 rps. The aperture was in the focus of the final objective (focal distance 25 cm, 10 cm in dia.) which formed a slightly divergent beam modulated at 15,000 Hz. The modulation of the beam followed by a tuned amplifier at the resonant frequency of the signal eliminated the effect of external unmodulated sources and made it possible to simplify the receiving equipment (d. c. amplifiers were not necessary).

The light detector (Figure 53) was comprised of two FEU-19 photomultipliers behind two apertures (g_1 and g_2) set at right angles to the beam followed by two prisms. The distance ρ between the apertures could be varied from 0.5 to 50 cm. The diameter of the receiving aperture was 2 mm, so that the effect of "aperture averaging" was completely eliminated.

The variable voltage components at the photomultiplier outputs with amplitudes proportional to $I(M_1)$ and $I(M_2)$ ($I(M)$ is the instantaneous value of the light flux through the aperture at M) are amplified by tuned amplifiers with a passband of about 2000 cps and then detected. The detector output voltages V_1 and V_2 are proportional to $I(M_1)$ and $I(M_2)$. The amplifiers are provided with a special feedback system which maintains the equality $\bar{V}_1 = \bar{V}_2$ (averaging with a 100 sec time constant). After removing the constant components, we have the voltages $V_1' = V_1 - \bar{V}_1$ and $V_2' = V_2 - \bar{V}_2$ proportional to the fluctuations of the light flux $I'(M_1) = I(M_1) - \bar{I}(M_1)$ and $I'(M_2) = I(M_2) - \bar{I}(M_2)$. The voltages V_1' and V_2' underwent automatic statistical analysis using special equipment (see /128/).

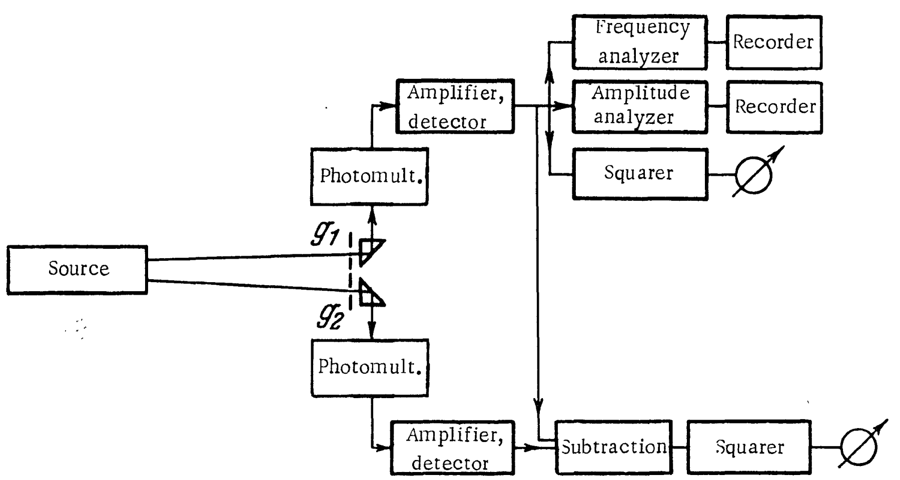


FIGURE 53. The block diagram of the experiment to investigate the scintillation of a light source close to the ground.

The measurements gave (in identical units) the probability distribution of the fluctuations $I'(M_1)$; the mean square of fluctuations $\langle I'^2(M_1)\rangle$; the mean value $\langle I(M_2)\rangle$; the correlation function $\langle I'(M_1)\cdot I'(M_2)\rangle = B_I(M_1, M_2)$; the frequency spectrum of the fluctuations $I'(M_1)$ in the frequency range between 0.05 and 1000 Hz.

At the same time as the scintillation measurements, meteorological observations were carried out along the ray path, so that C_ε^2 could be computed. Temperature profiles were made from 0.5 to 12 m above the ground, wind speed profiles in the same layer, and wind direction.

The results of these measurements were used to calculate the turbulence parameters ε, K, T_*.

Since the experiments were carried out over a very even region of the steppe and identical turbulent conditions prevailed over all parts of the ray path, meteorological measurements were made at a single point only.

The results of these measurements are briefly summarized in the following.

a. The probability distribution of the light intensity fluctuations. The theory shows that the log-amplitude of the light wave is expressed in terms of the refractive index fluctuations along the ray path by an integral of the form

$$\ln \frac{A}{A_0} = \iiint F(\mathbf{r}')\, n'(\mathbf{r}')\, d^3r'.$$

The entire integration region D can be divided into numerous subregions D_i with linear dimensions of the order of the outer scale of turbulence L_0, which in these experiments was determined by the height of the ray above the ground.

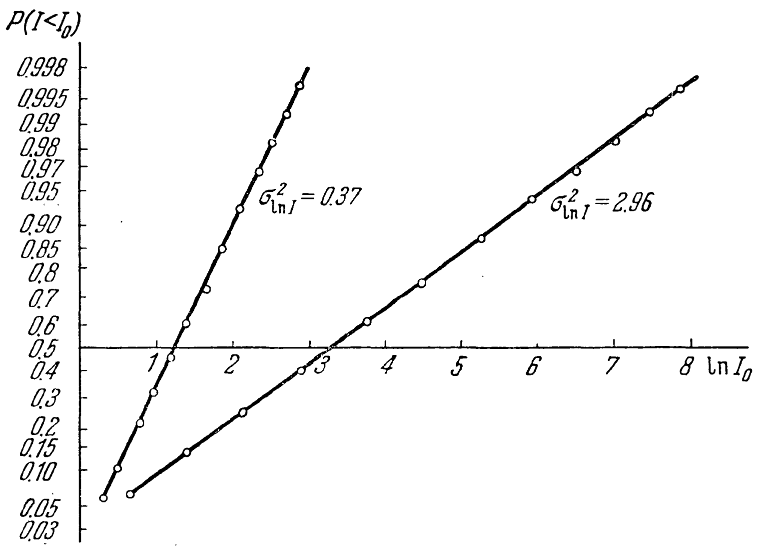


FIGURE 54. Empirical probability distributions of the log-intensity of light propagating close to the ground.

The distribution corresponding to $\sigma^2_{\ln I} = 2.96$ was obtained during daytime measurements /170/.

The fluctuations $n'(\mathbf{r})$ in these subregions are uncorrelated. Therefore, by the central limit theorem of probability theory, $\ln\, (A/A_0)$ is normally distributed.

Since $(I/I_0) = 2 \ln (A/A_0)$, $\ln (I/I_0)$ is also normally distributed and I has a log-normal distribution. Experimental findings confirm this conclusion. Figure 54 shows the empirical probability distributions of the fluctuations of I. The vertical axis gives the probability $P\ (I < I_0)$; the scale is linear since we actually laid off on this axis the function $\Phi^{-1}(P\ (I < I_0))$, where $\Phi^{-1}(x)$ is the inverse of the probability integral

$$\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-t^2/2}dt.$$

The abscissa axis gives $\ln I_0$. In these coordinates the log-normal distribution is represented by a straight line. All in all, some hundred empirical distribution functions $F(I)$ were reduced. They all show a good fit to a normal distribution for $\ln I$.

Using the log-normal distribution for I, we can relate the experimentally measured functions $\langle I\rangle$ and $\sigma_I^2=\langle[I-\langle I\rangle]^2\rangle$ to the theoretical parameter

$$\sigma^2=\langle[\ln I-\langle\ln I\rangle]^2\rangle=4\sigma_\chi^2=\ln\left[1+\frac{\sigma_I^2}{\langle I\rangle^2}\right].$$

This relation was used in further reduction of the experimental data.

b. The correlation function of the light intensity in a plane perpendicular to the beam. As we have already noted, for $\sqrt{\lambda L}\gg\lambda_0$ (close to the ground, where λ_0 is a millimeter or two, this condition is satisfied for light waves when L is of the order of a few tens of meters), the intensity correlation length is of the order $\sqrt{\lambda L}$ and the argument of the intensity correlation function is $\rho/\sqrt{\lambda L}$.

This similarity hypothesis was directly checked in the above experiments. The correlation coefficient R was measured for various values of $\sqrt{\lambda L}$, corresponding to $L=2000$, 1000, and 500 m. However, the distance ρ between the apertures was so chosen that $\rho/\sqrt{\lambda L}$ always assumed the same values 0.25, 0.5, 2, 4, and 8.

The measured correlation coefficients R showed a fairly large scatter due to insufficient accuracy of measurements. However, repeated measurements of R markedly reduced the error, so that the mean value of R obtained for constant $\rho/\sqrt{\lambda L}$ and various $\sqrt{\lambda L}$ show good agreement. Table 2 lists the values of R for various L, as well as the mean results for all L; n is the number of measurements.

TABLE 2.

$\frac{\rho}{\sqrt{\lambda L}}$	$L=2000$ m, $\sqrt{\lambda L}=3.2$ cm		$L=1000$ m, $\sqrt{\lambda L}=2.2$ cm		$L=500$ m, $\sqrt{\lambda L}=1.6$ cm		Mean		
	R	n	R	n	R	n	R	n	5% confidence limits
0.25	0.58	8	0.46	15	—	—	0.50	23	0.05
0.5	0.27	9	0.31	19	0.27	12	0.29	40	0.05
1.0	0.09	11	0.10	18	0.16	15	0.12	43	0.06
2	−0.05	7	−0.05	15	−0.07	14	−0.055	36	0.08
4	−0.08	6	−0.09	13	−0.03	14	−0.062	33	0.08
8	−0.08	7	−0.03	14	−0.13	9	−0.072	30	0.06

The data of Table 2 are graphically shown in Figure 55. Different symbols correspond to R values obtained for different L. We see from the graph that the scatter of R values obtained for different $\sqrt{\lambda L}$ falls within the measurement accuracy (the vertical bars in the figure mark the 5% confidence limits).

The results confirm convincingly the theoretical conclusion that the correlation function depends on $\rho/\sqrt{\lambda L}$ and that the correlation radius of the intensity fluctuations is of the order $\sqrt{\lambda L}$.

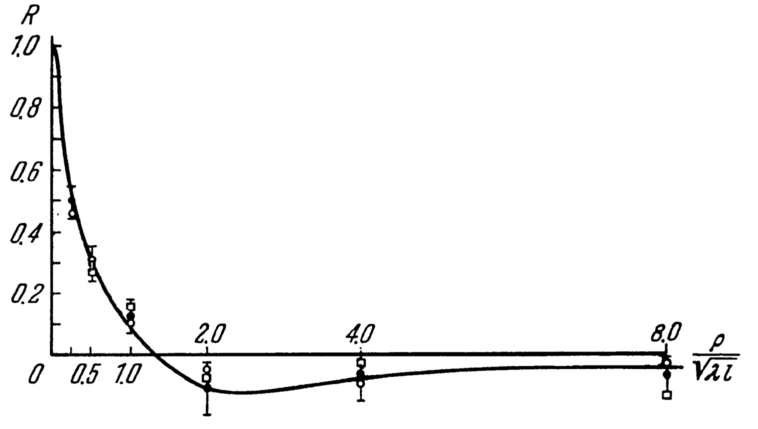


FIGURE 55. Empirical correlation function of light intensity fluctuations.

Thus, all attempts to determine the "mean size of inhomogeneities" from the correlation radius of the intensity fluctuations are doomed to failure, since these observations will give only the parameter $\sqrt{\lambda L}$.

c. Frequency spectrum of intensity fluctuations. The frequency spectrum of light intensity fluctuations was measured using a frequency analyzer with 30 filters each with 0.5 octave passband (the upper-to-lower frequency ratio of each filter was $\sqrt{2}$); the filters were spaced at 0.5 octave intervals from 0.05 to 1160 Hz.

Eighty scintillation spectra obtained over distances $L = 1000$ and 2000 m were analyzed. Simultaneous meteorological observations were used to compute the transverse component of the mean wind velocity $v_\perp$. The measurements for each L were divided into three categories according to the value of $v_\perp$: $1 < v_\perp < 2$ m/sec; $2 < v_\perp < 3$ m/sec; $3 < v_\perp < 4$ m/sec.

For each category, the mean spectral densities of the fluctuations $w(f)$ were obtained (the averaging was performed using a logarithmic scale). After that the "normalized" spectral densities were computed,

$$\frac{w(f)}{\int_0^\infty w(f)\,df}.$$

Figure 56 plots in log coordinates the function

$$u(f) = \frac{f w(f)}{\int_0^\infty w(f)\,df},$$

corresponding to various wind speeds $v_\perp$ (the average wind speed for each of the three categories was taken).

We see from the graph that the $u(f)$ curves are displaced toward higher frequencies as the mean wind speed increases. The frequency f_m corresponding to the peak of the $u(f)$ curve can be found from these graphs (f_m was actually calculated as half the sum of the frequencies corresponding to $u(f) = \frac{1}{2}[u(f)]_{max}$). Table 3 gives the mean wind velocities for various groups of f_m values, and the corresponding values of $f_m \sqrt{\lambda L}/v_\perp$.

TABLE 3.

	$L = 1000$ m			$L = 2000$ m		
$v_\perp$, m/sec	1.46	2.18	3.46	1.61	2.59	3.51
f_m, Hz	20	25.6	45.7	18.1	25.6	39.8
$\frac{f_m\sqrt{\lambda L}}{v_\perp}$	0.31	0.26	0.30	0.35	0.31	0.36

The ratio $f_m \sqrt{\lambda L}/v_\perp$ is approximately constant and its mean value is 0.32. The frequencies f_m are thus related to $v_\perp$ and $\sqrt{\lambda L}$ by the equality

$$f_m = 0.32 \frac{v_\perp}{\sqrt{\lambda L}}. \quad (1)$$

Note that calculation based on Taylor's hypothesis of frozen turbulence gives the relation $f_m = 0.55\frac{v_\perp}{\sqrt{\lambda L}}$, which differs from (1) only by a numerical coefficient.

However, the theoretical relation between the spatial correlation length R_0 $(B_A(R_0) = 0)$ and f_m, which has the form

$$R_0 = 0.44 \frac{v_\perp}{f_m},$$

shows good agreement with the experimental findings, since according to the experimental data $R_0 = 1.5\sqrt{\lambda L}$, and this gives $R_0 = 0.48 \frac{v_\perp}{f_m}$.

All the frequency spectra of Figure 56 are shown in Figure 57 as a function of the dimensionless argument $f\sqrt{\lambda L}/v_\perp$. As is evident from the graph the different spectra, when transformed in this manner, differ little from one another; this again confirms the previous conclusion that $u(f)$ is a function of the argument $f\sqrt{\lambda L}/v_\perp$ only,

$$\frac{f w(f)}{\int_0^\infty w(f)\,df} = u\left(\frac{f\sqrt{\lambda L}}{v_\perp}\right).$$

The function on the right-hand side of this equality was computed theoretically on pp. 262–263 using Taylor's hypothesis of frozen turbulence.

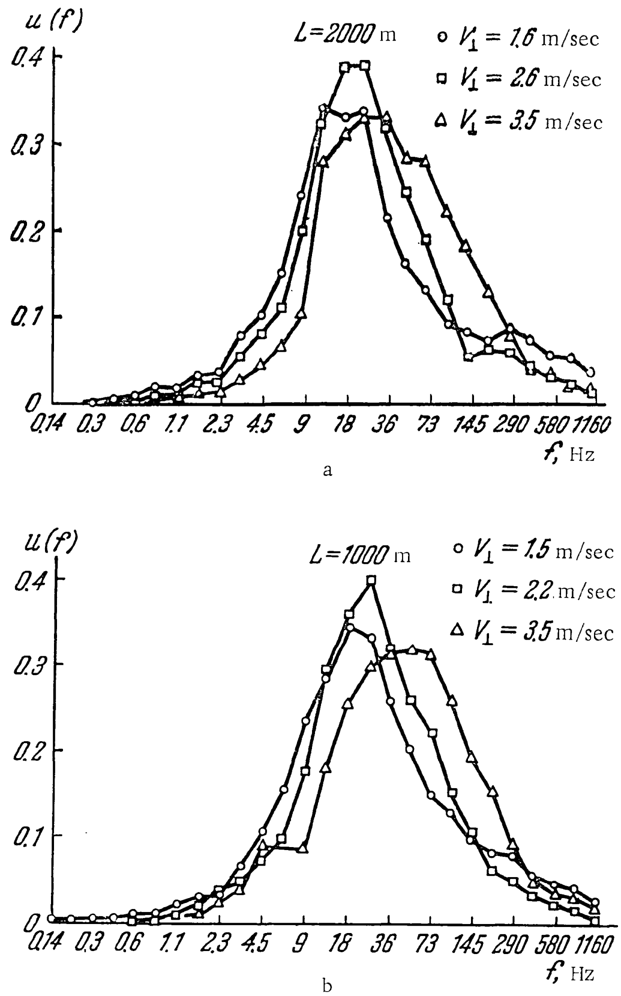


FIGURE 56. The empirical normalized frequency spectrum of light intensity fluctuations for various wind speeds.

The vertical axis gives $u = fw(f)/\langle\chi^2\rangle$. The horizontal axis gives the logarithm of the frequency f. In these coordinates, the area under each curve is unity.

Figure 58 gives a comparison of the theoretical curve with the experimental data obtained by averaging the spectra of Figure 57. We see that the theoretical curve is "narrower" than the experimental one; this is apparently due to the assumption in the calculations that the wind velocity is constant along the entire ray path.

Let us now consider the important results obtained by Gracheva and Gurvich /170/. They checked the previous expression $\langle\chi^2\rangle = 0.077 C_\varepsilon^2 k^{7/6} L^{11/6} \equiv \sigma_1^2(L)$ for the mean square of log-amplitude fluctuations.

In the comparison of the theoretical and experimental results it is necessary to remember that the source does not emit a plane wave but a weakly divergent beam of plane waves. Indeed, each point of the luminous body in the focal plane of the source lens gives a plane wave propagating at a certain angle to the optical axis. Different points of the luminous body act as incoherent light sources, so that the lens output is an ensemble of incoherent plane waves propagating inside a certain solid angle. The detector therefore receives simultaneously incoherent waves from several different directions, like the light from a finite-size planet. We have calculated previously the attenuation of the scintillations due to the finite angular size of the source. This calculation was experimentally checked in /171/, and the results were found in good agreement with the theoretical conclusions. In /170/ the averaging effect of the extended light source was taken into account and appropriate corrections were introduced in the comparison of theory with experiment: the results of the measurements were converted to a light source of infinitesimal dimensions.

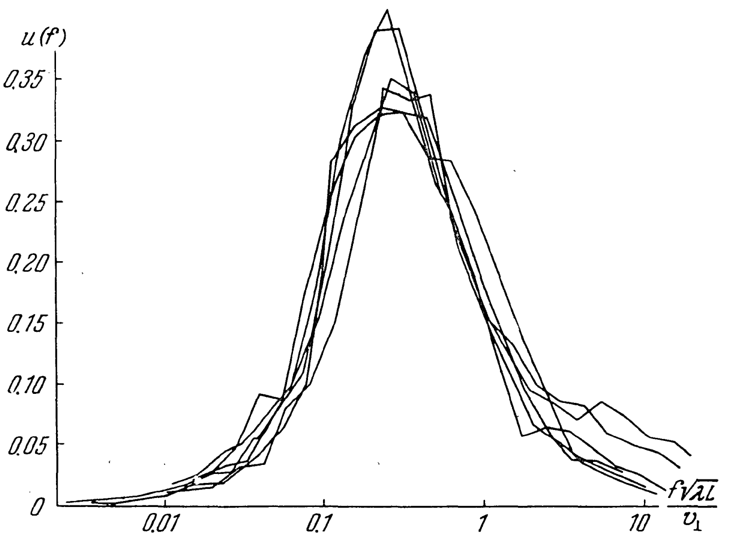


FIGURE 57. Empirical frequency spectra of light intensity fluctuations for the dimensionless frequency $\frac{f\sqrt{\lambda L}}{v_\perp}$.

The vertical axis gives $u(f) = fw(f)/\langle\chi^2\rangle$.

The measurements were carried out in daytime, under conditions of strong convective instability of the atmosphere, when the refractive index fluctuations are particularly pronounced. The light source was a 250-watt ultrahigh-pressure mercury lamp supplied by a 2500 Hz high-power generator. Since the lamp doubles the frequency, the emergent beam was modulated at 5000 Hz. The light source, located at the focus of the objective, was placed against a dark screen as background, in order to minimize the scattered light from other sources.

The distance between the source and the detector was varied from 125 to 1750 m. When the distance was changed, the outlet aperture of the source was so adjusted that the ratio γ/γ_0 remained constant ($\gamma_0 = \sqrt{2\lambda/\pi L}$ is the characteristic angle which determines the attenuation of scintillation due to averaging over the extended source). The angular diameter of the source γ in this experiment was equal to its apparent angular diameter D/L, since the beam width was much greater than this value. In changing the distance, the ratio of the aperture diameter D to the radius of the first Fresnel zone $\sqrt{\lambda L}$ was therefore maintained constant.

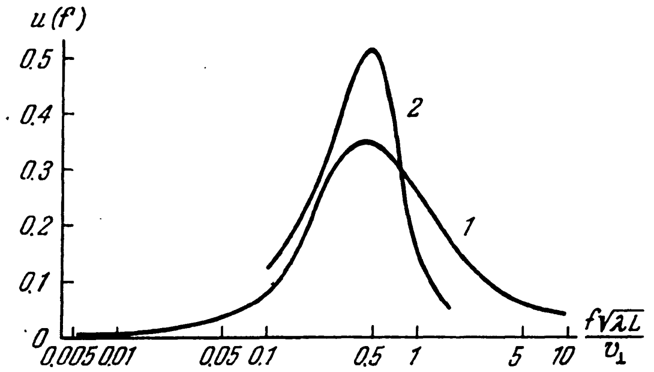


FIGURE 58. Comparison of empirical (curve 1) and theoretical (curve 2) frequency spectra of light intensity fluctuations $u(f) = fw(f)/\langle\chi^2\rangle$.

The light detector was a FEU-19 photomultiplier set at the end of a meter-long tube with darkened interior walls and also with stops to help reduce extraneous light. The apertures in the tube were 2 mm in diameter, which was quite sufficient to eliminate averaging over the inlet aperture. The photomultiplier signal was amplified by an amplifier with a passband from 4500 to 5500 Hz, detected by a linear detector, and then fed into special units which measured $\langle I\rangle$, $\langle|I-\langle I\rangle|\rangle$. The detector output signal was also recorded on a loop oscillograph, and was then analyzed to obtain the probability distribution of the intensity fluctuations (a typical recording is shown in Figure 59).

Vertical wind speed and temperature profiles were measured simultaneously with intensity fluctuations, and the results were used to compute C_ε^2. The calculations were carried out using the relations of Chapter 1 and allowing for the dependence of the turbulence fluctuation parameters on Richardson number.

Since the direct experimental results were $\langle|I-\langle I\rangle|\rangle$, $\langle I\rangle$, whereas comparison with the theory required the function $\sigma^2 = \langle[\ln I - \langle\ln I\rangle]^2\rangle$, a relation had to be established between these quantities. To obtain this relation, it suffices to know the probability distribution of I. A special study of the amplitude distribution was therefore undertaken. In the region of validity of the first approximation of the method of smooth perturbations, this distribution is log-normal, as is evident from the previous measurements. Under the conditions of the present experiments, $\sigma_1^2 = 0.077k^{7/6}C_\varepsilon^2L^{11/6}$ was often much greater than unity (reaching peak values of 25), but the distribution was nevertheless nearly log-normal.* An empirical distribution curve for the region of strong fluctuations is shown in Figure 54. In this example, $\sigma_1^2 = 9$.

Assuming a log-normal-distribution, the measured values of $\langle I\rangle$, $\langle|I-\langle I\rangle|\rangle$ can be used to calculate $\beta^2 = \langle[I-\langle I\rangle]^2\rangle/\langle I\rangle^2$ (we do not give here the corresponding formula, as it is very cumbersome). Having found β^2 for the scintillations of an extended source, one uses the curves in Figure 52 to reduce the results to a point source. After that the equality $\sigma_I^2 = \exp(\beta^2) - 1$ is applied to compute the fluctuations of the log intensity

* Sn all deviations from the log-normal distribution are sometimes observed for little-probable values of I (foı very high and very small I).

$$\sigma_I^2 = \langle [\ln I - \langle \ln I \rangle]^2 \rangle = 4 \langle [\ln A - \langle \ln A \rangle]^2 \rangle.$$

The dependence of the experimental parameter $\sigma_I = 2\sigma_\chi$ obtained in this way on the parameter $2\sigma_1$ (the corresponding parameter calculated on the basis of the first approximation of the method of smooth perturbations) is shown in Figure 60. A similar dependence was later obtained by the same authors using new equipment which measured directly the mean square of the intensity fluctuations $\langle [I - \langle I \rangle]^2 \rangle$ and $\langle I \rangle$ (rahter than $\langle I \rangle, \langle | I - \langle I \rangle | \rangle$ as before).

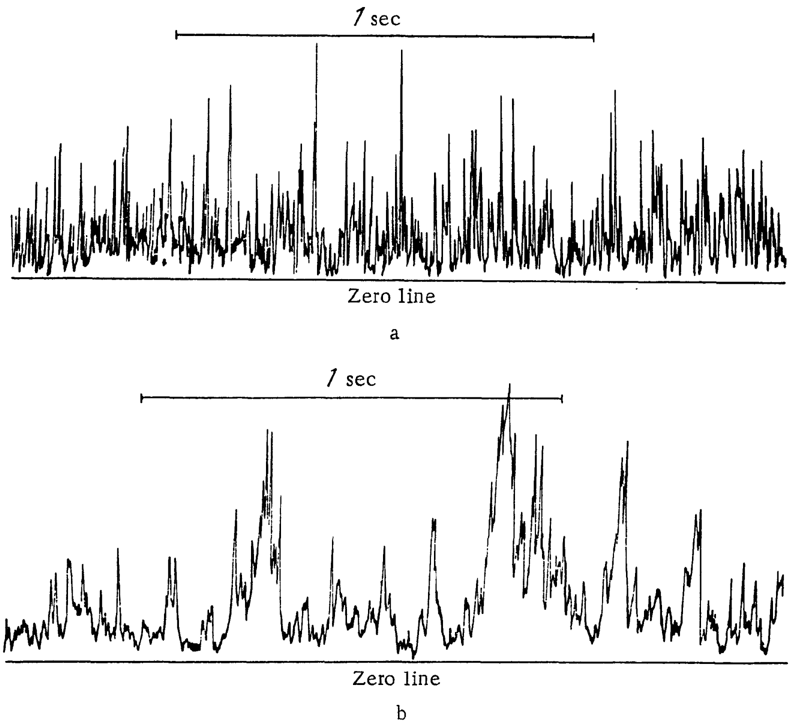


FIGURE 59. Typical recordings of strong fluctuations in light intensity:
(a) $\sigma_1 \approx 1$; (b) $\sigma_1 \approx 5$.
The wind speeds were approximately the same for the two cases /182/.

In Figure 60 we clearly distinguish between two regions: the region of weak fluctuations, corresponding to $\sigma_1 = 0.28\, C_\varepsilon k^{7/12} L^{11/12} < 0.8$, and the region of strong fluctuations, corresponding to values of this parameter that exceed 0.8. In the first of these regions the theoretical values obtained by using the first approximation of the method of smooth perturbations show a good fit with the experiment. In the strong-fluctuation region, on the other hand, the experimental values of the mean square of the log-intensity fluctuations show a definite "saturation." The results of /170/ thus indicate that the first approximation of the method of smooth perturbations applies to amplitude fluctuations up to $\sigma_1^2 = 0.077\, C_\varepsilon^2 k^{7/6} L^{11/6} < 0.64$.

The region of strong fluctuations was considered before in § 51. The treatment of that section did not permit finding a numerical value of the saturation level, but nevertheless explained the experimental findings of Gracheva and Gurvich. If we assume for the theoretical parameter σ_∞^2 the experimental numerical value of 0.64 and plot the theoretical curve in Figure 60 (the solid curve), we obtain a good fit between theory and experiment for all $\sigma_1^2 < 25$.

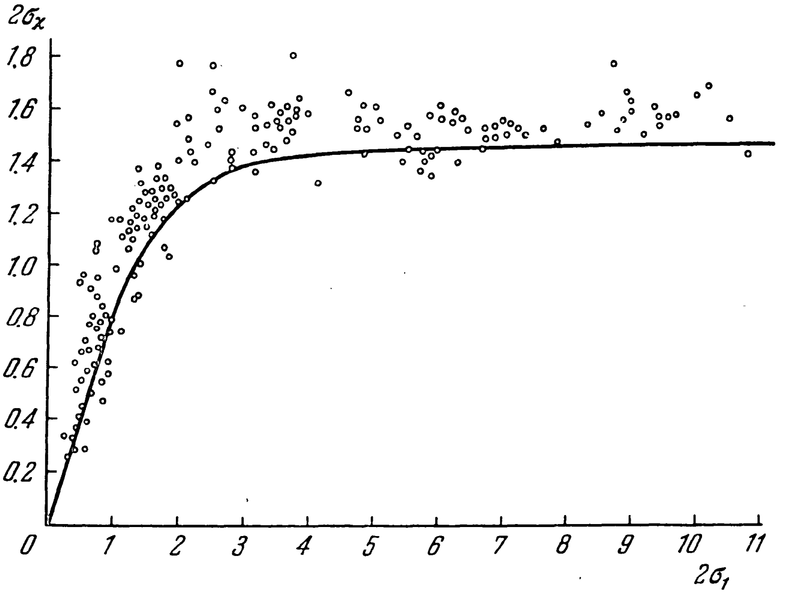


FIGURE 60. Comparison of the experimentally obtained values of the mean square of the log intensity fluctuation ($2\sigma_\chi$) with the corresponding quantity in the first approximation of the method of smooth perturbations ($2\sigma_1$).

In Chapter 5, § 66 we give an alternative approach to the theory of strong fluctuations which explains the experimentally observed dependence $\sigma_\chi = f(\sigma_1)$. The dependence $\sigma_\chi = f(\sigma_1)$ derived in § 66 is adequately fitted by the analytical expression

$$\sigma_\chi^2 = 1 - \frac{1}{[1 + 6\sigma_1^2]^{1/4}}$$

and shows good agreement with the experimental data of Figure 60 (see Figure 96 on p. 516).

Also note that the theory of § 51 and likewise that of § 66 show that the distribution of the intensity fluctuations in the strong-fluctuation region should be nearly log-normal.

d. M e a s u r e m e n t s o f a n g l e - o f - a r r i v a l f l u c t u a t i o n s. Angle-of-arrival fluctuations of a light wave (image "quivering") were measured in /129/.

The measurement circuit shown in Figure 61 was developed for determining the statistical characteristics of image "quivering" — the variance and the frequency spectrum. A light source with variable aperture (the angular diameter of the aperture was always about 2") was placed at a distance L from the telescope. The telescope objective had a focal length of 80 cm and a diameter of 8 cm. The mirror of a loop galvanometer was placed between the telescope objective and its focus, at a distance of 1 cm from the focus. The light reflected from the mirror was focused by a second lens on a vertical slit 30 μ wide (this is approximately half the size of the image of the source magnified by the second objective). A photomultiplier was located behind the slit when the distance to the light source was changed. System focusing was adjusted by moving the telescope objective.

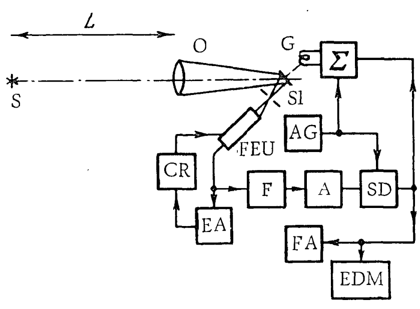


FIGURE 61. Block diagram of the experimental setup for measuring fluctuations in the angle of arrival of light.

S — light source, O — objective, G — loop galvanometer, Sl — slit, FEU — photomultiplier, F — 5 kHz filter, A — amplifier, SD — synchronous detector, Σ — summing bridge, EA — electrometer amplifier, CR — controlled high-voltage rectifier, AG — audio generator, FA — frequency analyzer, EDM — electrodynamic multiplier.

"Image quivering" was measured with a servo system tuned to the carrier frequency $f = 5000$ Hz, which eliminated the effect of "scintillation" (light intensity variation) on measurement results. The servo functioned as follows. The carrier frequency voltage from the audio generator is delivered through an adder device (a balanced bridge) to the galvanometer loop. The amplitude of the image oscillations across the slit was about 35 – 40 μ. The image oscillated in the horizontal plane at right angles to the slit. The photomultiplier load voltage is fed to an amplifier with a passband from 4800 to 5200 Hz. If the mean position of the source image (during the period $1/f$) is at the center of the slit, the photomultiplier output has the overtones $2f$, $4f$, etc., but the fundamental frequency f is suppressed. The amplitude of this fundamental component is proportional to the displacement of the mean position of the image from the center of the slit; its phase is equal to the phase of the oscillations of the loop or shows a phase shift of 180°, depending on whether the image is displaced to one side or the other relative to the center of the slit. The amplifier passes only frequencies close to f. The amplifier output is coupled to a synchronous detector, whose output current is fed through the adder to the loop. The phase of the control voltage applied to the synchronous detector is chosen to ensure negative feedback in the servo system, so that the change in the angle of arrival of the wave φ_0 is cancelled by an appropriate rotation of the mirror in the loop.

The detector output current is thus proportional to the angle of arrival of the wave averaged over several oscillations of the loop.

Vertical mean wind profiles and mean temperature profiles were measured simultaneously with the "image quivering" characteristics. A total of 60 variance and frequency spectrum measurements were carried out at distances of 125, 250, 500, 1000, and 2000 m. In each measurement, the variance and the spectral density were averaged over 10 min periods.

The graph in Figure 62 plots the values of σ^2_φ measured for various L under various meteorological conditions. This plot gives some idea as to the common values of σ^2_φ; on the average σ^2_φ is a linear function of L. It is further clear from this plot that the instrumental noise σ_n in angular units is approximately 1". The linear dependence on distance is in good agreement with the theoretical relation

$$\sigma^2_\varphi \sim d^{-1/3} L C^2_\varepsilon .$$

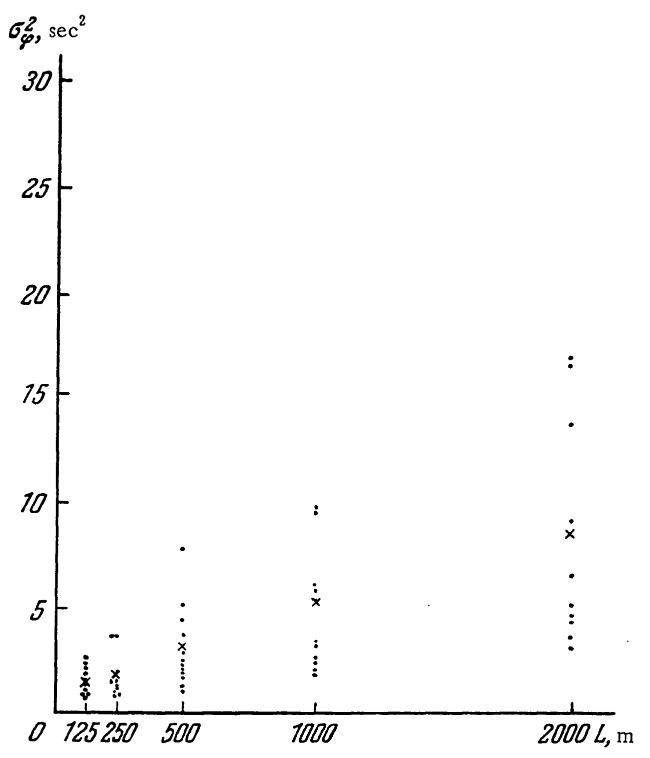


FIGURE 62. The mean square of the angle-of-arrival fluctuations vs. distance.

The large scatter of the experimental data is associated with considerable variations in meteorological conditions. The crosses mark the mean σ^2_φ for each distance.

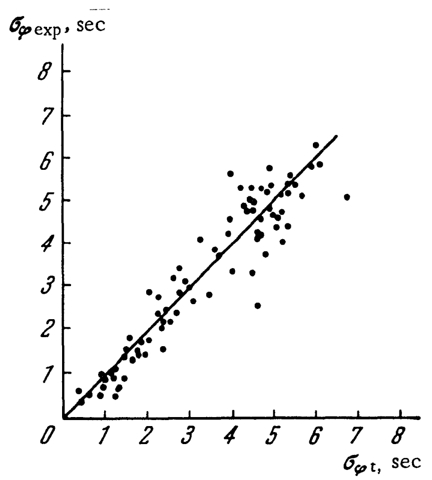


FIGURE 63. Experimentally measured mean square fluctuations ($\sigma_{\varphi\,\mathrm{exp}}$) plotted against the theoretical variance $\sigma_{\varphi t}$ computed from measurements of mean temperature and mean wind profiles.

Similar measurements were recently carried out by Gurvich and Kallistratova /184/ in daytime, when C_ε^2 is generally higher than at night. To ensure a more detailed comparison with the theory, C_ε^2 was calculated from the mean wind speed and temperature data using the equations of Chapter 1 (with allowance for the dependence on the Richardson number). Comparison of theoretical and experimental values of σ_φ is given in Figure 63, where the horizontal axis gives the theoretical values $\sigma_{\varphi t}$ and the vertical axis the experimentally measured variance $\sigma_{\varphi\,\mathrm{exp}}$ with allowance for instrumental noise. The straight line in Figure 63 corresponds to a perfect fit of the two sets of data. We see that the measured values fall along this straight line. The regression coefficient of $\sigma_{\varphi\,\mathrm{exp}}$ to the theoretical variance $\sigma_{\varphi t}$ is 0.98.

In Part A of this chapter the product

$$fw(f) = \frac{fF(f)}{\int\limits_0^\infty F(f)\,df}$$

($F(f)$ is the power spectral density of the fluctuations at frequency f) was shown to be a dimensionless function of the dimensionless frequency $fb/v_\perp$:

$$fw(f) = \mathrm{const}\,\sin^2\left(\frac{\pi fb}{v_\perp}\right)\left(\frac{fb}{v_\perp}\right)^{-8/3},$$

where $v_\perp$ is the transverse component of the mean wind velocity (at right angles to the beam) and b the aperture diameter. This equality was obtained assuming the "2/3 law" and Taylor's hypothesis of frozen turbulence.

Figure 64 shows a spectrum obtained by averaging the normalized spectra $fw(f)$ which has been transformed to the dimensionless frequency. For comparison with the theory the figure also shows the asymptotic behavior of the theoretical function $fw(f)$ for low frequencies f when $\frac{\pi f b}{v_\perp} \ll 1$:

$$fw(f) \sim \left(\frac{fb}{v_\perp}\right)^{1/3},$$

and at high frequencies when $\pi fb \gg v_\perp$:

$$fw(f) \sim \left(\frac{fb}{v_\perp}\right)^{-5/3}.$$

The dimensionless frequency $\frac{fb}{v_\perp} = 0.22$ corresponding to the peak in the theoretical spectrum is marked in the figure.

We see that the experimental spectrum has the same general characteristics as the theoretical curve. The spectral function is peaked around the same frequencies as the theoretical spectrum. The more rapid decrease at low frequencies of the experimental spectral function compared to the theoretical is apparently associated with the breakdown of the "2/3 law" in the large-scale region of the turbulence spectrum, which is responsible for the low-frequency region of the spectrum of the "quivering." The dashed curve in Figure 64 is the spectrum of angle of arrival fluctuations of sound waves (see below); its shape is fairly close to the "quivering" spectrum of a light source.

In conclusion, we summarize the main results which follow from the analysis of the above experiments.

In the region of weak fluctuations for $\sigma_1^2 = 0.077\, C_\varepsilon^2 k^{7/6} L^{11/6} < 0.6$:

1. Light intensity fluctuations due to atmospheric turbulence have a log-normal probability distribution.

2. The dependence

$$\langle [\ln A - \langle \ln A \rangle]^2 \rangle = 0.077 C_\varepsilon^2 k^{7/6} L^{11/6}$$

is confirmed quantitatively up to $0.077\, C_\varepsilon^2 k^{7/6} L^{11/6}$.

3. Direct measurements confirm the theoretical conclusion according to which the intensity correlation function depends on $\rho / \sqrt{\lambda L}$ and the correlation scale is of the order $\sqrt{\lambda L}$.

4. It is confirmed that the frequency spectrum of light intensity fluctuations depends on $f\sqrt{\lambda L}/v_\perp$ and a good fit between the space and the time correlation scales is observed.

5. The mean square fluctuations of the angle of arrival of light waves in a turbulent atmosphere is proportional to the distance traversed in the turbulent medium.

6. Experimental data on the mean square fluctuations of the angle of arrival are in good agreement with theoretical data calculated from vertical wind speed and temperature profiles in the atmospheric layer close to the ground.

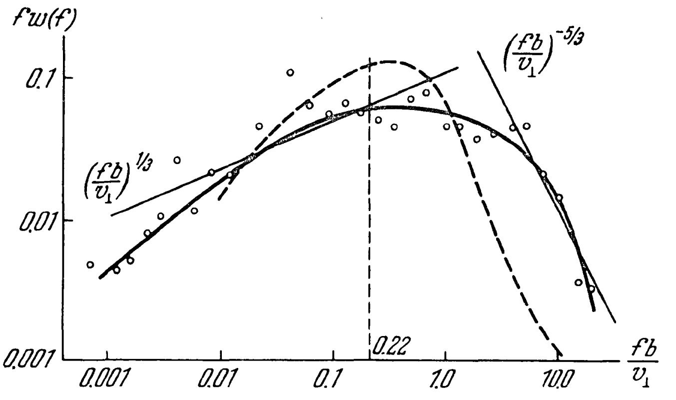


FIGURE 64. Frequency spectrum of the angle-of-arrival fluctuations of light.

The straight lines correspond to the theoretical asymptotes for low and high frequencies. The frequency $fb/v_\perp = 0.22$ marked by a vertical dashed line corresponds to the position of the peak of the theoretical function $fw\ (f)$. The dashed curve is the experimental spectrum of angle-of-arrival fluctuations for sound waves.

7. The frequency spectrum of angle of arrival fluctuations fits the theoretical curve, although to account for its width we should introduce the fluctuations in the transport velocities of the inhomogeneities.

In the region of strong fluctuations for $\sigma_1^2 > 2$:

1. The probability distribution of the light intensity fluctuations is nearly log-normal.

2. The mean square of the log intensity fluctuations is independent (or almost independent) of the distance travelled in the random medium and of the characteristic of the dielectric constant fluctuations, C_ε^2.

3. The value of $\langle [\ln I - \langle \ln I \rangle]^2 \rangle$ in the region of strong fluctuations is approximately 2.5, which corresponds to

$$\frac{\langle [I - \langle I \rangle]^2 \rangle}{\langle I \rangle^2} = 3.5.$$

§ 57. Propagation of sound in the atmospheric layer close to the ground

Experimental investigations of sound propagation in a turbulent atmosphere were undertaken by Krasil'nikov and Ivanov-Shits /130, 131/, Suchkov /132/, Golitsyn, Gurvich and the present author /133/.

Krasil'nikov measured the time structure function of phase fluctuations $\langle [S\ (t+\tau) - S\ (t)]^2 \rangle$ and the mean square of log amplitude fluctuations $\langle [\ln A - \langle \ln A \rangle]^2 \rangle$ of a sound wave.

Let us first consider phase fluctuations. If the wind velocity distribution and the temperature inhomogeneities do not change appreciably during the time τ, we may regard them as being transported (without "mixing") with the mean wind velocity.* If the wind direction is perpendicular to the direction of sound propagation and the wind velocity is $v_\perp$, the phase $S(t+\tau)$ at the point M is equal to the phase at the time t at a point distant $v_\perp\tau$ from M. Thus,

$$\langle [S(t+\tau)-S(t)]^2\rangle = D_S(\tau v_\perp).$$

By (47.37), for $\lambda_0 \ll \rho$ we have

$$D_S(\rho) \sim k^2 L C_n^2 \rho^{5/3}.$$

In this case we should have the relation

$$\sigma_S = \sqrt{\langle [S(t+\tau)-S(t)]^2\rangle} \sim C_n k L^{1/2} (v_\perp\tau)^{5/6},$$

i.e., the phase variance is proportional to the structure constant C_n, the frequency of the sound wave, the square root of the distance travelled by the sound wave, and the time interval to the 5/6 power.

Figure 65 plots the dependence of σ_S on the distance L as obtained by Krasil'nikov and Ivanov-Shits /130/; Figure 66 gives their plot for σ_S vs. $v_\perp\tau$ (the sound frequency is 3000 Hz, the distance $L = 22$, 45, and 67 m; $v_\perp =$ m/sec; $\tau = 0.04$, 0.08, and 0.2 sec).

We see from the graphs that the dependence of σ_S on L and $v_\perp\tau$ shows good agreement with the above relation. Experiments carried out with ultrasonic waves at frequencies up to 50 kHz also give satisfactory agreement between experimental and theoretical results /131/. The relation (47.37) is thus experimentally confirmed in the frequency range from 1 to 50 kHz.

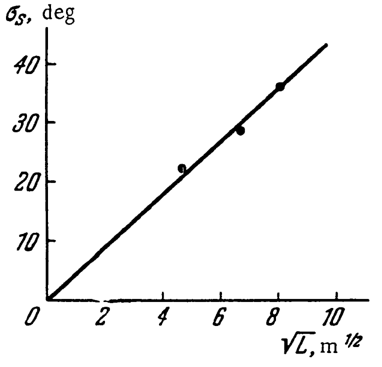


FIGURE 65. The rms time variance of the phase of sound waves σ_S as a function of path length L.

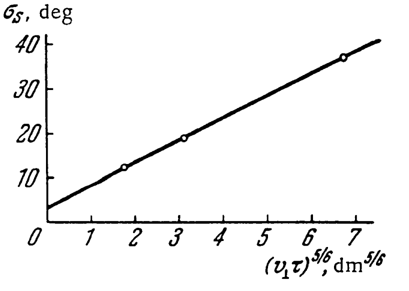


FIGURE 66. The rms time variance of the phase of sound waves σ_S as a function of $v_\perp\tau$.

The abscissa gives $(v_\perp\tau)^{5/6}$; the variance σ_S plotted on the vertical axis is the average for various L values.

* Taylor's hypothesis of "frozen turbulence" was treated in detail in Chapter 1, where we also gave some experimental data supporting this hypothesis.

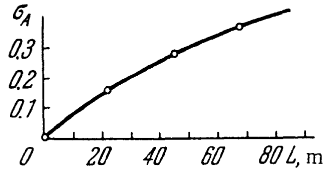


FIGURE 67. The rms fluctuations of the log amplitude of sound waves as a function of distance.

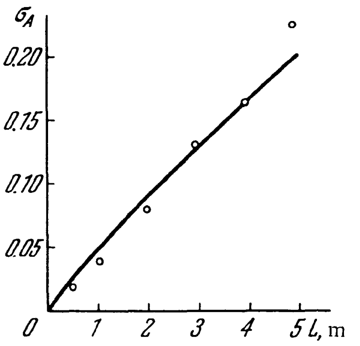


FIGURE 68. Rms fluctuations of the log-amplitude of ultrasonic waves vs. distance.

Figure 67 plots the dependence of $\sigma_A = \sqrt{\left\langle \left[\ln \frac{A}{A_0}\right]^2 \right\rangle}$ on distance (all data reduced to a distance of 22 m). The dependence of σ_A on L is satisfactorily approximated by the expression $\sigma_A = AL^{\alpha}$, where $\alpha \approx 0.8$. Note that according to (47.31) we should expect $\alpha \approx 0.92$ (in the relevant experiments $\sqrt{\lambda L} \gg \lambda_0$). The experiments of Krasil'nikov and Ivanov-Shits are thus in satisfactory agreement with the theoretical expression (47.31).

The measurements of σ_S and σ_A can be used to estimate C_n, characterizing the intensity of the sound velocity fluctuations, which enters (47.31) and (47.37).

C_n, calculated from the expression

$$C_n = \frac{\sigma_S}{1.7kL^{1/2}(v_\perp \tau)^{5/6}}$$

and taking $\sigma_S = 46° = 0.8$ rad, $k = 58\ \text{m}^{-1}$ ($f = 3$ kHz), $L = 67$ m, $v_\perp = 5$ m/sec $\tau = 0.2$ sec, was found to be equal to $0.0010\ \text{m}^{-1/3}$. The same parameter calculated from the relation

$$C_n = \frac{\sigma_A}{0.56k^{7/12}L^{11/12}}$$

taking $\sigma_A = 0.44$ and the same values of k and L as before was found to be equal to $0.0016\ \text{m}^{-1/3}$ (σ_A and σ_S were taken from /130/). If we note that σ_S and σ_A were obtained by reducing phase and amplitude fluctuation records of different duration, the agreement between these C_n values is clearly satisfactory. In Chapter 2 we derived an expression relating the C_n of sound waves to the temperature and wind velocity characteristics C_T^2 and C_v^2:

$$C_n^2 = \frac{C_T^2}{4\langle T\rangle^2} + \frac{C_v^2}{c_0^2},$$

where c_0 is the average speed of sound. Using (16.26) and (16.27), which express C_T^2 and C_v^2 in terms of the mean profile characteristics $\langle T(z)\rangle$, $\langle v(z)\rangle$, we obtain

$$C_n^2 = 2.3\cdot 10^{-6} f_1(\mathrm{Ri})\, z^{4/3}\left(\frac{d\langle v(z)\rangle}{dz}\right)^2 + 3\cdot 10^{-6} f_2(\mathrm{Ri})\, z^{4/3}\left(\frac{d\langle T(z)\rangle}{dz}\right)^2, \qquad (1)$$

where we used $\langle T\rangle = 290$ °K, $c_0 = 340$ m/sec, C_n^2 is expressed in $\mathrm{m}^{-2/3}$, and $\frac{dT}{dz}$ and $\frac{dv}{dz}$ are expressed in units of °C and m/sec per meter, respectively. The functions $f_1(\mathrm{Ri})$ and $f_2(\mathrm{Ri})$ are shown in Figures 17 and 18.

The value $C_n = 0.0010\ \mathrm{m}^{-1/3}$, obtained above, corresponds to a velocity difference $\Delta v = 1$ m/sec between the heights $z_2 = 8$ m and $z_1 = 4$ m. This is a typical value, so that the above expressions for the mean square of phase and amplitude fluctuations give correct results in terms of order of magnitude.

Highly detailed measurements of sound wave amplitude fluctuations were carried out by Suchkov /132/ in 1954. The sound amplitude measurements were made simultaneously with measurements of the mean temperature and mean wind speed profiles, so that C_n could be computed using (1).

Figure 68 plots the dependence of

$$\sigma_A = \sqrt{\left\langle\left[\ln\left(\frac{A}{A_0}\right)\right]^2\right\rangle}$$

on L (at 76 kHz) as obtained by Suchkov. We see from the graph that the experimental results show good agreement with the theoretical relation

$$\sigma_A^2 = 0.31 C_n^2 k^{7/6} L^{11/6}$$

(in all the experiments $\sqrt{\lambda L} \gg \lambda_0$, $\sigma_A^2 < 1$).

Suchkov carried out 28 series of experiments in which he measured the dependence $\sigma_A^2 = f(L)$ at frequencies from 3 to 76 kHz. The experimental findings were approximated using the relation $\sigma_A = AL^\alpha$. The average α for different sound frequencies (18 measurement series) was found to be 1.1; the average α at ultrasonic frequencies (30 – 76 kHz) was 0.95. These values of α are close to the theoretical figure $\alpha = \frac{11}{12} = 0.92$.

Suchkov used an expression of the same type as (1) (without taking into account $f(\mathrm{Ri})$) to compute σ_A from temperature and wind speed profiles.

Figure 69 plots directly the measured values of σ_A^2 against those calculated from simultaneous mean temperature and mean wind speed profiles. The correlation coefficient between $\log \sigma_A$ and $\log \sigma_{\mathrm{met}}$ was 0.90 (the graph shows 97 points).*

Suchkov also measured the time autocorrelation function of sound amplitude fluctuations. If the wind velocity is perpendicular to the direction of sound propagation and the correlation time is considerably less than z/v_*, we have the following approximate relation

$$\left\langle \ln\frac{A(t+\tau)}{A_0}\cdot\ln\frac{A(t)}{A_0}\right\rangle = B_A(v_\perp \tau).$$

* Note that all the experimental points in Figure 69 fall in the region of weak fluctuations.

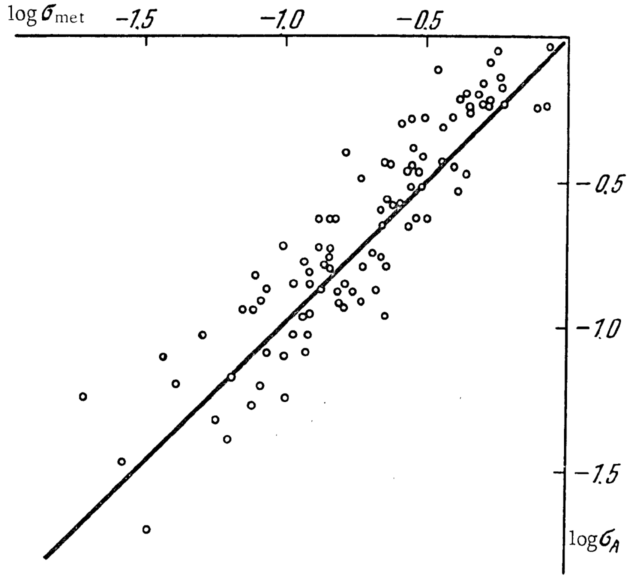


FIGURE 69. Directly measured rms fluctuations of log amplitude (σ_A) versus the same quantity computed from wind speed and temperature profiles (σ_{met}).

The measurements were carried out between 3 and 76 kHz.

The function $B_A(\rho)$ for $\sqrt{\lambda L} \gg \lambda_0$ was calculated previously (see Figure 45, p. 239). The amplitude correlation length is of the order of $\sqrt{\lambda L}$. Hence it follows that the amplitude correlation time is of the order $\sqrt{\lambda L}/v_\perp$.

Figure 70 plots the amplitude correlation functions obtained by Suchkov with $\tau v_\perp/\sqrt{\lambda L}$ plotted on the abscissa. The various curves correspond to different distances between the transmitter and the receiver (4, 8, and 16 m). If

$$\left\langle \ln \frac{A}{A_0} \ln \frac{A'}{A'_0} \right\rangle = f(\tau)$$

is plotted in the natural scale, i.e., as a function of τ, the curves for various L have a different appearance. After changing over to the variable $\tau_0 = \frac{\sqrt{\lambda L}}{v_\perp}$, all three curves approach one another, especially for small values of $v_\perp \tau/\sqrt{\lambda L}$.

Suchkov's experiments are in good agreement with the theory of Chapter 3. Comparison of measured and calculated σ_A shows that quantitative estimates of sound amplitude fluctuations can be obtained from simple measurements of wind speed and temperature profiles in the atmosphere.

Frequency spectra of sound amplitude and phase difference fluctuations were measured in /133/.

The measurements were carried out at the end of summer in 1958 on a flat stretch of open steppe near the village Tsimlyanskoe (Scientific Station of the Institute of Atmospheric Physics, USSR Academy of Sciences). A block diagram of the experiment is shown in Figure 71.

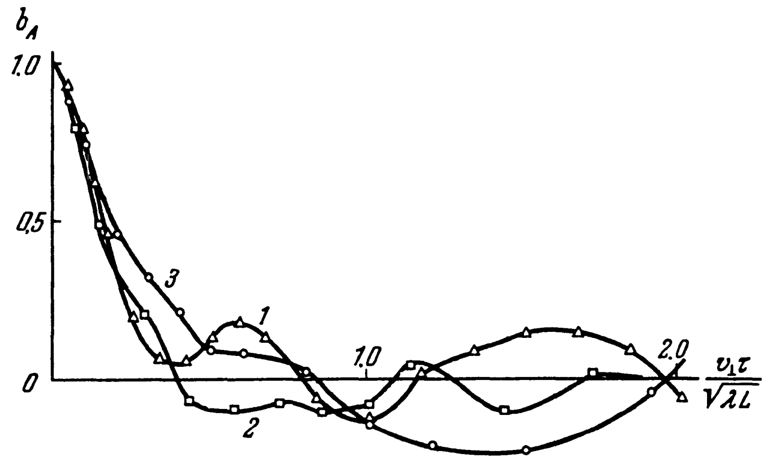


FIGURE 70. Empirical normalized autocorrelation functions of the log amplitude fluctuations of sound waves.

Curves 1, 2, 3 were obtained for L = 4 m, 8 m, and 16 m, respectively.

The signal from the audio generator 1 is amplified by a 25 watt amplifier 2, and then fed to a conical loudspeaker 3. The loudspeaker was set 1.5 m above the ground. The signal was picked up by microphones M_1 and M_2 held 9 m above the ground on a horizontal truss; they could be moved on special trolleys. The use of an "inclined" path and a horn loudspeaker greatly suppressed reflections from the ground, since the "reflecting zone" was outside the main lobe of the horn's radiation pattern. The amplitude reflection coefficient from the earth at an angle of 10°, measured with the pulse equipment described in /88/, was less than 0.1. The signals from the microphones were fed into a start-stop phasemeter 4 and the signal from one of the microphones was additionally fed to an amplifier with a linear detector at its output 5. The output voltage of the phasemeter or the detector was delivered to a 30-channel frequency analyzer 6. The detector voltage was also delivered to an integrating voltmeter, whose readings were proportional to the mean sound amplitude.

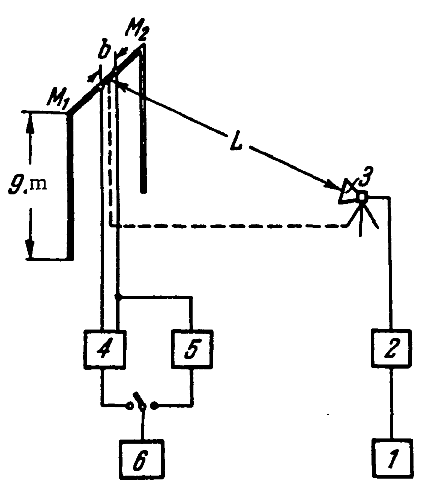


FIGURE 71. Block diagram for measurements of the frequency spectra of amplitude and phase difference:

1) audio generator; 2) amplifier; 3) loudspeaker; M_1, M_2 microphones; 4) phase meter; 5) amplifier with detector; 6) frequency analyzer.

Each frequency spectrum measurement involved averaging over 10 min (5 successive spectrum recordings averaged by the frequency analyzer integrating circuits over 100 sec each).

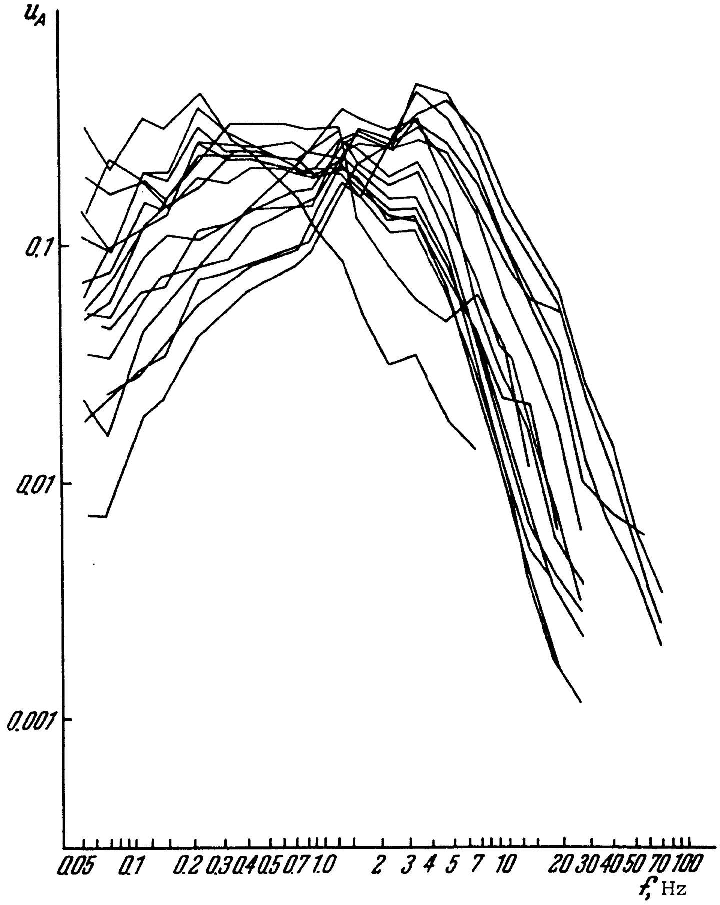


FIGURE 72. Empirical frequency spectra of sound amplitude fluctuations obtained for various distances, frequencies, and meteorological conditions.

Simultaneously with the acoustic measurements, wind speed and direction and the temperature were measured at heights of 0.5, 1, 2, 4, 8, and 12 m. These measurements were used to determine the atmospheric turbulence characteristics required for the analysis of the results.

The measurements were carried out at three frequencies, 2, 6, and 8.5 kHz. The path length L was varied from 21 to 80 m, and the baseline (the separation between the microphones) b from 0.4 to 3 m.

The phasemeter was capable of recording phase fluctuations within $\pm\pi$. Therefore, when the index of refraction fluctuations were large, the measurements were carried out over small distances, small baselines, and low frequencies. The relative amplitude fluctuations $\sigma_A/\langle A\rangle$ were 0.1 – 0.42.

A total of 16 amplitude spectra and 21 phase difference spectra obtained mainly at nighttime were reduced. Spectra corresponding to wind velocities directed almost along the ray (within 30°) or to wind speeds of less than 1 m/sec (weak wind is measured with low accuracy) were omitted from the treatment.

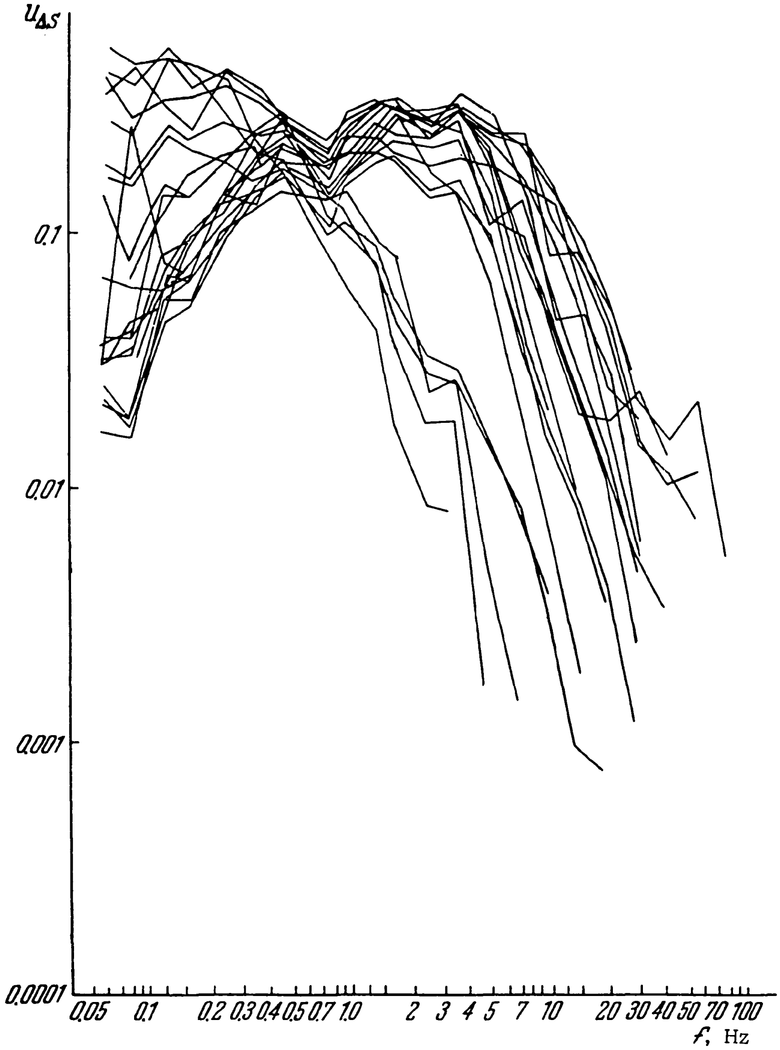


FIGURE 73. Empirical frequency spectra of sound phase difference fluctuations obtained for various distances, frequencies, and meteorological conditions.

The results of the frequency spectrum measurements are shown in Figures 72 and 73 in log-log coordinates, the horizontal axis giving the frequency and the vertical axis the dimensionless quantities.

$$u_A(f) = \frac{f w_A(f)}{\int\limits_{0.05}^{1160} w_A(f)\,df}, \quad u_{\Delta S}(f) = \frac{f w_{\Delta S}(f)}{\int\limits_{0.05}^{1160} w_{\Delta S}(f)\,df},$$

where $w_A(f)$ and $w_{\Delta S}(f)$ are the spectral densities of the sound amplitude and phase difference fluctuations. The quantities u_A and $u_{\Delta S}$ are normalized so that

$$\int\limits_0^\infty u_A(f)\,d\ln f = \int\limits_0^\infty u_{\Delta S}(f)\,d\ln f = 1,$$

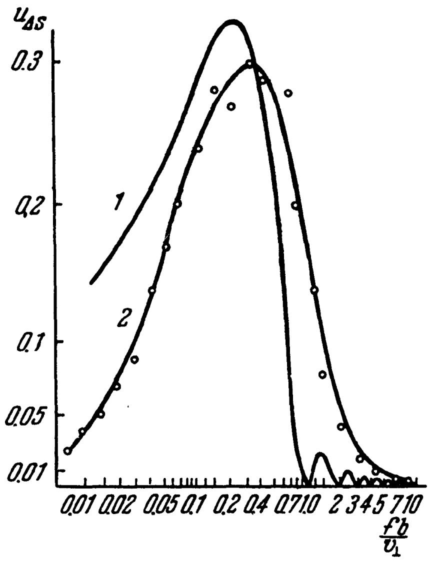


FIGURE 74. The theoretical form of the function $u_{\Delta S}$ which characterizes the phase difference frequency spectrum (curve 1); curve 2 is the result of reducing the experimental data shown in Figure 73 (see also Figure 77).

and represent the normalized spectral density of the "power" of the fluctuations on a logarithmic scale. In Figures 72 and 73 frequency spectra corresponding to various distances, frequencies, baselines, and meteorological conditions are combined; this explains the considerable scatter of the curves.

A theoretical calculation of the phase difference frequency spectrum was carried out in Part A of this chapter. It was established that the dimensionless function

$$u_{\Delta S}(f) = \frac{f w_{\Delta S}(f)}{\langle \Delta S^2 \rangle}$$

for $\frac{bf}{v_\perp} \ll 1$ has the form

$$\frac{f w_{\Delta S}(f)}{\langle \Delta S^2 \rangle} = u_{\Delta S}(f) = 0.44\left(\frac{bf}{v_\perp}\right)^{1/3},$$

i. e., in the low-frequency region $u_{\Delta S} \sim f^{1/3}$. For $\frac{bf}{v_\perp} \gg 1$ the function $u_{\Delta S}(f)$ decreases as $f^{-5/3}$. The characteristic frequency f_1 at which $u_{\Delta S}(f)$ has its maximum is $f_1 = 0.22\frac{v_\perp}{b}$, i.e., it depends on the baseline and the wind speed. The function $u_{\Delta S}(f)$ is shown in Figure 74 (the theoretical curve is 1; the method by which curve 2 was obtained will be explained in the following). We see from this figure that the maximum contribution to the energy of phase difference fluctuations is from inhomogeneities with dimensions of the order of the baseline. Note that when the phase difference spectrum is analyzed by frequency analyzers with passbands greater than $v_\perp/b$, the valleys in the spectrum corresponding to the zeros of the factor $\sin^2(\pi bf/v_\perp)$ are not observed. Moreover, fluctuations in the transport velocity will affect the position of the zeros, which on the average will lead to smoothing of the spectra. Therefore the form of the spectrum in the high-frequency region can be obtained if $\sin^2(\pi bf/v_\perp)$ is replaced by its mean value of 1/2:

$$u_{\Delta S}(f) = \frac{f w_{\Delta S}(f)}{\langle \Delta S^2 \rangle} = 0.022\left(\frac{fb}{v_\perp}\right)^{-5/3} \text{ for } f \gg f_1,$$

We now return to the analysis of the measurement results. The quantities $u_A(f)$ and $u_{\Delta S}(f)$ are functions of the dimensionless frequencies f/f_A and f/f_S, where $f_A = \frac{v_\perp}{\sqrt{\lambda L}}$, $f_S = \frac{v_\perp}{b}$ are the characteristic frequencies for the fluctuations in amplitude and phase difference of the wave. To check this "similarity hypothesis" the frequency spectra shown in Figures 72 and 73 in the coordinates (u_A, f) and $(u_{\Delta S}, f)$ are replotted in Figures 75 and 76 in the coordinates $(u_A, f/f_A)$ and $(u_{\Delta S}, f/f_S)$. Comparison of Figures 72, 73 with Figures 75, 76 readily shows that the frequency spectra, which in Figures 72 and 73 display considerable spread, are confined to a fairly narrow "band" in Figures 75 and 76. This indeed confirms that the frequency spectra of sound amplitude and phase fluctuations may be treated as functions of the dimensionless arguments $fb/v_\perp$ for phase difference and $f\sqrt{\lambda L}/v_\perp$ for amplitude. If we remember that the measurements were carried out at different times of the day under different meteorological conditions, at different frequencies and for different path lengths, the results clearly provide a satisfactory confirmation of the similarity law. The similarity law with respect to the parameter $f_A = \frac{v_\perp}{\sqrt{\lambda L}}$ for amplitude fluctuations was previously confirmed in experiments with light (see the previous section).

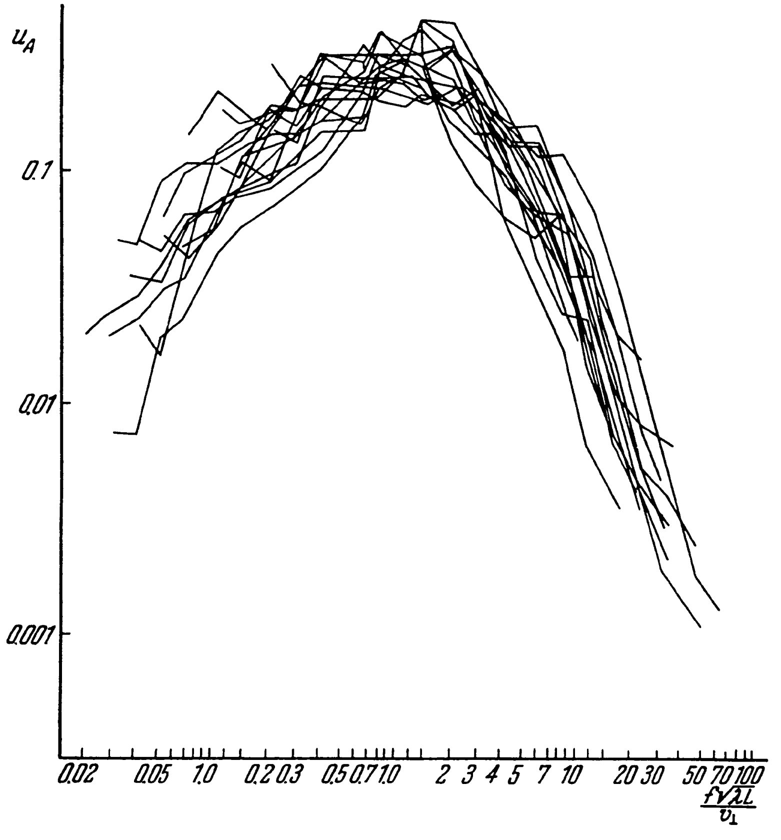


FIGURE 75. The empirical frequency spectra of amplitude fluctuations plotted as a function of $f\sqrt{\lambda L}/v_\perp$ (also see Figure 72, where these frequency spectra are plotted as a function of the frequency f).

The phase difference spectrum obtained by averaging all the spectra in Figure 76 is shown in Figure 74 (curve 2). The experimental and theoretical curves show fairly good agreement. The experimental curve does not show any dips in the high-frequency region of the spectrum for the reasons explained above. The averaged spectral density of the phase difference fluctuations plotted in log-log coordinates in Figure 77 decreases with increasing frequency as $(fb/v_\perp)^{-1.9}$. Here the exponent in the power law is fairly close to the theoretical value of −1.67. At low frequencies $fw_{\Delta S} \sim f^{0.8}$, and the exponent is different from the theoretical value of 0.33. This is quite understandable, since in this region the major contribution is from the low-frequency components of the turbulent spectrum, where the "2/3 law" breaks down. Indeed, we see from Figure 72 that the frequency spectrum is peaked at frequencies of about 1 Hz, which for mean wind speeds of 3 — 5 m/sec corresponds to scales of 3 — 5 m, i. e., scales comparable with the ray height above the ground at which point the "2/3 law" definitely breaks down. Therefore the exponent $\frac{1}{3} = 1 - \mu$ $\left(\mu = \frac{2}{3}\right)$ for the power law of $u_{\Delta S}$ is replaced by an exponent corresponding to a smaller μ.

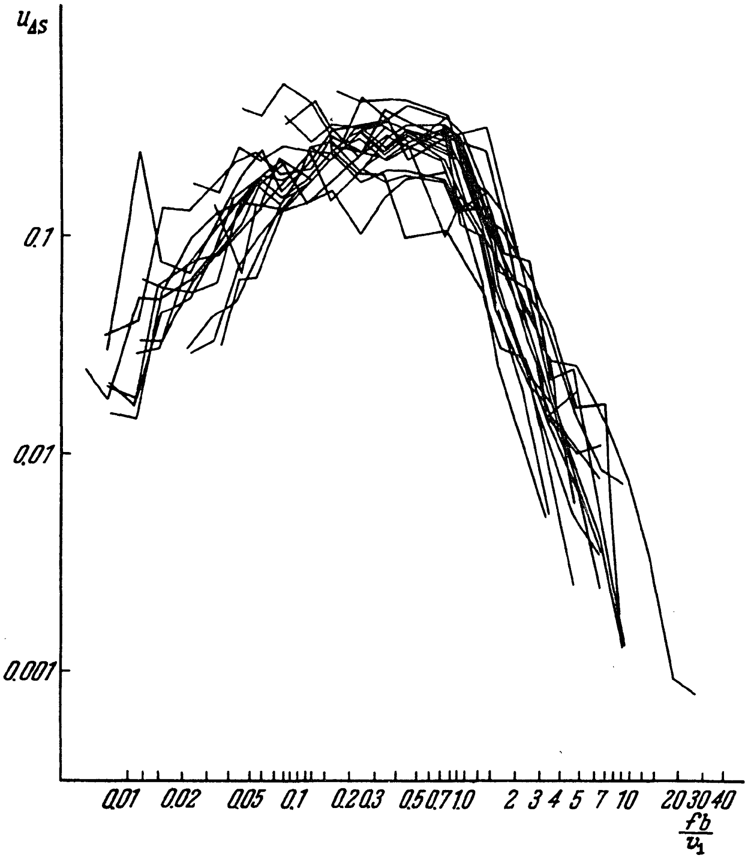


FIGURE 76. The empirical frequency spectra of phase difference fluctuations of sound plotted as a function of $fb/v_\perp$ (also see Figure 73, where these frequency spectra are plotted as a function of the frequency f).

Figure 78 shows the amplitude frequency spectrum averaged over all the spectra in Figure 75 (1), the corresponding theoretical spectrum (2), and the spectrum obtained in scintillation measurements with a light source near the ground (3). The amplitude spectrum of sound, like that of light, is broader and lower than the theoretical spectrum. The spreading and the

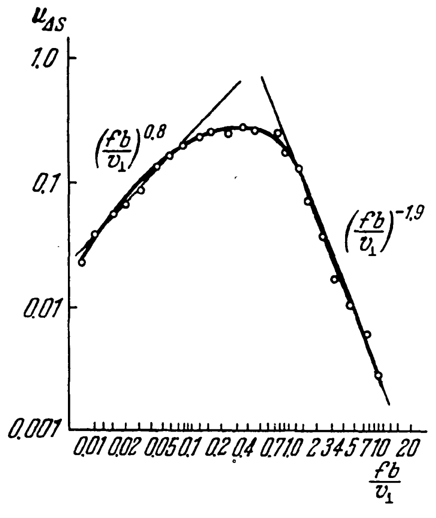


FIGURE 77. The averaged frequency spectrum of phase difference fluctuations of sound in log-log coordinates.

lowering of the peak may be attributed to transport velocity fluctuations along the path. The displacement of the light spectrum to the left and the displacement of the sound spectrum to the right relative to the theoretical curve can also be understood without much difficulty. The size of the Fresnel zone for light waves is of the order of a few centimeters, which is close to the inner scale of turbulence. For sound, on the other hand, the Fresnel zone extends over 1 – 2 m, and is thus comparable with the outer scale of turbulence (in our case, the mean height of the ray above the ground).

In Figure 79 the averaged experimental spectrum of sound amplitude fluctuations is shown on log-log scales. We see that the shape of the averaged experimental curve is very close to the theoretical. For $f \gg v_\perp/\sqrt{\lambda L}$ the experimental spectrum falls off as $f^{-1.6}$, and the theoretical as $f^{-1.67}$. For $f \ll v_\perp/\sqrt{\lambda L}$ the experimental exponent for the power law is 0.9; the theoretical exponent is 1.

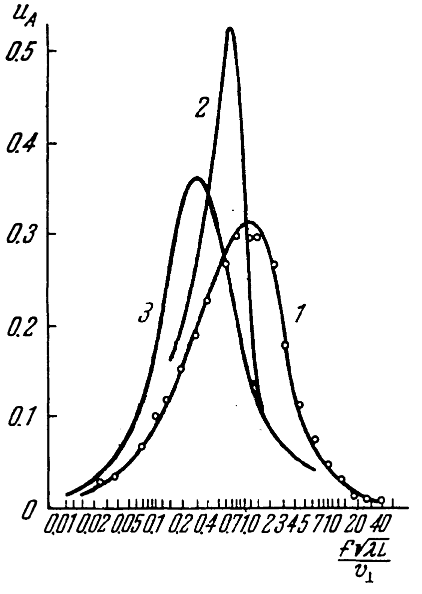


FIGURE 78. The averaged frequency spectrum of amplitude fluctuations of sound (1), theoretical curve (2), and the frequency spectrum of light intensity fluctuations (3), as a function of $f\sqrt{\lambda L}/v_\perp$.

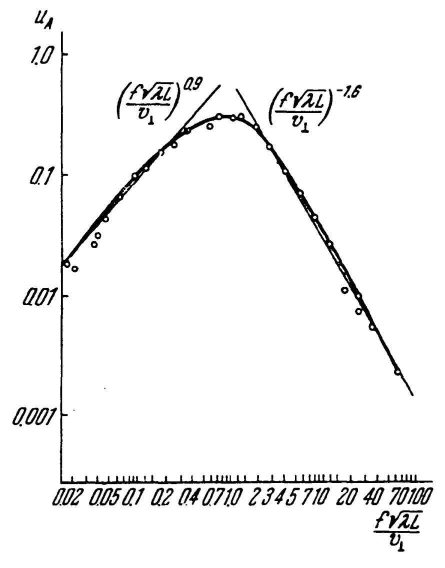


FIGURE 79. The averaged frequency spectrum of amplitude fluctuations of sound in log-log coordinates.

The theoretical value of the last exponent is independent of the exponent entering the index of refraction structure function, and therefore in the case of amplitude fluctuations departure from the "2/3 law" in the low-frequency region does not affect the exponent in the spectrum of amplitude fluctuations.

Analysis of data from sound experiments leads to the following conclusions.

1. The variation of amplitude fluctuations with distance satisfactorily agrees with the theoretical relation, $\sigma_\chi^2 \sim L^{11/6}$.
2. The correlation scale is of the order $\sqrt{\lambda L}$, the corresponding time scale is $\sqrt{\lambda L}/v_\perp$.
3. Phase fluctuations increase with distance as $\sigma_{\Delta S}^2 \sim L$.
4. The characteristic scale for phase difference fluctuations over the baseline (the separation over which the phase difference is measured) is equal to b. The corresponding time scale is of the order $b/v_\perp$.
5. There is a good agreement between the frequency spectra of light and sound fluctuations when reduced to a dimensionless form by using appropriate coordinates.
6. The assumption of a constant transport velocity does not fully account for the "width" of the amplitude and phase spectra, and requires consideration of the fluctuations in the transport velocity for its explanation.

§ 58. Dielectric constant fluctuations in the troposphere and propagation of ultrashort radio waves

The small-scale structure of the tropospheric index of refraction has been investigated in a large number of experimental works. Initially the main efforts were devoted to the determination of the crudest characteristics: the mean square of the refractive index fluctuations $\langle \delta n^2 \rangle$ and the correlation radius L_0. This was primarily due to the fact that these two parameters or more precisely their ratio $\langle \delta n^2 \rangle / L_0$ entered Booker and Gordon's pioneering theory of radio scattering by turbulent inhomogeneities in the troposphere /63/. The correlation radius L_0 (determined by approximating the correlation function $\langle \delta n \delta n' \rangle$ with an exponential expression of the form $\langle \delta n^2 \rangle e^{-\rho/L_0}$) was found to be of the order of 50 to 100 m. From the viewpoint of turbulence theory, this value of L_0 can be interpreted as the outer scale of turbulence. The mean square $\langle \delta n^2 \rangle$ is highly variable from one case to another and varies markedly with height.

These two parameters, however, provide very little information on the structure of turbulence. If they are translated into "spectral" language, we can write

$$\langle \delta n^2 \rangle = \iiint\limits_{-\infty}^{\infty} \Phi_n(\varkappa)\, d^3\varkappa,$$

and $2\pi/L_0$ characterizes the width of the spectral region where the main part of the "energy" of the fluctuations is contained. In reality, the main fraction of the "energy" of turbulent fluctuations is contained in large-scale inhomogeneities ($\varkappa \leqslant \frac{2\pi}{L_0}$). The parameters $\langle \delta n^2 \rangle$ and L_0 thus do not describe the shape of the spectrum $\Phi_n(\varkappa)$ for $\varkappa > \frac{2\pi}{L_0}$. Indeed, we can easily construct correlation functions with equal $\langle \delta n^2 \rangle$ and L_0 but entirely different $\Phi_n(\varkappa)$ for $\varkappa > \frac{2\pi}{L_0}$.

In later experimental studies of refractive index fluctuations more attention was paid to measurements of the frequency spectrum of the dielectric constant. Figures 19 and 20 show the frequency spectra of ε obtained from aircraft /35, 134/ and Figures 21 and 22 give the same spectra from ground observations /135/. As is evident from the figures, the experimental frequency spectra are adequately approximated using a power law. The exponent m of the power law f^{-m} is close to 5/3, although in individual measurements a different value is obtained.

Fifteen frequency spectra of refractive index fluctuations in the free troposphere at heights from 2 to 6 km were investigated in /61/. It is shown that the frequency spectra are well fitted by a power law of the form const f^{-m}, where m ranges between the limits

$$1.52 < m < 1.62.$$

This value of m, as noted in /61/, is in good agreement with the Kolmogorov — Obukhov theory of turbulence. From the frequency spectra of /61/ we can also estimate the parameter C_n. (C_n and C_ε are related by the simple equality $C_\varepsilon = 2\,C_n$). It is found to be approximately equal to 0.020 N-units $\cdot$ cm$^{-1/3}$.*

Note that aircraft measurements of δn reveal the horizontal structure of the inhomogeneities and "horizontal" spectra. However, in radio scattering the vector $\boldsymbol{K} = \boldsymbol{k}_0 - \boldsymbol{k}_s$ generally points vertically, so that the vertical structure of inhomogeneities and the "vertical" spectrum are the important quantities in radio scattering studies. In general, turbulence may be anisotropic, having different spectra in different directions. Such anisotropy can be expected, however, only for the large-scale components of the turbulence spectrum.

The quantity C_n can be found from the graphs in Figures 21 and 22; it is equal to $5 \cdot 10^{-8}$ cm$^{-1/3}$ = 0.05 N-units $\cdot$ cm$^{-1/3}$ for $v = 18$ m/sec (Figure 21) and 0.09 N-units $\cdot$ cm$^{-1/3}$ for $v = 1.2$ m/sec (Figure 23).

Let us compare this result with other ground measurements. Numerous measurements of temperature structure functions in the ground layer were carried out in /44/. The parameter C_T entering the structure function of the temperature field ($\langle (T_1 - T_2)^2 \rangle = C_T^2 r^{2/3}$) was found to be equal from zero to about 0.150 deg $\cdot$ cm$^{-1/3}$ under various meteorological conditions. The refractive index n of air in the atmosphere is related to its temperature T (in °K), pressure p (in millibars), and humidity e (in millibars) by the equation

$$(n-1)\,10^6 = \frac{79}{T}\left(p + \frac{4800e}{T}\right).$$

If we calculate the value of C_n attributable to these temperature fluctuations, we find $C_n = (0 - 0.15)$ N-units $\cdot$ cm$^{-1/3}$. Thus the C_n calculated from individual measurements of the index of refraction frequency spectra near

* N-units are used in measuring the deviation of the refractive index from unity. One N-unit is equal to 10^{-6}.

the ground /135/ lies between the limits indicated by several hundreds of measurements carried out under a variety of meteorological conditions.

Note, however, that the refractive index fluctuations near the ground (in the lower tens of meters) are much stronger than in the free troposphere. The function $C_n(z)$ (z is the height above the ground) decreases with height as $z^{-1/3}$ for $z \leqslant 50$ m; for large z, however, C_n may follow a different function. (For instance, under free convection conditions $C_n^2 \sim z^{-4/3}$.) It is very clear, however, that the typical values of C_n in the free troposphere are much less than those in the ground layer.

Tsvang /52, 59/ published measurements of temperature frequency spectra at heights up to 5 km. His spectra (a total of 26) are in good agreement with the "2/3" law, and the C_n calculated from these findings ranges from 0.004 to 0.010 N-units · cm$^{-1/3}$. Actually, these values may even be higher when humidity fluctuations are taken into account.

Let us now try to assess the fluctuations of the refractive index in the troposphere from phase and amplitude fluctuations of radio waves and light.

Fairly numerous observations of star scintillation and image quivering are currently available. Analysis of these observations (see next section) gives $C_n \sim 0.01$ N-units · cm$^{-1/3}$.

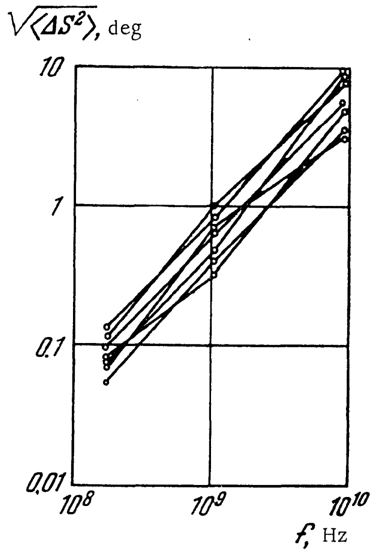


FIGURE 80. The root mean square of the phase difference fluctuations over a baseline of 150 m as a function of frequency.

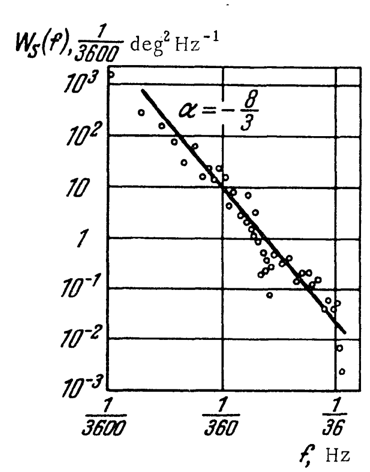


FIGURE 81. Frequency spectrum of phase fluctuations of decimeter radio waves (f = 1046 MHz) over a path length of 18.5 km for a wind velocity of 2.7 m/sec.

We shall now consider measurements of phase fluctuation characteristics of centimeter radio waves. These measurements were carried out in March – June 1955 near Colorado Springs, USA /135, 136/ and in November 1956 on the island of Maui (Hawaii) /137/.

Herbstreit and Thompson /135/ give the mean square of the phase difference fluctuations over a baseline of 150 m as a function of frequency (Figure 80). We see from the graph that $\langle \Delta S^2 \rangle \sim k^2$, and this relationship which should be satisfied for any turbulence spectra, is experimentally verified.

Figure 81 gives the frequency spectrum of phase fluctuations for decimeter radio waves ($f = 1046$ MHz) over a path length $L = 18.5$ km. The path was from Pikes Peak (4300 m elevation) to the Garden of the Gods (1950 m elevation). The straight line on the graph corresponds to a power law for the spectrum $W_S(f)$; its slope represents the relation $W_S(f) \sim f^{-8/3}$. The plot indicates that expression (52.18) is closely followed over the given frequency range.

This result hardly could have been predicted beforehand. Indeed, the "2/3 law" for refractive index fluctuations is in general applicable to scales below the outer scale of turbulence L_0, i.e., to frequencies $f > \frac{v}{L_0}$. In the case on hand $v \approx 3$ m/sec (see p. 321). Therefore the frequency range from 1/3600 to 1/36 Hz corresponds to distances from 100 m to 10 km, which are much greater than L_0. Thus, the frequency spectrum in Figure 81 throws light on the comparatively large-scale components of the turbulence spectrum, for which there is no reliable theory available. However, on the basis of the results of this experiment it is possible to assert that the structure of the turbulent inhomogeneities is described by the "2/3 law" in this spectral region too.

Since the frequency spectrum $W_S(f)$ closely follows the "2/3 law", expression (47.37) derived from this law will in all probability apply to Herbstreit and Thompson' phase fluctuation measurements over some baseline b. C_n can thus be estimated from the data in Figure 80. These data were obtained over a path 6.5 km long with a baseline of 150 m. Taking $k = 0.22$ cm^{-1} and $6.1 \cdot 10^{-3}$ rad $< \sqrt{\langle \Delta S^2 \rangle} < 1.7 \cdot 10^{-2}$ rad (which corresponds to the scatter of the experimental points in Figure 80 for $f = 1046$ MHz), we find from (47.37) that C_n varies from 0.007 N-units$\cdot$cm$^{-1/3}$ to 0.019 N-units$\cdot$cm$^{-1/3}$.

Herbstreit and Thompson /135/ also studied the functions

$$\Delta_1(r, \tau) = S(r, t+\tau) - S(r, t)$$

and

$$\Delta_2(r, \tau) = \Delta_1(r, t+\tau) - \Delta_1(r, t)$$

(the first and the second time differences in the phase at a certain point). These functions can be readily expressed in terms of the frequency spectrum of the phase fluctuations $W_S(f)$:

$$\langle \Delta_1^2 \rangle = 2 \int_0^\infty (1 - \cos 2\pi f\tau) W_S(f)\, df, \qquad (1)$$

$$\langle \Delta_2^2 \rangle = 2 \int_0^\infty [3 - 4\cos(2\pi f\tau) + \cos(4\pi f\tau)] W_S(f)\, df. \qquad (2)$$

If $W_S(f) = \text{const} \cdot f^{-m}$, the integrals are readily evaluated, and we get

$$\frac{\langle \Delta_2^2 \rangle}{\langle \Delta_1^2 \rangle} = 4(1 - 2^{m-3}). \qquad (3)$$

In this case the ratio of $\langle\Delta_2^2\rangle$ to $\langle\Delta_1^2\rangle$ depends only on the form of the frequency spectrum $W_S(f)$. The numerical values from /135/, $\sqrt{\langle\Delta_1^2\rangle} = 4.92°$, $\sqrt{\langle\Delta_2^2\rangle} = 2.83°$ ($\tau = 300$ sec), can be inserted in (3) to find the exponent m. It proves to be equal to 2.87, which is close to $8/3 = 2.67$.

Herbstreit and Thompson also consider correlation functions of the form

$$\rho_1(b) = \frac{\langle\Delta_1(r+b,\tau)\,\Delta_1(r,\tau)\rangle}{\langle\Delta_1^2\rangle}, \quad \rho_2(b) = \frac{\langle\Delta_2(r+b,\tau)\,\Delta_2(r,\tau)\rangle}{\langle\Delta_2^2\rangle}.$$

Adopting Taylor's hypothesis of "frozen" turbulence, expressions for ρ_1 and ρ_2 may easily be obtained in terms of the phase structure functions:

$$D_S(r) = \langle[S(\boldsymbol{r}+\boldsymbol{r}_1) - S(\boldsymbol{r}_1)]^2\rangle,$$

$$\rho_1(b) = \frac{D_S(|v\tau+b|) + D_S(|v\tau-b|) - 2D_S(b)}{2D_S(v\tau)}, \tag{4}$$

$$\rho_2(b) = \frac{D_S(|2v\tau+b|) + D_S(|2v\tau-b|) - 4D_S(|v\tau+b|)}{2D_S(2v\tau) - 8D_S(v\tau)} - \\ - \frac{4D_S(|v\tau-b|) - 6D_S(b)}{2D_S(2v\tau) - 8D_S(v\tau)}. \tag{5}$$

Here v is the effective transport velocity of the refractive index inhomogeneities.

Herbstreit and Thompson /135/ compare their experimental findings with the theory of /138/ where the exponential function e^{-r/L_0} is used as an approximation to the correlation function. To ensure good fit with the theory, they took $L_0 = 360$ m for the $\rho_1(b)$ curve. In order to fit the $\rho_2(b)$ curve, however, they were forced to take $L_0 = 146$ m, which is 1/2.5 of the previous figure. This indicates that e^{-r/L_0} is a poor choice for the refractive index correlation function. If we take $D_S(\rho) = \text{const}\cdot\rho^{5/3}$, which corresponds to the phase spectrum measured in this study and is consistent with the theory, expressions (4) and (5) take the form

$$\rho_1(b) = \frac{1}{2}\left[\left|1+\frac{b}{v\tau}\right|^{5/3} + \left|1-\frac{b}{v\tau}\right|^{5/3} - 2\left(\frac{b}{v\tau}\right)^{5/3}\right], \tag{6}$$

$$\rho_2(b) = \frac{1}{4(2^{2/3}-2)}\left[\left|2+\frac{b}{v\tau}\right|^{5/3} + \left|2-\frac{b}{v\tau}\right|^{5/3} - 4\left|1+\frac{b}{v\tau}\right|^{5/3} - \right. \\ \left. -4\left|1-\frac{b}{v\tau}\right|^{5/3} + 6\left(\frac{b}{v\tau}\right)^{5/3}\right]. \tag{7}$$

Figure 82 plots the experimental data of Herbstreit and Thompson and the curves based on (6) and (7). To match the experimental points with the theoretical curves, we took $v\tau = 320$ m. For this single value of $v\tau$, the experimental points and the theoretical curves show good agreement for both ρ_1 and ρ_2, in contrast to the analogous curves in /135/, where the authors had to assume different values of L_0. From this numerical value of $v\tau$, taking $\tau = 300$ sec, we can determine the effective transport velocity, $v = 2.7$ m/sec. It is found to lie between the wind speed at the highest point of the path ($v \approx 18$ m/sec) and the wind speed at the lowest point ($v \approx 2$ m/sec).

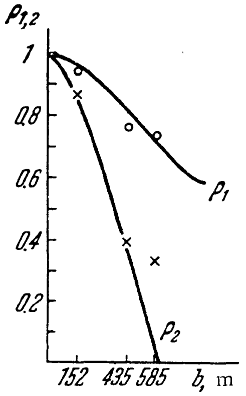


FIGURE 82. Spatial correlation functions for the first and the second time differences in phase.

The solid lines are the theoretical curves, circles and crosses — experimental data.

Using this v, we can find C_n from the frequency spectrum $W_S(f)$ (Figure 81) and also from the measured value of $\langle\Delta_1^2\rangle$.

Using (52.36) and inserting $k = 0.22\ \mathrm{cm}^{-1}$, $L = 18.5$ km, $v = 2.7$ m/sec, we obtain $C_n = 0.021$ N-units $\cdot\ \mathrm{cm}^{-1/3}$. To find C_n from the measured values of $\langle\Delta_1^2\rangle$, we rewrite (47.37) putting $\rho = v\tau$:

$$\langle\Delta_1^2\rangle = 2.9k^2L(v\tau)^{5/3}C_n^2.$$

Taking $\sqrt{\langle\Delta_1^2\rangle} = 4.92°$, $v\tau = 820$ m, and the previous values of k and L, we obtain $C_n = 0.015$ N-units $\cdot\ \mathrm{cm}^{-1/3}$, which is within 30% of the previous results.

Simultaneously with Herbstreit and Thompson's phase fluctuation measurements, Deam and Fannin measured phase difference fluctuations over several baselines from 5.5 m to 150 m at a wavelength of 3.2 cm /136/. Their experimental phase structure functions show a relatively wide scatter, and it is difficult to determine the corresponding structure function of phase fluctuations. In any case, it does not contradict expression (47.37). The rms phase difference over a 150 m baseline varies from 1° to 8°. In these experiments $L = 5.5$ km, $\lambda = 3.2$ cm. The value of C_n determined from (47.37) varies from 0.002 N-units $\cdot\ \mathrm{cm}^{-1/3}$ to 0.018 N-units $\cdot\ \mathrm{cm}^{-1/3}$. This estimate for C_n is of the same order of magnitude as the preceding figures.

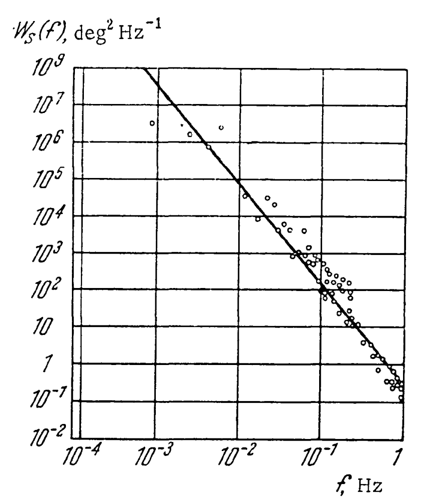


FIGURE 83. The phase spectrum of centimeter radio waves (f = 9414 MHz) over a path 30 km long with wind speed 2.2 m/sec.

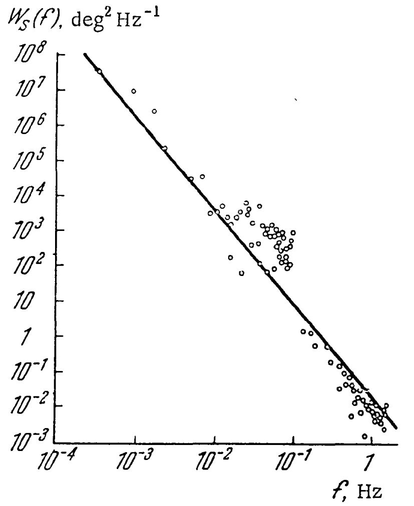


FIGURE 84. The phase spectrum of centimeter radio waves (f = 9414 MHz) over a path 30 km long with wind speed 0.5 m/sec.

Norton /137/ gives phase spectra for centimeter radio waves ($\lambda = 3.2$ cm), measured over a path 30 km long on the island of Maui (Hawaii) (Figures 83, 84).

The straight lines in the figures correspond to the power law $W_S(f) \sim f^{-8/3}$ and show a good fit to the experimental data. The spectra in Figures 83 and 84 thus give the same result as the spectrum of /135/ (Figure 81). The spectra in Figures 83 and 84, however, correspond to higher frequencies (scales from 1 m to 2 km) than the spectrum in Figure 81 and encompass a wider range of scales. The high-frequency part of these spectra clearly belongs to the inertial subrange of the turbulence spectrum (where the "2/3 law" is satisfied); for this range the exponent $-8/3$ in the expression for $W_S(f)$ is more natural. That the function $W_S(f)$ does not change appreciably in the low-frequency region either indicates that the "2/3 law" in this case is satisfied up to relatively large scales, as we previously saw in Figure 81.

In concluding this section, Table 4 summarizes the values of C_n obtained from various experiments /139/.

TABLE 4.

No.	C_n, N-units$\cdot$cm$^{-1/3}$	Method
1	0.004 — 0.010	Direct measurements of temperature spectra from aircraft /52, 59/
2	0.020	Direct measurements of refractive index spectra from aircraft /61/
3	0.050*	Refractive index spectrum (Garden of the Gods, Colorado, June, 0500 — 0600 /135/)
4	0.090*	Refractive index spectrum (Pikes Peak, Colorado, June, 0500 — 0600 /135/
5	0 — 0.150*	Measurements of the temperature structure function in the atmospheric layer close to the ground /44/
6	0.007	Scintillation and "quivering" of star images /140/
7	0.007 — 0.019	Phase difference fluctuations over 150 m baseline at frequencies from 173 to 9350 MHz (Colorado, USA, March, 1500 hrs /135/)
8	0.002 — 0.018	Phase difference fluctuations over baselines of up to 150 m, $\lambda = 3$ cm (Colorado, USA, March /136/)
9	0.021	Phase frequency spectrum, $\lambda = 30$ cm (Colorado, USA, June /135/)
10	0.015	Phase variations during 5 min at $\lambda = 30$ cm (Colorado, USA, June /135/)
11	0.130 — 0.180	Phase frequency spectrum at $\lambda = 3.2$ cm (Maui Isl., Hawaii /137/)

* C_n values for the atmospheric layer close to the ground.

§59. Scintillation and "quivering" of star images in telescopes

The scintillation ("twinkling") of stars is one of the atmospheric turbulence phenomena which has been studied in the greatest detail. It attracts the attention of astronomers as one of the factors which seriously hampers observations. For example, in measurements of stellar spectra, intensity fluctuations may substantially distort the results.

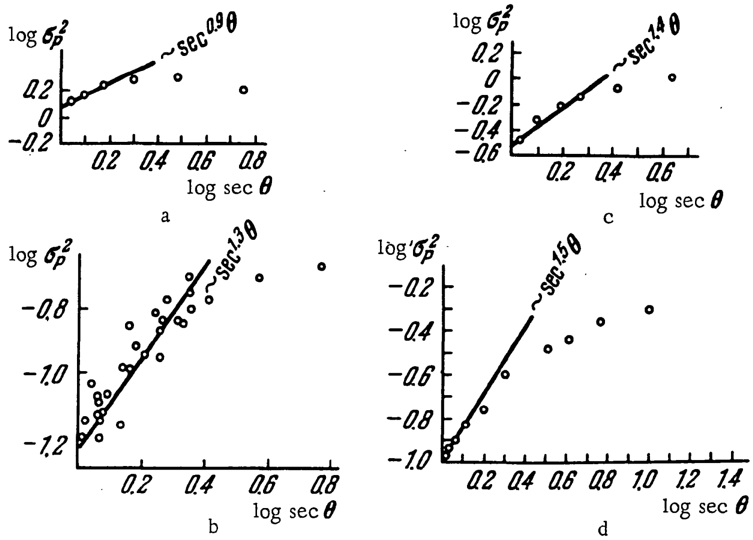


FIGURE 85. The mean square of light flux fluctuations through the aperture of a telescope as a function of zenith angle for various aperture diameters:

(a) $D = 7.6$ cm, (b) $D = 24$ cm, (c) $D = 32$ cm, (d) $D = 38$ cm.

Astronomical observations have revealed the following known facts. The scintillation σ_P^2 (the mean square of the relative fluctuations of the light flux through the telescope) is a function of the star's zenith angle θ. This function is different for different diameters of the telescope objective /141 — 143/. Figure 85a, b, c, d shows four plots for diameters of the objective equal to $D = 7.6$ cm, 24 cm, 32 cm, and 38 cm. We see from the figures that for small telescope objectives σ_P^2 at first increases with increasing θ, and then starts decreasing. For large D and small θ the function $\sigma_P^2(\theta)$ increases faster than it does for small D, but at large θ the curve reaches saturation and does not fall off, as for small D. Figure 86 plots $\sigma_P^2 = f(D)$ for small zenith angles /141/.

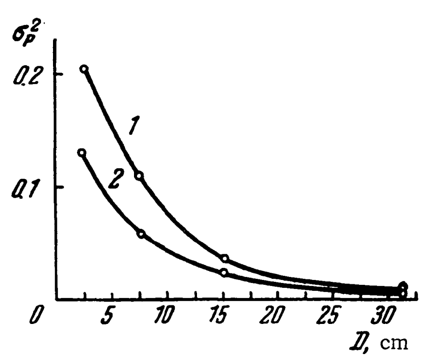


FIGURE 86. Empirical dependence of scintillation on objective diameter:

1) winter; 2) summer.

If we examine the distribution of illumination on the surface of the telescopic objective (this can be accomplished with a special lens), we will observe a succession of dark and light spots moving across the surface in a certain direction. These running shadows are called "schlieren." Each spot is about 10 cm. Figure 87 plots the empirical correlation function obtained in /144/ for the distribution of illumination over the objective. Some authors related the size of the schlieren to the size of actual refractive index inhomogeneities along the ray path.

Astronomical observations show that planets scintillate much less than stars at the same zenith distance (we considered this problem in Part A of the chapter /146/).

Finally, the so-called effect of chromatic scintillation should be mentioned. Stars observed near the horizon display random variations in their spectral characteristics — the spectral maximum jumps at random from one region to another. In visual observations this effect is manifested by color changes of the star. The dependence of scintillation on zenith distance θ for observations in a narrow region of the optical spectrum is different from the corresponding dependence in integrated light. Figure 88 shows two plots of $\sigma_P^2(\theta)$ obtained simultaneously for a spectral band 90 Å wide and for integrated light. We see from the plot that the $\sigma_P^2(\theta)$ curve for integrated light falls off for large θ, whereas no such downward trend is observed in the narrow frequency band /145/.

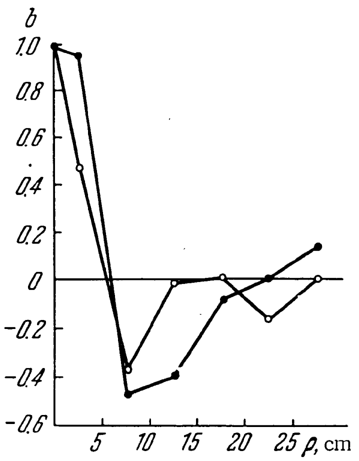


FIGURE 87. Empirical correlation functions of the illumination distribution over the surface of a telescope objective.

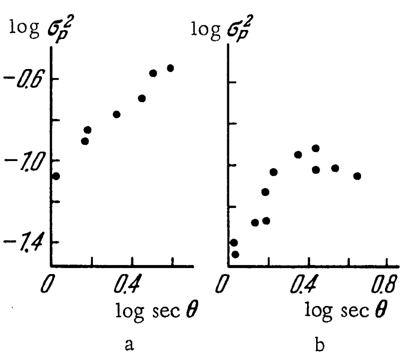


FIGURE 88. The effect of the spectral bandwidth on the scintillation vs. zenith angle curve:

(a) 90 Å passband, (b) integrated light.

Some authors (see /147/) compared the velocity of motion of the schlieren with wind velocities at various heights. Maximum correlation was observed between the velocity of the schlieren and the wind velocity at altitudes of about 10 km (near the tropopause).

All the above features of the scintillation are naturally accounted for by the preceding theory.

As we have noted before, the parameter $L_k = \lambda_0^2/\lambda$, which determines the distances over which geometrical optics is applicable, is of the order of 100 m in the atmosphere for $\lambda = 0.5\,\mu$. The mean square of the log amplitude fluctuations observed with a "point" receiver is therefore given by (48.24):

$$\langle\chi^2\rangle = 0.56k^{7/6}\int_0^L C_n^2(x)\,x^{5/6}dx. \quad (1)$$

Here C_ε^2 has been replaced by $C_n^2 = \frac{1}{4}C_\varepsilon^2$, which is regarded as a function of position. The origin of the coordinate in (1) is at the observation point and the integral is taken along the ray. If C_n^2 is only a function of height above the ground, i.e., of $z = x\cos\theta$, substitution of variables $x = z\sec\theta$ in (1) gives

$$\langle\chi^2\rangle = 0.56k^{7/6}(\sec\theta)^{11/6}\int_0^\infty C_n^2(z)\,z^{5/6}dz \quad (2)$$

(the upper integration limit can be extended to infinity, as $C_n^2(z)$ rapidly decreases at large z).

Thus, in telescope observations through an aperture with a small D (small compared to the correlation radius of fluctuations), we have for $\sigma_I^2 = 4\langle\chi^2\rangle$ the expression

$$\sigma_I^2(\theta) = 2.2k^{7/6}(\sec\theta)^{11/6}\int_0^\infty C_n^2(z)\,z^{5/6}dz, \quad (3)$$

i.e., $\sigma_I^2 \sim (\sec\theta)^{11/6}$. Figure 85a shows a straight line corresponding to the power law dependence $\sigma_I \sim (\sec\theta)^{0.9}$, which is very close to the theoretical $\sigma_I \sim (\sec\theta)^{11/12} = (\sec\theta)^{0.92}$. The saturation of the curve $\sigma_P^2 = f(\theta)$ will be explained below; here we will concentrate on the dependence of the fluctuations on the aperture diameter D. In § 53 we calculated the function $G(R)$, defined as the ratio of the mean square fluctuations of the total light flux through an aperture of radius R to the mean square fluctuations of the light intensity. This function is in fact dependent on the dimensionless parameter $R/\sqrt{\lambda L}$. Comparing the theoretical curve with the experimental dependence shown in Figure 86 we can choose the value of $\sqrt{\lambda L}$ for which the two virtually coincide. This matching is done in Figure 89, which gives $\sqrt{\lambda L} = 9.2$ cm ± 0.5 cm for winter measurements and $\sqrt{\lambda L} = 8.1 \pm 0.5$ cm for summer measurements. Taking $\lambda = 0.5\,\mu$, we can find the effective thickness of the layer responsible for the scintillations (some 13 – 16 km). The parameter L is related to the vertical thickness H of this layer by $L = H\sec\theta$. The value of $\sec\theta$ for the data in Figure 86 ranges from 1 to 1.5. Dividing our value of L by the mean value of $\sec\theta$ (1.25), we obtain for the effective thickness H of the scintillation layer 10 km (summer measurements) and 13 km (winter measurements). It follows from (3) that the factor $z^{5/6}$ in the integrand, vanishing at $z = 0$, markedly suppresses the contribution from the lower atmospheric layer. The product $C_n^2 z^{5/6}$ thus has a maximum at relatively high altitudes, which explains the large value of H. The above result for H is in good agreement with the results obtained by comparing the schlieren velocities with wind velocities at various altitudes.

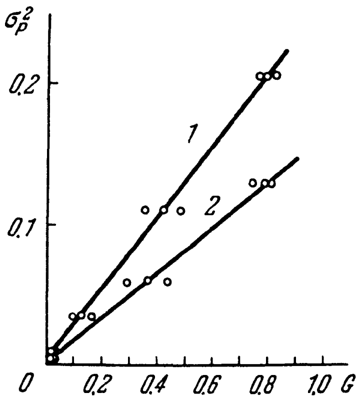


FIGURE 89. Matching of the empirical and the theoretical functions $\sigma_P^2 = f(D)$, where D is the telescope diameter.

The abscissa axis gives the values of the function $G\left(\frac{D}{\sqrt{\lambda L}}\right)$ shown in Figure 49, p.274, the ordinate gives the values of σ_P^2 taken from Figure 86. By an appropriate choice of $\sqrt{\lambda L}$, we can ensure a linear dependence between these functions. The two points left and right of the points joined by the straight line correspond to values of $\sqrt{\lambda L}$ incremented by ±1 cm. These points do not lie on a straight line. Line 1 corresponds to winter data, line 2 to summer data.

We now return to the function $\sigma_P^2 = f(\theta)$. For an objective of radius R it has the form

$$\sigma_P^2 = 2.2 \cdot k^{7/6} (\sec\theta)^{11/6} \times \times G\left(\frac{R}{\sqrt{\lambda H \sec\theta}}\right) \int_0^\infty C_n^2(z)\, z^{5/6} dz, \qquad (4)$$

where G is a function computed in § 53. We see from (4) that the dependence on θ is associated not only with the factor $(\sec\theta)^{11/6}$ but also with the function G. If $R \ll \sqrt{\lambda H}$, we may take $G = 1$ and this leads back to expression (3). If $R \gg \sqrt{\lambda H}$, we may use the asymptotic expansion of the function $G(x) \sim x^{-7/3}$ for $x \gg 1$, which gives

$$\sigma_P^2 \sim R^{-7/3} H^{7/6} \sec^3\theta \int_0^\infty C_n^2(z)\, z^{5/6} dz. \qquad (5)$$

Thus, for $D \gg \sqrt{\lambda H}$, the dependence of σ_P^2 on θ is expressed by a factor $\sec^3\theta$. For intermediate values of the ratio $D/\sqrt{\lambda H}$ the dependence $\sigma_P^2 = f(\theta)$ can be approximated by a relation $\sigma_P^2 = (\sec\theta)^\alpha$, where $11/6 < \alpha < 3$. This conclusion is in good agreement with the experimental dependence $\alpha(R)$ /141/: 5

2R, cm	2.5	7.6	15.2	32
α	1.8	2	2.4	3

In Figure 85c, the solid line is drawn using expression (4). The change in the initial part of the curve $\sigma_I^2 = f(\theta)$ with changing aperture radius R is thus attributable to aperture averaging. This effect also explains the distortion of the frequency spectra of the scintillations (see the theoretical curves in Figure 50) and the experimental data in Figure 90).

Let us now consider the curves $\sigma_P^2 = f(\theta)$ for large θ $(\theta > 60°)$.

First note that for observations in a narrow spectral band the curve $\sigma_P^2 = f(\theta)$ does not reach saturation for large θ (see Figure 88). The saturation effect is thus associated with chromatic scintillation.

Let us consider this problem in more detail /148/.

Since the atmosphere is a weakly dispersive medium, the refraction of rays of different wavelengths is different. Therefore, if two rays of wavelengths λ_1 and λ_2 converge at one point on the surface, then at a certain altitude z they are separated by a distance ρ, which is expressed with fair accuracy by the equality /149/

$$\rho(\Delta\lambda, \theta) = D_0(\theta)\,[n(\lambda_1) - n(\lambda_2)]\,(1 - e^{-z/h}). \qquad (6)$$

Here $\Delta\lambda = \lambda_1 - \lambda_2$; $D_0(\theta)$ is a function computed in /149/; $h = 8$ km is the scale height of the free atmosphere; and $n(\lambda)$ is the refractive index.* It follows from (6) that at a height of 10 — 15 km, in the layer important for scintillation, the distance $\rho(\Delta\lambda, \theta)$ between the two rays does not change much with z, so that we may assume with fair accuracy that the two rays are parallel propagating at a distance $\rho(\Delta\lambda, \theta)$ from one another; this distance is given by expression (6) with $z = 10$ km. Therefore, to compute the chromatic scintillation, it suffices to find the amplitude correlation coefficient for the following idealized case: two plane monochromatic waves (of wavelengths λ_1 and λ_2) propagating in the same direction through the atmosphere (without refraction), and the observation points are distant $\rho(\Delta\lambda, \theta)$ from each other in a plane perpendicular to the rays. The starting equation is taken from the method of smooth perturbations,

$$\frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} + 2ik\frac{\partial\Phi}{\partial x} + 2k^2 n_1(x, y, z) = 0, \tag{7}$$

and it describes the perturbation of amplitude A and phase S of a plane monochromatic wave propagating in the direction of the x axis. Here $\mathrm{Re}\,\Phi = \ln A - \langle \ln A \rangle \equiv \chi$.

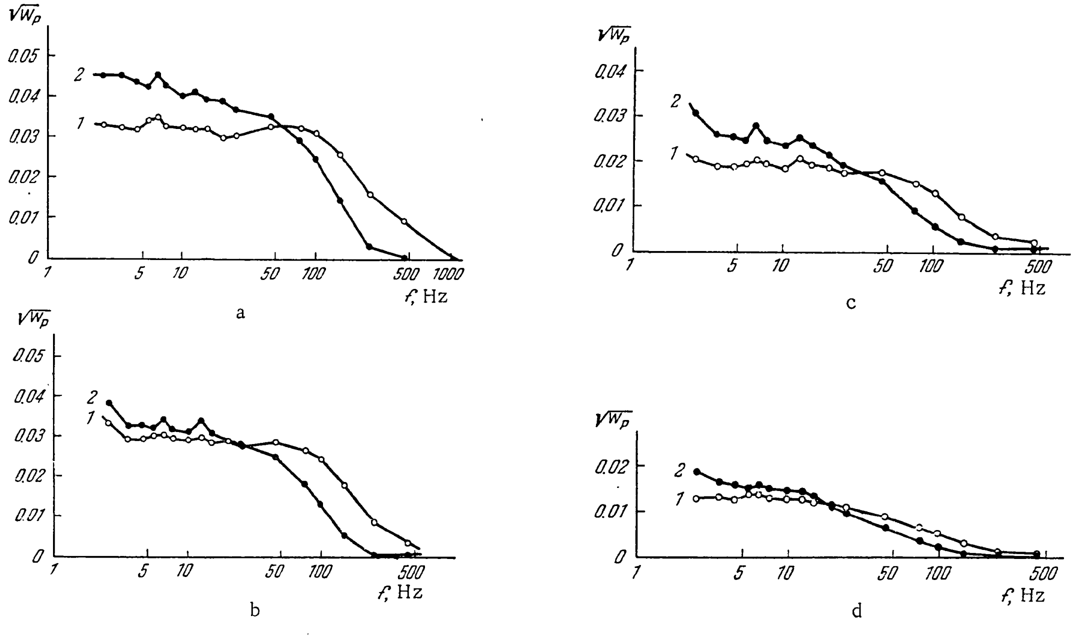


FIGURE 90. Frequency spectra of light flux fluctuations for telescopes with various aperture diameters: (a) $D = 2.5$ cm, (b) $D = 7.6$ cm, (c) $D = 15$ cm, (d) $D = 32$ cm; 1) winter, 2) summer.

* For $z \gg H$ and $n(\lambda_1) - n(\lambda_2) = 5.9 \cdot 10^{-6}$ (the extreme wavelengths in the visible spectrum), ρ takes on the following values: 0 for $\theta \leqslant 10°$, 0.2 cm for $\theta = 20°$, 0.8 cm for $\theta = 30°$, 2.1 cm for $\theta = 40°$, 4.9 cm for $\theta = 50°$, 11.6 cm for $\theta = 60°$, 31.2 cm for $\theta = 70°$, 128 cm for $\theta = 80°$, and 1283 cm for $\theta = 88°$.

Writing the solution of (7) in spectral form (see (46.8)) we have

$$\chi(L, y, z) = \int\int_{-\infty}^{\infty} e^{i(\varkappa_2 y + \varkappa_3 z)} \left\{ k_1 \int_0^L \left[\sin \frac{\varkappa^2 (L-x)}{2k_1} u_n (d\varkappa_2, d\varkappa_3, x) \right] dx \right\}. \qquad (8)$$

Multiplying (8) by the analogous equation for the other observation point and the other wavelength $\lambda_2 = \frac{2\pi}{k_2}$ and averaging, we obtain after certain manipulations (making use of the condition $\lambda \ll \lambda_0$)

$$\langle \chi(M_1)\chi(M_2)\rangle = \pi k_1 k_2 \int_0^{\infty} e^{i\varkappa\rho(\Delta\lambda, \theta)} \varkappa \, d\varkappa \times$$

$$\times \iint \left[\cos \frac{\varkappa^2 (L-\eta)}{2\tilde{k}} - \cos \frac{\varkappa^2 (L-\eta)}{2k} \right] F(\varkappa_2, \varkappa_3, \xi, \eta) \, d\xi d\eta, \qquad (9)$$

where $\frac{1}{\tilde{k}} = \frac{1}{k_1} - \frac{1}{k_2}$, $\frac{1}{k} = \frac{1}{2}\left(\frac{1}{k_1} + \frac{1}{k_2}\right)$, and $F(\varkappa_2, \varkappa_3, \xi, \eta)$ is the two-dimensional function of the refractive index fluctuations.

The integral in (9) can be expressed in terms of the amplitude correlation function of a monochromatic wave. Indeed, taking $k_1 = k_2 = k$ and $\tilde{k}^{-1} = 0$, we obtain

$$\langle \chi(M_1)\chi(M_2)\rangle \equiv B_A(\rho, k) =$$

$$= \pi k^2 \int_0^{\infty} e^{i\varkappa\rho(\Delta\lambda, \theta)} \varkappa \, d\varkappa \iint \left[1 - \cos \frac{\varkappa^2 (L-\eta)}{k} \right] F(\varkappa_2, \varkappa_3, \xi, \eta) \, d\xi \, d\eta, \qquad (10)$$

where $B_A(\rho, k)$ is the log-amplitude correlation function of a plane monochromatic wave with $\lambda = \frac{2\pi}{k}$. It is readily seen that the integral in (9) can be written as a difference of two functions of the form given in (10):

$$\langle \chi(M_1)\chi(M_2)\rangle = \frac{k_1 k_2}{k^2} B_A(\rho(\Delta\lambda, \theta), k) - \frac{k_1 k_2}{4\tilde{k}^2} B_A(\rho(\theta, \Delta\lambda), 2\tilde{k}). \qquad (11)$$

Here $B_A(\rho, 2\tilde{k})$ is the log-amplitude correlation function of some hypothetical wave of wavelength $\frac{2\pi}{2\tilde{k}} = \frac{1}{2}(\lambda_1 - \lambda_2)$. The function $B_A(\rho, k)$ for a monochromatic wave was determined in the preceding. If $\lambda_0 \ll \sqrt{\lambda L} \ll L_0$, where L_0 is the outer scale of turbulence (in star scintillation problems this condition almost always holds true), we have

$$B_A(\rho, k) = 0.56 \, k^{7/6} (\sec\theta)^{11/6} b_A \left(\frac{\rho}{\sqrt{\lambda H \sec\theta}} \right) \int_0^{\infty} C_n^2(z) z^{5/6} dz, \qquad (12)$$

where b_A is the log-amplitude correlation coefficient, and H is the vertical thickness of the layer causing the scintillations.

Let us determine the log-amplitude correlation coefficient $R(\Delta\lambda, \theta)$ for wavelengths λ_1 and λ_2. Using (6) and (11) we find

$$R(\Delta\lambda\ \theta) = (1-\varepsilon^2)^{-5/12} \left[b_A \left(\frac{\rho(\Delta\lambda, \theta)}{\sqrt{\lambda H \sec\theta}} \right) - \varepsilon^{5/6} b_A \left(\frac{1}{\sqrt{\varepsilon}} \frac{\rho(\Delta\lambda, \theta)}{\sqrt{\lambda H \sec\theta}} \right) \right], \qquad (13)$$

where $\varepsilon = \frac{\lambda_2 - \lambda_1}{\lambda_2 + \lambda_1}$.

Figure 91 plots $R(\Delta\lambda, \theta)$ for various θ according to expression (13). A function calculated assuming the "2/3 law" for the refractive index fluctuations was used for b_A. For $\sqrt{\lambda H}$ we took 10 cm, $\rho(\Delta\lambda, \theta)$ was calculated from (6) for $z = 10$ km. The same graph also shows some experimental points obtained at the Pulkovo Observatory for $\theta = 76°$, 65°, 50° and 40°. The experimental data are in satisfactory agreement with the theoretical curves.

Chromatic scintillation is also responsible for the fact that the scintillations of integrated (white) light observed with a sufficiently small aperture (up to 6 – 7 cm in diameter) decreases with increasing zenith distance for $\theta > 60°$.

Let $I(\lambda)$ be the spectral density of the light intensity. The integrated light flux J is then

$$J = \int_{\lambda_1}^{\lambda_2} I(\lambda)\, d\lambda,$$

and the mean square of the fluctuations in J is given by

$$\langle J'^2 \rangle = \int_{\lambda_1}^{\lambda_2}\int_{\lambda_1}^{\lambda_2} \langle I'(\lambda) I'(\lambda')\rangle \, d\lambda\, d\lambda',$$

i. e., it depends on the correlation function for fluctuations of the spectral intensity of the light flux in various spectral regions. For small fluctuations (which is nearly always the case), $\langle I'(\lambda) I'(\lambda')\rangle$ is proportional to $R(\Delta\lambda, \theta)$. Using (13), we can compute the function $F(\theta)$, which is equal to the ratio of the mean square fluctuations of the integrated light flux between λ_1 and λ_2 to the mean square fluctuations of monochromatic light of wavelength $\lambda_0 = \frac{1}{2}(\lambda_1 + \lambda_2)$. (The mean square fluctuation of monochromatic light is expressed by (4).)

$F(\theta)$ can be found from experimental data by two independent methods. On the one hand, given the function

$$G = f\left(\frac{D}{\sqrt{\lambda_0 H \sec\theta}}\right),$$

it can be computed from the experimental curve $\sigma_I^2 = \varphi(\sec\theta)$, comparing the latter with expression (4). On the other hand, this function can be found directly as the ratio of integrated-light scintillations to monochromatic-light scintillations.

In Figure 92 the solid curve plots the function $F(\theta)$ obtained from (13) by theoretical calculations for $\lambda_1 = 0.3\mu$ and $\lambda_2 = 0.6\mu$. The same graph shows some experimental points. The asterisks mark the ratio of integrated-light scintillations to scintillations in a narrow spectral band (90 Å wide) measured at the Pulkovo Observatory by L. N. Zhukova. Using expression (4) the other points were obtained by reduction of the dependence of integrated-light scintillation on zenith distance obtained by various authors:

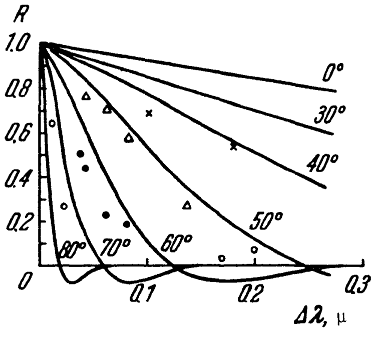


FIGURE 91. Correlation coefficient of light intensity fluctuations vs. wavelength difference for various zenith distances.

Experimental data:
○ $\theta = 76°$, ● $\theta = 65°$; △ $\theta = 50°$; × $\theta = 40°$.

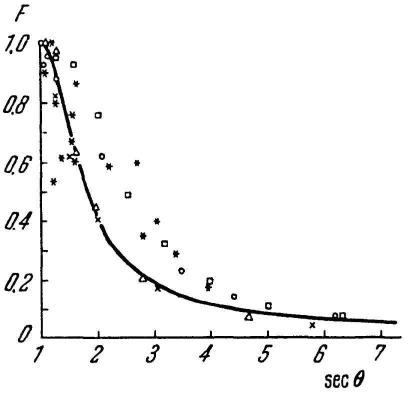


FIGURE 92. The ratio F of the mean square relative intensity fluctuations between wavelengths λ_1 and λ_2 to the mean square relative intensity fluctuations of monochromatic light of wavelength $\lambda_0 = \frac{\lambda_1 + \lambda_2}{2}$ as a function of zenith angle.

Solid curve — theoretical, other symbols — experimental measurements.

a) $D = 15''$ /142/;
b) $D = 10''$ /143/;
c) $D = 3''$ /141/;
d) $D = 12''$ /141/.

In $G(D/\sqrt{\lambda_0 H \sec\theta})$ we took $\sqrt{\lambda_0 H}$ equal to 10 cm. All the results were normalized to 1 for $\theta = 0$. The good fit between the theoretical curve and the experimental points obtained by a variety of independent methods confirms that the decrease of integrated-light scintillations with increasing zenith distance may indeed be due to chromatic scintillations.

All the main features of star scintillations are thus naturally accounted for by the theory.

Let us further consider image "quivering" experiments. In Part A we derived the expression

$$\sigma_\alpha^2 = 2.84 C_n^2 L (2R)^{-1/3}.$$

For inhomogeneous turbulence it takes the form (48.25)

$$\sigma_\alpha^2 = 2.84 (2R)^{-1/3} \int_0^\infty C_n^2(\boldsymbol{r})\, dx,$$

and when C_n^2 is a function of height only,

$$\sigma_\alpha^2 = 2.84 (2R)^{-1/3} \sec\theta \int_0^\infty C_n^2(z)\, dz. \tag{14}$$

It follows from this expression that inhomogeneities at different heights above the ground enter σ_α^2 with the same weight. However, $C_n^2(z)$ usually has a maximum for small z, so that "image quivering" is mainly due to the lower atmospheric layers (refractive index fluctuations near the observation point may also be quite significant). This in particular explains the lack of correlation between scintillations and "image quivering."

The dependence of σ_α and θ is given by the factor sec θ. This dependence is on the average confirmed experimentally (Figure 93), but individual observations show fairly frequent departures from this formula, which probably are explained by the strong effect of local conditions (see /151 – 153/).

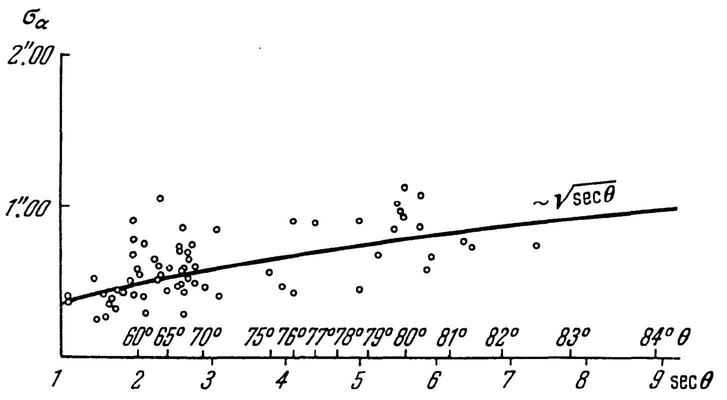


FIGURE 93. Rms fluctuations of the angle-of-arrival of light from stars vs. zenith angle.

A very comprehensive study of "quivering" of the edge of the Sun's disk was carried out in /173/. The equipment used was described in the preceding in connection with "quivering" measurements of an artificial source in the atmospheric layer close to the ground.

The measurements were made on hot summer days with highly developed convection. The temperature fluctuation profile is thus given by the expression from Chapter 1

$$C_n^2(z) = C_n^2(z_0)\left(\frac{z_0}{z}\right)^{4/3}.$$

Simultaneous measurements of angle-of-arrival fluctuations with wind velocity and temperature profiles in the ground layer thus give sufficient data to compute the expected mean square of the angle-of-arrival fluctuations. The measured and calculated values are plotted in Figure 94. The vertical axis gives the mean square fluctuations of the angle-of-arrival reduced to the zenith. The horizontal axis gives the same function (in arc seconds) calculated from meteorological observations in the atmospheric layer close to the ground. Crosses in the graph mark averages of groups of points close to that value.

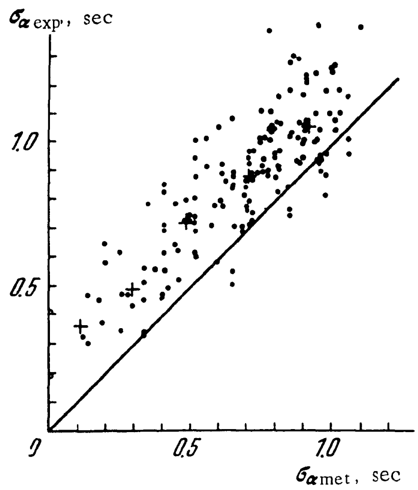


FIGURE 94. Measured mean square fluctuations of the angle of arrival of light from the edge of the Sun's disk ($\sigma_{\alpha\,\mathrm{exp}}$) vs. the fluctuations computed from meteorological data ($\sigma_{\alpha\,\mathrm{met}}$).

The graph shows that the theory provides an adequate quantitative description of "image quivering".

The frequency spectra of angle-of-arrival fluctuations were also studied. The output signal recording the instantaneous position of the edge of the solar disk was fed to a 30-channel frequency analyzer, similar to that used in experiments with a source close to the ground. The results of the measurements are shown in Figure 95. Here the vertical axis gives the product of the normalized spectral density of angle-of-arrival fluctuations and the frequency of the fluctuations. The horizontal axis gives the frequency of the fluctuations. The curve is plotted in log-log coordinates. The solid curve is the theoretical dependence from Part A of this chapter. The dots mark experimental data averaged over a few tens of frequency spectra obtained under various conditions. The averaged spectra were plotted in dimensionless coordinates $fw_\alpha(f)$, $\frac{fb}{v_\perp}$, where the peak falls at $\frac{fb}{v_\perp} = 0.22$. To give some idea of the actual frequencies of fluctuations, the frequency in Figure 95 is given in hertz (cps), but the position of the characteristic frequency $0.22\, v_\perp/b$ corresponding to the peak of the product $fw_\alpha(f)$ is nevertheless shown. (To derive the real spectrum for particular values of b and $v_\perp$ from the generalized spectrum in the figure, we should compute the characteristic frequency $0.22\, v_\perp/b$ and then shift the frequency scale so that this frequency occupies its proper position).

In conclusion of this section, let us estimate the value of C_n^2 from scintillation and "image quivering" measurements.

First we will obtain this estimate from scintillation data. We start with expression (3) taking

$$\int_0^\infty C_n^2(z)\, z^{5/6}\, dz = \frac{6}{11} C_{n\,\mathrm{av}}^2 H^{11/6},$$

where $C_{n\,\mathrm{av}}^2$ is the value of C_n^2 averaged over a layer of thickness H with a weight $z^{5/6}$. According to experimental data $\sigma_I^2 = 0.2$ for $\theta = 0$ and $D \ll \sqrt{\lambda H}$. Inserting the above value $H = 10$ km, $k = \frac{2\pi}{\lambda}$, $\lambda = 0.5\,\mu$, we find $C_n \approx 10^{-3} N$-units $\cdot$ cm$^{-1/3}$. This result refers to high atmospheric layers ($z \sim 10$ km).

"Image quivering" observations on stars in telescopes give the C_n^2 characterizing lower atmospheric layers. We should first establish, however, the effective thickness of the atmospheric layer responsible for this phenomenon. A suitable estimate is obtained by comparing the theory with experimental data on the angle-of-arrival correlation coefficient for light from stars with close angular coordinates. In Chapter 3 we derived an expression for the correlation coefficient of two rays diverging at an angle γ:

$$b(\gamma)=\begin{cases}\frac{3}{16}\frac{(1+x)^{5/3}-(1-x)^{5/3}-2x^{5/3}}{x} & \text{for } x<1,\\ \frac{3}{16}\frac{(x+1)^{5/3}+(x-1)^{5/3}-2x^{5/3}}{x} & \text{for } x>1,\end{cases}$$

where $x=\frac{2L}{b}\operatorname{tg}\frac{\gamma}{2}$, b is the telescope diameter (see (43.23), p. 208).

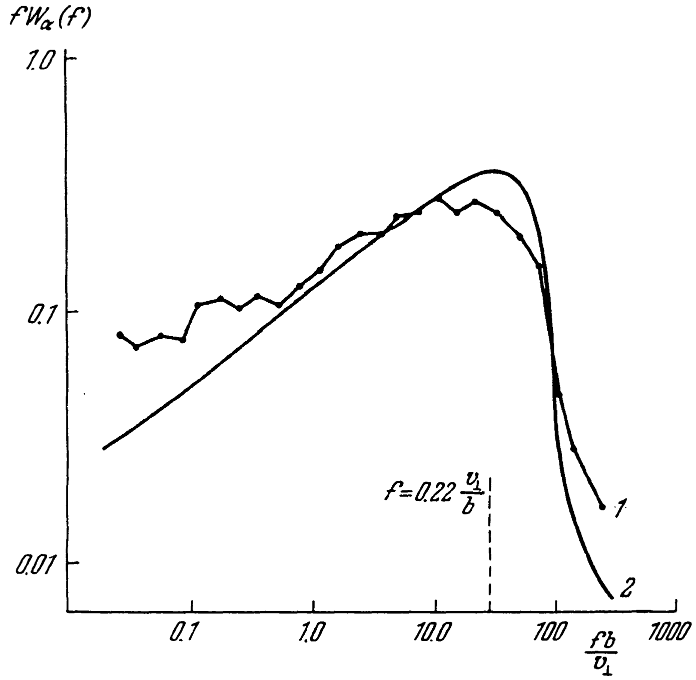


FIGURE 95. Empirical frequency spectrum of angle-of-arrival fluctuations for light from the edge of the solar disk (curve 1) and the theoretical curve (2).

We see from this expression that the characteristic correlation angle is determined by the ratio b/L of the telescope aperture diameter to the effective thickness of the atmospheric layer which produces image "quivering." Using Kolchinskii's experimental data on the angular correlation of the angle-of-arrival fluctuations /154/ we obtain for L values of the order of 0.5 — 1 km.

We now use expression (14) substituting $\sigma_\alpha \approx 1.7\cdot 10^{-6}$ rad, $2R=40$ cm, $L=0.5$ km. For the average of C_n over the layer we obtain the estimate $C_n \sim 10^{-2} N$-units $\cdot$ cm$^{-1/3}$. This result for C_n is in good agreement with the data tabulated in the previous section.

Chapter 5

APPLICATION OF METHODS OF QUANTUM FIELD THEORY TO WAVE PROPAGATION IN A RANDOM MEDIUM

A. WAVE PROPAGATION IN A MEDIUM WITH STRONG FLUCTUATIONS

The investigation of all the problems in the previous chapters was invariably based on some modification of the perturbation technique. The results of this treatment are naturally applicable only in the case of weak fluctuations of the refractive index.

We will now consider the application of alternative methods, which are not an outgrowth of perturbation theory. The problem of wave propagation in a random medium has much in common with quantum field theory. This analogy is based on the following fact: the problems of quantum field theory require for their solution finding the solutions of the field equations in a medium with arbitrary external sources, which interact with the field, and averaging the solutions over the quantum fluctuations of the sources. This problem clearly has many similar points to the subject we are investigating. However, the problem of wave propagation in a random medium is essentially simpler, as in quantum field theory the commutation functions, which are the analog of correlation functions in our treatment, are invariably singular and lead to a divergence. No such divergence arises in connection with wave propagation in a random medium.

The application of the mathematical tools of quantum field theory enables us to leave the limitations of the theory of small perturbations and to obtain a solution valid in the case of strong fluctuations. The presentation in this chapter does not assume any previous knowledge of quantum techniques on the part of the reader.

Note that the theory described in this chapter is far from being perfect, and at this stage we can only give a sketch of the method. However, in view of the great potentialities of these techniques, we decided on the inclusion of this topic in the book.

The treatment is largely based on the scalar equation. The generalization to the case of Maxwell's vector equations can be accomplished without any difficulty.

§ 60. Analysis of perturbation-theoretical series

Consider the equation

$$\Delta\psi + k^2(1 + \varepsilon_1(\boldsymbol{r}))\psi = \delta(\boldsymbol{r} - \boldsymbol{r}_0), \tag{1}$$

where $\varepsilon_1(\boldsymbol{r})$ is a random function of position. We define a new operator $L_0(\boldsymbol{r}) = \Delta(\boldsymbol{r}) + k^2$. The inverse of L_0, i.e., the operator $L_0^{-1} = M_0$, can be obtained by solving the equation $L_0(\boldsymbol{r})\varphi(\boldsymbol{r}) = f(\boldsymbol{r})$ in conjunction with the radiation condition. The solution of this equation has the standard form

$$\varphi(\boldsymbol{r}) = \int G_0(\boldsymbol{r}, \boldsymbol{\rho}) f(\boldsymbol{\rho})\, d^3\rho,$$

where

$$G_0(\boldsymbol{r}, \boldsymbol{\rho}) = -\frac{e^{ik|\boldsymbol{r}-\boldsymbol{\rho}|}}{4\pi|\boldsymbol{r}-\boldsymbol{\rho}|}. \tag{2}$$

The solution of the equation $L_0(\boldsymbol{r})\varphi(\boldsymbol{r}) = f(\boldsymbol{r})$ can be written in symbolic form using the operator $M_0 = L_0^{-1}$ as $\varphi(\boldsymbol{r}) = M_0(\boldsymbol{r}) f(\boldsymbol{r})$. Comparing this equality with the explicit expression of the solution for φ, we see that M_0 is an integral operator whose kernel is given by (2). Note that the symbolic notation $M_0 f(\boldsymbol{r})$ indicates that the function f transformed by the operator M_0, i.e., the function $M_0 f$, is considered at the point $\boldsymbol{r}$. A more proper notation would be $(M_0 f)(\boldsymbol{r})$, where $(M_0 f)$ is the new function. We will, however, prefer the simpler notation $M_0 f(\boldsymbol{r})$, which need not cause any confusion.

Clearly,

$$G_0(\boldsymbol{r}, \boldsymbol{\rho}) = M_0 \delta(\boldsymbol{r} - \boldsymbol{\rho}). \tag{3}$$

We write equation (1) in the form

$$L_0(\boldsymbol{r})\psi = -k^2\varepsilon_1(\boldsymbol{r})\psi(\boldsymbol{r}) + \delta(\boldsymbol{r} - \boldsymbol{r}_0)$$

and act on it with the operator M_0:

$$\psi(\boldsymbol{r}) = -k^2 M_0 \varepsilon_1(\boldsymbol{r})\psi(\boldsymbol{r}) + G_0(\boldsymbol{r} - \boldsymbol{r}_0) = \\ = -k^2 \int G_0(\boldsymbol{r}, \boldsymbol{\rho})\, \varepsilon_1(\boldsymbol{\rho})\, \psi(\boldsymbol{\rho})\, d^3\rho + G_0(\boldsymbol{r} - \boldsymbol{r}_0). \tag{4}$$

Equation (4) is an integral equation which is equivalent to (1). The solution to (4) can be found by successive iterations, repeatedly inserting for $\psi(\boldsymbol{\rho})$ in the right-hand side of (4) the expression given by (4).

A more convenient way, however, is to solve (4) by the operational technique. Writing this equation in the form

$$[1 + k^2 M_0 \varepsilon_1]\psi = G_0(\boldsymbol{r} - \boldsymbol{r}_0) = M_0\, \delta(\boldsymbol{r} - \boldsymbol{r}_0)$$

and introducing a new operator

$$[1 + k^2 M_0 \varepsilon_1]^{-1} \equiv \sum_{n=0}^{\infty} (-k^2 M_0 \varepsilon_1)^n,$$

we obtain

$$\psi(\boldsymbol{r}) = \sum_{n=0}^{\infty} (-k^2 M_0 \varepsilon_1)^n M_0 \delta(\boldsymbol{r} - \boldsymbol{r}_0) = \\ = [M_0 - k^2 M_0 \varepsilon_1 M_0 + k^4 M_0 \varepsilon_1 M_0 \varepsilon_1 M_0 - \ldots]\, \delta(\boldsymbol{r} - \boldsymbol{r}_0). \tag{5}$$

In (5) each operator M_0 acts on all the factors to the right of it. For example,

$$M_0\varepsilon_1 M_0 f(\boldsymbol{r}) = \iint G_0(\boldsymbol{r}, \rho)\,\varepsilon_1(\rho)\,G_0(\rho, \rho')\,f(\rho')\,d^3\rho\,d^3\rho',$$

etc. Series (5) is a perturbation-theoretical series. In what follows we will be concerned with the mean $\langle\psi(\boldsymbol{r})\rangle$ and the correlation function $\langle\psi(\boldsymbol{r}_1)\psi^*(\boldsymbol{r}_2)\rangle$. First we consider the mean $\langle\psi\rangle$. Suppose that the random field ε_1 has a normal probability distribution. This means that $\langle\varepsilon_1\rangle = 0$, $\langle\varepsilon_1(\boldsymbol{r}_1)\ \varepsilon_1(\boldsymbol{r}_2)\ \varepsilon_1(\boldsymbol{r}_3)\rangle = 0$ and, in general, $\langle\varepsilon_1(\boldsymbol{r}_1)\ldots\varepsilon_1(\boldsymbol{r}_{2n+1})\rangle = 0$; moreover, even order moments are expressible in terms of the second moments:

$$\begin{aligned}\langle\varepsilon_1(\boldsymbol{r}_1)\,\varepsilon_1(\boldsymbol{r}_2)\varepsilon_1(\boldsymbol{r}_3)\,\varepsilon_1(\boldsymbol{r}_4)\rangle &= \langle\varepsilon_1(\boldsymbol{r}_1)\varepsilon_1(\boldsymbol{r}_2)\rangle \times \\ \times \langle\varepsilon_1(\boldsymbol{r}_3)\ \varepsilon_1(\boldsymbol{r}_4)\rangle &+ \langle\varepsilon_1(\boldsymbol{r}_1)\varepsilon_1(\boldsymbol{r}_3)\rangle\cdot\langle\varepsilon_1(\boldsymbol{r}_2)\,\varepsilon_1(\boldsymbol{r}_4)\rangle + \\ &+ \langle\varepsilon_1(\boldsymbol{r}_1)\,\varepsilon_1(\boldsymbol{r}_4)\rangle\cdot\langle\varepsilon_1(\boldsymbol{r}_2)\ \varepsilon_1(\boldsymbol{r}_3)\rangle.\end{aligned} \tag{6}$$

A similar expression can be written for the sixth-order moments (it contains 15 terms, each of which is a product of three correlation functions), etc. A moment of order $2n$ is expressed as a sum of $(2n-1)!!$ terms, each of which is a product of n correlation functions; among these $(2n-1)!!$ terms, there are all possible permutations of the functions $\varepsilon_1(\boldsymbol{r}_k)$, $\varepsilon_1(\boldsymbol{r}_l)$. Using this rule, we can now average expression (5).

The mean $\langle\psi(\boldsymbol{r})\rangle = \widetilde{G}(\boldsymbol{r}, \boldsymbol{r}_0)$ is the averaged Green's function,*

$$\begin{aligned}\widetilde{G}(\boldsymbol{r}, \boldsymbol{r}_0) = [M_0 + k^4\langle M_0\varepsilon_1 M_0\varepsilon_1 M_0\rangle + \\ + k^8\langle M_0\varepsilon_1 M_0\varepsilon_1 M_0\varepsilon_1 M_0\varepsilon_1 M_0\rangle + \ldots]\ \delta(\boldsymbol{r}-\boldsymbol{r}_0).\end{aligned} \tag{7}$$

The first term in (7) is $M_0\delta(\boldsymbol{r}-\boldsymbol{r}_0) = G_0(\boldsymbol{r}, \boldsymbol{r}_0)$. Let us consider in more detail the second term:

$$\begin{aligned}&k^4\langle M_0\varepsilon_1 M_0\varepsilon_1 M_0\rangle\,\delta(\boldsymbol{r}-\boldsymbol{r}_0) = \\ &= \Big\langle k^4 \iiint G_0(\boldsymbol{r}, \rho_1)\,\varepsilon_1(\rho_1)\,G_0(\rho_1, \rho_2)\,\varepsilon_1(\rho_2)\,G_0(\rho_2, \rho_3)\,\delta(\rho_3-\boldsymbol{r}_0)\times \\ &\times d^3\rho_1\,d^3\rho_2\,d^3\rho_3\Big\rangle = k^4\iint G_0(\boldsymbol{r}, \rho_1)\,G_0(\rho_1, \rho_2)\,G_0(\rho_2, \boldsymbol{r}_0)\times \\ &\times\langle\varepsilon_1(\rho_1)\,\varepsilon_1(\rho_2)\rangle\,d^3\rho_1\,d^3\rho_2.\end{aligned} \tag{8}$$

The expressions entering (7) can be made to correspond to various diagrams, which are constructed according to the following rules. The function $G_0(\rho_1, \rho_2)$ is represented by a linear segment whose end points are ρ_1 and ρ_2, thus:

$$G_0(\rho_1, \rho_2) \sim \overline{\rho_1 \qquad \rho_2}.$$

The factor k^2 is represented by dots placed on the diagram at the points where the factor $\varepsilon_1(\rho_i)$ is evaluated, thus:

$$k^2 \sim \cdot$$

(the vertices of the diagram); a dashed line joins those vertices for which the functions $\varepsilon_1(\rho_1)$ and $\varepsilon_1(\rho_2)$ appear inside the same averaging bracket:

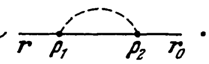
$$k^4\langle\varepsilon_1(\rho_1)\,\varepsilon_1(\rho_2)\rangle \sim \underset{\rho_1}{\bullet}\text{- - - -}\underset{\rho_2}{\bullet}\,.$$

* We recall that $\psi(\boldsymbol{r})$, being the solution of equation (1) with $\delta(\boldsymbol{r}-\boldsymbol{r}_0)$ on the right, is by definition the Green's function of this equation.

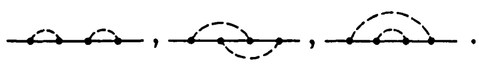
Expression (8) is thus represented graphically by the following diagram:

Integration is performed over all the interior vertices of a diagram (ρ_1 and ρ_2). The integral in (8) can thus be uniquely represented in graphical form (by a Feynman diagram) and vice versa: a Feynman diagram is uniquely represented by the corresponding integral. The same diagram may be used to represent the kernel of the integral operator acting on $\delta(\rho_0 - r_0)$:

$$k^4 M_0(r, \rho_1)\langle \varepsilon_1(\rho_1) M_0(\rho_1, \rho_2)\varepsilon_1(\rho_2)\rangle M_0(\rho_2, r_0) \sim$$



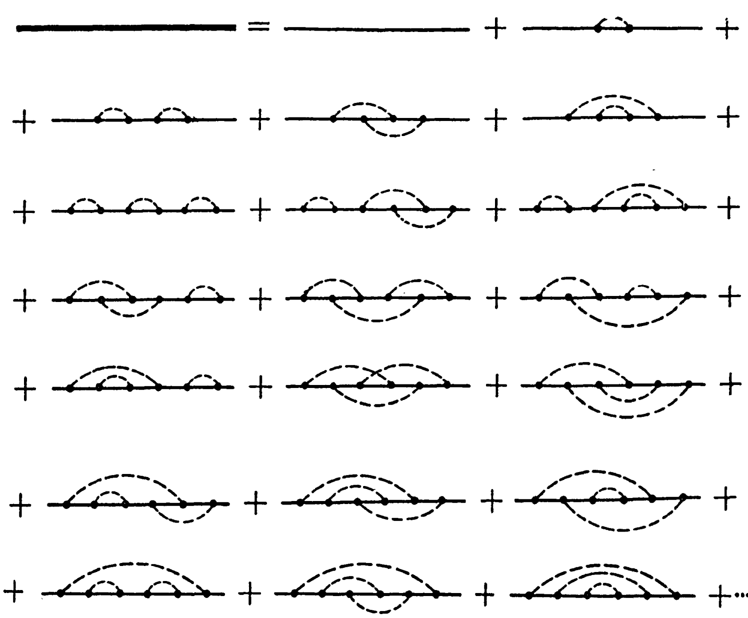
Now consider the next term in (7). Here the integrand contains four factors ε_1 taken at four different points. Using expression (6), we break the integral into three terms which are represented by three fourth-order diagrams (each with four vertices):



If the averaged Green's function $\tilde G(r, r_0)$ is represented by a thick line, thus:

$$\tilde G(r, r_0) \sim \overline{r \qquad r_0},$$

then expansion (7) may be represented as the following series of diagrams:



The rules for constructing higher terms of the series is clear from the figure. In diagrams of order $2n$ we join the pairs of vertices in all the possible ways and add up the various diagrams.

The diagrams appearing in $\tilde G$ can be divided into two groups: weakly coupled diagrams and strongly coupled diagrams. Weakly coupled diagrams separate into two parts when one of the G_0 lines is broken. These diagrams include

etc.

All the other diagrams are strongly coupled. The subseries of weakly coupled diagrams can now be separated from (7). This is done simply if $\psi(\boldsymbol{r})$ is written in the form

$$\psi(\boldsymbol{r}) = \langle\psi(\boldsymbol{r})\rangle + \varphi(\boldsymbol{r}), \quad \langle\varphi\rangle = 0. \tag{9}$$

Inserting (9) in equation (1), we find

$$L_0(\boldsymbol{r})\langle\psi(\boldsymbol{r})\rangle + L_0(\boldsymbol{r})\varphi + k^2\varepsilon_1\langle\psi\rangle + k^2\varepsilon_1\varphi = \delta(\boldsymbol{r}-\boldsymbol{r}_0). \tag{10}$$

Averaging (and seeing that $\langle\varepsilon_1\rangle = \langle\varphi\rangle = 0$), we obtain

$$L_0(\boldsymbol{r})\langle\psi\rangle + k^2\langle\varepsilon_1\varphi\rangle = \delta(\boldsymbol{r}-\boldsymbol{r}_0). \tag{11}$$

Subtracting (11) from (10), we get

$$L_0(\boldsymbol{r})\varphi(\boldsymbol{r}) + k^2\varepsilon_1\langle\psi\rangle + k^2[\varepsilon_1\varphi - \langle\varepsilon_1\varphi\rangle] = 0. \tag{12}$$

Equations (11) and (12) are equivalent to the original equation (1).

We act on (12) with the operator M_0:

$$\varphi(\boldsymbol{r}) = -k^2M_0\varepsilon_1\langle\psi\rangle - k^2M_0[\varepsilon_1\varphi - \langle\varepsilon_1\varphi\rangle]. \tag{13}$$

The integral equation (13) can be solved by successive iterations. This leads to the series

$$\varphi = -k^2M_0\varepsilon_1\langle\psi\rangle + k^4M_0(\varepsilon_1M_0\varepsilon_1 - \langle\varepsilon_1M_0\varepsilon_1\rangle)\langle\psi\rangle - \\ - k^6(M_0\varepsilon_1M_0\varepsilon_1M_0\varepsilon_1 - M_0\varepsilon_1M_0\langle\varepsilon_1M_0\varepsilon_1\rangle)\langle\psi\rangle + \dots \tag{14}$$

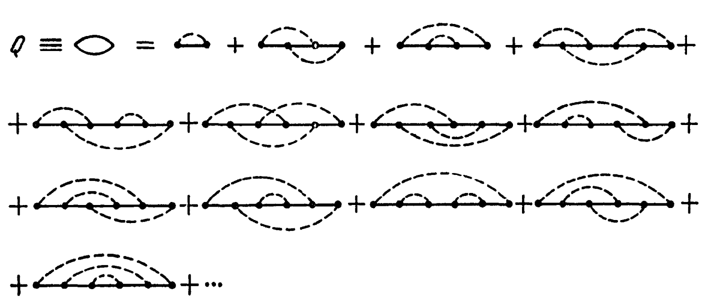
Multiplying (14) by $k^2\varepsilon_1$ and averaging gives

$$k^2\langle\varepsilon_1\varphi\rangle = [-k^4\langle\varepsilon_1M_0\varepsilon_1\rangle - k^8(\langle\varepsilon_1M_0\varepsilon_1M_0\varepsilon_1M_0\varepsilon_1\rangle - \\ - \langle\varepsilon_1M_0\varepsilon_1\rangle M_0\langle\varepsilon_1M_0\varepsilon_1\rangle) - \dots]\langle\psi\rangle = -Q\langle\psi\rangle, \tag{15}$$

where Q is an integral operator (with a kernel which is likewise denoted $Q = Q(\boldsymbol{r}_1, \boldsymbol{r}_2)$):

$$Q = k^4\langle\varepsilon_1M_0\varepsilon_1\rangle + k^8[\langle\varepsilon_1M_0\varepsilon_1M_0\varepsilon_1M_0\varepsilon_1\rangle - \langle\varepsilon_1M_0\varepsilon_1\rangle M_0\langle\varepsilon_1M_0\varepsilon_1\rangle] + \dots$$

The operator Q (the analog of the so-called mass operator in quantum field theory) is represented by the following diagram:



As we have separated from ψ its mean $\langle\psi\rangle$, the operator Q contains no weakly coupled diagram. Insertion of (15) in equation (11) gives

$$L_0\langle\psi\rangle - Q\langle\psi\rangle = (L_0 - Q)\langle\psi\rangle = \delta(\boldsymbol{r}-\boldsymbol{r}_0). \tag{16}$$

This equation has the form $L\langle\psi\rangle = \delta(\boldsymbol{r}-\boldsymbol{r}_0)$, where

$$L = L_0 - Q. \tag{17}$$

Let $M = L^{-1}$, so that

$$M\delta(\boldsymbol{r}-\boldsymbol{r}_0) = \langle\psi\rangle = \tilde{G}(\boldsymbol{r}, \boldsymbol{r}_0). \tag{18}$$

By (17),

$$Q = L_0 - L = M_0^{-1} - M^{-1}. \tag{19}$$

Consider the expression

$$M_0 + M_0QM = M_0 + M_0(M_0^{-1} - M^{-1})M = M_0 + M - M_0 = M.$$

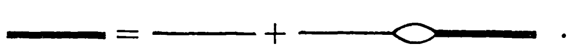
We have thus derived a relation between M and Q (Dyson's equation):

$$M = M_0 + M_0QM. \tag{20}$$

Acting with (20) on the δ-function, we obtain the equation

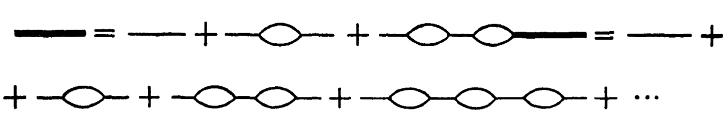
$$\tilde{G}(\boldsymbol{r}, \boldsymbol{r}_0) = G_0(\boldsymbol{r}, \boldsymbol{r}_0) + M_0Q\,\tilde{G}(\boldsymbol{r}, \boldsymbol{r}_0). \tag{21}$$

The graphical representation of (20) is


(20')

If Q is known, equation (21) can be solved. Thus, $\tilde{G}$ can be expressed only in terms of the strongly coupled diagrams appearing in Q.

Let us apply the diagram technique to prove that the solution of equation (20') does indeed give $\tilde{G}$. Equation (20') can be solved by diagram iterations, i.e., repeatedly inserting (20') in the right-hand side of (20'):

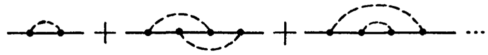


Here the first term is G_0, the second term includes all the strongly coupled diagrams, the third all the weakly coupled diagrams composed of two parts (each of which is strongly coupled), the fourth all the weakly coupled diagrams which are composed of three parts, etc.

Dyson's equation can also be derived by the diagram technique (an analogous derivation is given in what follows for the correlation functions).

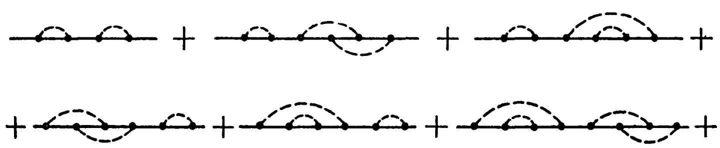
All the diagrams entering the expression for $\widetilde{G}$ can be divided into the following subclasses:

1. Strongly coupled diagrams

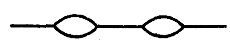


This subseries can be compactly represented as

2. Weakly coupled diagrams comprising only two strongly coupled elements:



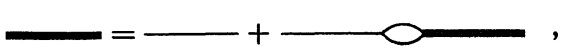
This subseries can clearly be represented in the form



3. Weakly coupled diagrams comprising three strongly coupled elements. This subseries may be represented in the form

etc. Clearly,

This expression, however, is the solution of the equation



if it is solved by iterations.

Let us now return to equation (21). Let the correlation function $B_\varepsilon(\boldsymbol{r}_1, \boldsymbol{r}_2)$ depend only on $\boldsymbol{r}_1 - \boldsymbol{r}_2$ (the statistically homogeneous case). Since $G_0(\boldsymbol{r}_1, \boldsymbol{r}_2) = G_0(\boldsymbol{r}_1 - \boldsymbol{r}_2)$, the kernel of the operator Q also depends only on $\boldsymbol{r}_1 - \boldsymbol{r}_2$: $Q(\boldsymbol{r}_1, \boldsymbol{r}_2) = Q(\boldsymbol{r}_1 - \boldsymbol{r}_2)$. In this case equation (21) takes the form

$$\widetilde{G}(\boldsymbol{r}_1 - \boldsymbol{r}_2) = G_0(\boldsymbol{r}_1 - \boldsymbol{r}_2) + \iint G_0(\boldsymbol{r}_1 - \boldsymbol{\rho}_1)\, Q(\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2)\, \widetilde{G}(\boldsymbol{\rho}_2 - \boldsymbol{r}_2)\, d^3\rho_1\, d^3\rho_2. \quad (22)$$

This equation can be solved using Fourier transforms. Let

$$\begin{aligned} \widetilde{G}(\boldsymbol{r}) &= \int \widetilde{g}(\boldsymbol{\varkappa})\, e^{i\boldsymbol{\varkappa r}}\, d^3\varkappa, \\ G_0(\boldsymbol{r}) &= \int g_0(\boldsymbol{\varkappa})\, e^{i\boldsymbol{\varkappa r}}\, d^3\varkappa, \\ Q(\boldsymbol{r}) &= \int q(\boldsymbol{\varkappa})\, e^{i\boldsymbol{\varkappa r}}\, d^3\varkappa. \end{aligned} \quad (23)$$

Let us find the integral on the right of (22):

$$\iint G_0(\boldsymbol{r}_1-\boldsymbol{\rho}_1)\, Q(\boldsymbol{\rho}_1-\boldsymbol{\rho}_2)\, \widetilde{G}(\boldsymbol{\rho}_2-\boldsymbol{r}_2)\, d^3\rho_1\, d^3\rho_2 =$$
$$= \iiiint\!\!\int e^{i[\varkappa_1(\boldsymbol{r}_1-\boldsymbol{\rho}_1)+\varkappa_2(\boldsymbol{\rho}_1-\boldsymbol{\rho}_2)+\varkappa_3(\boldsymbol{\rho}_2-\boldsymbol{r}_2)]} g_0(\varkappa_1)\, q(\varkappa_2)\, \widetilde{g}(\varkappa_3) \times$$
$$\times\, d^3\rho_1\, d^3\rho_2\, d^3\varkappa_1\, d^3\varkappa_2\, d^3\varkappa_3 = (8\pi^3)^2 \iiint \delta(\varkappa_2-\varkappa_1)\, \delta(\varkappa_3-\varkappa_2) \times$$
$$\times\, e^{i(\varkappa_1\boldsymbol{r}_1-\varkappa_3\boldsymbol{r}_2)} g_0(\varkappa_1)\, q(\varkappa_2)\, \widetilde{g}(\varkappa_3)\, d^3\varkappa_1\, d^3\varkappa_2\, d^3\varkappa_3 =$$
$$= (8\pi^3)^2 \int g_0(\varkappa)\, q(\varkappa)\, \widetilde{g}(\varkappa)\, e^{i\varkappa(\boldsymbol{r}_1-\boldsymbol{r}_2)}\, d^3\varkappa. \tag{24}$$

Inserting (23) and (24) in equation (22), we find

$$\widetilde{g}(\varkappa) = g_0(\varkappa) + (8\pi^3)^2 g_0(\varkappa)\, q(\varkappa)\, \widetilde{g}(\varkappa), \tag{25}$$

hence

$$\widetilde{g}(\varkappa) = \frac{g_0(\varkappa)}{1-(2\pi)^6 g_0(\varkappa)\, q(\varkappa)}. \tag{26}$$

Insertion of (26) in (23) gives

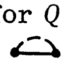
$$\widetilde{G}(\boldsymbol{r}) = \int \frac{g_0(\varkappa)\, e^{i\varkappa \boldsymbol{r}} d^3\varkappa}{1-(2\pi)^6 g_0(\varkappa)\, q(\varkappa)}. \tag{27}$$

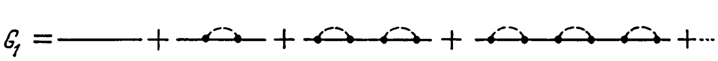
The function $g_0(\varkappa)$ has the form

$$g_0(\varkappa) = \frac{1}{8\pi^3(k^2-\varkappa^2)} \tag{28}$$

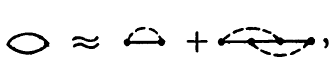
(the simplest derivation of (28) is by solving the equation $\Delta G_0 + k^2 G_0 = \delta(\boldsymbol{r})$ using Fourier transforms). In (28) k is regarded as a complex quantity ($\mathrm{Im}\, k > 0$); this gives a prescription for bypassing the poles in (23) and ensures the radiation condition. Thus,

$$\widetilde{G}(\boldsymbol{r}) = \frac{1}{8\pi^3} \int \frac{e^{i\varkappa \boldsymbol{r}} d^3\varkappa}{k^2-\varkappa^2-8\pi^3 q(\varkappa)} = \frac{1}{8\pi^3} \int \frac{e^{i\varkappa \boldsymbol{r}} d^3\varkappa}{k^2-\varkappa^2-\int Q(\boldsymbol{\rho})\, e^{-i\varkappa\boldsymbol{\rho}}\, d^3\rho}. \tag{29}$$

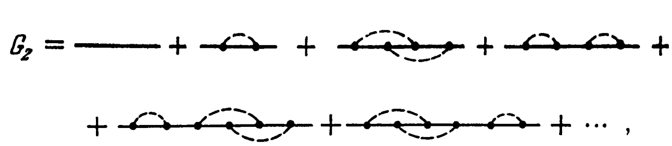
We have thus derived a formal solution for $\widetilde{G}$. However, it includes the unknown function Q, which has been defined only as an infinite series. If in the expression for Q we retain only the first term of the expansion, i.e., replace  by  in the previous diagram, we obtain the solution $G_1(\boldsymbol{r})$ in the form



Thus, even if only one term is retained in the expansion for Q, the approximate solution obtained from (29) for $Q(\boldsymbol{r}) \approx Q_1(\boldsymbol{r}) = k^4 G_0(\boldsymbol{r})\, B_\varepsilon(\boldsymbol{r})$ is a sum of an infinite subseries of the perturbation theory series. Retaining in Q the next term of the expansion, i.e., taking



we obtain



i.e., a more extensive combination of diagrams. Using (29), we can find explicit expressions for G_1, G_2,.... The properties of the functions G_1, G_2, etc., are considered in more detail later on, and now we turn to the correlation function

$$\langle\psi(\boldsymbol{r}_1)\psi^*(\boldsymbol{r}_2)\rangle=\langle\psi(\boldsymbol{r}_1)\rangle\langle\psi^*(\boldsymbol{r}_2)\rangle+\langle\varphi(\boldsymbol{r}_1)\varphi^*(\boldsymbol{r}_2)\rangle.$$

The product of means is of no interest in this case, and we shall only consider the function $\langle\varphi(\boldsymbol{r}_1)\ \varphi^*(\boldsymbol{r}_2)\rangle$. Averaging (5) and subtracting the result from the original expression, we find

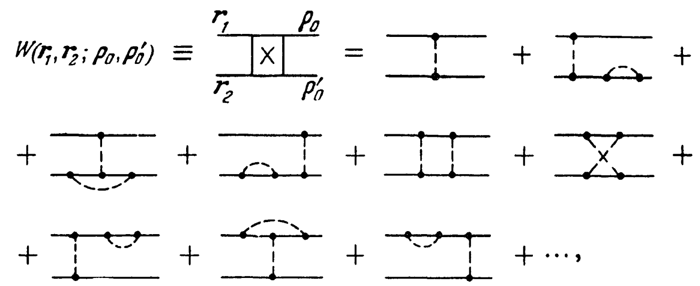
$$\varphi(\boldsymbol{r}_1)=\{-k^2M_0\varepsilon_1M_0+k^4[M_0\varepsilon_1M_0\varepsilon_1M_0-\langle M_0\varepsilon_1M_0\varepsilon_1M_0\rangle]-\\-k^6M_0\varepsilon_1M_0\varepsilon_1M_0\varepsilon_1M_0+\ldots\}\,\delta(\boldsymbol{r}_1-\boldsymbol{r}_0),\qquad(30)$$

$$\varphi^*(\boldsymbol{r}_2)=\{-k^{*2}M_0^*\varepsilon_1M_0^*+k^{*4}[M_0^*\varepsilon_1M_0^*\varepsilon_1M_0^*-\langle M_0^*\varepsilon_1M_0^*\varepsilon_1M_0^*\rangle]-\\-k^{*6}M_0^*\varepsilon_1M_0^*\varepsilon_1M_0^*\varepsilon_1M_0^*+\ldots\}\,\delta(\boldsymbol{r}_2-\boldsymbol{r}_0).\qquad(31)$$

Multiplying (30) by (31) and averaging, we get

$$\langle\varphi(\boldsymbol{r}_1)\varphi^*(\boldsymbol{r}_2)\rangle=\{|k|^4\langle(M_0\varepsilon_1M_0)(M_0^*\varepsilon_1M_0^*)\rangle+\\+k^2k^{*6}\langle(M_0\varepsilon_1M_0)(M_0^*\varepsilon_1M_0^*\varepsilon_1M_0^*\varepsilon_1M_0^*)\rangle+\\+|k|^8\langle(M_0\varepsilon_1M_0\varepsilon_1M_0)(M_0^*\varepsilon_1M_0^*\varepsilon_1M_0^*)\rangle-\\-|k|^8\langle(M_0\varepsilon_1M_0\varepsilon_1M_0)\rangle\langle(M_0^*\varepsilon_1M_0^*\varepsilon_1M_0^*)\rangle+\\+k^6k^{*2}\langle(M_0\varepsilon_1M_0\varepsilon_1M_0\varepsilon_1M_0)(M_0^*\varepsilon_1M_0^*)\rangle+\ldots\}\,\delta(\boldsymbol{r}_1-\boldsymbol{r}_0)\,\delta(\boldsymbol{r}_2-\boldsymbol{r}_0).\qquad(32)$$

In (32) the group of operators enclosed in parentheses does not act on the functions ε_1 in the other pair of parentheses. The groups of operators M_0 act on $\delta(\boldsymbol{r}_1-\boldsymbol{r}_0)$ and the operators M_0^* act on $\delta(\boldsymbol{r}_2-\boldsymbol{r}_0)$. We represent expression (32) by a diagram where the upper lines and vertices represent the functions G_0 and factors k^2, and the lower represent G_0^* and k^{*2}, respectively. Then (32) is represented by the diagram (the dashed line, as before, represents $B_\varepsilon(\boldsymbol{r}_1,\ \boldsymbol{r}_2)$)



(33)

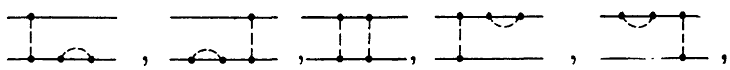
and

$$\langle\varphi(\boldsymbol{r}_1)\varphi^*(\boldsymbol{r}_2)\rangle=\iint W(\boldsymbol{r}_1,\ \boldsymbol{r}_2;\ \rho_0,\ \rho_0')\,\delta(\rho_0-\boldsymbol{r}_0)\,\delta(\rho_0'-\boldsymbol{r}_0)\,d^3\rho_0\,d^3\rho_0'=W(\boldsymbol{r}_1,\ \boldsymbol{r}_2;\ \boldsymbol{r}_0,\ \boldsymbol{r}_0).$$

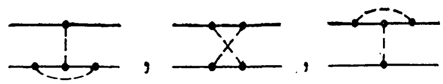
Since we have isolated the mean $\langle\psi\rangle$, all the so-called uncoupled diagrams of the form

,

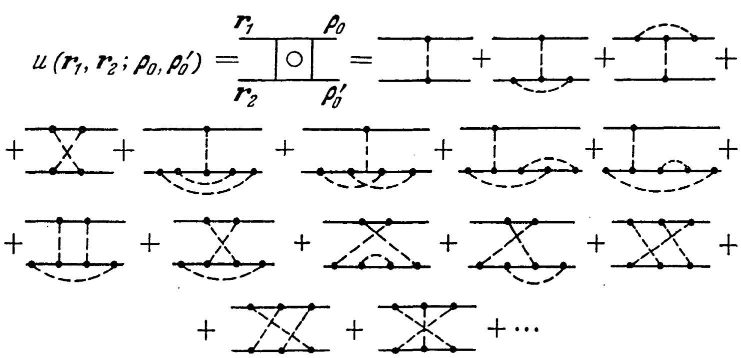
containing separate, unjoined elements, drop out from this diagram series. The diagram for $\langle\varphi(r_1)\,\varphi^*(r_2)\rangle$ clearly shows the rules for constructing the higher terms of the expansion. To construct diagrams of order $2n$, we should place one vertex on the top line, $(2n-1)$ vertices on the bottom line, and join them in all the possible ways: then we take two vertices on the top line and $(2n-2)$ vertices on the bottom line, etc.; at every stage only coupled diagrams are retained. The diagrams representing $\langle\varphi(r_1)\varphi^*(r_2)\rangle$ can also be classified into weakly coupled and strongly coupled. A diagram is said to be weakly coupled if it can be separated into two elements, each with at least two vertices, by a single break in the two solid lines (G_0 and G_0^*). Otherwise, the diagram is called strongly coupled. Of the fourth-order diagrams, the weakly coupled diagrams are



and strongly coupled diagrams are



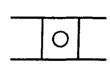
From W we can isolate a subseries of strongly coupled diagrams:

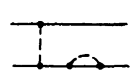


(34)

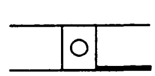
(some of the strongly coupled sixth-order diagrams can be obtained from those shown in (34) by rotating the plane of the drawing through 180° about the horizontal or the vertical axis).

Weakly coupled diagrams can be obtained from strongly coupled diagrams in one of the following ways.

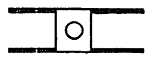
1. By replacing one of the outer lines in  with the thick line of the averaged Green's function . For example, the diagram



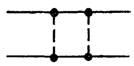
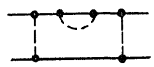
is one of the elements in the combination

 ,

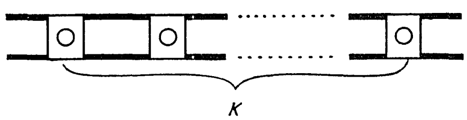
if in  we choose the element  and in the element . All weakly coupled diagrams containing a single strongly coupled element can be represented as one of the terms in the series



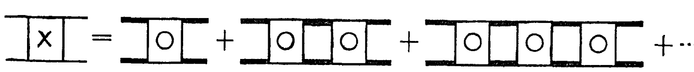
2. Diagrams containing only two strongly coupled elements are elements of the series

For example, the diagram  is obtained if in  we choose  and in the line . The diagram  is obtained if for the upper averaged line we choose the diagram  and for the other elements the simple line .

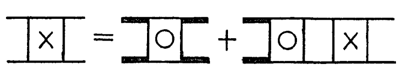
3. Similarly, if the diagram contains k strongly coupled elements, it belongs to



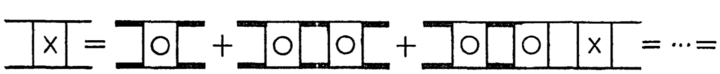
The above considerations lead to the expansion

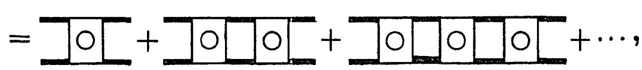
 (35)

From expansion (35) we can readily obtain an analog of Dyson's equation:

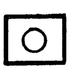
 (36)

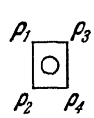
Indeed, solving (36) by successive iterations, we find





i.e., equality (35).

Let us write equation (36) in analytical form. We introduce the function , which has already been used in deriving (36). If

 $= P(\rho_1,\rho_2;\rho_3,\rho_4),$ (37)

then P is found from the equality

$$u(r_1, r_2; \rho_0, \rho_0') = \iiiint G_0(r_1, \rho_1)\, G_0^*(r_2, \rho_2)\, P(\rho_1, \rho_2; \rho_3, \rho_4) \times$$
$$\times G_0(\rho_3, \rho_0)\, G_0^*(\rho_4, \rho_0')\, d^3\rho_1\, d^3\rho_2\, d^3\rho_3\, d^3\rho_4.$$

Since by definition (34)

$$u(\boldsymbol{r}_1, \boldsymbol{r}_2; \rho_0, \rho_0') =$$
$$= |k|^4 \iint G_0(\boldsymbol{r}_1, \rho_1) G_0^*(\boldsymbol{r}_2, \rho_2) B_\varepsilon(\rho_1, \rho_2) G_0(\rho_1, \rho_0) G_0^*(\rho_2, \rho_0') d^3\rho_1 d^3\rho_2 +$$
$$+ k^{*6} k^2 \iiiint G_0(\boldsymbol{r}_1, \rho_1) G_0^*(\boldsymbol{r}_2, \rho_2) G_0^*(\rho_2, \rho_3) G_0^*(\rho_3, \rho_4) G_0^*(\rho_4, \rho_0') \times$$
$$\times G_0(\rho_1, \rho_0) B_\varepsilon(\rho_1, \rho_3) B_\varepsilon(\rho_2, \rho_4) d^3\rho_1 d^3\rho_2 d^3\rho_3 d^3\rho_4 + \ldots, \tag{34a}$$

we have

$$P(\boldsymbol{r}_1, \boldsymbol{r}_2; \boldsymbol{r}_3, \boldsymbol{r}_4) = |k|^4 \delta(\boldsymbol{r}_1 - \boldsymbol{r}_3) \delta(\boldsymbol{r}_2 - \boldsymbol{r}_4) B_\varepsilon(\boldsymbol{r}_1, \boldsymbol{r}_2) +$$
$$+ k^2 k^{*6} \int \delta(\boldsymbol{r}_1 - \boldsymbol{r}_3) G_0^*(\boldsymbol{r}_2, \rho) G_0^*(\boldsymbol{r}_4, \rho) B_\varepsilon(\boldsymbol{r}_1, \rho) B_\varepsilon(\boldsymbol{r}_2, \boldsymbol{r}_4) d^3\rho + \ldots \tag{37a}$$

Using (37), we write equation (36) in analytical form:

$$W(\boldsymbol{r}_1, \boldsymbol{r}_2; \rho_0, \rho_0') = \iiiint \widetilde{G}(\boldsymbol{r}_1, \rho_1) \widetilde{G}^*(\boldsymbol{r}_2, \rho_2) P(\rho_1, \rho_2; \rho_3, \rho_4) \times$$
$$\times \widetilde{G}(\rho_3, \rho_0) \widetilde{G}^*(\rho_4, \rho_0') d^3\rho_1 d^3\rho_2 d^3\rho_3 d^3\rho_4 +$$
$$+ \iiiint \widetilde{G}(\boldsymbol{r}_1, \rho_1) \widetilde{G}^*(\boldsymbol{r}_2, \rho_2) P(\rho_1, \rho_2; \rho_3, \rho_4) W(\rho_3, \rho_4; \rho_0, \rho_0') d^3\rho_1 \ldots d^3\rho_4. \tag{36a}$$

This equation is the analog of the Bethe – Salpeter equation in quantum field theory.

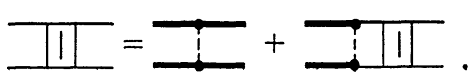
In contrast to the analogous equation (22), equation (36a) in general cannot be solved analytically. Both equation (22) and (36a) are essentially unclosed, as they contain a function which is defined as an infinite series (Q or P). If we can manage, as in solving Dyson's equation, to express the solution W of equation (36a) in terms of a given function P, then even by retaining only the first term in the series for P we end up with a summation over some infinite subseries from (33). If, e.g., in (36a) we take

$$P(\rho_1, \rho_2; \rho_3, \rho_4) \approx P_1(\rho_1, \rho_2; \rho_3, \rho_4) = |k|^4 \delta(\rho_1 - \rho_3) \delta(\rho_2 - \rho_4) B_\varepsilon(\rho_1 - \rho_2)$$

(the first term of the series (37a)), then (36a) takes the form

$$W_1(\boldsymbol{r}_1, \boldsymbol{r}_2; \rho_0, \rho_0') =$$
$$= |k|^4 \iint \widetilde{G}(\boldsymbol{r}_1, \rho_1) \widetilde{G}^*(\boldsymbol{r}_2, \rho_2) B_\varepsilon(\rho_1 - \rho_2) \widetilde{G}(\rho_1, \rho_0) \widetilde{G}^*(\rho_2, \rho_0') d^3\rho_1 d^3\rho_2 +$$
$$+ |k|^4 \iint \widetilde{G}(\boldsymbol{r}_1, \rho_1) \widetilde{G}^*(\boldsymbol{r}_2, \rho_2) B_\varepsilon(\rho_1 - \rho_2) W_1(\rho_1, \rho_2; \rho_0, \rho_0') d^3\rho_1 d^3\rho_2. \tag{38}$$

Using diagrams



(38a)

Solving (38a) by diagram iterations, we obtain

(39)

The solution of equation (38) is thus a sum of the diagrams of the "ladder" approximation to the Bethe—Salpeter equation. Equation (38) will be solved in what follows for a particular form of the function $B_\varepsilon(\boldsymbol{r})$.

§ 61. Equations in variational derivatives for the mean Green's function and the correlation function. The vertex function

The analysis of the previous section applies to the particular case of a normal probability distribution for ε_1. Let us now consider a more general case and establish its relation to the method of the previous section.

A random function $\varepsilon_1(\boldsymbol{r})$ is fully defined if its characteristic functional is given

$$\Phi[v(\boldsymbol{\rho})] = \left\langle \exp i \int v(\boldsymbol{\rho})\, \varepsilon_1(\boldsymbol{\rho})\, d^3\rho \right\rangle. \tag{1}$$

All the moments of the random field $\varepsilon_1(\boldsymbol{\rho})$ can be obtained as the variational derivatives* of (1) for $v = 0$:

$$\langle \varepsilon_1(\boldsymbol{r})\rangle = \frac{1}{i}\frac{\delta\Phi[v(\rho)]}{\delta v(\boldsymbol{r})}\bigg|_{v=0}, \quad \langle \varepsilon_1(\boldsymbol{r}_1)\,\varepsilon_1(\boldsymbol{r}_2)\rangle = \frac{1}{i^2}\frac{\delta^2\Phi[v(\rho)]}{\delta v(\boldsymbol{r}_1)\,\delta v(\boldsymbol{r}_2)}\bigg|_{v=0}.$$

Consider the functional

$$G[v; \boldsymbol{r}] \equiv \langle \psi(\boldsymbol{r})\, e^{i\int v(\rho)\,\varepsilon_1(\rho)\, d^3\rho}\rangle. \tag{2}$$

The function $\psi(\boldsymbol{r})$, appearing as the solution of equation (60.1) in which $\varepsilon_1(\boldsymbol{r})$ also appears, is statistically related to ε_1 and therefore cannot be taken outside the averaging brackets in (2). The mean $\langle\psi(\boldsymbol{r})\rangle$ can be defined using (2) as $G[0; \boldsymbol{r}]$.

Multiplying the equation

$$L_0(\boldsymbol{r})\,\psi = -k^2\varepsilon_1(\boldsymbol{r})\,\psi(\boldsymbol{r}) + \delta(\boldsymbol{r}-\boldsymbol{r}_0)$$

by $\exp\left\{i\int v(\boldsymbol{\rho})\,\varepsilon_1(\boldsymbol{\rho})\,d^3\rho\right\}$ and averaging, we obtain

$$L_0(\boldsymbol{r})\, G[v; \boldsymbol{r}] = -k^2\langle \varepsilon_1(\boldsymbol{r})\, e^{i\int v(\rho)\,\varepsilon_1(\rho)\,d^3\rho}\psi(\boldsymbol{r})\rangle + \delta(\boldsymbol{r}-\boldsymbol{r}_0)\,\Phi[v]. \tag{3}$$

Differentiation of (2) gives

$$\frac{1}{i}\frac{\delta G[v; \boldsymbol{r}]}{\delta v(\boldsymbol{r})} = \langle \varepsilon_1(\boldsymbol{r})\,\psi(\boldsymbol{r})\, e^{i\int v\varepsilon_1 d^3\rho}\rangle. \tag{4}$$

Inserting (4) in (3), we obtain the equation

$$L_0(\boldsymbol{r})\, G[v; \boldsymbol{r}] = ik^2\frac{\delta G[v; \boldsymbol{r}]}{\delta v(\boldsymbol{r})} + \delta(\boldsymbol{r}-\boldsymbol{r}_0)\,\Phi[v] \tag{5}$$

for the functional $G[v; \boldsymbol{r}]$.

* The definition and some fundamental operations with variational derivatives will be found in the Appendix at the end of this chapter.

Let

$$G[v; \boldsymbol{r}] = g[v; \boldsymbol{r}]\,\Phi[v], \quad G[0; \boldsymbol{r}] = g[0; \boldsymbol{r}] = \langle\psi(\boldsymbol{r})\rangle. \tag{6}$$

Using the equalities

$$L_0(\boldsymbol{r})\,G[v; \boldsymbol{r}] = \Phi[v]\,L_0(\boldsymbol{r})\,g[v; \boldsymbol{r}],$$
$$\frac{\delta\Phi g[v; \boldsymbol{r}]}{\delta v(\boldsymbol{r})} = \Phi\frac{\delta g[v; \boldsymbol{r}]}{\delta v(\boldsymbol{r})} + g[v; \boldsymbol{r}]\frac{\delta\Phi}{\delta v(\boldsymbol{r})},$$
$$\frac{1}{\Phi}\frac{\delta\Phi[v]}{\delta v(\boldsymbol{r})} = \frac{\delta\ln\Phi}{\delta v(\boldsymbol{r})},$$

and dividing (5) by Φ, we obtain

$$ik^2\frac{\delta g[v; \boldsymbol{r}]}{\delta v(\boldsymbol{r})} + ik^2 g[v; \boldsymbol{r}]\frac{\delta\ln\Phi}{\delta v(\boldsymbol{r})} - L_0(\boldsymbol{r})\,g[v; \boldsymbol{r}] + \delta(\boldsymbol{r}-\boldsymbol{r}_0) = 0. \tag{7}$$

Equation (7) is the analog of Schwinger's equation. Unlike the equations of the preceding section, (7) is a closed equation which contains only one unknown functional.

The equation for the correlation function is derived in a similar manner. We introduce the functional

$$H[v; \boldsymbol{r}_1, \boldsymbol{r}_2] = \langle\psi(\boldsymbol{r}_1)\,\psi^*(\boldsymbol{r}_2)\,e^{i\int v\varepsilon_1 d^3\rho}\rangle. \tag{8}$$

Differentiating (8), we find

$$\frac{1}{i}\frac{\delta H[v; \boldsymbol{r}_1, \boldsymbol{r}_2]}{\delta v(\boldsymbol{r}_1)} = \langle\varepsilon_1(\boldsymbol{r}_1)\,\psi(\boldsymbol{r}_1)\,\psi^*(\boldsymbol{r}_2)\,e^{i\int v\varepsilon_1 d^3\rho}\rangle. \tag{9}$$

Multiplying the equation for $\psi(\boldsymbol{r})$ by

$$\psi^*(\boldsymbol{r}')\,e^{i\int v(\rho)\,\varepsilon_1(\rho)\,d^3\rho},$$

averaging and using (8) and (9), we find

$$L_0(\boldsymbol{r})\,H[v; \boldsymbol{r}, \boldsymbol{r}'] = ik^2\frac{\delta H[v; \boldsymbol{r}, \boldsymbol{r}']}{\delta v(\boldsymbol{r})} + \delta(\boldsymbol{r}-\boldsymbol{r}_0)\,\langle\psi^*(\boldsymbol{r}')\,e^{i\int v(\rho)\,\varepsilon_1(\rho)\,d^3\rho}\rangle. \tag{10}$$

From (2)

$$G^*[-v; \boldsymbol{r}'] = \langle\psi^*(\boldsymbol{r}')\,e^{i\int v(\rho)\,\varepsilon_1(\rho)\,d^3\rho}\rangle.$$

Therefore equation (10) can be written in the form

$$ik^2\frac{\delta H[v; \boldsymbol{r}, \boldsymbol{r}']}{\delta v(\boldsymbol{r})} - L_0(\boldsymbol{r})\,H[v; \boldsymbol{r}, \boldsymbol{r}'] + \delta(\boldsymbol{r}-\boldsymbol{r}_0)\,G^*[-v; \boldsymbol{r}'] = 0. \tag{11}$$

Let

$$H[v; \boldsymbol{r}, \boldsymbol{r}'] = \Phi[v]\cdot h[v; \boldsymbol{r}, \boldsymbol{r}'];$$
$$H[0; \boldsymbol{r}, \boldsymbol{r}'] = h[0; \boldsymbol{r}, \boldsymbol{r}'] = \langle\psi(\boldsymbol{r})\,\psi^*(\boldsymbol{r}')\rangle. \tag{12}$$

Inserting (12) in (11) and seeing that

$$G^*[-v; \boldsymbol{r}'] = g^*[-v; \boldsymbol{r}']\,\Phi^*[-v] = g^*[-v; \boldsymbol{r}']\,\Phi[v],$$

we obtain after dividing through by Φ

$$ik^2 \frac{\delta h[v; \boldsymbol{r}, \boldsymbol{r}']}{\delta v(\boldsymbol{r})} + ik^2 h[v; \boldsymbol{r}, \boldsymbol{r}'] \frac{\delta \ln \Phi}{\delta v(\boldsymbol{r})} - L_0(\boldsymbol{r}) h[v; \boldsymbol{r}, \boldsymbol{r}'] + \delta(\boldsymbol{r} - \boldsymbol{r}_0) g^*[-v; \boldsymbol{r}'] = 0. \tag{13}$$

In equations (7) and (13) the characteristic functional enters only as a logarithmic derivative. This is one of the main advantages of equations (7) and (13) as compared to equations (5) and (11). Indeed, according to definition (1) we have $|\Phi| \leqslant 1$, since we are averaging a function whose absolute value is 1. Therefore the expansion of Φ in powers of v should always contain an infinite number of terms, as a finite degree polynomial is always unbounded for $v \to \infty$. On the other hand, $\ln \Phi$ is not required to remain bounded.

As an example, consider the case of a normally distributed random field $\varepsilon_1(\rho)$. In this case,

$$z = i \int \varepsilon_1(\rho) v(\rho) d^3\rho$$

is also a normally distributed random variable. Let us calculate $\langle \exp z \rangle$. For any normally distributed random variable z with zero mean we have $\langle \exp z \rangle = \exp\left(\frac{1}{2}\langle z^2 \rangle\right)$. But

$$\langle z^2 \rangle = -\iint v(\rho_1) v(\rho_2) \langle \varepsilon_1(\rho_1) \varepsilon_1(\rho_2) \rangle d^3\rho_1 d^3\rho_2 = -\iint B_\varepsilon(\rho_1, \rho_2) v(\rho_1) v(\rho_2) d^3\rho_1 d^3\rho_2. \tag{14}$$

Therefore, for a normally distributed random field

$$\Phi[v] = e^{-\frac{1}{2}\iint B_\varepsilon(\rho_1, \rho_2) v(\rho_1) v(\rho_2) d^3\rho_1 d^3\rho_2}, \tag{15}$$

$$\ln \Phi[v] = -\frac{1}{2}\iint B_\varepsilon(\rho_1, \rho_2) v(\rho_1) v(\rho_2) d^3\rho_1 d^3\rho_2. \tag{16}$$

If the random field ε_1 is not normally distributed, $\ln \Phi$ may contain, besides the second order term, higher powers of v (expansion in cumulants).

Let us consider some approximate methods for solving equation (7). The simplest method is that of successive iterations. We act on (7) with the operator M_0:

$$g[v; \boldsymbol{r}] = G_0(\boldsymbol{r}, \boldsymbol{r}_0) + ik^2 \int G_0(\boldsymbol{r}, \rho) \left\{ g[v, \rho] \frac{\delta \ln \Phi}{\delta v(\rho)} + \frac{\delta g[v; \rho]}{\delta v(\rho)} \right\} d^3\rho. \tag{17}$$

Solution of equation (17) by successive iterations is equivalent to the ordinary perturbation theory; However, in contrast to the treatment of the previous section, we can now readily deal with the case of ε_1 with an arbitrary probability distribution.

Consider another approximate technique based on a series expansion of the desired function $g[v; \boldsymbol{r}]$:

$$g[v; \boldsymbol{r}] = g_0(\boldsymbol{r}) + \int g_1(\boldsymbol{r}, \rho) v(\rho) d^3\rho + \iint g_2(\boldsymbol{r}; \rho_1, \rho_2) v(\rho_1) v(\rho_2) d^3\rho_1 d^3\rho_2 + \ldots \tag{18}$$

In the expansion $\delta \ln \Phi / \delta v(r)$, besides the term contributed by expression (16) for a normally distributed field, we also retain the higher terms in the expansion, using the correlation function of the third (F) and higher orders:

$$\frac{\delta \ln \Phi}{\delta v(r)} = -\int B_\varepsilon(r, \rho) v(\rho) d^3\rho + i \iint F(r, \rho_1, \rho_2) v(\rho_1) v(\rho_2) d^3\rho_1 d^3\rho_2 + \ldots \quad (19)$$

Inserting expansions (18) and (19) in equation (7), we equate to zero the collected coefficients of equal powers of v (in higher terms of the expansion we need consider only that part of the integrand which is symmetric in all the variables entering the factors $v(\rho_i)$). We thus obtain a set of equations for g_i:

$$L_0(r) g_0(r) = \delta(r - r_0) + ik^2 g_1(r, r), \quad (20)$$

$$L_0(r) g_1(r, \rho) = -ik^2 B_\varepsilon(r, \rho) g_0(r) + 2ik^2 g_2(r; r, \rho), \quad (21)$$

$$L_0(r) g_2(r; \rho_1, \rho_2) = -ik^2 [B_\varepsilon(r, \rho_1) g_1(r, \rho_2) + B_\varepsilon(r, \rho_2) g_1(r, \rho_1)] + 3ik^2 g_3(r; r, \rho_1, \rho_2) - k^2 g_0(r) F(r, \rho_1, \rho_2), \quad (22)$$

. . . .

Equation (20) for g_0 contains the function g_1, equation (21) for g_1 contains g_2, and so on. We have thus obtained an infinite chain of coupled equations. The simplest method of obtaining an approximate solution is by quite arbitrarily setting one of the functions g_k equal to zero. The higher the index k, the more accurate is the approximation. Note that since we have the expansion

$$G[v; r] = \left[g_0(r) + \int g_1(r, \rho) v(\rho) d^3\rho + \ldots\right] \Phi[v],$$

the assumption $g_k = 0$ does not mean that we take the correlation $\langle \varepsilon_1^k(r) \psi(r) \rangle$ and higher order correlations equal to zero: we only impose a certain additional constraint on these correlation functions.

Setting $g_1 = 0$, we obtain from (20) $g_0 = G_0(r, r_0)$, i.e., a solution for a medium without fluctuations. Setting $g_2 = 0$, we obtain the set of equations

$$L_0(r) g_0(r) = \delta(r - r_0) + ik^2 g_1(r, r), \quad (20a)$$

$$L_0(r) g_1(r, \rho) = -ik^2 B_\varepsilon(r, \rho) g_0(r). \quad (21a)$$

By (21a)

$$g_1(r, \rho) = -ik^2 \int G_0(r, \rho') B_\varepsilon(\rho', \rho) g_0(\rho') d^3\rho',$$

$$g_1(\rho, \rho) = -ik^2 \int G_0(\rho, \rho') B_\varepsilon(\rho', \rho) g_0(\rho') d^3\rho'. \quad (23)$$

Writing (20a) in integral form (using the operator M_0)

$$g_0(r) = G_0(r - r_0) + ik^2 \int G_0(r, \rho) g_1(\rho, \rho) d^3\rho, \quad (24)$$

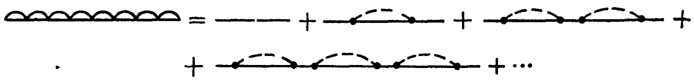
and inserting (23) in (24), we find

$$g_0(r, r_0) = G_0(r - r_0) + k^4 \int\int G_0(r, \rho) G_0(\rho, \rho') B_\varepsilon(\rho', \rho) g_0(\rho', r_0) d^3\rho d^3\rho'. \quad (25)$$

Here r_0 has been incorporated in the argument of the function $g_0(r)$, since clearly g_0 also depends on r_0. According to the diagram technique, taking $g_0(r, r_0) \sim$ , we obtain for equation (25)

 (25a)

Solving (25a) by successive iterations, we get

 (26)

Comparing this diagram with the diagram of the function G_1 on p. 343, we see that $g_0 = G_1$, i.e.,

$$g_0(\boldsymbol{r}, \boldsymbol{r}_0) = G_1(\boldsymbol{r} - \boldsymbol{r}_0) = \frac{1}{8\pi^3} \int \frac{e^{i\varkappa(\boldsymbol{r}-\boldsymbol{r}_0)} d^3\varkappa}{k^2 - \varkappa^2 - k^4 \int G_0(\rho) B_\varepsilon(\rho) e^{-i\varkappa\rho} d^3\rho}. \qquad (26a)$$

(Clearly (26a) can also be found as the solution of equation (25).)

Thus, having taken $g_2 = 0$, we arrived at the same approximation as in the previous section for $Q = k^4 G_0 B_\varepsilon$. Note, however, that the next approximation, corresponding to $g_3 = 0$, is no longer equivalent to retaining still another term in the mass operator Q; it rather leads to a summation over a more extensive set of diagrams and, in general, accounts for deviations from the normal distribution, as the solution now depends on F.

We should consider still another point. In deriving (26a) we did not assume a normal probability distribution of the random field ε_1, since the deviation of ε_1 from a normally distributed field, described by the function $F(\boldsymbol{r}, \rho_1, \rho_2)$, did not affect the solution in any way. However, the conditions of applicability of the solution (26), which will be considered below, depend on whether or not the probability distribution of ε_1 is close to the normal distribution.

The same technique, i.e., the expansion of $h[v; \boldsymbol{r}, \boldsymbol{r}']$ in powers of v, can be applied to solve equation (13). Taking $h_2 = 0$, we obtain a set of equations that can be solved by Fourier transforms. However, the corresponding solution does not satisfy the obvious requirement

$$H^*[v; \boldsymbol{r}_1, \boldsymbol{r}_2] = H[-v; \boldsymbol{r}_2, \boldsymbol{r}_1]$$

and should therefore be further improved. The correlation of the field will be considered later on in the "ladder" approximation of the Bethe – Salpeter equation.

In concluding this section we give a derivation of the equation relating the mass operator to the so-called vertex function. To this end, we replace the functional argument $v(\rho)$ in equation (17) by a new argument $u(\rho)$ defined by

$$u(\rho) = \frac{1}{i} \frac{\delta \ln \Phi[v]}{\delta v(\rho)}. \qquad (27)$$

Note that the equality $\langle \varepsilon_1(\rho) \rangle = 0$ shows that $u(\rho) = 0$ for $v(\rho) = 0$. For a normal probability distribution, u is a linear functional of v:

$$u(\rho) = i \int B_\varepsilon(\rho, \rho') v(\rho') d^3\rho'. \qquad (28)$$

In general, however, $u(\rho)$ is a nonlinear function of v.

Let us transform the variational derivative with respect to v in (17) to a derivative with respect to u. Using the general relation

$$\frac{\delta g[v;\rho,r_0]}{\delta v(\rho)}=\int\frac{\delta g[u;\rho,r_0]}{\delta u(\rho')}\frac{\delta u[v;\rho']}{\delta v(\rho)}d^3\rho'$$

and substituting (27), we obtain

$$\frac{\delta g[v;\rho,r_0]}{\delta v(\rho)}=\frac{1}{i}\int\frac{\delta^2\ln\Phi[v]}{\delta v(\rho)\,\delta v(\rho')}\frac{\delta g[u;\rho,r_0]}{\delta u(\rho')}d^3\rho'. \tag{29}$$

The functional $\delta^2\ln\Phi/\delta v(\rho)\;\delta v(\rho')$ in this equation will be denoted

$$-\frac{\delta^2\ln\Phi[v]}{\delta v(\rho)\,\delta v(\rho')}=B_1[v;\rho,\rho'].$$

For $v=0$ it reduces to the correlation function $B_\varepsilon(\rho,\rho')$ of the random field $\varepsilon_1(\rho)$. For a normal probability distribution $B_1[v;\rho,\rho']$ is simply a correlation function and is independent of v. If using (27) we change over to a new argument u in $B_1[v;\rho,\rho']$, the resulting functional will be denoted $B[u;\rho,\rho']$. Like $B_1[v;\rho,\rho']$, it reduces to the correlation function $B_\varepsilon(\rho,\rho')$ for $u=0$. After substitution of the new argument, equation (17) takes the form

$$g[u;r,r_0]=G_0(r,r_0)-k^2\int G_0(r,\rho)g[u;\rho,r_0]u(\rho)d^3\rho-$$
$$-k^2\iint G_0(r,\rho)\frac{\delta g[u;\rho,r_0]}{\delta u(\rho')}B[u;\rho,\rho']d^3\rho\;d^3\rho'. \tag{30}$$

Let $g^{-1}[u;\rho,r_0]$ be the inverse functional of $g[u;r,\rho]$, i.e.,

$$\int g[u;r,\rho]g^{-1}[u;\rho,r_0]d^3\rho=\delta(r-r_0). \tag{31}$$

Moreover, let

$$\int g'^{-1}[u;r,\rho']g[u;\rho',\rho]d^3\rho'=\delta(r-\rho).$$

Multiplying the last relation by $g^{-1}[u;\rho,r_1]$, integrating over ρ, and using (31), we get

$$g^{-1}[u;r_1,r_2]=g'^{-1}[u;r_1,r_2],$$

from which it follows that

$$\int g^{-1}[u;r,\rho']g[u;\rho',\rho]d^3\rho'=\delta(r-\rho). \tag{31a}$$

Differentiation of equality (31) gives

$$\int\frac{\delta g[u;r,\rho]}{\delta u(\rho')}g^{-1}[u;\rho,r_0]d^3\rho=-\int g[u;r,\rho]\frac{\delta g^{-1}[u;\rho,r_0]}{\delta u(\rho')}d^3\rho.$$

We multiply this relation by $g[u;r_0,r_1]$ and integrate over r_0. Using (31a), we obtain for $\delta g[u;\rho,r_0]/\delta u(\rho')$ the expression

$$\frac{\delta g[u;\rho,r_0]}{\delta u(\rho')}=-\iint g[u;\rho,r']\frac{\delta g^{-1}[u;r',\rho'']}{\delta u(\rho')}g[u;\rho'',r_0],d^3r'\,d^3\rho''. \tag{32}$$

We define a vertex operator $\Gamma[u;\ \boldsymbol{r}',\ \rho'';\ \rho']$ by the relation

$$\Gamma[u;\ \boldsymbol{r}';\ \rho'';\ \rho'] = \frac{\delta g^{-1}[u;\ \boldsymbol{r}',\ \rho'']}{\delta u(\rho')}. \tag{33}$$

Inserting (32) and (33) in (30), we obtain the equation

$$\begin{gathered} g[u;\ \boldsymbol{r},\ \boldsymbol{r}_0] = G_0(\boldsymbol{r},\ \boldsymbol{r}_0) - k^2 \int G_0(\boldsymbol{r},\ \rho)\, u(\rho)\, g[u;\ \rho,\ \boldsymbol{r}_0]\, d^3\rho + \\ + k^2 \iiiint G_0(\boldsymbol{r},\ \rho)\, g[u;\ \rho,\ \rho_1]\, \Gamma[u;\ \rho_1,\ \rho'';\ \rho']\, g[u;\ \rho'',\ \boldsymbol{r}_0] \times \\ \times B[u;\ \rho,\ \rho']\, d^3\rho\, d^3\rho'\, d^3\rho_1\, d^3\rho''. \end{gathered} \tag{34}$$

Setting $u = 0$, we obtain the relation

$$\begin{gathered} \widetilde{G}(\boldsymbol{r},\ \boldsymbol{r}_0) = G_0(\boldsymbol{r},\ \boldsymbol{r}_0) + k^2 \iiiint G_0(\boldsymbol{r},\ \rho)\, \widetilde{G}(\rho,\ \rho_1) \times \\ \times \Gamma[0;\ \rho_1,\ \rho;\ \rho']\, \widetilde{G}(\rho'',\ \boldsymbol{r}_0)\, B_\varepsilon(\rho,\ \rho')\, d^3\rho\, d^3\rho'\, d^3\rho_1\, d^3\rho''. \end{gathered} \tag{35}$$

The vertex operator (33) with its functional argument equal to zero is called a vertex function, and for it we introduce the notation

$$\Gamma(\boldsymbol{r}_1,\ \boldsymbol{r}_2;\ \boldsymbol{r}_3) = \Gamma[0;\ \boldsymbol{r}_1,\ \boldsymbol{r}_2;\ \boldsymbol{r}_3].$$

Comparing equation (35) with equation (60.22), which was derived earlier by the diagram technique, we obtain a relation between the mass operator $Q(\rho_1,\ \rho_2)$ and the vertex function $\Gamma(\boldsymbol{r}_1, \boldsymbol{r}_2;\ \boldsymbol{r}_3)$:

$$Q(\rho_1,\ \rho_2) = k^2 \iint \widetilde{G}(\rho_1,\ \rho')\, \Gamma(\rho',\ \rho_2;\ \rho'')\, B_\varepsilon(\rho_1,\ \rho'')\, d^3\rho'\, d^3\rho''. \tag{36}$$

It is also advisable to define the mass operator for values of u other than zero. To this end, we write equation (34) in a form analogous to (60.22) but for $u \neq 0$:

$$g[u;\boldsymbol{r},\ \boldsymbol{r}_0] = G_0(\boldsymbol{r},\ \boldsymbol{r}_0) + \iint G_0(\boldsymbol{r},\rho)\, q[u;\rho,\rho']\, g[u;\ \rho',\ \boldsymbol{r}_0]\, d^3\rho\, d^3\rho'. \tag{37}$$

Comparing (37) with (34) we obtain for $q[u;\ \boldsymbol{r},\ \boldsymbol{r}_1]$:

$$q[u;\ \boldsymbol{r},\ \boldsymbol{r}_1] = k^2 \iint g[u;\boldsymbol{r},\rho]\, \Gamma[u;\rho,\boldsymbol{r}_1;\rho']\, B[u;\boldsymbol{r},\rho']\, d^3\rho\, d^3\rho' - k^2 u(\boldsymbol{r}_1)\delta(\boldsymbol{r}-\boldsymbol{r}_1). \tag{38}$$

Multiplying equation (37) by $g^{-1}[u;\ \boldsymbol{r}_0,\ \boldsymbol{r}_1]$, integrating over $\boldsymbol{r}_0$, and making use of (31), we obtain

$$\delta(\boldsymbol{r} - \boldsymbol{r}_1) = \int G_0(\boldsymbol{r},\ \boldsymbol{r}_0)\, g^{-1}[u;\ \boldsymbol{r}_0,\ \boldsymbol{r}_1]\, d^3 r_0 + \int G_0(\boldsymbol{r},\ \rho)\, q[u;\ \rho,\ \boldsymbol{r}_1]\, d^3\rho.$$

We now act on this equation with the operator $L_0(\boldsymbol{r}) = \Delta(\boldsymbol{r}) + k^2$; taking into account that $L_0(\boldsymbol{r})\, G_0(\boldsymbol{r},\ \boldsymbol{r}') = \delta(\boldsymbol{r} - \boldsymbol{r}')$, we obtain

$$L_0(\boldsymbol{r})\, \delta(\boldsymbol{r} - \boldsymbol{r}_1) = g^{-1}[u;\ \boldsymbol{r},\ \boldsymbol{r}_1] + q[u;\ \boldsymbol{r},\ \boldsymbol{r}_1]. \tag{39}$$

This relation between $g^{-1}[u;\boldsymbol{r},\ \boldsymbol{r}_1]$ and $q[u;\boldsymbol{r},\ \boldsymbol{r}_1]$ is a generalization of (60.19). Differentiation of (39) and substitution in (33) give another expression for the vertex operator:

$$\Gamma[u;\ \boldsymbol{r}_1,\ \boldsymbol{r}_2;\ \boldsymbol{r}_3] = -\frac{\delta q[u;\ \boldsymbol{r}_1,\ \boldsymbol{r}_2]}{\delta u(\boldsymbol{r}_3)}. \tag{40}$$

Expression (40) leads to a simple relation between the diagrams of the vertex function and the mass operator in the case of a normally distributed dielectric constant. To derive this relation, we first construct the perturbation theory series for $g\,[u;\ \boldsymbol{r},\ \boldsymbol{r}_0]$. Using (60.5), we write the expression for the nonaveraged Green's function (we denote it here by $G(\boldsymbol{r},\ \boldsymbol{r}_0)$, and not by $\psi(\boldsymbol{r})$ as before):

$$G(\boldsymbol{r},\ \boldsymbol{r}_0)=G_0(\boldsymbol{r},\ \boldsymbol{r}_0)+\sum_{n=1}^{\infty}(-k^2)^n\int\ldots\int G_0(\boldsymbol{r},\ \rho_1)\,G_0(\rho_1,\ \rho_2)\ldots$$
$$\ldots G_0(\rho_{n-1},\ \rho_n)\,G_0(\rho_n,\ \boldsymbol{r}_0)\,\varepsilon_1(\rho_1)\ldots\varepsilon_1(\rho_n)\,d^3\rho_1\ldots d^3\rho_n. \qquad (41)$$

Multiplying (41) by $\exp\{i\int\varepsilon_1(\rho)\,v(\rho)\,d^3\rho\}$ and averaging, we obtain a series for $G\,[v;\ \boldsymbol{r},\ \boldsymbol{r}_0]$. Making use of the equality

$$\left\langle\varepsilon_1(\rho_1)\ldots\varepsilon_1(\rho_n)\exp\left\{i\int\varepsilon_1(\rho)\,v(\rho)\,d^3\rho\right\}\right\rangle=\frac{1}{i^n}\,\frac{\delta^n\Phi\,[v]}{\delta v(\rho_1)\ldots\delta v(\rho_n)},$$

we also obtain

$$G\,[v;\ \boldsymbol{r},\ \boldsymbol{r}_0]=G_0(\boldsymbol{r},\ \boldsymbol{r}_0)\,\Phi\,[v]+\sum_{n=1}^{\infty}(ik^2)^n\int\ldots\int G_0(\boldsymbol{r},\ \rho_1)\ldots G_0(\rho_n,\ \boldsymbol{r}_0)\times$$
$$\times\frac{\delta^n\Phi}{\delta v(\rho_1)\ldots\delta v(\rho_n)}\,d^3\rho_1\ldots d^3\rho_n.$$

Dividing by $\Phi\,[v]$, we find for $g\,[v;\ \boldsymbol{r},\ \boldsymbol{r}_0]$

$$g\,[v;\ \boldsymbol{r},\ \boldsymbol{r}_0]=G_0(\boldsymbol{r},\ \boldsymbol{r}_0)+$$
$$+\sum_{n=1}^{\infty}(ik^2)^n\int\ldots\int G_0(\boldsymbol{r}\ \rho_1)\ldots G_0(\rho_n,\ \boldsymbol{r}_0)\,\frac{1}{\Phi}\,\frac{\delta^n\Phi}{\delta v(\rho_1)\ldots\delta v(\rho_n)}\,d^3\rho_1\ldots d^3\rho_n \qquad (42)$$

Now, using relation (28), we should change over to a new argument $u(\rho)$ in (42). The simplest method is to invert (28). Suppose that a function $S(\boldsymbol{r},\ \rho)$ exists, which satisfies the equation

$$\int S(\boldsymbol{r},\ \rho)\,B_\varepsilon(\rho,\ \boldsymbol{r}')\,d^3\rho=\delta(\boldsymbol{r}-\boldsymbol{r}'). \qquad (43)$$

We should emphasize from the outset that this function does not enter the final expression, and therefore the question of its existence and its actual form are of no consequence for our treatment. By symmetry of the correlation function we see that $S(\boldsymbol{r},\ \rho)$ is also symmetric and it is thus a right, as well as a left, inverse of $B_\varepsilon(\rho,\ \boldsymbol{r}')$. Multiplying (28) by $S(\boldsymbol{r},\rho)$ and integrating (making use of (43)) we find

$$v(\boldsymbol{r})=-i\int S(\boldsymbol{r},\ \rho)\,u(\rho)\,d^3\rho. \qquad (44)$$

Inserting (44) in expression (15) for the characteristic functional, we obtain after integration

$$\Phi\,[v\,[u]]=\Phi_1\,[u]=\exp\left[\frac{1}{2}\iint S(\rho_1,\ \rho_2)\,u(\rho_1)\,u(\rho_2)\,d^3\rho_1\,d^3\rho_2\right]. \qquad (45)$$

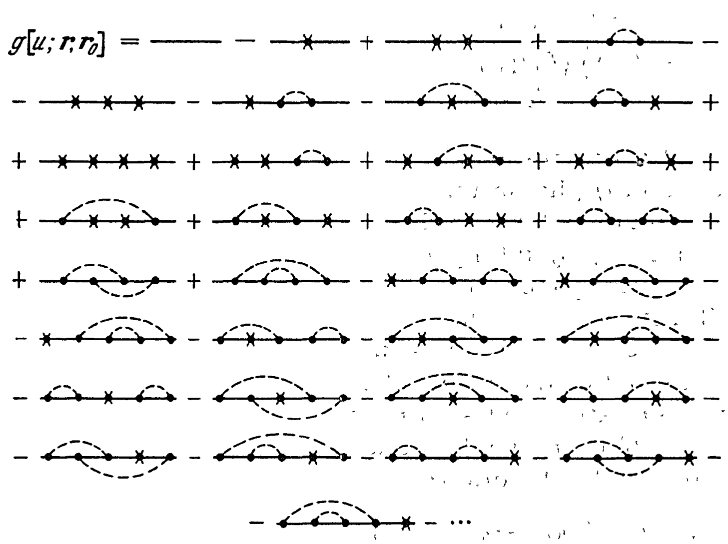
We again use equality (29) in the form

$$\frac{\delta}{\delta v(\rho_k)} = i\int d^3\rho_k' B_\varepsilon(\rho_k, \rho_k')\frac{\delta}{\delta u(\rho_k')},$$

and change from differentiation with respect to v to differentiation with respect to u. We thus obtain the following expression for $g[u;\ \boldsymbol{r},\ \boldsymbol{r}_0]$:

$$g[u;\ \boldsymbol{r},\ \boldsymbol{r}_0] = G_0(\boldsymbol{r},\ \boldsymbol{r}_0) + \sum_{n=1}^{\infty}(-k^2)^n\int\ldots\int G_0(\boldsymbol{r},\ \rho_1)G_0(\rho_1,\ \rho_2)\ldots$$
$$\ldots G_0(\rho_n,\ \boldsymbol{r}_0)B_\varepsilon(\rho_1,\ \rho_1')B_\varepsilon(\rho_2,\ \rho_2')\ldots B_\varepsilon(\rho_n,\ \rho_n')\times$$
$$\times\frac{1}{\Phi_1[u]}\frac{\delta^n\Phi_1[u]}{\delta u(\rho_1')\ldots\delta u(\rho_n')}d^3\rho_1\ldots d^3\rho_n d^3\rho_1'\ldots d^3\rho_n'. \tag{46}$$

The n-th term in series (46) is a polylinear functional of n-th degree; it contains terms independent of u, terms linear in u, etc. Therefore, if $g[u;\ \boldsymbol{r},\ \boldsymbol{r}_0]$ is expanded in a power series in functionals, each coefficient in this expansion (they are analogous to the functions $g_0,\ g_1,\ g_2,\ldots$ in (18)) will be an infinite series. We are concerned only with those terms in (46) which are independent of u and linear in u. To write out these terms in explicit form, we should carry out the differentiation in (46) and collect all the terms without u and linear in u, respectively. Differentiation of the exponential function in Φ_1 introduces the functions $S(\rho_i',\ \rho_k')$ in (46). Using (43), we carry out the corresponding integrations each time, so that the functions $S(\boldsymbol{r},\ \rho)$ drop out from the final expression. To represent the functional $g[u;\ \boldsymbol{r},\ \boldsymbol{r}_0]$ in a more visual form, we again use Feynman diagrams. A solid line corresponds as before to $G_0(\rho_i,\ \rho_j)$, a dashed line to $B_\varepsilon(\rho_i,\ \rho_j)$, a dot corresponds to a factor k^2. The factor $u(\rho)$ will be represented by a cross placed at the vertex with the corresponding argument. The functional $g[u;\ \boldsymbol{r},\ \boldsymbol{r}_0]$ is then represented by the sum of the following diagrams:



(46a)

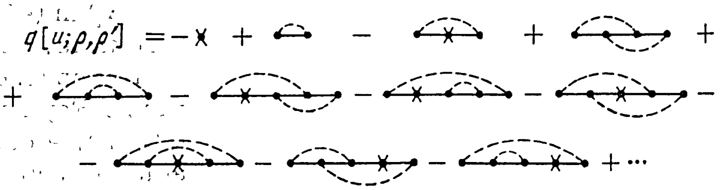
The sign in front of each diagram is $(-1)^n$, where n is the number of vertices.

Here we wrote out all terms up to the fourth order, inclusive; of the fifth-order terms, we give only those which are linear in u. The rule for drawing diagrams of any order is clear from the figure. One diagram contains n factors $u(\rho)$. Then, any two factors $u(\rho_i)$, $u(\rho_j)$ are replaced by a dashed line joining the corresponding vertices and a sum of all possible diagrams formed in this way is taken. Then another pair from among the remaining factors $u(\rho_i)$, $u(\rho_j)$ is replaced by a dashed line, and so on. Starting with an even-order diagram we eventually obtain diagrams containing only dashed lines; odd-order diagrams are reduced in this way to diagrams containing a single factor $u(\rho)$.

Let us now return to equation (37). Comparing its solution, obtained by successive iterations, with diagram (46a), we readily see that the subseries of strongly coupled diagrams corresponds to the term

$$\iint G_0(\boldsymbol{r}, \rho)\, q[u;\, \rho,\, \rho']\, G_0(\rho', \boldsymbol{r}_0)\, d^3\rho\, d^3\rho'$$

in the iteration series. The diagrams corresponding to $q[u;\, \rho,\, \rho']$ can therefore be obtained from (46a) by taking the set of all strongly coupled diagrams with the solid lines on the left and the right omitted. In what follows, we will require only the first two terms in the expansion of $q[u;\, \rho,\, \rho']$ in powers of u, which are represented by the following diagram formula:


(47)

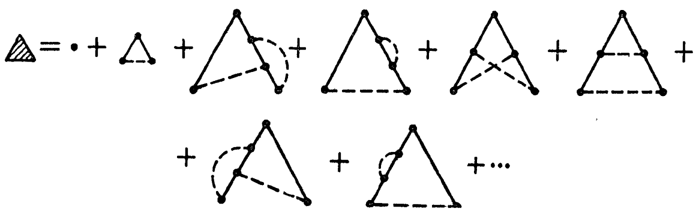
The analytical form of this expression is

$$\begin{gathered} q[u;\, \rho,\, \rho'] = Q(\rho,\, \rho') - k^2\delta(\rho-\rho')\, u(\rho) - \\ - k^6 B_\varepsilon(\rho,\, \rho') \int G_0(\rho,\, \rho_1)\, u(\rho_1)\, G_0(\rho_1,\, \rho')\, d^3\rho_1 - \\ - k^{10} \iiint G_0(\rho,\, \rho_1)\, G_0(\rho_1,\, \rho_2)\, G_0(\rho_2,\, \rho_3)\, G_0(\rho_3,\, \rho') \times \\ \times B_\varepsilon(\rho,\, \rho_3)\, B_\varepsilon(\rho_2,\, \rho')\, u(\rho_1)\, d^3\rho_1\, d^3\rho_2\, d^3\rho_3 + \ldots, \end{gathered} \tag{47a}$$

where the sum of all terms independent of u is denoted as before by $Q(\rho,\, \rho')$. Inserting (47a) in (40) (the definition of the vertex operator), we carry out the differentiation and put $u(\rho) = 0$. This gives the vertex function $\Gamma(\boldsymbol{r}_1,\, \boldsymbol{r}_2;\, \boldsymbol{r}_3)$. Clearly, by taking $u = 0$, we dropped the contribution from all diagrams containing more than a single factor $u(\rho)$. In terms of diagrams, differentiation amounts to the following: an internal vertex having a cross becomes an external uncrossed vertex with argument corresponding to that of the differential operator. The second term in (47a) after differentiation and sign reversal gives a contribution $k^2\delta(\boldsymbol{r}_1 - \boldsymbol{r}_3)\,\delta(\boldsymbol{r}_2 - \boldsymbol{r}_3)$ to the vertex function. We thus obtain the following expansion for the vertex function:

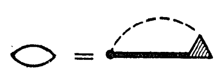
$$\begin{gathered} \Gamma(\boldsymbol{r}_1, \boldsymbol{r}_2; \boldsymbol{r}_3) = k^2\delta(\boldsymbol{r}_1-\boldsymbol{r}_3)\delta(\boldsymbol{r}_2-\boldsymbol{r}_3) + k^6 B_\varepsilon(\boldsymbol{r}_1, \boldsymbol{r}_2) G_0(\boldsymbol{r}_1, \boldsymbol{r}_3) G_0(\boldsymbol{r}_2, \boldsymbol{r}_3) + \\ + k^{10} \iint G_0(\boldsymbol{r}_1,\, \boldsymbol{r}_3)\, G_0(\boldsymbol{r}_3,\, \rho_1)\, G_0(\rho_1,\, \rho_2)\, G_0(\rho_2,\, \boldsymbol{r}_2)\, B_\varepsilon(\boldsymbol{r}_1,\, \rho_2) \times \\ \times B_\varepsilon(\rho_1,\, \boldsymbol{r}_2)\, d^3\rho_1\, d^3\rho_2 + \ldots, \end{gathered} \tag{48}$$

Graphically we have



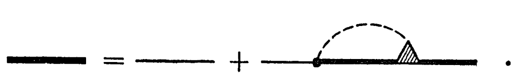
Let us now extablish a relation between diagrams for the mass operator $Q(\rho, \rho')$ and the vertex function $\Gamma(\rho, \rho'; r)$. Clearly, diagrams representing terms linear in u in the expansion of $g[u, r, r_0]$ differ from the diagrams representing $Q(r, r_0)$ by the presence of one additional crossed vertex on each solid line in the diagram for $Q(r, r_0)$. This follows from an analysis of (46a). Since differentiation of diagrams amounts to replacing a crossed internal vertex by an uncrossed external vertex, all the diagrams for the vertex function, except the first in (48a), can be derived from the diagrams of the mass operator $Q(r, r_0)$ by adding a new vertex on each line corresponding to the function $G_0(\rho_i, \rho_j)$. Thus, diagram No. 2 in (48a) is obtained from diagram No. 2 in (47), diagrams Nos. 3, 5, 7 in (48a) are generated by diagram No. 4 in (47), diagrams Nos. 4, 6, 8 in (48a) by diagram No. 5 in (47), etc.

Let us now return to expression (36), which relates the mass operator to the vertex function and the averaged Green's function. Using appropriate diagrams we may write for this expression



(36a)

We have previously derived an expression for the averaged Green's function in terms of the mass operator. Replacing the mass operator by the first term in its series expansion, we obtained the function $G_1(R)$. This approximation to G is satisfactory if the next terms in the expansion of Q may may be neglected (the necessary conditions are analyzed in some detail in the next section). If, however, the higher terms in the expansion of Q cannot be ignored, partial summation of this series is unavoidable. This problem is best solved using equation (36). Indeed, even if we retain only the first term in (48) and replace the vertex function $\Gamma(r_1, r_2; r_3)$ in (36a) by a simple vertex $k^2\delta(r_1 - r_3)\,\delta(r_2 - r_3)$, the result for the mass operator is an infinite series, and not the first term of an expansion. Actually, this cannot be accomplished, since the expression for the mass operator again contains the sought function $\widetilde{G}(r, r_0)$. Therefore, to obtain a closed system of equations, we should insert (36) in Dyson's equation, i.e., in the final account we use equation (35). In terms of diagrams, this equation takes the form



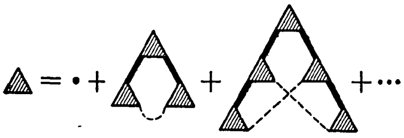
(35a)

Even if we use only the first term of (48) in equation (35), i.e., consider the equation

$$G_I(r, r_0) = G_0(r, r_0) + k^4 \iint G_0(r, \rho)\, G_I(\rho, \rho')\, G_I(\rho', r_0)\, B_\varepsilon(\rho, \rho')\, d^3\rho\, d^3\rho', \qquad (35b)$$

the result is a nonlinear equation; however the solution of equation (35b) sums over a much more extensive set of diagrams than does the function $G_1(\boldsymbol{r}, \boldsymbol{r}_0)$.

Further improvement of the solution can be achieved by constructing an equation for $\Gamma(\boldsymbol{r}_1, \boldsymbol{r}_2; \boldsymbol{r}_3)$. However, unlike (36a), this equation is not closed and it has the form

 (49)

Equations (35a) and (49) constitute an extremely complicated system of nonlinear integral equations, which are hardly applicable to practical calculations.

We should make another last remark in connection with equation (35). The derivation in this section does not assume a normal probability distribution of the fluctuations in ε_1. The probability distribution only affects the actual form of the vertex function. In particular, representation of the mass operator in terms of the averaged Green's function and the vertex operator is universal (it is independent of the distribution function), but the relation between the diagrams for the vertex function and for the mass operator follows from (40), which was previously formulated as a method for drawing the vertex function diagrams, and depends on the probability distribution. We thus see that if the conditions of applicability of the approximate theory permit replacing the vertex function by a simple vertex, the results are independent of the particular form of the probability distribution of the dielectric constant fluctuations.

§ 62. Wave propagation in a medium with small-scale fluctuations

We now return to expression (60.29) for the averaged Green's function $\tilde{G}(\boldsymbol{R})$:

$$\tilde{G}(\boldsymbol{R}) = \frac{1}{8\pi^3}\int \frac{e^{i\varkappa\boldsymbol{R}}d^3\varkappa}{k^2 - \varkappa^2 - \int Q(\rho)\, e^{-i\varkappa\rho}d^3\rho}\,. \qquad (1)$$

Consider the case when the correlation radii are small compared to the wavelength λ, i.e.,

$$ka \ll 1. \qquad (2)$$

In this case we need only retain the first term in the expansion of Q, namely $Q_1 = k^4 B_\varepsilon(\rho)\, G_0(\rho)$, since Q is expanded in powers of the small parameter k (the exact conditions when only the first term in Q needs to be retained will be given below). We make the calculations for a correlation function $B_\varepsilon(\rho) = \sigma^2 e^{-\alpha\rho}$, where $\sigma^2 = \langle\varepsilon_1^2\rangle$ and α^{-1} is the correlation radius of the fluctuations.

Performing the integration in the denominator in (1), we obtain after simple manipulations

$$\int Q_1(\rho)\, e^{-i\varkappa\rho}\, d^3\rho = -\frac{k^4\sigma^2}{(\alpha - ik)^2 + \varkappa^2}. \tag{3}$$

Inserting (3) in (1), we change over to spherical coordinates in $\varkappa$ and carry out the integration over the angular variables. This gives

$$\tilde{G}(\boldsymbol{R}) \approx G_1(\boldsymbol{R}) = \frac{1}{2\pi^2 R}\int_0^\infty \frac{\varkappa \sin \varkappa R}{k^2 - \varkappa^2 + \frac{k^4\sigma^2}{(\alpha - ik)^2 + \varkappa^2}}\, d\varkappa =$$

$$= -\frac{1}{8\pi^2 iR}\left[\int_{-\infty}^{\infty} \frac{\varkappa e^{i\varkappa R} d\varkappa}{k^2 - \varkappa^2 + \frac{k^4\sigma^2}{(\alpha - ik)^2 + \varkappa^2}} - \int_{-\infty}^{\infty} \frac{\varkappa e^{-i\varkappa R}\, d\varkappa}{k^2 - \varkappa^2 + \frac{k^4\sigma^2}{(\alpha - ik)^2 + \varkappa^2}}\right]. \tag{4}$$

The poles of the integrand are the roots of the equation

$$(k^2 - \varkappa^2)\,[(\alpha - ik)^2 + \varkappa^2] + k^4\sigma^2 = 0 \tag{5}$$

i.e.,

$$\varkappa_{1,2} = \frac{1}{\sqrt{2}}\sqrt{k^2 - (\alpha - ik)^2 \pm \sqrt{\alpha^2(\alpha - 2ik)^2 + 4k^4\sigma^2}}, \tag{6}$$
$$\varkappa_{3,4} = -\varkappa_{1,2}.$$

Closing the integration contour in the first integral upward and in the second integral downward and evaluating the residues at the poles given by (6) (remembering that $\operatorname{Im} k > 0$), we obtain after elementary but tedious calculations

$$G_1(R) = C_1 \frac{e^{i\varkappa_1 R}}{R} + C_2 \frac{e^{i\varkappa_2 R}}{R}, \tag{7}$$
$$C_{1,2} = -\frac{1}{8\pi}\left[1 \pm \left(1 + \frac{4k^4\sigma^2}{\alpha^2(\alpha - 2ik)^2}\right)^{-\frac{1}{2}}\right].$$

Seeing that by (2) $\alpha \gg k$, we may write the expression for $\varkappa_2$ in the form

$$\varkappa_2 \approx i\alpha\sqrt{\frac{1 + \sqrt{1 + \frac{4k^4\sigma^2}{\alpha^4}}}{2}}. \tag{8}$$

We see that the second term in (7) is small even for $R \sim \frac{1}{\alpha} \sim a$. Therefor, if we are dealing with the region $R \gg a$, it can be dropped. In other words, if we are dealing with the region $R \gg a$, we need not take into account the appearance of the new poles in (4); it suffices to consider only the displacement of the old poles $\varkappa = \pm k$. This is equivalent to replacing the term $\frac{k^4\sigma^2}{(\alpha - ik)^2 + \varkappa^2}$ in the denominator in (4) by the simpler term $\frac{k^4\sigma^2}{(\alpha - ik)^2}$. In the general case of an arbitrary correlation function, its behavior in the region of large $R \gg a$ is also determined by the asymptotic spectrum for small $\varkappa$, so that again, as in this example, we may ignore the appearance of new poles in the integrand of (1). Taking in (1) $e^{-i\varkappa\rho}=1$, we thus obtain an asymptotic expression of $\widetilde{G}(R)$ for $R \gg a$:

$$\widetilde{G}(R) \approx \frac{1}{8\pi^3}\int \frac{e^{i\varkappa R} d^3\varkappa}{\left[k^2-\int Q(\rho)\, d^3\rho\right]-\varkappa^2}. \tag{9}$$

Let

$$k_1^2 = k^2 - \int Q(\rho)\, d^3\rho. \tag{10}$$

Expression (9) differs from the spectral expansion of the function $G_0(R)$ only in that k_1 has been substituted for k. The asymptotic behavior of $\widetilde{G}(R)$ for $R \gg a$ thus has the form

$$\widetilde{G}(R) \approx -\frac{e^{ik_1R}}{4\pi R}. \tag{11}$$

Inserting expression (60.15) for Q and seeing that in virtue of the equality $Q(\rho_1, \rho_2) = Q(\rho_1-\rho_2)$ integration over $\rho_1-\rho_2$ is equivalent to integration over ρ_1, we obtain

$$\begin{gathered} k_1^2-k^2 = -k^4\int G_0(\rho) B_\varepsilon(\rho)\, d^3\rho - \\ -k^8 \iiint G_0(\rho_1-\rho_2) G_0(\rho_2-\rho_3) G_0(\rho_3-\rho_4) [B_\varepsilon(\rho_1-\rho_3) \times \\ \times B_\varepsilon(\rho_2-\rho_4) + B_\varepsilon(\rho_1-\rho_4) B_\varepsilon(\rho_2-\rho_3)]\, d^3\rho_1\, d^3\rho_2\, d^3\rho_3 - \ldots \end{gathered} \tag{12}$$

Let us evaluate the integrals in this expression for the case $ka \ll 1$:

$$\begin{gathered} \int G_0(\rho) B_\varepsilon(\rho)\, d^3\rho = 4\pi \int_0^\infty G_0(\rho) B_\varepsilon(\rho) \rho^2\, d\rho = -\int_0^\infty e^{-ik\rho} B_\varepsilon(\rho) \rho\, d\rho = \\ = -\int_0^\infty [1+ik\rho+\ldots] B_\varepsilon(\rho)\rho\, d\rho = -\int_0^\infty \rho B_\varepsilon(\rho)\, d\rho - ik\int_0^\infty B_\varepsilon(\rho)\rho^2 d\rho + \ldots \end{gathered} \tag{13}$$

The integral correlation scale a is defined by

$$\int B_\varepsilon(\rho)\, d^3\rho = \sigma^2 a^3. \tag{14}$$

Then

$$\int_0^\infty B_\varepsilon(\rho)\rho^2\, d\rho = \frac{\sigma^2 a^3}{4\pi}.$$

For the first integral in the right-hand side of (13) we may write

$$\int_0^\infty B_\varepsilon(\rho)\rho\, d\rho = \frac{\mu}{4\pi}\sigma^2 a^2, \tag{15}$$

where μ is a numerical constant dependent on the particular form of the correlation function. Thus

$$\int G_0(\rho) B_\varepsilon(\rho)\, d^3\rho = -\frac{\mu\sigma^2 a^2}{4\pi} - \frac{ik\sigma^2 a^3}{4\pi} + \ldots \tag{16}$$

The series (16) is arranged in powers of the small parameter ka, and therefore it is only necessary to retain the first terms.

Let us now estimate the higher terms of (12). First consider the integral

$$I=\iiint G_0(\rho_1-\rho_2)\,G_0(\rho_2-\rho_3)\,G_0(\rho_3-\rho_4)\,B_\varepsilon(\rho_2-\rho_3)\,B_\varepsilon(\rho_1-\rho_4)\times \\ \times\, d^3\rho_1\,d^3\rho_2\,d^3\rho_3=\iiint G_0(\boldsymbol{r}_1-\boldsymbol{r}_2)\,G_0(\boldsymbol{r}_2-\boldsymbol{r}_3)\,G_0(\boldsymbol{r}_3)\,B_\varepsilon(\boldsymbol{r}_2-\boldsymbol{r}_3)\times \\ \times\, B_\varepsilon(\boldsymbol{r}_1)\,d^3r_1\,d^3r_2\,d^3r_3. \tag{17}$$

Since $ka\ll 1$, the functions $B_\varepsilon(\rho)$ in the integrands can be written in the form $B_\varepsilon(\rho)=\sigma^2a^3\ \delta_a(\rho)$, where $\delta_a(\rho)$ is a δ-function "smeared" over a region of width a. If integration with the "unsmeared" δ-function does not lead to divergences, δ_a can be replaced by an ordinary δ-function; otherwise (when divergences do arise), we should use the rule

$$\int f(\boldsymbol{r})\,\delta_a(\boldsymbol{r})\,d^3r\sim f(a).$$

Applying this rule to (17), we get

$$\int G_0(\boldsymbol{r}_2-\boldsymbol{r}_3)\,\delta_a(\boldsymbol{r}_2-\boldsymbol{r}_3)\,d^3r_3\sim G_0(a),$$
$$I\sim(\sigma^2a^3)^2\,G_0\ (a)\int G_0^2\,(\boldsymbol{r}_2)\,d^3r_2\sim\frac{\sigma^4a^5}{k}$$

(we made use of the fact that for $ka\ll 1$, $G_0\,(a)\sim a^{-1}$, and evaluation of the integral gives $\int G_0^2\,(\boldsymbol{r})\,d^3r\sim k^{-1}$).

Similarly we can estimate the integral

$$\mathcal{J}=\iiint G_0(\boldsymbol{r}_1-\boldsymbol{r}_2)\,G_0(\boldsymbol{r}_2-\boldsymbol{r}_3)\,G_0(\boldsymbol{r}_3)\,B_\varepsilon(\boldsymbol{r}_1-\boldsymbol{r}_3)\,B_\varepsilon(\boldsymbol{r}_2)d^3\boldsymbol{r}_1d^3\,\boldsymbol{r}_2\times \\ \times d^3\boldsymbol{r}_3\sim(\sigma^2a^3)^2\int_{|\boldsymbol{r}_1|>a}[G_0(\boldsymbol{r}_1)]^3\,d^3r_1$$

(we made use of the fact that after integration with $\delta_a\,(\boldsymbol{r})$ the arguments coincide only to the nearest a, and therefore the singularities of the functions $G_0\,(\boldsymbol{r})$ are all distinct). Seeing that

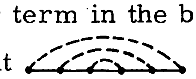
$$\int_{|\boldsymbol{r}|>a}[G_0(\boldsymbol{r})]^3d^3r\sim\ln\frac{1}{ka},$$

we have

$$\mathcal{J}\sim\sigma^4a^6\ln\frac{1}{ka}.$$

Inserting these results in (12), we find

$$k_1^2=k^2\left[1+\frac{\mu}{4\pi}\sigma^2k^2a^2+\frac{i}{4\pi}\sigma^2k^3a^3+\text{const}\cdot\sigma^4k^5a^5+\text{const}\cdot\sigma^4k^6a^6\ln\frac{1}{ka}+\ldots\right]. \tag{18}$$

The next higher term in the brackets is of the order $\sigma^6\,(ka)^8$ (it is associated with the element  of the operator Q).

First note that $ka\ln\frac{1}{ka}\ll 1$ for $ka\ll 1$, so that the last term in (18) is small compared to $\sigma^4k^5a^5$. The condition for retaining only the first term in the expansion of Q can thus be obtained from (18):

$$\sigma^4 k^5 a^5 \ll 1 + \frac{\mu}{4\pi}\sigma^2 k^2 a^2 \qquad (19)$$

(the term $\sigma^2 k^3 a^3$ is always small compared to $\sigma^2 k^2 a^2$). If $\sigma^2 k^2 a^2 \ll 1$ condition (19) is always satisfied. If now $\sigma^2 k^2 a^2 \gg 1$, we obtain from (19) the constraint

$$\sigma^2 (ka)^3 \ll 1. \qquad (20)$$

If this inequality is satisfied, the higher terms of the expansion are also small, since the ratio of any term in (18) to its predecessor is of the order $\sigma^2 (ka)^3$.

Since $ka \ll 1$, condition (20) imposes a weak constraint on $\sigma^2 k^2 a^2$:

$$\sigma^2 k^2 a^2 \ll \frac{1}{ka}, \qquad (20a)$$

from which it follows that $\sigma^2 k^2 a^2$ may be quite a large number. Using (20), we write k_1 in the form

$$k_1 = k\sqrt{1 + \frac{\mu}{4\pi}\sigma^2 k^2 a^2}\left[1 + \frac{i\sigma^2 k^3 a^3}{8\pi\left(1 + \frac{\mu}{4\pi}\sigma^2 k^2 a^2\right)} + \text{const}\cdot\frac{\sigma^4 k^5 a^5}{1 + \frac{\mu}{4\pi}\sigma^2 k^2 a^2} + \ldots\right]. \qquad (21)$$

The quantity k_1 enters (11) as an exponential function, $\exp(ik_1 R)$. Therefore the last term in (21) can be dropped only if

$$kR\frac{\sigma^4 k^5 a^5}{\sqrt{1 + \frac{\mu}{4\pi}\sigma^2 k^2 a^2}} \ll 1. \qquad (22)$$

If $\sigma^2 k^2 a^2 \ll 1$, (22) gives

$$kR \ll \frac{1}{\sigma^4 k^5 a^5} \quad (\sigma^2 k^2 a^2 \ll 1). \qquad (22a)$$

If, however, $\sigma^2 k^2 a^2 \gg 1$, we find

$$kR \ll \frac{1}{\sigma^3 k^4 a^4} \quad (\sigma^2 k^2 a^2 \gg 1). \qquad (22b)$$

If condition (22b) is satisfied, large kR can be considered only if $\sigma^3 k^4 a^4 \ll 1$, which is equivalent to the condition

$$\sigma^2 k^2 a^2 \ll \frac{1}{(ka)^{2/3}}. \qquad (23)$$

Constraint (23) is more rigid than (20a), but again it does not require that $\sigma^2 k^2 a^2$ be small.

Thus, if conditions (19) and (22) are satisfied, we have the following expression for k_1:

$$k_1 = k\sqrt{1 + \frac{\mu}{4\pi}\sigma^2 k^2 a^2}\left[1 + i\frac{\sigma^2 k^3 a^3}{8\pi\left(1 + \frac{\mu}{4\pi}\sigma^2 k^2 a^2\right)}\right]. \qquad (24)$$

The propagation constant k_1 may markedly differ from k if the mean square of the fluctuations σ^2 is sufficiently large.

Let $k = p_0 + i\gamma_0$, where $\gamma_0 \ll p_0$. Let $k_1 = n_e p_0 + i\gamma'$, where n_e is the effective index of refraction, and γ' the extinction coefficient. Then, ignoring the effect of γ_0 on n_e and writing out γ' only up to terms linear in γ_0, we obtain

$$n_e = \sqrt{1 + \frac{\mu}{4\pi}\sigma^2 a^2 p_0^2}, \tag{25}$$

$$\gamma' = \frac{2n_e^2 - 1}{n_e}\gamma_0 + \frac{1}{n_e}\frac{\sigma^2 p_0^4 a^3}{8\pi}. \tag{26}$$

It follows from (25) that $n_e \geqslant 1$, and quite possibly $n_e \gg 1$. The increase of n_e describes the average increase in the optical path due to multiple scattering by the refractive index fluctuations. The extinction coefficient is composed of two terms. The first is proportional to the absorption γ_0, though differing from the latter by a factor $(2n_e^2 - 1)/n_e$.

If $n_e \gg 1$, this factor may be quite substantial. The increase of absorption in a random medium is also a consequence of the lengthening of the optical path due to multiple scattering.

The second term in (26) is associated with the backscattering from the fluctuations. Indeed, the effective cross section for single scattering from a unit volume in a solid angle $d\Omega$ is given by (see Chapter 2)

$$d\sigma = \frac{\pi}{2} p_0^4 \Phi_\varepsilon\left(2k\sin\frac{\theta}{2}\right) d\Omega,$$

where $\Phi_\varepsilon(\varkappa)$ is the spectral density of ε_1. If $ka \ll 1$, we have for all θ

$$\Phi_\varepsilon\left(2k\sin\frac{\theta}{2}\right) \approx \Phi_\varepsilon(0) = (2\pi)^{-3}\int B_\varepsilon(r)\, d^3r = (2\pi)^{-3}\sigma^2 a^3.$$

Thus

$$d\sigma(\theta) = \frac{\sigma^2 a^3 p_0^4}{16\pi^2} d\Omega.$$

The effective scattering cross section per unit volume for the entire back hemisphere is

$$\alpha = \frac{\sigma^2 p_0^4 a^3}{8\pi}.$$

Thus (26) may be written in the form

$$\gamma' = \frac{2n_e^2 - 1}{n_e}\gamma_0 + \frac{\alpha}{n_e}. \tag{26a}$$

The calculation of the attenuation by scattering in the first approximation of the perturbation technique would give α. Thus, the introduction of multiple scattering in effect reduces the attenuation. This is quite natural, as multiple scattering restores part of the backscattered energy in the initial direction.

In concluding this section, we will briefly discuss the possible improvements of our solution, the aim of which should be to weaken the constraints (19) and (22).

It is perfectly clear that the introduction of a few higher terms in the expansion for Q will not markedly improve our position, since if condition (19) is broken the ratio of any higher term in the expansion to its predecessors will not be small for a whole range of higher order diagrams. Therefore, a substantial improvement of the results can be accomplished only if we sum over the principal diagrams entering the mass operator Q. To this end, we have equation (61.36a).

In this section we used the simplest diagram from the set in (61.36a), which is obtained when the averaged Green's function is replaced by $G_0(\boldsymbol{r}, \boldsymbol{r}_0)$ and the vertex function is reduced to a simple vertex. We also estimated the corrections in the first approximation to the mass-operator due to the higher terms of the expansion. We found that the correction $I\sim\sigma^4a^5k^{-1}$ associated with the substitution of G_0 for the averaged Green's function is much greater than the correction $\mathcal{J}\sim\sigma^4a^6\ln\frac{1}{ka}$ (their ratio being $\mathcal{J}/I\sim ka\ln\frac{1}{ka}$)) associated with the substitution of a simple vertex for the vertex function. Therefore, if we wish to improve on our approximation, we should primarily introduce the correction to the mass operator associated with the difference between $\tilde{G}$ and G_0; the correction due to the vertex function can be neglected at the first stage. We thus arrive at equation (61.35b):

$$G_I(\boldsymbol{r}, \boldsymbol{r}_0) = G_0(\boldsymbol{r}, \boldsymbol{r}_0) + k^4\iint G_0(\boldsymbol{r}, \boldsymbol{\rho})\,G_I(\boldsymbol{\rho}, \boldsymbol{\rho}')\,G_I(\boldsymbol{\rho}', \boldsymbol{r}_0)\,B_\varepsilon(\boldsymbol{\rho}, \boldsymbol{\rho}')\,d^3\rho\,d^3\rho'. \qquad (27)$$

How to solve this equation is a fairly complex problem, and we will not try to tackle it here with any rigor. However, we can assess the changes in the effective wavenumber introduced by equation (27). To this end we use expression (10), inserting for Q its expression from (61.36) with the vertex function replaced by a simple vertex and using for the unknown function $G_I(\boldsymbol{r}, \boldsymbol{r}_0)$ its asymptotic expression for large $|\boldsymbol{r}-\boldsymbol{r}_0|$, i.e., $-(4\pi|\boldsymbol{r}-\boldsymbol{r}_0|)^{-1}\exp(ik_1|\boldsymbol{r}-\boldsymbol{r}_0|)$. The estimate will be obtained for an exponential correlation function of the form $B_\varepsilon(\rho)=\sigma^2\exp(-\alpha\rho)$. In this case, the integral in (10) is readily evaluated and we obtain for k_1 a fourth order equation:

$$\left(\frac{k_1}{k}\right)^2 = 1 + \frac{\sigma^2\nu^2}{\left(1 - i\nu\frac{k_1}{k}\right)^2}, \qquad \nu = \frac{k}{\alpha} \ll 1. \qquad (28)$$

If $|\nu\frac{k_1}{k}|\ll 1$ this equation coincides with the previous expression for k_1. Otherwise, if this condition is violated, simple analysis of the solution of equation (28) shows that the dependence of k_1 on σ and ν is nc longer expressed by the previous formula and we now have

$$k_1 \sim k\sqrt{\sigma}.$$

We will not go into the various aspects of this solution in any more detail than this.

§ 63. The coherence function of the field, the optical theorem, and the equation of radiative transfer*

We have derived above the Bethe – Salpeter equation (60.36) for the function

$$W(\mathbf{r}_1, \mathbf{r}_2; \vec{\rho}_0, \vec{\rho}_0') = \langle \psi(\mathbf{r}_1, \vec{\rho}_0)\, \psi^*(\mathbf{r}_2, \vec{\rho}_0') \rangle - \widetilde{G}(\mathbf{r}_1, \vec{\rho}_0)\, \widetilde{G}^*(\mathbf{r}_2, \vec{\rho}_0').$$

Here the function $\psi(\mathbf{r}, \vec{\rho})$ satisfies the equation

$$\{\Delta(\mathbf{r}) + k^2[1 + \epsilon_1(\mathbf{r})]\}\ \psi(\mathbf{r}, \vec{\rho}) = \delta(\mathbf{r} - \vec{\rho}) \tag{1}$$

and is a particular case of the Green's function.

Now consider sources of the field with volume density $j(\vec{\rho})$. In a random medium the field $u(\mathbf{r})$ resulting from these sources is then expressed in terms of $\psi(\mathbf{r}, \vec{\rho})$ by the integral

$$u(\mathbf{r}) = \int \psi(\mathbf{r}, \vec{\rho})\, j(\vec{\rho})\, d^3\rho \tag{2}$$

and it satisfies the equation

$$\{\Delta(\mathbf{r}) + k^2[1 + \epsilon_1(\mathbf{r})]\}\ u(\mathbf{r}) = j(\mathbf{r}).$$

In what follows, we consider the field $u(\mathbf{r})$ only in that part of space which is free from any sources, i.e., the region where $j(\mathbf{r}) = 0$. In this region, $u(\mathbf{r})$ satisfies the equation

$$\{\Delta + k^2[1 + \epsilon_1(\mathbf{r})]\}\ u(\mathbf{r}) = 0. \tag{3}$$

The mean value of the Green's function $\langle \psi(\mathbf{r}, \vec{\rho}) \rangle = \widetilde{G}(\mathbf{r}, \vec{\rho})$, as we have seen before, satisfies Dyson's equation

$$\widetilde{G}(\mathbf{r}, \vec{\rho}) = G_0(\mathbf{r}, \vec{\rho}) + \iint G_0(\mathbf{r}, \vec{\rho}_1)\, Q(\vec{\rho}_1, \vec{\rho}_2)\, \widetilde{G}(\vec{\rho}_2, \vec{\rho})\, d^3\rho_1 d^3\rho_2. \tag{4}$$

Acting on (4) with the operator $\Delta(\mathbf{r}) + k^2$ and noting that

$$[\Delta(\mathbf{r}) + k^2]\, G_0(\mathbf{r}, \vec{\rho}_a) = \delta(\mathbf{r} - \vec{\rho}_a),$$

we obtain

$$[\Delta(\mathbf{r}) + k^2]\, \widetilde{G}(\mathbf{r}, \vec{\rho}) = \delta(\mathbf{r} - \vec{\rho}) + \int Q(\mathbf{r}, \vec{\rho}')\, \widetilde{G}(\vec{\rho}', \vec{\rho})\, d^3\rho'. \tag{5}$$

Letting $\hat{Q}$ denote the integral operator with the kernel $Q(\vec{\rho}', \vec{\rho})$, we can rewrite the last equation in the form of Dyson's integro-differential equation

$$\hat{D}\widetilde{G} \equiv [\Delta(\mathbf{r}_1) + k^2 - \hat{Q}]\, \widetilde{G}(\mathbf{r}_1, \mathbf{r}_2) = \delta(\mathbf{r}_1 - \mathbf{r}_2). \tag{5a}$$

The operator $\hat{D} = \Delta + k^2 - \hat{Q}$ will be referred to as Dyson's operator.

* The presentation of this section closely follows that of Yu. N. Barabanenkov, A. G. Vinogradov, and Yu. A. Kravtsov /202/.

Averaging (2), we obtain

$$\langle u(\mathrm{r})\rangle = \int \widetilde{G}(\mathrm{r}, \vec{\rho}) j(\vec{\rho}) d^3\rho.$$

Multiplying (5a) by $j(\mathrm{r}_2)$ and integrating over r_2, we obtain

$$\hat{D}\langle u(\mathrm{r})\rangle = j(\mathrm{r})$$

and in the region without sources, where $j(\mathrm{r}) = 0$,

$$\hat{D}\langle u(\mathrm{r})\rangle = 0. \tag{6}$$

We now define the coherence function of second order as

$$\overline{\Gamma}_2(\mathrm{r}_1, \mathrm{r}_2) = \langle u(\mathrm{r}_1) u^*(\mathrm{r}_2)\rangle. \tag{7}$$

Using (2), we obtain

$$\overline{\Gamma}_2(\mathrm{r}_1, \mathrm{r}_2) = \iint \langle \psi(\mathrm{r}_1, \vec{\rho}_1) \psi^*(\mathrm{r}_2, \vec{\rho}_2)\rangle j(\vec{\rho}_1) j^*(\vec{\rho}_2) d^3\rho_1 d^2\rho_2. \tag{8}$$

An equation for $\overline{\Gamma}_2$ can be obtained from the Bethe – Salpeter equation (60.36a). Inserting in (60.36a) $W = \langle \psi\psi^*\rangle - \widetilde{G}\widetilde{G}^*$ and collecting similar terms, we obtain

$$\langle \psi(\mathrm{r}_1, \vec{\rho}_0) \psi^*(\mathrm{r}_2, \vec{\rho}_0')\rangle = \widetilde{G}(\mathrm{r}_1, \vec{\rho}_0) \widetilde{G}^*(\mathrm{r}_2, \vec{\rho}_0') + \iiiint \widetilde{G}(\mathrm{r}_1, \vec{\rho}_1) \widetilde{G}^*(\mathrm{r}_2, \vec{\rho}_2) \cdot$$
$$\cdot P(\vec{\rho}_1, \vec{\rho}_2; \vec{\rho}_3, \vec{\rho}_4) \langle \psi(\vec{\rho}_3, \vec{\rho}_0) \psi^*(\vec{\rho}_4, \vec{\rho}_0')\rangle d^3\rho_1 d^3\rho_2 d^3\rho_3 d^3\rho_4. \tag{9}$$

Multiplying (9) by $j(\vec{\rho}_0) j^*(\vec{\rho}_0')$ and integrating over $\vec{\rho}_0, \vec{\rho}_0'$, we obtain the following equation for $\overline{\Gamma}_2$:

$$\overline{\Gamma}_2(\mathrm{r}_1, \mathrm{r}_2) = \langle u(\mathrm{r}_1)\rangle \langle u^*(\mathrm{r}_2)\rangle + \iiiint \widetilde{G}(\mathrm{r}_1, \vec{\rho}_1) \widetilde{G}^*(\mathrm{r}_2, \vec{\rho}_2) \cdot$$
$$P(\vec{\rho}_1, \vec{\rho}_2; \vec{\rho}_3, \vec{\rho}_4) \overline{\Gamma}_2(\vec{\rho}_3, \vec{\rho}_4) d^3\rho_1 d^3\rho_2 d^3\rho_3 d^3\rho_4. \tag{10}$$

Let the observation points $\mathrm{r}_1, \mathrm{r}_2$ lie outside the region containing the field sources $j(\mathrm{r})$, so that equation (6) is valid. Acting on (10) with Dyson's operator $\hat{D}(\mathrm{r}_1)$ and using (5) and (6), we get

$$\hat{D}(\mathrm{r}_1) \overline{\Gamma}_2(\mathrm{r}_1, \mathrm{r}_2) = \iiint \widetilde{G}^*(\mathrm{r}_2, \vec{\rho}_2) P(\mathrm{r}_1, \vec{\rho}_2, \vec{\rho}_3, \vec{\rho}_4) \overline{\Gamma}_2(\vec{\rho}_3, \vec{\rho}_4) d^3\rho_2 d^3\rho_3 d^3\rho_4. \tag{11}$$

Let us now act on (10) with the operator $\hat{D}^*(\mathrm{r}_2)$. Using the complex conjugates of equations (5) and (6), we obtain

$$\hat{D}^*(\mathrm{r}_2) \overline{\Gamma}_2(\mathrm{r}_1, \mathrm{r}_2) = \iiint \widetilde{G}(\mathrm{r}_1, \vec{\rho}_1) P(\vec{\rho}_1, \mathrm{r}_2; \vec{\rho}_3, \vec{\rho}_4) \overline{\Gamma}_2(\vec{\rho}_3, \vec{\rho}_4) d^3\rho_1 d^3\rho_3 d^3\rho_4. \tag{12}$$

Subtracting (12) from (11) and making use of the equality $\hat{D}(\mathrm{r}_1) - \hat{D}^*(\mathrm{r}_2) = \Delta_1 - \Delta_2 - \hat{Q}_1 + \hat{Q}^*_2$ we obtain the relation

$$[\Delta(\mathrm{r}_1) - \Delta(\mathrm{r}_2)] \overline{\Gamma}_2(\mathrm{r}_1, \mathrm{r}_2) - \int [Q(\mathrm{r}_1, \vec{\rho}) \overline{\Gamma}_2(\vec{\rho}, \mathrm{r}_2) - Q^*(\mathrm{r}_2, \vec{\rho}) \overline{\Gamma}_2(\mathrm{r}_1, \vec{\rho})] d^3\rho =$$
$$= \iiint [\widetilde{G}^*(\mathrm{r}_2, \vec{\rho}) P(\mathrm{r}_1, \vec{\rho}; \vec{\rho}_3, \vec{\rho}_4) - \widetilde{G}(\mathrm{r}_1, \vec{\rho}) P(\vec{\rho}, \mathrm{r}_2; \vec{\rho}_3, \vec{\rho}_4)] \overline{\Gamma}_2(\vec{\rho}_3, \vec{\rho}_4) d^3\rho d^3\rho_3 d^3\rho_4. \tag{13}$$

Let us introduce new variables $R=\frac{1}{2}(r_1+r_2)$, $r=r_1-r_2$. Then $\Delta(r_1)-\Delta(r_2)=$ $=2\nabla_R\nabla_r$. We also write

$$\overline{\Gamma}_2(R+\frac{1}{2}r, R-\frac{1}{2}r)\equiv\Gamma_2(R, r)=\langle u(R+\frac{r}{2})u^*(R-\frac{r}{2})\rangle.$$

Since for a statistically homogeneous medium $\widetilde{G}(r_a, r_b)=\widetilde{G}(r_a-r_b)$, $Q(r_a, r_b)=$ $=Q(r_a-r_b)$, in the new variables equation (13) takes the form

$$2\nabla_R\nabla_r\Gamma_2(R, r)-\int[Q(R+\frac{r}{2}-\vec{\rho})\Gamma_2(\frac{R}{2}-\frac{r}{4}+\frac{\vec{\rho}}{2}, \vec{\rho}-R+\frac{r}{2})-Q^*(R-\frac{r}{2}-\vec{\rho})\cdot$$
$$\cdot\Gamma_2(\frac{R}{2}+\frac{r}{4}+\frac{\vec{\rho}}{2}, R+\frac{r}{2}-\vec{\rho})]d^3\rho=\iiint[\widetilde{G}^*(R-\frac{r}{2}-\vec{\rho})P(R+\frac{r}{2}, \vec{\rho}, \vec{\rho}_3, \vec{\rho}_4)-$$
$$-\widetilde{G}(R+\frac{r}{2}-\vec{\rho})P(\vec{\rho}, R-\frac{r}{2}, \vec{\rho}_3, \vec{\rho}_4)]\Gamma_2(\frac{\vec{\rho}_3+\vec{\rho}_4}{2}, \vec{\rho}_3-\vec{\rho}_4)d^3\rho d^3\rho_3 d^3\rho_4. \quad (14)$$

Using the arguments of the functions $Q, Q^*, \widetilde{G}^*, \widetilde{G}$ as the independent variables we change over to new integration variables in the various terms of equation (14). Furthermore, setting $R'=\frac{1}{2}(\vec{\rho}_3+\vec{\rho}_4)$, $r'=\vec{\rho}_3-\vec{\rho}_4$ we obtain

$$2\nabla_R\nabla_r\Gamma_2(R, r)-\int[Q(\vec{\rho})\Gamma_2(R-\frac{\vec{\rho}}{2}, r-\vec{\rho})-Q^*(\vec{\rho})\Gamma_2(R-\frac{\vec{\rho}}{2}, r+\vec{\rho})]d^3\rho=$$
$$=\iiint[\widetilde{G}^*(\vec{\rho})P(R+\frac{r}{2}, R-\frac{r}{2}-\vec{\rho}, R'+\frac{r'}{2}, R'-\frac{r'}{2})-\widetilde{G}(\vec{\rho})\cdot$$
$$\cdot P(R+\frac{r}{2}-\vec{\rho}, R-\frac{r}{2}, R'+\frac{r'}{2}, R'-\frac{r'}{2})]\Gamma_2(R', r')d^3\rho d^3R' d^3r'. \quad (15)$$

Equation (15) is an exact consequence of the Dyson and Bethe — Salpeter equations.

Let us consider the relation of this equation to the law of conservation of energy. Multiplying the equation

$$\Delta u+k^2[1+\epsilon_1(r)]u(r)=j(r)$$

by $u^*(r)$ and subtracting from the result its complex conjugate, we obtain the equality

$$u^*\Delta u-u\Delta u^*=ju^*-j^*u.$$

In the region where $j(r)=0$, this equality leads to the conservation principle

$$u^*\Delta u-u\Delta u^*\equiv\operatorname{div}[u^*\nabla u-u\nabla u^*]=0. \quad (16)$$

The vector $S=(2ik)^{-1}(u^*\nabla u-u\nabla u^*)$ is the energy flux density. Relation (16), which is applicable to points where $j(r)=0$, may be written in the form

$$\operatorname{div} S(r)=0. \quad (16a)$$

Note that $S(r)$ is a random variable. Averaging (16a), we obtain

$$\operatorname{div}\langle S(r)\rangle=0. \quad (17)$$

The vector $\langle \mathbf{S} \rangle$ may be expressed in terms of the function $\Gamma_2(\mathbf{R}, \mathbf{r})$. Indeed, differentiating the relation

$$\Gamma_2(\mathbf{R}, \mathbf{r}) = \langle u(\mathbf{R} + \frac{\mathbf{r}}{2})\, u^*(\mathbf{R} - \frac{\mathbf{r}}{2}) \rangle$$

with respect to $\mathbf{r}$, we obtain

$$\nabla_r \Gamma_2(\mathbf{R}, \mathbf{r}) = \frac{1}{2} \langle u^*(\mathbf{R} - \frac{\mathbf{r}}{2}) \nabla u(\mathbf{R} + \frac{\mathbf{r}}{2}) - u(\mathbf{R} + \frac{\mathbf{r}}{2}) \nabla u^*(\mathbf{R} - \frac{\mathbf{r}}{2}) \rangle.$$

Setting $\mathbf{r} = 0$, we have

$$\langle u^*(\mathbf{R}) \nabla u(\mathbf{R}) - u(\mathbf{R}) \nabla u^*(\mathbf{R}) \rangle = [2 \nabla_r \Gamma_2(\mathbf{R}, \mathbf{r})]_{\mathbf{r}=0}.$$

Comparison of this relation with the expression for $\mathbf{S}$ shows that

$$\mathbf{S}(\mathbf{R}) = \frac{1}{ik} [\nabla_r \Gamma_2(\mathbf{R}, \mathbf{r})]_{\mathbf{r}=0}. \tag{18}$$

Let us now return to equation (15). The first term on the left is $2\nabla_R \nabla_r \Gamma_2 = \operatorname{div}(2\Delta_r \Gamma_2(\mathbf{R}, \mathbf{r}))$. If we set $\mathbf{r} = 0$ in (15), relations (18) and (17) give $[2\nabla_R \nabla_r \Gamma_2(\mathbf{R}, \mathbf{r})]_{\mathbf{r}=0} = 0$, so that (15) reduces to the equality

$$\begin{aligned} &\int [Q(\vec{\rho})\, \Gamma_2(\mathbf{R} - \frac{\vec{\rho}}{2}, -\vec{\rho}) - Q^*(\vec{\rho})\, \Gamma_2(\mathbf{R} - \frac{\vec{\rho}}{2}, \vec{\rho})]\, d^3\rho = \\ &= \iiint [\widetilde{G}(\vec{\rho}) P(\mathbf{R} - \vec{\rho}, \mathbf{R}, \mathbf{R}' + \frac{\mathbf{r}'}{2}, \mathbf{R}' - \frac{\mathbf{r}'}{2}) - \\ &- \widetilde{G}^*(\vec{\rho}) P(\mathbf{R}, \mathbf{R} - \vec{\rho}, \mathbf{R}' + \frac{\mathbf{r}'}{2}, \mathbf{R}' - \frac{\mathbf{r}'}{2})]\, \Gamma_2(\mathbf{R}', \mathbf{r}')\, d^3\rho\, d^3R'\, d^3r'. \end{aligned} \tag{19}$$

In this equality, the functions Γ_2 depend on the particular choice of the field sources $j(\mathbf{r})$, whereas the functions $Q, \widetilde{G}, P$ are independent of the sources. Since an appropriate choice of the field source will modify the function Γ_2 to any desired extent, without altering the functions $Q, \widetilde{G}$, and P, equation (19) should hold true identically for all Γ_2. Writing the left-hand side of (19) as a triple integral

$$\iiint [Q(\vec{\rho})\, \delta(\mathbf{R} - \mathbf{R}' - \frac{\vec{\rho}}{2})\, \delta(\mathbf{r}' + \vec{\rho}) - Q^*(\vec{\rho})\, \delta(\mathbf{R} - \mathbf{R}' - \frac{\vec{\rho}}{2})\, \delta(\mathbf{r}' - \vec{\rho})]\, \Gamma_2(\mathbf{R}', \mathbf{r}')\, d^3\rho\, d^3R'\, d^3r',$$

inserting this expression in (19), and equating the coefficients of $\Gamma_2(\mathbf{R}', \mathbf{r}')$, we obtain

$$\begin{aligned} &Q(-\mathbf{r}')\, \delta(\mathbf{R} - \mathbf{R}' + \frac{\mathbf{r}'}{2}) - Q^*(\mathbf{r}')\, \delta(\mathbf{R} - \mathbf{R}' - \frac{\mathbf{r}'}{2}) = \\ &= \int [\widetilde{G}(\vec{\rho}) P(\mathbf{R} - \vec{\rho}, \mathbf{R}, \mathbf{R}' + \frac{\mathbf{r}'}{2}, \mathbf{R}' - \frac{\mathbf{r}'}{2}) - \widetilde{G}^*(\vec{\rho}) P(\mathbf{R}, \mathbf{R} - \vec{\rho}, \mathbf{R}' + \frac{\mathbf{r}'}{2}, \mathbf{R}' - \frac{\mathbf{r}'}{2})]\, d^3\rho. \end{aligned} \tag{20}$$

This relation between the mass operator, the mean Green's function $\widetilde{G}$, and the intensity operator P is known as the optical theorem /201/. In physical terms, the optical theorem states that the attenuation of the mean field, described by the imaginary part of the mass operator, is attributed to the scattering described by the operator P. Therefore between these quantities there must exist a relationship, expressed by the "optical theorem." Note that if approximate expressions are used for Q and P, these functions should still satisfy relation (20), since otherwise we arrive at a contradiction with the law of energy conservation.

Let us return to equation (15). Consider the case when the total vertex can be replaced by a simple vertex in the representation (61.36a) of the mass operator. Then

$$Q(\vec{\rho}) = k^4 B_\epsilon(\vec{\rho})\,\widetilde{G}(\vec{\rho}). \tag{21}$$

For the intensity operator we use the first term of the expansion

$$P(\mathbf{r}_1, \mathbf{r}_2; \mathbf{r}_3, \mathbf{r}_4) = k^4 B_\epsilon(\mathbf{r}_1 - \mathbf{r}_2)\,\delta(\mathbf{r}_1 - \mathbf{r}_3)\,\delta(\mathbf{r}_2 - \mathbf{r}_4). \tag{22}$$

Inserting (21) and (22) in (15), we obtain

$$\begin{gathered} 2\nabla_R \nabla_r \Gamma_2(\mathbf{R}, \mathbf{r}) - k^4 \int B_\epsilon(\vec{\rho})\,[\widetilde{G}(\vec{\rho})\,\Gamma_2(\mathbf{R} - \tfrac{\vec{\rho}}{2}, \mathbf{r} - \vec{\rho}) - \widetilde{G}^*(\vec{\rho})\,\Gamma_2(\mathbf{R} - \tfrac{\vec{\rho}}{2}, \mathbf{r} + \vec{\rho})]\,d^3\rho = \\ = k^4 \int [\widetilde{G}^*(\vec{\rho})\,B_\epsilon(\mathbf{r} + \vec{\rho})\,\Gamma_2(\mathbf{R} - \tfrac{\vec{\rho}}{2}, \mathbf{r} + \vec{\rho}) - \widetilde{G}(\vec{\rho})\,B_\epsilon(\mathbf{r} - \vec{\rho})\,\Gamma_2(\mathbf{R} - \tfrac{\vec{\rho}}{2}, \mathbf{r} - \vec{\rho})]\,d^3\rho. \end{gathered} \tag{23}$$

Note that this equation may also be written in the form

$$\begin{gathered} 2\nabla_R \nabla_r \Gamma_2(\mathbf{R}, \mathbf{r}) = k^4 \int \Big\{ \widetilde{G}(\vec{\rho})\,[B_\epsilon(\vec{\rho}) - B_\epsilon(\mathbf{r} - \vec{\rho})]\,\Gamma_2(\mathbf{R} - \tfrac{\vec{\rho}}{2}, \mathbf{r} - \vec{\rho}) - \\ - \widetilde{G}^*(\vec{\rho})\,[B_\epsilon(\vec{\rho}) - B_\epsilon(\mathbf{r} + \vec{\rho})]\,\Gamma_2(\mathbf{R} - \tfrac{\vec{\rho}}{2}, \mathbf{r} + \vec{\rho}) \Big\}\, d^3\rho. \end{gathered} \tag{23a}$$

We see from the last equation, in particular, that for $\mathbf{r} = 0$ the right-hand side of (23a) vanishes, i.e., in this approximation, the energy conservation principle and the optical theorem are both satisfied (in other words, the approximations (21) and (22) are consistent from the standpoint of energy conservation). Moreover, as we see from (23a), the equation for Γ_2 only contains the differences $B_\epsilon(\vec{\rho}) - B_\epsilon(\mathbf{r} \pm \vec{\rho})$ which may be expressed in terms of the structure functions $D_\epsilon(\vec{\rho})$. This signifies that the equation for the coherence function may be written also in the case when the spectral density of the dielectric constant has a singularity at the origin and the correlation function does not exist.

In what follows, to simplify the analysis, we will replace the mean Green's function $\widetilde{G}$ by the Green's function of free space G_0, although the general case can be considered without difficulty /202/. Our approach will yield the same expression as before for the extinction (attenuation) coefficient, so that this substitution is justified under the particular conditions specified above.

We define the spectral density of the function $\Gamma_2(\mathbf{R}, \mathbf{r})$ relative to the variable $\mathbf{r}$:

$$\Gamma_2(\mathbf{R}, \mathbf{r}) = \int e^{i\vec{\kappa}\mathbf{r}} f(\mathbf{R}, \vec{\kappa})\,d^3\kappa. \tag{24}$$

Inserting (24) in (23) and using the spectral representation (4.8) for the correlation function $B_\epsilon(\vec{\rho})$, after simple manipulations we obtain

$$\begin{gathered} 2i\vec{\kappa}\nabla_R f(\mathbf{R}, \vec{\kappa}) - k^4 \int \Big\{ G_0(\vec{\rho})\,e^{-i\vec{\kappa}\vec{\rho}} - G_0^*(\vec{\rho})\,e^{i\vec{\kappa}\vec{\rho}} \Big\} \cdot B_\epsilon(\vec{\rho}) \cdot \\ \cdot f(\mathbf{R} - \tfrac{\vec{\rho}}{2}, \vec{\kappa})\,d^3\rho - k^4 \iint \Big\{ G_0^*(\vec{\rho})\,e^{i\vec{\kappa}\vec{\rho}} - G_0(\vec{\rho})\,e^{-i\vec{\kappa}\vec{\rho}} \Big\} \cdot \\ \cdot \phi_\epsilon(\vec{\kappa} - \vec{\kappa}')\,f(\mathbf{R} - \tfrac{\vec{\rho}}{2}, \vec{\kappa}')\,d^3\rho\,d^3\kappa' = 0. \end{gathered} \tag{25}$$

We define the operators $\hat{B}$ and $\hat{B}_0$ as follows:

$$\hat{B}\varphi(\mathrm{R},\vec{\kappa}) = 2i\vec{\kappa}\nabla_R\varphi(\mathrm{R},\vec{\kappa}) - 2ik^4\int B_\epsilon(\vec{\rho})\,[\mathrm{Im}\,(G_0(\vec{\rho})\,e^{-i\vec{\kappa}\vec{\rho}})]\,\cdot$$

$$\cdot\,\varphi(\mathrm{R}-\frac{\vec{\rho}}{2},\vec{\kappa})\,d^3\rho + 2ik^4\iint[\mathrm{Im}\,(G_0(\vec{\rho})\,e^{-i\vec{\kappa}\vec{\rho}})]\,\phi_\epsilon(\vec{\kappa}-\vec{\kappa}')\,\varphi(\mathrm{R}-\frac{\vec{\rho}}{2},\vec{\kappa}')\,d^3\rho\,d^3\kappa' \quad (26)$$

$$\hat{B}_0\varphi(\mathrm{R},\vec{\kappa}) = 2i\vec{\kappa}\nabla_R\varphi(\mathrm{R},\vec{\kappa}) - 2ik^4\int B_\epsilon(\vec{\rho})\,[\mathrm{Im}\,(G_0(\vec{\rho})\,e^{-i\vec{\kappa}\vec{\rho}})]\,d^3\rho\cdot\varphi(\mathrm{R},\vec{\kappa}) +$$

$$+\,2ik^4\int\phi_\epsilon(\vec{\kappa}-\vec{\kappa}')\,\varphi(\mathrm{R},\vec{\kappa}')\,d^3\kappa'\int[\mathrm{Im}\,(G_0(\vec{\rho})\,e^{-i\vec{\kappa}\vec{\rho}})]\,d^3\rho. \quad (27)$$

The operator $\hat{B}_0$ differs from $\hat{B}$ in that the argument of the function φ in the integrand is evaluated at the point R, and not $\mathrm{R}-\frac{1}{2}\vec{\rho}$.

Equation (25) takes the form

$$\hat{B}f(\mathrm{R},\vec{\kappa}) = 0. \quad (25a)$$

Subtracting $\hat{B}_0 f$ from the two sides of (25a), we obtain

$$\hat{B}_0 f(\mathrm{R},\vec{\kappa}) = (\hat{B}_0-\hat{B})f \equiv \delta\hat{B}\cdot f. \quad (28)$$

If $f(\mathrm{R},\vec{\kappa})$ is a sufficiently smooth function of the first argument, $\delta\hat{B}\cdot f$ is small. In this case, equation (28) can be solved by the following iteration method. We seek a solution in the series form

$$f(\mathrm{R},\vec{\kappa}) = \sum_{n=0}^{\infty} f^{(n)}(\mathrm{R},\vec{\kappa}) \quad (29)$$

where the functions $f^{(n)}$ obey the equations

$$\hat{B}_0 f^{(0)}(\mathrm{R},\vec{\kappa}) = 0 \quad (30)$$

$$\hat{B}_0 f^{(n)}(\mathrm{R},\vec{\kappa}) = \delta\hat{B}f^{(n-1)}(\mathrm{R},\vec{\kappa}). \quad (31)$$

Consider equation (30) in some detail. Dropping the common factor $2i$, we write this equation in the form

$$\vec{\kappa}\nabla_R f^{(0)}(\mathrm{R},\vec{\kappa}) - k^4\left\{\int B_\epsilon(\vec{\rho})\,[\mathrm{Im}\,(G_0(\vec{\rho})\,e^{-i\vec{\kappa}\vec{\rho}})]\,d^3\rho\right\}\cdot f^{(0)}(\mathrm{R},\vec{\kappa}) +$$

$$+\,k^4\int\phi_\epsilon(\vec{\kappa}-\vec{\kappa}')\,[\int\mathrm{Im}\,(G_0(\vec{\rho})\,e^{-i\vec{\kappa}\vec{\rho}})\,d^3\rho]\,f^{(0)}(\mathrm{R},\vec{\kappa}')\,d^3\kappa' = 0. \quad (32)$$

The integrals over $\vec{\rho}$ entering (32) are readily evaluated:

$$\int\mathrm{Im}\,(G_0(\vec{\rho})\,e^{-i\vec{\kappa}\vec{\rho}})\,d^3\rho = -\frac{1}{4\pi}\,\mathrm{Im}\int\frac{e^{i\kappa\rho}}{\rho}\,e^{-i\vec{\kappa}\vec{\rho}}\,d^3\rho =$$

$$= -\,\mathrm{Im}\int_0^\infty\frac{e^{i\kappa\rho}}{\rho}\,\frac{\sin\kappa\rho}{\kappa\rho}\,\rho^2 d\rho = -\frac{1}{2\kappa}\int_0^\infty[\cos(k-\kappa)\rho-\cos(k+\kappa)\rho]\,d\rho =$$

$$= -\frac{\pi}{2\kappa}\,[\delta(\kappa-k)-\delta(\kappa+k)] = -\frac{\pi}{2\kappa}\,\delta(\kappa-k)$$

(since $k>0$ and $\kappa=\sqrt{\vec{\kappa}^2}\geqslant 0$, we have $\delta(\kappa+k)=0$);

$$A = \int B_\epsilon(\vec{\rho})\,\mathrm{Im}\,[G_0(\vec{\rho})\,e^{-i\vec{\kappa}\vec{\rho}}]\,d^3\rho = \mathrm{Im}\iint\phi_\epsilon(\vec{\kappa}')\,e^{i(\vec{\kappa}'-\vec{\kappa})\vec{\rho}}\,G_0(\vec{\rho})\,\cdot$$

$$d^3\rho\,d^3\kappa' = \mathrm{Im}\iint\phi_\epsilon(\vec{\kappa}_1+\vec{\kappa})\,e^{i\vec{\kappa}_1\vec{\rho}}\,G_0(\vec{\rho})\,d^3\rho\,d^3\kappa_1 =$$

$$= -4\pi\,\mathrm{Im}\int\phi_\epsilon(\vec{\kappa}+\vec{\kappa}')\,d^3\kappa'\int_0^\infty\frac{\sin\kappa'\rho}{\kappa'\rho}\,\frac{e^{ik\rho}}{4\pi\rho}\,\rho^2 d\rho =$$

$$= -\frac{1}{2}\int\frac{\phi_\epsilon(\vec{\kappa}'+\vec{\kappa})}{\kappa'}\,d^3\kappa'\int_0^\infty[\cos(\kappa'-k)\rho-\cos(\kappa'+k)\rho]\,d\rho.$$

Inserting $\int\limits_0^\infty \cos\kappa\rho\, d\rho = \pi\delta(\kappa)$ and introducing spherical coordinates $\kappa', \mathrm{s}' = \frac{\vec{\kappa}'}{\kappa'}$ in the integral over $\vec{\kappa}'$, so that $d^3\kappa' = \kappa'^2 d\kappa'\, d\Omega_{s'}$, we obtain

$$A = -\frac{\pi}{2}\iint d\Omega_{s'} \int\limits_0^\infty \kappa' d\kappa'\, \phi_\epsilon(\vec{\kappa} + \kappa'\mathrm{s}')\,[\delta(\kappa' - k) - \delta(\kappa' + k)] = -\frac{\pi k}{2}\iint \phi_\epsilon(\vec{\kappa} + k\mathrm{s}')\, d\Omega_{s'}.$$

Insertion of the resulting integrals in (32) gives

$$\vec{\kappa}\nabla_R f^{(0)}(\mathrm{R}, \vec{\kappa}) + k\cdot\frac{\pi k^4}{2}\iint \phi_\epsilon(\vec{\kappa} + k\mathrm{s}')\, d\Omega_{s'}\cdot f^{(0)}(\mathrm{R}, \vec{\kappa}) = \\ = \frac{1}{k}\,\frac{\pi k^4}{2}\,\delta(\kappa - k)\int \phi_\epsilon(\vec{\kappa} - \vec{\kappa}')\, f^{(0)}(\mathrm{R}, \vec{\kappa}')\, d^3\kappa'. \tag{33}$$

Since the right-hand side of this equation is proportional to $\delta(\kappa - k)$, its solution may be sought in the form

$$f^{(0)}(\mathrm{R}, \vec{\kappa}) = \frac{\delta(\kappa - k)}{k^2} J(\mathrm{R}, \mathrm{s}) \tag{34}$$

where $\mathrm{s} = \frac{\vec{\kappa}}{\kappa}$. Inserting (34) in (33) and equating the coefficients before the factor $k^{-1}\delta(\kappa - k)$, we obtain an equation for the function J:

$$\mathrm{s}\nabla_R J(\mathrm{R}, \mathrm{s}) + \frac{\pi k^4}{2}\iint \phi_\epsilon(k(\mathrm{s} + \mathrm{s}'))\, d\Omega_{s'}\cdot J(\mathrm{R}, \mathrm{s}) = \frac{\pi k^4}{2}\iint \phi_\epsilon(k(\mathrm{s} - \mathrm{s}'))\, J(\mathrm{R}, \mathrm{s}')\, d\Omega_{s'}. \tag{35}$$

Let

$$\sigma_0(\mathrm{s}, \mathrm{s}') = \frac{\pi k^4}{2}\,\phi_\epsilon(k\mathrm{s} - k\mathrm{s}') \tag{36}$$

$$\alpha = \iint \sigma_0(\mathrm{s}, \mathrm{s}')\, d\Omega_{s'}. \tag{37}$$

Here $\sigma_0(\mathrm{s}, \mathrm{s}')$ is the effective scattering cross section of a wave propagating in the direction of the unit vector s into a unit solid angle in the direction s′ (see (26.27)). Using this notation, we may write (35) in the form

$$\mathrm{s}\nabla_R J(\mathrm{R}, \mathrm{s}) + \alpha J(\mathrm{R}, \mathrm{s}) = \iint \sigma_0(\mathrm{s}, \mathrm{s}')\, J(\mathrm{R}, \mathrm{s}')\, d\Omega_{s'}. \tag{35a}$$

Equation (35a) is the well known equation of radiative transfer which is derived phenomenologically, for example, in /203, 204/. The same sources also describe reliable methods for the solution of this equation.

Considerable attention is being paid lately to the justification of the equation of radiative transfer from the standpoint of the statistical methods applied to the wave equation (see /205—209/). The above method of derivation of this equation, borrowed from /202/, is convenient in that it can be generalized without much difficulty to more complex situations (for example, the case of a statistically inhomogeneous medium is considered in /202/).

Relation (37) between the extinction (attenuation) coefficient α and the effective scattering cross section σ_0 is a consequence of the conservation of energy. It implies that the attenuation of a wave propagating in a certain direction is entirely attributable to scattering in other directions.

Inserting (34) in (24), we obtain a relation between the ray intensity $J(\mathrm{R}, \mathrm{s})$ and the coherence function:

$$\Gamma_2(\mathrm{R}, \mathrm{r}) = \iint e^{ik\mathrm{sr}}\, J(\mathrm{R}, \mathrm{s})\, d\Omega_s. \tag{38}$$

Setting r=0, we obtain the mean value of the energy density

$$\langle I(\mathbf{R})\rangle = \iint J(\mathbf{R}, \mathbf{s})\, d\Omega_s. \tag{39}$$

Inserting (38) in the right-hand side of (18), we obtain an expression for the mean value of the energy flux density:

$$\mathbf{S}(\mathbf{R}) = \iint \mathbf{s} J(\mathbf{R}, \mathbf{s})\, d\Omega_s. \tag{40}$$

Relations (38)—(40) link the parameters describing the propagation of the radiation, using the "photometric" and the "field" approach. They were first derived for one particular case by Dolin /186/.

Note that a general coherence function may not always be represented in the form (38). For this representation to apply, the spectral density should have the form (34), i.e., it should be possible to represent the random field as a superposition of plane waves, each with a wave vector of magnitude κ. This is one of the conditions that is necessary in order to derive the equation of radiative transfer.

Let us now consider the corrections to the first approximation that we obtained. They should be estimated using equation (31):

$$\hat{B}_0 f^{(1)}(\mathbf{R}, \vec{\kappa}) = (\hat{B}_0 - \hat{B}) f^{(0)}(\mathbf{R}, \vec{\kappa}). \tag{41}$$

It is more convenient, however, to transform (41) to the corresponding equation for the functions $\Gamma_2^{(1)}, \Gamma_2^{(0)}$, which is equivalent to solving equation (23a) by the same iteration scheme. The equation for $\Gamma_2^{(1)}$ will take the form $\hat{B}_0\Gamma_2^{(1)} = (\hat{B}_0 - \hat{B})\Gamma_2^{(0)}$, or

$$2\nabla_R \nabla_r \Gamma_2^{(1)}(\mathbf{R}, \mathbf{r}) - k^4 \int \{G_0(\vec{\rho})\,[B_\epsilon(\vec{\rho}) - B_\epsilon(\mathbf{r}-\vec{\rho})]\,\Gamma_2^{(1)}(\mathbf{R}, \mathbf{r}-\vec{\rho}) - G_0^*(\vec{\rho})\,[B_\epsilon(\vec{\rho}) - B_\epsilon(\mathbf{r}+\vec{\rho})]\,\Gamma_2^{(1)}(\mathbf{R}, \mathbf{r}+\vec{\rho})\}\, d^3\rho =$$
$$= k^4 \int \Big\{G_0(\vec{\rho})\,[B_\epsilon(\vec{\rho}) - B_\epsilon(\mathbf{r}-\vec{\rho})]\cdot[\Gamma_2^{(0)}(\mathbf{R}, \mathbf{r}-\vec{\rho}) - \Gamma_2^{(0)}(\mathbf{R}-\frac{\vec{\rho}}{2}, \mathbf{r}-\vec{\rho})] - G_0^*(\vec{\rho})\,[B_\epsilon(\vec{\rho}) - B_\epsilon(\mathbf{r}+\vec{\rho})]\cdot[\Gamma_2^{(0)}(\mathbf{R}, \mathbf{r}+\vec{\rho}) - \Gamma_2^{(0)}(\mathbf{R}-\frac{\vec{\rho}}{2}, \mathbf{r}+\vec{\rho})]\Big\}\, d^3\rho. \tag{42}$$

We will only estimate the orders of magnitude of the various terms, without actually solving the equation. The characteristic scale of the function $\Gamma_2^{(0)}$ with respect to the first argument R is $L = \alpha^{-1}$ (the extinction length). A characteristic value of the variable $\vec{\rho}$ in (42) is limited by the factor $B_\epsilon(\vec{\rho})$, so that $\rho \sim a$, where a is the scale of the inhomogeneities. Assume $a \ll L$ (the conditions under which this inequality is valid will be described later on). Then

$$\Gamma_2^{(0)}(\mathbf{R}, \mathbf{r} \pm \vec{\rho}) - \Gamma_2^{(0)}(\mathbf{R} - \frac{\vec{\rho}}{2}, \mathbf{r} \pm \vec{\rho}) \approx \frac{\vec{\rho}}{2} \nabla_R \Gamma_2^{(0)}(\mathbf{R}, \mathbf{r} \pm \vec{\rho})$$

and for the right-hand side of (42) we obtain the estimate

$$(\hat{B}_0 - \hat{B})\,\Gamma_2^{(0)} \lesssim k^4 \nabla_R \Gamma_2^{(0)}(\mathbf{R}, \mathbf{r}) \cdot \int \frac{1}{4\pi\rho} \cdot \rho\, B_\epsilon(\vec{\rho})\, d^3\rho \sim \frac{k^4}{L}\Gamma_2^{(0)} a^3 \sigma_\epsilon^2. \tag{43}$$

Similarly, taking l_1 for the characteristic scale of $\Gamma_2^{(1)}$ with respect to r, we obtain an estimate for the left-hand side of (42):

$$\hat{B}_0 \Gamma_2^{(1)} \sim \frac{\Gamma_2^{(1)}}{L l_1} + k^4 \Gamma_2^{(1)} \int \frac{1}{\rho} B_\epsilon(\vec{\rho})\, d^3\rho \sim \left(\frac{1}{L l_1} + k^4 a^2 \sigma_\epsilon^2\right)\Gamma_2^{(1)}. \tag{44}$$

Equating (43) and (44), we obtain an estimate for $\Gamma_2^{(1)}$:

$$\Gamma_2^{(1)} \sim \frac{k^4 L^{-1} a^3 \sigma_\epsilon^2}{(L l_1)^{-1} + k^4 a^2 \sigma_\epsilon^2} \Gamma_2^{(0)} \qquad (45)$$

which indicates that the term $\Gamma_2^{(0)}$ is sufficient when

$$\frac{(ka)^3 \sigma_\epsilon^2}{(kl_1)^{-1} + (ka)^2 (kL)\sigma_\epsilon^2} \ll 1.$$

The scale l_1 is apparently larger (or even much larger) than the wavelength, i.e., $(kl_1)^{-1} \lesssim 1$. Setting 1 for $(kl_1)^{-1}$, we thus obtain a somewhat stronger inequality

$$\frac{\sigma_\epsilon^2 (ka)^3}{1 + (ka)^2 (kL) \sigma_\epsilon^2} \ll 1. \qquad (46)$$

We have previously assumed that $a \ll L$, i.e., $\alpha a \ll 1$. Let us find α. If we take, for example,

$$B_\epsilon(\vec{\rho}) = \sigma_\epsilon^2 \exp(-\rho/a),$$

the function $\phi_\epsilon(\vec{\kappa})$ according to (4.16) is expressed in the form

$$\phi_\epsilon(\vec{\kappa}) = \frac{\sigma_\epsilon^2 a^3}{\pi^2 (1 + \kappa^2 a^2)^2}.$$

Inserting this expression in (36), (37), we obtain after a simple integration

$$\alpha = \frac{2k\sigma_\epsilon^2 (ka)^3}{1 + 4(ka)^2} \qquad (47)$$

so that

$$\alpha a = \frac{2\sigma_\epsilon^2 (ka)^4}{1 + 4(ka)^2}. \qquad (48)$$

The condition $\alpha a \ll 1$ takes the form

$$\begin{aligned} &\text{a)} \quad \sigma_\epsilon^2 (ka)^4 \ll 1 \quad \text{for} \quad ka \ll 1, \\ &\text{b)} \quad \sigma_\epsilon^2 (ka)^2 \ll 1 \quad \text{for} \quad ka \gg 1. \end{aligned} \qquad (49)$$

Let us now return to condition (46). If $ka \ll 1$, we have $\alpha = 2k\sigma_\epsilon^2 (ka)^3$, $1 + \sigma_\epsilon^2 (ka)^2 kL = 1 + (2ka)^{-1} \approx (2ka)^{-1}$, so that (46) takes the form

$$\sigma_\epsilon^2 (ka)^4 \ll 1 \qquad (ka \ll 1).$$

This condition coincides with (49a).

If, on the other hand, $ka \gg 1$, we have $\alpha = \frac{1}{2} k\sigma_\epsilon^2 \cdot ka$, $1 + \sigma_\epsilon^2 (ka)^2 kL_1 =$ $= 1 + 2ka \approx 2ka$, so that (46) may be written in the form

$$\sigma_\epsilon^2 (ka)^2 \ll 1, \qquad (ka \gg 1).$$

This condition coincides with (49b). We thus conclude that when

$$\frac{\alpha}{L} = \alpha a \sim \frac{G_{\epsilon}^{2}(ka)^{4}}{1+(2ka)^{2}} \ll 1 \tag{50}$$

inequality (46) is satisfied automatically, i.e., $|\Gamma_2^{(1)}| \ll |\Gamma_2^{(0)}|$.

Note, however, that the above estimates for $\Gamma_2^{(1)}$ are extremely crude and probably ignore cumulative errors. A more detailed analysis may possibly reveal the existence of a certain constraint on the distances over which the equation of radiative transfer is applicable.

B. PROPAGATION OF SHORT WAVES IN A MEDIUM WITH RANDOM INHOMOGENEITIES AND THE APPROXIMATION OF MARKOV RANDOM PROCESSES

The calculations of the amplitude fluctuations, assuming the wavelength was small, carried out in Chapter 3 were based on the first approximation of geometrical optics or on the first approximation of Rytov's method of smooth perturbations. Both these approximations are based on the first order of the perturbation theory, and therefore their applicability is limited to the region of small fluctuations in the log amplitudes. In a number of recent publications, similar calculations were carried out either by the summation of the series of the perturbation theory /179, 180, 188, 189, 190, 196/ or by some alternative methods which permit constructing an analogous theory for the region of strong fluctuations /186, 187, 191–195/. We will use here the results of /191–195/, which apply the methods used in Markov processes and construct a general theory for the computation of statistical moments of any order.

§ 64. Physical justification of the application of the Markov approximation to the propagation of short waves in an inhomogeneous random medium

Wave propagation will be described by means of Leontovich's parabolic equation (the so-called "quasi-optical" approximation

$$2ik\frac{\partial u(x,\vec{\rho})}{\partial x} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + k^2 \epsilon_1(x,\vec{\rho})u = 0 \tag{1}$$

(here x is the longitudinal coordinate and $\vec{\rho}$ is the transverse coordinate).

Equation (1) is derived from the complete wave equation

$$\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} + \frac{\partial^2 \Psi}{\partial z^2} + k^2[1+\epsilon_1(x,\vec{\rho})]\,\Psi = 0 \tag{2}$$

using the substitution

$$\Psi = u \cdot \exp(ikx) \tag{3}$$

and dropping the term $\frac{\partial^2 u}{\partial x^2}$. If the complex amplitude u varies markedly over distances of the order of an inhomogeneity scale l, the second derivative $\frac{\partial^2 u}{\partial x^2}$ is of the order of u/l^2. On the other hand, the term $2ik\partial u/\partial x$ in (1) is of the order of $u/\lambda l$. Therefore, for $\lambda \ll l$, the term $\frac{\partial^2 u}{\partial x^2}$ is small compared to the first term in (1). The condition $\lambda \ll l$ is clearly necessary, though by no means sufficient for transforming to the parabolic equation. Another condition for applying the parabolic equation can be derived by transforming (1) into an integral equation. Writing (1) in the form

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + 2ik\frac{\partial u}{\partial x} = -k^2 \epsilon_1 u$$

and comparing the result with equation (45.32), whose solution is (45.30), we obtain an integral equation equivalent to (1):

$$u(x, \vec{\rho}) = u(0, \vec{\rho}) + \frac{k^2}{4\pi}\int_0^x dx' \iint_{\infty}^{\infty} d^2\rho' \frac{\exp \dfrac{ik(\vec{\rho}-\vec{\rho}')^2}{2(x-x')}}{x-x'} \epsilon_1(x', \vec{\rho}') \cdot u(x', \vec{\rho}'). \quad (4)$$

The kernel of equation (4) contains exp $(ik\rho^2/2(x-x'))$, which is obtained from the exponential of the exact kernel $ik|\mathbf{r}-\mathbf{r}'|$ when $\lambda^3 x \ll l^4$(see (45.29)). This restriction (together with $\lambda \ll l$), however, still does not provide a sufficient condition for the application of the parabolic equation. After all, equation (1) is of the first order in x, and its solution in the plane $x =$ const is therefore completely determined only by those inhomogeneities which are located for $x' < x$ (see integral equation (4)). Thus equation (1) ignores the reflection of the wave, whereas the reflected component is invariably present in the exact solution. If the inhomogeneities ϵ are sufficiently strong, reflected waves may be quite large despite the two conditions $\lambda \ll l, \lambda^3 x \ll l^4$. The applicability of the parabolic equation is therefore dependent on some condition on the smallness of the fluctuations ϵ. These conditions will be considered in §68.

Let us now consider the relation of equation (1) to the equations of the method of smooth perturbations. Setting $u = \exp(\phi)\phi = ln u$, we obtain from (1)

$$2ik\frac{\partial \phi}{\partial x} + \Delta\phi + (\nabla_\perp \phi)^2 + k^2\epsilon_1 = 0, \quad \Delta = \partial^2/\partial y^2 + \partial^2/\partial z^2, \nabla_\perp = (\frac{\partial}{\partial y}, \frac{\partial}{\partial z}) \cdot \quad (5)$$

The equation for the first approximation of the method of smooth perturbations, (45.32), is obtained from (5) if the term $(\nabla_\perp\phi)^2$ is dropped from the latter equation. If we expand the function ϕ in a power series of order ϵ_1, i.e., letting $\phi = \phi_0 + \phi_1 + \phi_2 + \cdots$, we obtain from (5) equation (45.32) for ϕ_1; for ϕ_2 we obtain the equation satisfied by function (45.31), etc.

Thus the solution of the nonlinear equation (5) sums the entire series of the perturbation theory for the method of smooth perturbations (an attempt to sum this series, based on equation (5), was undertaken in /196/, but the results obtained in that study are much less rigorous than those presented below).

The parabolic equation (1) is equivalent to the nonlinear equation (5) of the method of smooth perturbations. The exact solutions of equation (1) are therefore equivalent to the results obtained from the exact summation of the entire series $\phi = \phi_0 + \phi_1 + \phi_2 + \cdots$.

It turns out that, without any additional assumptions regarding the fluctuations ϵ, besides the assumptions introduced in the solution of the linearized equation (45.32), we can obtain from (1) exact closed equations for the moments $\langle u(x, \vec{\rho}_1) \ldots u^*(x, \vec{\rho}_n)\rangle$, and derive explicit solutions of these equations for the first- and second-order moments.

Let us analyze in more detail the assumptions regarding the fluctuations adopted in the method of smooth perturbations. The two-dimensional spectral density $F_\epsilon(\kappa_2, \kappa_3, \xi)$ of the fluctuations ϵ, as we have seen before, is a "peaked" function confined in the narrow region where $|\kappa\xi| \lesssim 1$. This property of the function F_ϵ is associated with the isotropy (or the very slight anisotropy) of the fluctuations ϵ (see §5).

Let us consider in some detail the derivation of the spectral densities F_1 and F_2 of the field ϕ_1 (see §46). Using the above mentioned property of the function F_ϵ, we may transform expression (46.22) for F_1 into the approximate expression (46.31):

$$F_1(\kappa_2, \kappa_3, x) = \frac{k^2}{4} \int_0^x dx' \int_0^x dx'' \exp\left[\frac{i\kappa^2(x'-x'')}{2k}\right] F_\epsilon(\kappa_2, \kappa_3, x'-x'') \approx \approx \frac{k^2 x}{2} \int_0^\infty F_\epsilon(\kappa_2, \kappa_3, \xi)\, d\xi = \frac{\pi k^2 x}{2} \phi_\epsilon(0, \kappa_2, \kappa_3). \qquad \text{(a)}$$

Expression (46.26) for F_2 is similarly transformed:

$$F_2(\kappa_2, \kappa_3, x) = -\frac{k^2}{4} \int_0^x dx' \int_0^x dx'' \exp\left[\frac{i\kappa^2}{2\kappa}(2x - x' - x'')\right] \cdot F_\epsilon(\kappa_2, \kappa_3, x'-x'') \approx -\frac{\pi\kappa^2}{2} \phi_\epsilon(0, \kappa_2, \kappa_3) \int_0^x \exp\left[-\frac{i\kappa^2}{k}(x-\eta)\right] d\eta. \qquad \text{(b)}$$

Transformations of this type, which only introduce negligible errors in the final results, were actively used in all the preceding calculations. Note that these transformations are equivalent to the following approximation of the function $F_\epsilon(\kappa_2, \kappa_3, x)$:

$$F_\epsilon(\kappa_2, \kappa_3, \xi) \approx \delta(\xi) \int_{-\infty}^{\infty} F_\epsilon(\kappa_2, \kappa_3, \xi')\, d\xi' = \delta(\xi) \cdot 2\pi \phi_\epsilon(0, \kappa_2, \kappa_3). \qquad (6)$$

Indeed, if (6) is inserted on the left in (a) or (b), we automatically recover the right-hand sides of these expressions.

All the results of the method of smooth perturbations thus in fact correspond to approximation (6) for the spectral density F_ϵ.

The relative error associated with the approximation (6) can be assessed without difficulty. For the average of the square of the fluctuations of the phase difference in the inertial subrange of the turbulence spectrum, this error is proportional to

$$\frac{\delta D_s(\rho)}{D_s(\rho)} \sim \left(\frac{\rho}{x}\right)^{1/3}$$

and the error for the average of the square of the fluctuations of the log amplitude is proportional to

$$\frac{\delta \sigma_\rho^2}{\sigma_\rho^2} \sim \left(\frac{\sqrt{\lambda x}}{x}\right)^{1/3}.$$

The errors associated with approximation (6) thus diminish with increasing x (in the method of smooth perturbations, we actually ignored these errors).

Let us now consider the physical meaning of approximation (6). The fluctuations of the phase difference between two points separated by a distance ρ, as we have seen before, are mainly caused by inhomogeneities whose transverse scale is ρ. Because of the statistical isotropy of the fluctuations ϵ (and also in cases of slight anisotropy), the longitudinal extent of the most significant inhomogeneities is also of the order of magnitude ρ. Therefore, ignoring terms of the order ρ/x in the longitudinal correlation of the inhomogeneities ϵ, we in effect consider only inhomogeneities which have a delta correlation function (are delta-correlated) along the x axis. Similarly, the main contribution to amplitude fluctuations is from inhomogeneities of scale of the order $\sqrt{\lambda x}$, and by neglecting terms of the order $\frac{\sqrt{\lambda x}}{x}$ in the longitudinal correlation of ϵ, we in effect adopt approximation (6).

Note, however, that if the fluctuations of some physical quantity are determined by large-scale inhomogeneities (e.g., fluctuations of the total phase increment over a path length x), approximation (6) may lead to erroneous results (the transformations in (a) and (b) in this case cannot be justified as in §46).

Model (6) is formally equivalent to the following assumption regarding the delta-correlation of the fluctuations ϵ along the x axis:

$$\langle \epsilon_1(x, \vec{\rho})\epsilon_1(x', \vec{\rho}')\rangle = \delta(x-x')A(\vec{\rho}-\vec{\rho}'). \tag{7}$$

The function $A(\vec{\rho})$ may be obtained from the requirement that the integral over x' of the true correlation function be equal to the corresponding integral of the right-hand side of (7). Integration of (7) over x' gives

$$A(\vec{\rho}-\vec{\rho}') = \int_{-\infty}^{\infty} \langle \epsilon_1(x, \vec{\rho})\epsilon_1(x', \vec{\rho}')\rangle dx'. \tag{8}$$

Inserting in (8) the spectral representation

$$B_\epsilon(x-x', \vec{\rho}-\vec{\rho}') = \iiint_{-\infty}^{\infty} \phi_\epsilon(p, \vec{\kappa}) \exp[ip(x-x') + i\vec{\kappa}(\vec{\rho}-\vec{\rho}')]\, dp\, d^2\kappa$$

we obtain

$$A(\vec{\rho}) = 2\pi \iint_{-\infty}^{\infty} \phi_\epsilon(\vec{\rho}, \vec{\kappa}) \exp(i\vec{\kappa}\vec{\rho})\, d^2\kappa. \tag{9}$$

The assumption of inhomogeneities delta-correlated along the x axis, in conjunction with equation (1), enables us to apply the methods of Markov processes to our problem, although in a more elaborate form than for ordinary differential equations.

§ 65. The derivation of equations for the mean value of the field and the mutual coherence function of second-order

In accordance with the previous section, we assume that the random field $\epsilon_1(x, \vec{\rho})$ is delta-correlated along the x axis. Moreover in this paragraph we assume (as in some of the following sections) that ϵ_1 is a Gaussian random field.

The field ϵ_1 thus satisfies the following conditions:

$$\epsilon_1(x, \vec{\rho}) \text{ is a Gaussian random field} \tag{1}$$

$$\langle \epsilon_1(x, \vec{\rho}) \rangle = 0 \tag{2}$$

$$\langle \epsilon_1(x, \vec{\rho}) \epsilon_1(x' \vec{\rho}') \rangle = \delta(x - x') A(\vec{\rho} - \vec{\rho}') \tag{3}$$

where

$$A(\vec{\rho}) = 2\pi \iint_{-\infty}^{\infty} \phi_\epsilon(0, \vec{\kappa}) \exp(i \vec{\kappa} \vec{\rho}) d^2\kappa. \tag{4}$$

In what follows, we will require a relationship between the random function $f(\mathbf{R})$ ($\mathbf{R}$ is an n-dimensional vector) satisfying the conditions

(a) $f(\mathbf{R})$ is a Gaussian random field,

(b) $\langle f(\mathbf{R}) \rangle = 0$

and the functional $\phi[f]$ of this random function. A relationship of this kind was derived by Furutsu /157/ and Novikov /197/ and has the form

$$\langle f(\mathbf{R}) \phi[f] \rangle = \int \langle f(\mathbf{R}) f(\mathbf{R}') \rangle \langle \frac{\delta \phi[f]}{\delta f(\mathbf{R}')} \rangle d^n R'. \tag{5}$$

We will now prove this relation. Expanding the functional $\phi[f]$ in a functional Taylor series (see Appendix I, equation (12)), we obtain

$$\phi[f] = \phi[0] + \int \varphi_1(\mathbf{R}_1) f(\mathbf{R}_1) d^n\mathbf{R}_1 + \ldots +$$
$$+ \int \cdots \int \varphi_k(\mathbf{R}_1, \mathbf{R}_2, \ldots, \mathbf{R}_k) f(\mathbf{R}_1) \cdots f(\mathbf{R}_k) d^n R_1 \cdots d^n R_k + \ldots \tag{6}$$

The functions $\varphi_k(\mathbf{R}_1, \ldots, \mathbf{R}_k)$ clearly may be regarded as symmetric in all their arguments.

We multiply (6) by $f(\mathbf{R})$ and average. Remembering that $\langle f_1 \cdot f_2 \cdots f_{2k+1} \rangle = 0$, we obtain

$$\langle f(\mathbf{R}) \phi[f] \rangle = \int \varphi_1(\mathbf{R}_1) \langle f(\mathbf{R}) f(\mathbf{R}_1) \rangle d^n R_1 + \cdots +$$
$$+ \int \overset{(2k+1)}{\cdots} \int \varphi_{2k+1}(\mathbf{R}_1, \ldots, \mathbf{R}_{2k+1}) \langle f(\mathbf{R}) f(\mathbf{R}_1) \cdots f(\mathbf{R}_{2k+1}) \rangle d^n R_1 \cdots d^n R_{2k+1} + \ldots \tag{7}$$

We can now apply the known formula for the moments of a Gaussian random field, equivalent to (60.6):

$$\langle f(\mathbf{R}) f(\mathbf{R}_1) \cdots f(\mathbf{R}_{2k+1}) \rangle = \langle f(\mathbf{R}) f(\mathbf{R}_1) \rangle \cdot \langle f(\mathbf{R}_2) \cdots f(\mathbf{R}_{2k+1}) \rangle$$
$$+ \langle f(\mathbf{R}) f(\mathbf{R}_2) \rangle \cdot \langle f(\mathbf{R}_1) f(\mathbf{R}_3) \cdots f(\mathbf{R}_{2k+1}) \rangle + \cdots + \langle f(\mathbf{R}) f(\mathbf{R}_{2k+1}) \rangle \langle f(\mathbf{R}_1) \cdots f(\mathbf{R}_{2k}) \rangle \tag{8}$$

(a total of $(2k+1)$ terms). This gives

$$\langle f(\mathbf{R}) \phi[f] \rangle = \sum_{k=0}^{\infty} \int_{(2k+1)}^{\cdots} \int \varphi_{2k+1}(\mathbf{R}_1, \ldots, \mathbf{R}_{2k+1}) \cdot$$
$$\cdot [\langle f(\mathbf{R}) f(\mathbf{R}_1) \rangle \langle f(\mathbf{R}_2) \cdots f(\mathbf{R}_{2k+1}) \rangle + \langle f(\mathbf{R}) f(\mathbf{R}_2) \rangle \cdot$$
$$\cdot \langle f(\mathbf{R}_1) f(\mathbf{R}_3) \cdots f(\mathbf{R}_{2k+1}) \rangle + \langle f(\mathbf{R}) f(\mathbf{R}_{2k+1}) \rangle \langle f(\mathbf{R}_1) \cdots f(\mathbf{R}_{2k}) \rangle] d^n R_1 \cdots d^n R_{2k+2}. \tag{9}$$

The integral in (9) is partitioned into a sum of $(2k+1)$ integrals. In the second of these integrals we make the substitution $\mathbf{R}_2 \leftrightarrow \mathbf{R}_1$, in the third integral the substitution $\mathbf{R}_3 \leftrightarrow \mathbf{R}_1, \ldots,$ and in the $(2k+1)$-th integral the substitution $\mathbf{R}_{2k+1} \leftrightarrow \mathbf{R}_1$. Since the function $\varphi_{2k+1}(\mathbf{R}_1, \ldots, \mathbf{R}_{2k+1})$ is symmetric in all its arguments, all the integrals after this substitution of variables coincide with the first integral in (9). Hence,

$$\langle f(\mathbf{R})\phi[f]\rangle = \sum_{k=0}^{\infty}(2k+1)\int \underset{(2k+1)}{\cdots} \int \langle f(\mathbf{R})f(\mathbf{R}_1)\rangle \cdot$$

$$\varphi_{2k+1}(\mathbf{R}_1, \ldots, \mathbf{R}_{2k+1})\langle f(\mathbf{R}_2)\cdots f(\mathbf{R}_{2k+1})\rangle d^nR_1 \cdots d^nR_{2k+1} = \int \langle f(\mathbf{R})f(\mathbf{R}_1)\rangle d^nR_1$$

$$\left[\sum_{k=0}^{\infty}(2k+1)\int \underset{(2k)}{\cdots} \int \varphi_{2k+1}(\mathbf{R}_1, \ldots, \mathbf{R}_{2k+1})\ \langle f(\mathbf{R}_2)\cdots f(\mathbf{R}_{2k+1})\rangle d^nR_2 \cdots d^nR_{2k+1}\right]. \quad (10)$$

Let us now apply the operator $\delta/\delta f(\mathbf{R}')$ to (6). Using the symmetry of φ_k, we obtain

$$\frac{\delta\phi[f]}{\delta f(\mathbf{R}')} = \varphi_1(\mathbf{R}') + 2\int \varphi_2(\mathbf{R}', \mathbf{R}_2)f(\mathbf{R}_2)d^nR_2 + \cdots +$$

$$+ m\int \underset{(m-1)}{\cdots} \int \varphi_m(\mathbf{R}', \mathbf{R}_2, \ldots, \mathbf{R}_m)f(\mathbf{R}_2)\cdots f(\mathbf{R}_m)d^nR_2 \cdots d^nR_m. \quad (11)$$

Averaging (11) and remembering that for an odd number of factors, i.e., for $m = 2k$, $\langle f(\mathbf{R}_2)\cdots f(\mathbf{R}_{2k})\rangle = 0$, we obtain

$$\left\langle \frac{\delta\phi[f]}{\delta f(\mathbf{R}')}\right\rangle = \sum_{k=0}^{\infty}(2k+1)\int \underset{(2k)}{\cdots} \int \varphi_{2k+1}(\mathbf{R}', \mathbf{R}_2, \ldots, \mathbf{R}_{2k+1}) \cdot$$

$$\cdot \langle f(\mathbf{R}_2)\cdots f(\mathbf{R}_{2k+1})\rangle d^nR_2 \cdots d^nR_{2k+1}. \quad (12)$$

Comparison of (12) with the expression in brackets in (10) proves the validity of (5). Let us now turn to the equation for u:

$$2ik\frac{\partial u(x, \vec{\rho})}{\partial x} + \Delta u(x, \vec{\rho}) + k^2\epsilon_1(x, \vec{\rho})\,u(x, \vec{\rho}) = 0 \quad (13)$$

where $\Delta = \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$. This equation is considered in conjunction with the initial condition

$$u(0, \vec{\rho}) = u_0(\vec{\rho}). \quad (14)$$

Averaging (13) and setting $\langle u(x, \vec{\rho})\rangle \equiv \bar{u}(x, \vec{\rho})$, we obtain

$$2ik\frac{\partial \bar{u}(x, \vec{\rho})}{\partial x} + \Delta\bar{u}(x, \vec{\rho}) + k^2\langle \epsilon_1(x, \vec{\rho})u(x, \vec{\rho})\rangle = 0. \quad (15)$$

To find the last term in (15), we use (5). Indeed, the function $u(x, \vec{\rho})$ defined by equation (13) is a functional of $\epsilon_1(x, \vec{\rho})$. The solution of the integral equation (64.4) for u, which is equivalent to (13) and (14), can be written as an iterative series:

$$u(x, \vec{\rho}) = u_0(\vec{\rho}) + \int_0^x dx_1 \iint_{-\infty}^{\infty} d^2\rho_1 G_0(x, \vec{\rho}; x_1, \vec{\rho}_1)\epsilon_1(x_1, \vec{\rho}_1)u_0(\vec{\rho}_1) +$$

$$+ \int_0^x dx_1 \iint_{-\infty}^{\infty} d^2\rho_1 G_0(x, \vec{\rho}; x_1, \vec{\rho}_1)\epsilon_1(x_1, \vec{\rho}_1)\int_0^{x_1} dx_2 \iint_{-\infty}^{\infty} d^2\rho_2 G_0(x, \vec{\rho}_1; x_2, \vec{\rho}_2)$$

$$\cdot \epsilon_1(x_2, \vec{\rho}_2)u_0(\vec{\rho}_2) + \ldots \quad (16)$$

Here

$$G_0(x,\vec{\rho};x_1,\rho_1)=\frac{k^2}{4\pi(x-x_1)}\exp\left\{\frac{ik(\vec{\rho}-\vec{\rho}_1)^2}{2(x-x_1)}\right\}. \tag{17}$$

It is clear from (16) that $u(x,\vec{\rho})$ is a functional of $\epsilon_1(x',\vec{\rho}')$, and $u(x,\vec{\rho})$ depends on $\epsilon_1(x',\vec{\rho}')$ only for $0\leqslant x'\leqslant x$.

Hence it follows that

$$\frac{\delta u(x,\vec{\rho})}{\delta\epsilon_1(x',\vec{\rho}')}=0 \text{ if } x'>x \text{ or } x'<0. \tag{18}$$

We will use (5) to compute $\langle\epsilon_1(x,\vec{\rho})u(x,\vec{\rho})\rangle$:

$$\langle\epsilon_1(x,\vec{\rho})u(x,\vec{\rho})\rangle=\int_0^x dx'\iint_{-\infty}^{\infty}d^2\rho'\langle\epsilon_1(x,\vec{\rho})\epsilon_1(x',\vec{\rho}')\rangle\left\langle\frac{\delta u(x,\vec{\rho})}{\delta\epsilon_1(x',\vec{\rho}')}\right\rangle. \tag{19}$$

(The limits of integration over x' in (19) are determined by (18).) Inserting (3) in (19) and remembering that*

$$\int_0^x dx'\delta(x-x')=\tfrac{1}{2}$$

we obtain

$$\langle\epsilon_1(x,\vec{\rho})u(x,\vec{\rho})\rangle=\tfrac{1}{2}\iint_{-\infty}^{\infty}A(\vec{\rho}-\vec{\rho}')\left\langle\frac{\delta u(x,\vec{\rho})}{\delta\epsilon_1(x,\vec{\rho}')}\right\rangle d^2\rho'. \tag{20}$$

Note that the variational derivative of u in (20) is evaluated in the plane through the observation point.

To find $\delta u(x,\vec{\rho})/\delta\epsilon_1(x,\vec{\rho}')$, we integrate (13) over x from 0 to x:

$$2ik[u(x,\vec{\rho})-u_0(\vec{\rho})]+\int_0^x\Delta u(\xi,\vec{\rho})d\xi+k^2\int_0^x\epsilon_1(\xi,\vec{\rho})u(\xi,\vec{\rho})d\xi=0. \tag{21}$$

Acting on this equation with the operator

$$\frac{\delta}{\delta\epsilon_1(x',\vec{\rho}')},\quad\text{where } 0<x'<x,$$

and taking into account that

$$\frac{\delta\epsilon_1(\xi,\vec{\rho})}{\delta\epsilon_1(x',\vec{\rho}')}=\delta(\xi-x')\delta(\vec{\rho}-\vec{\rho}') \tag{21}$$

we obtain

$$2ik\frac{\delta u(x,\vec{\rho})}{\delta\epsilon_1(x',\vec{\rho}')}+\int_0^x\Delta(\vec{\rho})\frac{\delta u(\xi,\vec{\rho})}{\delta\epsilon_1(x',\vec{\rho}')}d\xi+$$
$$+k^2\delta(\vec{\rho}-\vec{\rho}')u(x',\vec{\rho})+k^2\int_0^x\epsilon_1(\xi,\vec{\rho})\frac{\delta u(\xi,\vec{\rho})}{\delta\epsilon_1(x',\vec{\rho}')}d\xi=0. \tag{22}$$

By (18), $\partial u(\xi,\vec{\rho})/\delta\epsilon_1(x',\vec{\rho}')=0$ if $\xi<x'$. Therefore, the lower integration limit in (22) may be replaced with x':

$$2ik\frac{\delta u(x,\vec{\rho})}{\delta\epsilon_1(x',\vec{\rho}')}+k^2\delta(\vec{\rho}-\vec{\rho}')u(x',\vec{\rho}')+\int_{x'}^x[k^2\epsilon_1(\xi,\vec{\rho})+\Delta(\vec{\rho})]\frac{\delta u(\xi,\vec{\rho})}{\delta\epsilon_1(x',\vec{\rho}')}d\xi=0. \tag{23}$$

* Since the initial correlation function $\langle\epsilon_1(x,\vec{\rho})\epsilon_1(x',\vec{\rho}')\rangle$ is even in $(x-x')$, the delta-function in (3) should be regarded as even, thus leading to a factor of 1/2.

Since (20) contains the value of $\delta u(x,\vec{\rho})/\delta\epsilon_1(x,\vec{\rho}')$ for $x'=x$, we may take $x'=x$ in (23). The integral vanishes and we obtain

$$\frac{\delta u(x,\vec{\rho})}{\delta\epsilon_1(x,\vec{\rho}')}=\frac{ik}{2}\,\delta(\vec{\rho}-\vec{\rho}')u(x,\vec{\rho}). \tag{24a}$$

In what follows, we will also require the complex conjugate of this equality

$$\frac{\delta u^*(x,\vec{\rho})}{\delta\epsilon_1(x,\vec{\rho}')}=\frac{ik}{2}\,\delta(\vec{\rho}-\vec{\rho}')u^*(x,\vec{\rho}). \tag{24b}$$

Averaging (24a) and inserting in (20), we obtain

$$\langle\epsilon_1(x,\vec{\rho})u(x,\vec{\rho})\rangle=\frac{ik}{4}A(0)\bar{u}(x,\vec{\rho}). \tag{25}$$

Inserting (25) in (15), we obtain a closed differential equation for $\bar{u}(x,\vec{\rho})$:

$$2ik\frac{\partial\bar{u}(x,\vec{\rho})}{\partial x}+\Delta\bar{u}(x,\vec{\rho})+\frac{ik^3}{4}A(0)\bar{u}(x,\vec{\rho})=0\,. \tag{26}$$

Together with the initial condition which follows from (14)

$$\bar{u}(0,\vec{\rho})=u_0(\vec{\rho}) \tag{27}$$

this equation fully determines the mean field $\bar{u}$.

Let us now derive an equation for the mutual coherence function:

$$\Gamma_2(x,\vec{\rho}_1,\vec{\rho}_2)=\langle\gamma_2(x,\vec{\rho}_1,\vec{\rho}_2)\rangle,\quad \gamma_2(x,\vec{\rho}_1,\vec{\rho}_2)=u(x,\vec{\rho}_1)u^*(x,\vec{\rho}_2). \tag{28}$$

Multiplying (13) by $u^*(x,\vec{\rho}_2)$, we obtain

$$2iku^*(x,\vec{\rho}_2)\frac{\partial u(x,\vec{\rho}_1)}{\partial x}+\Delta_1 u(x,\vec{\rho}_1)u^*(x,\vec{\rho}_2)+k^2\epsilon_1(x,\vec{\rho}_1)u(x,\vec{\rho}_1)u^*(x,\vec{\rho}_2)=0$$

Interchanging the subscripts 1 and 2 in this equation and taking the complex conjugate, we obtain

$$-2iku(x,\vec{\rho}_1)\frac{\partial u^*(x,\vec{\rho}_2)}{\partial x}+\Delta_2 u(x,\vec{\rho}_1)u^*(x,\vec{\rho}_2)+k^2\epsilon_1(x,\vec{\rho}_2)u(x,\vec{\rho}_1)u^*(x,\vec{\rho}_2)=0\,.$$

Subtracting, we find

$$2ik\frac{\partial\gamma_2(x,\vec{\rho}_1,\vec{\rho}_2)}{\partial x}+(\Delta_1-\Delta_2)\gamma_2(x,\vec{\rho}_1,\vec{\rho}_2)+$$
$$+k^2[\epsilon_1(x,\vec{\rho}_1)-\epsilon_1(x,\vec{\rho}_2)]\gamma_2(x,\vec{\rho}_1,\vec{\rho}_2)=0. \tag{29}$$

Averaging (29), we obtain

$$2ik\frac{\partial\Gamma_2(x,\vec{\rho}_1,\vec{\rho}_2)}{\partial x}+(\Delta_1-\Delta_2)\,\Gamma_2(x,\vec{\rho}_1,\vec{\rho}_2)+$$
$$+k^2\langle[\epsilon_1(x,\vec{\rho}_1)-\epsilon_1(x,\vec{\rho}_2)]\,u(x,\vec{\rho}_1)\,u^*(x,\vec{\rho}_2)\rangle=0. \tag{30}$$

Consider the factor $\langle\epsilon_1(x,\vec{\rho}_1)u(x,\vec{\rho}_1)u^*(x,\vec{\rho}_2)\rangle$. Let $\phi=u(x,\vec{\rho}_1)u^*(x,\vec{\rho}_2)$ in (5). Proceeding along the same lines as before, and making use of (3),

we obtain

$$\langle \epsilon_1(x,\vec{\rho}_1)u(x,\vec{\rho}_1)u^*(x,\vec{\rho}_2)\rangle = \int_0^x dx' \iint_{-\infty}^{\infty} d^2\rho' \langle \epsilon_1(x,\vec{\rho}_1)\epsilon_1(x,\vec{\rho}\,')\rangle \cdot$$
$$\cdot \langle \frac{\delta u(x,\vec{\rho}_1)u^*(x,\vec{\rho}_2)}{\delta\epsilon_1(x',\vec{\rho}\,')}\rangle = \frac{1}{2}\iint_{-\infty}^{\infty} d^2\rho' A(\vec{\rho}_1-\vec{\rho}\,')\langle\frac{\delta u(x,\vec{\rho}_1)u^*(x,\vec{\rho}_2)}{\delta\epsilon_1(x,\vec{\rho}\,')}\rangle.$$

Using (24a) and (24b), we find

$$\langle\frac{\delta u(x,\vec{\rho}_1)u^*(x,\vec{\rho}_2)}{\delta\epsilon_1(x,\vec{\rho}\,')}\rangle = \langle\frac{\delta u(x,\vec{\rho}_1)}{\delta\epsilon_1(x,\vec{\rho}\,')}u^*(x,\vec{\rho}_2)\rangle + \langle u(x,\vec{\rho}_1)\frac{\delta u^*(x,\vec{\rho}_2)}{\delta\epsilon_1(x,\vec{\rho}\,')}\rangle =$$

$$= \frac{ik}{2}\delta(\vec{\rho}_1-\vec{\rho}\,')\Gamma_2(x,\vec{\rho}_1,\vec{\rho}_2) - \frac{ik}{2}\delta(\vec{\rho}_2-\vec{\rho}\,')\Gamma_2(x,\vec{\rho}_1,\vec{\rho}_2)$$

so that

$$\langle\epsilon_1(x,\vec{\rho}_1)u(x,\vec{\rho}_1)u^*(x,\vec{\rho}_2)\rangle = \frac{ik}{4}[A(0)\Gamma_2(x,\vec{\rho}_1,\vec{\rho}_2) - A(\vec{\rho}_1-\vec{\rho}_2)\Gamma_2(x,\vec{\rho}_1,\vec{\rho}_2)].$$

Similarly,

$$\langle\epsilon_1(x,\vec{\rho}_2)u(x,\vec{\rho}_1)u^*(x,\vec{\rho}_2)\rangle = \frac{ik}{4}[A(\vec{\rho}_2-\vec{\rho}_1) - A(0)]\Gamma_2(x,\vec{\rho}_1,\vec{\rho}_2)$$

and using the evenness of the function $A(\vec{\rho}_1-\vec{\rho}_2)$, we find

$$\langle[\epsilon_1(x,\vec{\rho}_1)-\epsilon_1(x,\vec{\rho}_2)]u(x,\vec{\rho}_1)u^*(x,\vec{\rho}_2)\rangle = \frac{ik}{2}[A(0)-A(\vec{\rho}_1-\vec{\rho}_2)]\Gamma_2(x,\vec{\rho}_1,\vec{\rho}_2).$$

Inserting the last expression in (30) we obtain a closed differential equation

$$2ik\frac{\partial\Gamma_2(x,\vec{\rho}_1,\vec{\rho}_2)}{\partial x} + \Delta_1\Gamma_2(x,\vec{\rho}_1,\vec{\rho}_2) - \Delta_2\Gamma_2(x,\vec{\rho}_1,\vec{\rho}_2) + \frac{i\pi k^3}{2}H(\vec{\rho}_1-\vec{\rho}_2)\Gamma_2(x,\vec{\rho}_1,\vec{\rho}_2) = 0 \tag{31}$$

where

$$H(\vec{\rho}) = \frac{1}{\pi}[A(0)-A(\vec{\rho})] = 2\iint_{-\infty}^{\infty}[1-\cos\vec{\kappa}\vec{\rho}]\,\phi_\epsilon(0,\vec{\kappa})d^2\kappa \tag{32}$$

with the initial condition

$$\Gamma_2(0,\vec{\rho}_1,\vec{\rho}_2) = u_0(\vec{\rho}_1)u_0^*(\vec{\rho}_2).$$

Equations (26) and (31) may be solved in a general form (see below).

The above derivation of the equations for $\bar{u}$ and Γ_2 /191/ does not make use of the smallness of the fluctuations ϵ_1. It only assumes that ϵ_1 is delta-correlated and has a normal distribution. In fact, however, the parabolic equation for u is applicable only when certain restrictions imposed on the magnitude of the fluctuations ϵ_1 are met. These restrictions will be considered at a later stage. Equations (26) and (31) were also derived by other authors (see, e.g., /186, 187/), but all the other derivations essentially use the assumption that the fluctuations of ϵ_1 are small.

Equation (31) for the coherence function Γ_2 is closely linked with the equation of radiation transfer, as was first noted by Dolin /186/. To demonstrate this connection, we replace $\vec{\rho}_1$ and $\vec{\rho}_2$ in (31) by new variables $\mathbf{R} = \frac{1}{2}(\vec{\rho}_1 + \vec{\rho}_2)$, $\vec{\rho} = \vec{\rho}_1 - \vec{\rho}_2$. Then

$\Delta_1 \Gamma_2 - \Delta_2 \Gamma_2 = 2\nabla_R \nabla_\rho \Gamma_2(x, \mathbf{R}, \vec{\rho})$, where $\nabla_R = (\frac{\partial}{\partial R_y}, \frac{\partial}{\partial R_z})$ and $\nabla_\rho = (\frac{\partial}{\partial \rho_y}, \frac{\partial}{\partial \rho_z})$.

Equation (31) thus takes the form

$$2ik \frac{\partial \Gamma_2(x, \mathbf{R}, \vec{\rho})}{\partial x} + 2\nabla_R \nabla_\rho \Gamma_2(x, \mathbf{R}, \vec{\rho}) + \frac{i\pi k^3}{2} H(\vec{\rho}) \Gamma_2(x, \mathbf{R}, \vec{\rho}) = 0. \quad (33)$$

We define the spectral density $J(x, \mathbf{R}, \vec{\kappa})$ of the function Γ_2 with respect to the difference variable $\vec{\rho}$ by the equality

$$\Gamma_2(x, \mathbf{R}, \vec{\rho}) = \iint_{-\infty}^{\infty} J(x, \mathbf{R}, \vec{\kappa}) \exp(i\vec{\kappa}\vec{\rho}) d^2\kappa. \quad (34)$$

Moreover we introduce the notation

$$f(\vec{\kappa}) = \frac{\pi k^2}{2} \phi_\epsilon(0, \vec{\kappa}), \ \alpha = \iint_{-\infty}^{\infty} f(\vec{\kappa}) d^2\kappa. \quad (35)$$

Inserting (34) in (33), we obtain an equation for J /186/:

$$\frac{\partial J(x, \mathbf{R}, \vec{\kappa})}{\partial x} + \frac{\vec{\kappa}}{k} \nabla_R J(x, \mathbf{R}, \vec{\kappa}) + \alpha J(x, \mathbf{R}, \vec{\kappa}) = \iint_{-\infty}^{\infty} f(\vec{\kappa} - \vec{\kappa}') J(x, \mathbf{R}, \vec{\kappa}') d^2\kappa'. \quad (36)$$

This equation coincides with the small-angle approximation of the equation of radiative transfer, where $J(x, \mathbf{R}, \vec{\kappa})$ is the beam intensity at the point $x, \mathbf{R}$, for radiation propagating along the vector $(k, \vec{\kappa})$, α is the extinction coefficient, $f(\vec{\kappa})$ is the angular scattering function. The quantities $A(0)$ and α are related by the equality $\alpha = \frac{1}{4} k^2 A(0)$.

The above model for the dielectric constant fluctuations thus leads to the small-angle approximation of the equation of radiative transfer when applied to the field description in the parabolic equation approximation. Equation (36), like (31), can be solved in general form for any angular scattering function and for arbitrary initial conditions.

§ 66. Equations of the Einstein — Fokker — Planck type for the characteristic functional of the wave field and the equations for the higher moments

We have so far derived closed equations for the first two moments of the field $u(x, \vec{\rho})$. A complete statistical description of this field can be achieved with the aid of the characteristic functional

$$\Psi_x[\nu, \nu^*] = \langle \exp(iR_x) \rangle = \langle \exp\left\{ i \iint_{-\infty}^{\infty} [u(x, \vec{\rho}')\nu(\vec{\rho}') + u^*(x, \vec{\rho}')\nu^*(\vec{\rho}')]\, d^2\rho' \right\} \rangle. (1)$$

Since u is a complex field, i.e., it is described by two real functions Reu and Imu, we may introduce two conjugate real functions Reν and Imν. More convenient variables, however, are the linear combinations ν and ν^* of these functions, which are treated as linear independent functions.

We will derive an equation for the functional $\Psi_x[\nu, \nu^*]$ following the method proposed by Novikov /197/. Differentiating (1) with respect to x and using the parabolic equation (65.13) for $\frac{\partial u}{\partial x}$, we find

$$\frac{\partial \Psi_x[\nu,\nu^*]}{\partial x} = \langle e^{iR_x} \cdot i \iint_{-\infty}^{\infty} [\nu(\vec{\rho})(-\frac{1}{2ik})(\Delta u(x,\vec{\rho}) + k^2\epsilon_1(x,\vec{\rho})u(x,\vec{\rho}) +$$

$$+ \nu^*(\vec{\rho})\frac{1}{2ik}(\Delta u^*(x,\vec{\rho}) + k^2\epsilon_1(x,\vec{\rho})u^*(x,\vec{\rho})]d^2\rho\rangle =$$

$$= -\frac{1}{2k}\iint d^2\rho \Big\{ \nu(\vec{\rho})[\langle \Delta u(x,\vec{\rho})e^{iR_x}\rangle + k^2\langle \epsilon_1(x,\vec{\rho})u(x,\vec{\rho})e^{iR_x}\rangle] -$$

$$- \nu^*(\vec{\rho})[\langle \Delta u^*(x,\vec{\rho})e^{iR_x}\rangle + k^2\langle \epsilon_1(x,\vec{\rho})u^*(x,\vec{\rho})e^{iR_x}\rangle]. \qquad (2)$$

Acting on (1) with the operators $\frac{\delta}{i\delta\nu(\vec{\rho})}, \frac{\delta}{i\delta\nu^*(\vec{\rho})}$, we obtain

$$\langle u(x,\vec{\rho})\exp(iR_x)\rangle = \frac{1}{i}\frac{\delta\Psi_x}{\delta\nu(\vec{\rho})}$$

$$\langle u^*(x,\vec{\rho})\exp(iR_x)\rangle = \frac{1}{i}\frac{\delta\Psi_x}{\delta\nu^*(\vec{\rho})} \qquad (3)$$

and acting on these equalities with the operator Δ, we express the terms entering (2) in terms of Ψ_x:

$$\langle \Delta u(x,\vec{\rho})\exp(iR_x)\rangle = \frac{1}{i}\Delta(\vec{\rho})\frac{\delta\Psi_x[\nu,\nu^*]}{\delta\nu(\vec{\rho})}$$

$$\langle \Delta u^*(x,\vec{\rho})\exp(iR_x)\rangle = \frac{1}{i}\Delta(\vec{\rho})\frac{\delta\Psi_x[\nu,\nu^*]}{\delta\nu^*(\vec{\rho})}. \qquad (3a)$$

To derive an equation for Ψ_x from (2), it now remains to express the terms containing ϵ_1 in terms of Ψ_x. To this end, consider the functional

$$g[\nu,\nu^*;x,\vec{\rho}] = \langle \epsilon_1(x,\vec{\rho})\exp(iR_x)\rangle. \qquad (4)$$

Applying (65.5), (65.3), (65.24a), (65.24b), and (3), we obtain

$$g[\nu,\nu^*,x,\vec{\rho}] = \int_0^x dx' \iint_{-\infty}^{\infty} d^2\rho''\langle \epsilon_1(x,\vec{\rho})\epsilon_1(x',\vec{\rho}'')\rangle\langle \frac{\delta e^{iR_x}}{\delta\epsilon_1(x',\vec{\rho}'')}\rangle =$$

$$= \frac{1}{2}\iint_{-\infty}^{\infty} d^2\rho'' A(\vec{\rho}''-\vec{\rho})\langle e^{iR_x}\cdot\iint_{-\infty}^{\infty} i[\nu(\vec{\rho}')\frac{\delta u(x,\vec{\rho}')}{\delta\epsilon_1(x',\vec{\rho}'')} + \nu^*(\vec{\rho}')\frac{\delta u^*(x,\vec{\rho}')}{\delta\epsilon_1(x,\vec{\rho}'')}]d^2\rho'\rangle =$$

$$= -\frac{k}{4}\iint_{-\infty}^{\infty} d^2\rho'' A(\vec{\rho}-\vec{\rho}'')[\nu(\vec{\rho}'')\langle u(x,\vec{\rho}'')e^{iR_x}\rangle - \nu^*(\vec{\rho}'')\langle u^*(x,\vec{\rho}'')e^{iR_x}\rangle] =$$

$$= \frac{ik}{4}\iint_{-\infty}^{\infty} d^2\rho'' A(\vec{\rho}-\vec{\rho}'')[\nu(\vec{\rho}'')\frac{\delta\Psi_x}{\delta\nu(\vec{\rho}'')} - \nu^*(\vec{\rho}'')\frac{\delta\Psi_x}{\delta\nu^*(\vec{\rho}'')}]. \qquad (5)$$

We define a new operator

$$\hat{M}(\vec{\rho}) = \nu(\vec{\rho})\frac{\delta}{\delta\nu(\vec{\rho})} - \nu^*(\vec{\rho})\frac{\delta}{\delta\nu^*(\vec{\rho})}. \qquad (6)$$

Then the expression for $g[\nu, \nu^*; x, \vec{\rho}]$ can be written in the form

$$g[\nu, \nu^*; x, \vec{\rho}] = \frac{ik}{4} \iint_{-\infty}^{\infty} A(\vec{\rho} - \vec{\rho}\,') \hat{M}(\vec{\rho}\,') \Psi_x[\nu, \nu^*] d^2\rho'. \quad (7)$$

Acting on (4) with the operators $\frac{\delta}{i\delta\nu(\vec{\rho})}, \frac{\delta}{i\delta\nu^*(\vec{\rho})}$, we obtain

$$\frac{1}{i} \frac{\delta g[\nu, \nu^*; x, \vec{\rho}]}{\delta\nu(\vec{\rho})} = \langle \epsilon_1(x, \vec{\rho}) u(x, \vec{\rho}) \exp(iR_x) \rangle \quad (8a)$$

$$\frac{1}{i} \frac{\delta g[\nu, \nu^*; x, \vec{\rho}]}{\delta\nu^*(\vec{\rho})} = \langle \epsilon_1(x, \vec{\rho}) u^*(x, \vec{\rho}) \exp(iR_x) \rangle. \quad (8b)$$

Multiplying (8a) by $\nu(\vec{\rho})$ and (8b) by $\nu^*(\vec{\rho})$ and subtracting, we form the combination appearing in (2):

$$\nu(\vec{\rho}) \langle \epsilon_1(x, \vec{\rho}) u(x, \vec{\rho}) e^{iR_x} \rangle - \nu^*(\vec{\rho}) \langle \epsilon_1(x, \vec{\rho}) u^*(x, \vec{\rho}) e^{iR_x} \rangle =$$

$$= \frac{1}{i} [\nu(\vec{\rho}) \frac{\delta}{\delta\nu(\vec{\rho})} - \nu^*(\vec{\rho}) \frac{\delta}{\delta\nu^*(\vec{\rho})}] g = \frac{1}{i} \hat{M}(\vec{\rho}) g[\nu, \nu^*; x, \vec{\rho}]. \quad (9)$$

Inserting (3a), (9), and (7) in (2), we obtain an equation for the characteristic functionals Ψ_x /191/:

$$\frac{\partial \Psi_x[\nu, \nu^*]}{\partial x} = \frac{i}{2k} \iint_{-\infty}^{\infty} [\nu(\vec{\rho}) \Delta \frac{\delta\Psi_x}{\delta\nu(\vec{\rho})} - \nu^*(\vec{\rho}) \Delta \frac{\delta\Psi_x}{\delta\nu^*(\vec{\rho})}] d^2\rho -$$

$$- \frac{k^2}{8} \iint_{-\infty}^{\infty} d^2\rho \iint_{-\infty}^{\infty} d^2\rho' A(\vec{\rho} - \vec{\rho}\,') \hat{M}(\vec{\rho}) \hat{M}(\vec{\rho}\,') \Psi_x[\nu, \nu^*]. \quad (10)$$

Let us now consider some properties of equation (10). This is an equation with variational derivatives of second order, since it contains products of the operators $\hat{M}(\vec{\rho})$ and $\hat{M}(\vec{\rho}\,')$, each of which is a first-order differential operator.

The operator $\hat{M}$, as is readily seen by replacing ν, ν^* with the new variables $a = \frac{1}{2}(\nu + \nu^*)$, $b = \frac{1}{2i}(\nu - \nu^*)$, is Hermitian, i.e., $\hat{M}^* = \hat{M}$. The function $A(\vec{\rho} - \vec{\rho}\,')$ is a positive-definite kernel, and we thus conclude that the operator

$$\hat{K} = -\frac{k^2}{8} \iint_{-\infty}^{\infty} d^2\rho \iint_{-\infty}^{\infty} d^2\rho' A(\vec{\rho} - \vec{\rho}\,') \hat{M}(\vec{\rho}) \hat{M}(\vec{\rho})$$

is negative-definite.

Equation (10) is clearly an analog of the Einstein—Fokker—Planck equation. It differs from the conventional EFP equation in some respects, however. First, the EFP equation is written for the probability density, and not for the characteristic functional, so that (10) is an analog of the Fourier transform of this equation. Second, the probability density, as is known, does not exist in the infinite function space $u(x, \vec{\rho})$, but the characteristic functional does exist in this case. Therefore, we may write the Fourier transform of the EFP equation in this case, but not the equation as such. The presence of the second derivatives in (10) corresponds to a quadratic (in u) diffusion coefficient. The negative-definiteness of the operator $\hat{K}$ corresponds to the negative-definiteness of the Laplace operator in the EFP equation.

Since in our approximation the field $u(x, \vec{\rho})$ is described by an analog of the EFP equation, we will refer to this approximation as the Markov approximation.

Equation (10) has one further property, which is of the greatest importance for the following. In the first term on the right in (10) and in the operator $\hat{M}$, each differentiation $\delta/\delta\nu(\vec{\rho})$ (or $\delta/\delta\nu^*(\vec{\rho})$) is accompanied by multiplication by the function $\nu(\vec{\rho})$ (or $\nu^*(\vec{\rho})$). This property of equation (10) enables us to seek its solutions in the form of powers of functionals. Let $K_{n,m}[\nu, \nu^*]$ be a functional of the form

$$K_{n,m}[\nu, \nu^*] = \int_{-\infty}^{\infty}\cdots\int M_{n,m}(x; \vec{\rho}_1, \ldots, \vec{\rho}_n; \vec{\rho}_1{}', \ldots, \vec{\rho}_m{}') \cdot$$
$$\cdot \nu(\vec{\rho}_1) \cdots \nu(\vec{\rho}_n) \nu^*(\vec{\rho}_1{}') \cdots \nu^*(\vec{\rho}_m{}') d^2\rho_1 \cdots d^2\rho_n d^2\rho_1' \cdots d^2\rho_m' . \tag{11}$$

The function $M_{n,m}$ is clearly symmetric in all its arguments $\vec{\rho}_i$ and $\vec{\rho}_j{}'$. Then,

$$\frac{\delta K_{n,m}}{\delta\nu(\vec{\rho})} = n \int_{-\infty}^{\infty}\cdots\int M_{n,m}(x, \vec{\rho}, \vec{\rho}_2, \ldots, \vec{\rho}_n; \vec{\rho}_1{}', \ldots, \rho_m')$$
$$\nu(\vec{\rho}_2) \cdots \nu(\vec{\rho}_n) \nu^*(\vec{\rho}_1{}') \cdots \nu^*(\vec{\rho}_m{}') d^2\rho_2 \cdots d^2\rho_n d^2\rho_1' \cdots d^2\rho_m'$$

and

$$\iint_{-\infty}^{\infty} \nu(\vec{\rho}) \Delta \frac{\delta K_{n,m}}{\delta\nu(\vec{\rho})} d^2\rho = \int_{-\infty}^{\infty}\cdots\int [n \Delta(\vec{\rho}_1) M_{n,m}(x, \vec{\rho}_1, \ldots, \vec{\rho}_n, \vec{\rho}_1{}', \ldots, \vec{\rho}_m{}')$$
$$\nu(\vec{\rho}_1) \cdots \nu(\vec{\rho}_n) \nu^*(\vec{\rho}_1{}') \cdots \nu^*(\vec{\rho}_m{}') d^2\rho_1 \cdots d^2\rho_n d^2\rho_1' \cdots d^2\rho_m' . \tag{12}$$

If we make the kernel of this functional symmetric with respect to the variables $\vec{\rho}_1, \ldots, \vec{\rho}_n$, we get

$$\iint_{-\infty}^{\infty} \nu(\vec{\rho}) \Delta \frac{\delta K_{n,m}}{\delta\nu(\vec{\rho})} d^2\rho = \int_{-\infty}^{\infty}\cdots\int \Big\{ [\Delta_1 + \Delta_2 + \ldots + \Delta_n] M_{n,m}(\nu, \vec{\rho}_n, \vec{\rho}_n, \vec{\rho}_n{}'$$
$$\cdots \vec{\rho}_m{}') \Big\} \nu(\vec{\rho}_1) \cdots \nu(\vec{\rho}_n) \nu^*(\vec{\rho}_1{}') \cdots \nu^*(\vec{\rho}_m{}') d^2\rho_1 \cdots d^2\rho_n d^2\rho_1' \cdots d^2\rho_n' . \tag{13}$$

The functional (13) is of the same type as the functional (11) but its kernel is obtained from $M_{n,m}$ by differentiation. Repeating the same manipulations with the operator $\iint_{-\infty}^{\infty} d^2\rho \nu^*(\vec{\rho}) \Delta \frac{\delta}{\partial\nu^*(\vec{\rho})}$ and subtracting the resulting equation from (13), we obtain

$$\frac{i}{2k} \iint_{-\infty}^{\infty} [\nu(\vec{\rho}) \Delta \frac{\delta K_{n,m}}{\delta\nu(\vec{\rho})} - \nu^*(\vec{\rho}) \Delta \frac{\delta K_{n,m}}{\delta\nu(\vec{\rho})}] d^2\rho =$$
$$= \frac{i}{2k} \int_{-\infty}^{\infty}\cdots\int \Big\{ [\Delta_1 + \ldots + \Delta_n - \Delta_1' - \ldots - \Delta_m'] M_{n,m} \Big\} \nu(\vec{\rho}_1) \cdots \nu^*(\vec{\rho}_m{}') d^2\rho_1 \cdots d^2\rho_m' . \tag{14}$$

The second term on the right in (10) can be found in the same way, although the intermediate stages are more tedious and therefore are omitted:

$$\hat{K} \cdot K_{n,m}[\nu, \nu^*] = -\frac{k^2}{8} \int_{-\infty}^{\infty}\cdots\int [\sum_{i=1}^{n} \sum_{j=1}^{n} A(\vec{\rho}_i - \vec{\rho}_j) - \sum_{k=1}^{n} \sum_{j=1}^{n} A(\vec{\rho}_k{}' - \vec{\rho}_j) -$$
$$- \sum_{i=1}^{n} \sum_{k=1}^{n} A(\vec{\rho}_i - \vec{\rho}_k{}') + \sum_{k=1}^{m} \sum_{e=1}^{m} A(\vec{\rho}_k{}' - \vec{\rho}_e{}')] M_{n,m}(x, \vec{\rho}_1, \ldots, \vec{\rho}_n, \vec{\rho}_1{}', \ldots, \vec{\rho}_m{}') .$$
$$\cdot \nu(\vec{\rho}_1) \cdots \nu(\vec{\rho}_n) \nu^*(\vec{\rho}_1{}') \cdots \nu^*(\vec{\rho}_m{}') d^2\rho_1 \cdots d^2\rho_m d^2\rho_1' \cdots d^2\rho_m' . \tag{15}$$

Inserting (14) and (15) in (10) and using the derivative of (11) with respect to x, we obtain an equation which only contains functionals of the form $(\nu^n \nu^{*m})$. Equating the coefficients of ν, ν^* in this equation, we obtain an equation for the function $M_{n,m}$:

$$\frac{\partial M_{n,m}}{\partial x} = \frac{i}{2k}[\Delta_1 + \Delta_2 + \dots + \Delta_n - \Delta_1' - \Delta_2' - \dots - \Delta_m'] M_{n,m} - \\ - \frac{k^2}{8} Q_{n,m}(\vec{\rho}_1, \dots, \vec{\rho}_n, \vec{\rho}_1', \dots, \vec{\rho}_m') M_{n,m}(x, \vec{\rho}_1, \dots, \vec{\rho}_n, \vec{\rho}_1', \dots, \vec{\rho}_m') \tag{16}$$

where

$$Q_{n,m} = \sum_{i=1}^{n}\sum_{j=1}^{n} A(\vec{\rho}_i - \vec{\rho}_j) - \sum_{k=1}^{m}\sum_{j=1}^{n} A(\vec{\rho}_k' - \vec{\rho}_j) - \sum_{i=1}^{n}\sum_{k=1}^{m} A(\vec{\rho}_i - \vec{\rho}_k') + \sum_{k=1}^{m}\sum_{l=1}^{m} A(\vec{\rho}_k' - \vec{\rho}_l'). \tag{17}$$

The functional $K_{n,m}[\nu, \nu^*]$ clearly is not a characteristc functional ($K_{n,m}$ indefinitely increases for $|\nu| \to \infty$, whereas $|\Psi_x| \leqslant 1$). However, $\Psi_x[\nu, \nu^*]$ can be expanded in a series in $K_{n,m}$:

$$\Psi_x[\nu, \nu^*] = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{i^{n+m}}{n!m!} K_{n,m}[\nu, \nu^*]. \tag{18}$$

If the functions $M_{n,m}$ satisfy equation (16), then the series (18) satisfies equation (10). To clarify the meaning of the functions $M_{n,m}$, we expand the exponential in (1) in a double Taylor series:

$$\Psi_x[\nu, \nu^*] = \left\langle \left\{ \sum_{n=0}^{\infty} \frac{i^n}{n!} \left[\iint_{-\infty}^{\infty} u(x, \vec{\rho}) \nu(\vec{\rho}) d^2\rho \right]^n \right\} \cdot \left\{ \sum_{m=0}^{\infty} \frac{i^m}{m!} \cdot \right. \right. \\ \left. \cdot \left[\iint_{-\infty}^{\infty} u^*(x, \vec{\rho}') \nu^*(\vec{\rho}') d^2\rho' \right]^m \right\} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{i^{n+m}}{n!m!} \int_{-\infty}^{\infty}\!\!\cdots\!\int d^2\rho_1 \cdots d^2\rho_n \cdot \\ \cdot \int_{-\infty}^{\infty}\!\!\cdots\!\int d^2\rho_1' \cdots d^2\rho_m' \langle u(x, \vec{\rho}_1) \cdots u(x, \vec{\rho}_n) u^*(x, \vec{\rho}_1') \cdots u^*(x, \vec{\rho}_m') \rangle \cdot \\ \cdot \nu(\vec{\rho}_1) \cdots \nu(\vec{\rho}_n) \nu^*(\vec{\rho}_1') \cdots \nu^*(\vec{\rho}_m'). \tag{19}$$

Comparing this expression with (18), where $K_{n,m}$ is expressed by (11), we see that

$$M_{n,m}(x, \vec{\rho}_1, \dots, \vec{\rho}_n; \vec{\rho}_1', \dots, \vec{\rho}_m') = \langle u(x, \vec{\rho}_1) \cdots u(x, \vec{\rho}_n) u^*(x, \vec{\rho}_1') \cdots u^*(x, \vec{\rho}_m') \rangle$$

is a moment of order $(n+m)$ of the field u. We have thus derived a closed differential equation (16) for a moment of any order of the field.

Equation (16) should be considered in conjunction with the initial condition

$$M_{n,m}(0; \vec{\rho}_1, \dots, \vec{\rho}_n; \vec{\rho}_1', \dots, \vec{\rho}_m') = u_0(\vec{\rho}_1) \cdots u_0(\vec{\rho}_n) u_0^*(\vec{\rho}_1') \cdots u_0^*(\vec{\rho}_m'). \tag{20}$$

Taking $n=1, m=0$, we obtain $Q_{1,0}(\vec{\rho}_1) = A(0), M_{1,0} = \bar{u}$, and (16) reduces to (65.26). Taking $n=1, m=1$, we obtain

$$Q_{1,1}(\vec{\rho}, \vec{\rho}') = A(0) - A(\vec{\rho}' - \vec{\rho}) - A(\vec{\rho} - \vec{\rho}') + A(0) = 2\pi H(\vec{\rho} - \vec{\rho}')$$
$$M_{1,1}(x, \vec{\rho}, \vec{\rho}') = \Gamma_2(x, \vec{\rho}, \vec{\rho}')$$

and (16) reduces to (65.31).

Let us also write out the equation for the fourth moment $M_{2,2}(x, \vec{\rho}_1, \vec{\rho}_2; \vec{\rho}_1', \vec{\rho}_2') = \rho_4(x, \vec{\rho}_1, \vec{\rho}_2; \vec{\rho}_1', \vec{\rho}_2')$ (the fourth-order coherence function), which determines the intensity fluctuations of the field u. Taking $n=2$, $m=2$, we obtain

$$\begin{aligned} Q_{2,2}(\vec{\rho}_1, \vec{\rho}_2; \vec{\rho}_1', \vec{\rho}_2') &= 2[2A(0) + A(\vec{\rho}_2 - \vec{\rho}_1) + A(\vec{\rho}_2' - \vec{\rho}_1') - \\ &- A(\vec{\rho}_1 - \vec{\rho}_1') - A(\vec{\rho}_2 - \vec{\rho}_2') - A(\vec{\rho}_2 - \vec{\rho}_1') - A(\vec{\rho}_1 - \vec{\rho}_2')] = \\ &= 2\pi[H(\vec{\rho}_1 - \vec{\rho}_1') + H(\vec{\rho}_2 - \vec{\rho}_2') + H(\vec{\rho}_2 - \vec{\rho}_1') + H(\vec{\rho}_1 - \vec{\rho}_2') - \\ &- H(\vec{\rho}_2 - \vec{\rho}_1) - H(\vec{\rho}_2' - \vec{\rho}_1')] \equiv 2\pi F(\vec{\rho}_1, \vec{\rho}_2; \vec{\rho}_1', \vec{\rho}_2') \end{aligned} \tag{21}$$

and the equation for Γ_4 takes the form *

$$\frac{\partial \Gamma_4(x, \vec{\rho}_1, \vec{\rho}_2; \vec{\rho}_1', \vec{\rho}_2')}{\partial x} = \frac{i}{2k}(\Delta_1 + \Delta_2 - \Delta_1' - \Delta_2')\Gamma_4 - \frac{\pi k^2}{4} F(\vec{\rho}_1, \vec{\rho}_2; \vec{\rho}_1', \vec{\rho}_2')\Gamma_4$$

$$\Gamma_4(0, \vec{\rho}_1, \vec{\rho}_2; \vec{\rho}_1', \vec{\rho}_2') = u_0(\vec{\rho}_1)\, u_0(\vec{\rho}_2)\, u_0^*(\vec{\rho}_1')\, u_0^*(\vec{\rho}_2'). \tag{22}$$

Equation (22) also can be obtained directly from the parabolic equation for u, like the equation for Γ_2. Note that once the function Γ_4 has been determined, the expression

$$\Gamma_4(x, \vec{\rho}_1, \vec{\rho}_2; \vec{\rho}_1, \vec{\rho}_2) = \langle u(x, \vec{\rho}_1)\, u(x, \vec{\rho}_2) u^*(x, \vec{\rho}_1) u^*(x, \vec{\rho}_2) \rangle =$$

$$= \langle I(x, \vec{\rho}_1) I(x, \vec{\rho}_2) \rangle \qquad (I(x, \vec{\rho}) = |u(x, \vec{\rho})|^2)$$

considered together with Γ_2 enables us to find the correlation function of the intensity fluctuations in the plane x = const:

$$\begin{aligned} B_I(x, \vec{\rho}_1, \vec{\rho}_2) &= \langle I(x, \vec{\rho}_1) I(x, \vec{\rho}_2) \rangle - \langle I(x, \vec{\rho}_1) \rangle \langle I(x, \vec{\rho}_2) \rangle = \\ &= \Gamma_4(x, \vec{\rho}_1, \vec{\rho}_2; \vec{\rho}_1, \vec{\rho}_2) - \Gamma_2(x, \vec{\rho}_1, \vec{\rho}_1)\Gamma_2(x, \vec{\rho}_2, \vec{\rho}_2). \end{aligned} \tag{23}$$

In particular, the mean square intensity fluctuation is given by

$$\sigma_I^2(x, \vec{\rho}) = B_I(x, \vec{\rho}, \vec{\rho}) = \Gamma_4(x, \vec{\rho}, \vec{\rho}; \vec{\rho}, \vec{\rho}) - [\Gamma_2(x, \vec{\rho}, \vec{\rho})]^2. \tag{24}$$

However, unlike the equations for $\bar{u}$ and Γ_2, equation (22) cannot be solved in a general form.

Equation (22), like the previous equations for $\bar{u}$ and Γ_2, was derived without applying a perturbation theory. Therefore it is also valid in the region of strong intensity fluctuations, provided the reflected waves in this region are sufficiently weak and the parabolic equation applies. This is apparently the situation corresponding to the propagation of light in a turbulent atmosphere, where the intensity fluctuations reach large magnitudes over relatively short distances (of the order of 1 km), whereas

* A particular case of equation (22) was derived by Shishov /190/ by means of selective summation of Feynman diagrams. De Wolf /188, 189/ also tried to derive an expression for Γ_4, but his results for Γ_4 are definitely erroneous (in particular, the conclusion regarding the Rayleigh distribution of the intensity fluctuations stems from the unjustified omission of the connected diagrams describing the fourth cumulants of the field u (see /196/).

back-scattering of light is still insignificant over these distances (the reflected waves may be neglected).

In the following two sections we will consider the limits of application of the Markov approximation and of the parabolic equation. The computations that follow are fairly tedious and those readers who are not particularly interested in the exact details of the mathematical procedure may safely skip the next two sections.

§ 67. The limits of application of the Markov approximation

In this paragraph, following Klyatskin's work /192/* we will consider here the corrections to the solution using the Markov-approximation which arise when the finite radius of correlation in the longitudinal direction is taken into consideration. As before, we regard $\epsilon_1(x, \vec{\rho})$ as a Gaussian (normal) random field, but the correlation function

$$\langle \epsilon_1(x, \vec{\rho})\epsilon_1(x', \vec{\rho}')\rangle = B_\epsilon(x - x', \vec{\rho} - \vec{\rho}')$$

is no longer a delta-function, although it is still a "sharp" peaked function of the argument $(x - x')$. As in §65, we will derive the equations for $\bar{u}(x, \vec{\rho})$ and $\Gamma_2(x, \vec{\rho}_1, \vec{\rho}_2)$.

As before, we start with the parabolic equation (65.13) and apply the Furutsu—Novikov formula (65.5), since $\epsilon_1(x, \vec{\rho})$ is treated at this stage as a normal field.

The mathematics is considerably simplified in the Fourier representation, and the original equation

$$2ik\frac{\partial u(x, \vec{\rho})}{\partial x} + \Delta u(x, \vec{\rho}) + k^2\epsilon_1(x, \vec{\rho})u(x, \vec{\rho}) = 0 \tag{1}$$

is therefore replaced with the equation for the spectral components $u(x, \vec{\kappa})$, $\epsilon_1(x, \vec{\kappa})$, defined by the relations**

$$u(x, \vec{\rho}) = \iint\limits_{-\infty}^{\infty} u(x, \vec{\kappa})\exp(i\vec{\kappa}\vec{\rho})d^2\kappa,\ \epsilon_1(x, \vec{\rho}) = \iint\limits_{-\infty}^{\infty} \epsilon_1(x, \vec{\kappa})\exp(i\vec{\kappa}\vec{\rho})d^2\kappa;$$

$$\frac{\partial u(x, \vec{\kappa})}{\partial x} + \frac{i\kappa^2}{2k}u(x, \vec{\kappa}) = \frac{ik}{2}\iint\limits_{-\infty}^{\infty} \epsilon_1(x, \mathrm{p})u(x, \vec{\kappa} - \mathrm{p})d^2\mathrm{p} \tag{2}$$

$$u(0, \vec{\kappa}) = u_0(\vec{\kappa}).$$

Averaging this equation, we obtain for $\bar{u}(x, \vec{\kappa}) \equiv \langle u(x, \vec{\kappa})\rangle$:

$$\frac{\partial \bar{u}(x, \vec{\kappa})}{\partial x} + \frac{i\kappa^2}{2k}\bar{u}(x, \vec{\kappa}) = \frac{ik}{2}\iint\limits_{-\infty}^{\infty}\langle \epsilon_1(x, \mathrm{p})u(x, \vec{\kappa} - \mathrm{p})\rangle d^2\mathrm{p}. \tag{3}$$

The quantity $u(x, \vec{\kappa} - \mathrm{p})$ is a functional of $\epsilon_1(x, \mathrm{p})$, and by (2) $u(x, \vec{\kappa} - \mathrm{p})$ is dependent on the values of $\epsilon_1(\xi, \vec{\kappa})$ for $0 \leqslant \xi \leqslant x$. Since $\epsilon_1(x, \vec{\mathrm{p}})$ is a Gaussian

* Certain inaccuracies detected in /192/ have been corrected.

** For the sake of simplicity, we use the conventional form of the Fourier integral (rather than the Fourier—Stieltjes integral), always remembering that this formula can be generalized by passing to the limit (see Chapter 1).

random field, we have by (65.5)

$$\langle \epsilon_1(x, \mathrm{p})u(x, \vec{\kappa}-\mathrm{p})\rangle = \int_0^x dx' \iint_{-\infty}^{\infty} d^2q \langle \epsilon_1(x, \mathrm{p})\epsilon_1(x', \mathrm{q})\rangle \langle \frac{\delta u(x, \vec{\kappa}-\mathrm{p})}{\delta \epsilon_1(x', \mathrm{q})}\rangle. \quad (4)$$

The field $\epsilon_1(x, \vec{\mathrm{p}})$ is statistically homogeneous, so that its two-dimensional spectral density satisfies the relation

$$\langle \epsilon_1(x, \mathrm{p})\epsilon_1(x', \mathrm{q})\rangle = \delta(\mathrm{p}+\mathrm{q})F_\epsilon(x-x', \mathrm{p}). \quad (5)$$

Inserting (5) in (4), we find

$$\langle \epsilon_1(x, \mathrm{p})u(x, \vec{\kappa}-\mathrm{p})\rangle = \int_0^x dx' F_\epsilon(x-x', \mathrm{p}) \langle \frac{\delta u(x, \vec{\kappa}-\mathrm{p})}{\delta \epsilon_1(x', -\mathrm{p})}\rangle. \quad (6)$$

Since $\epsilon_1(x, \mathrm{p})$ is no longer regarded as a delta-correlated field, equation (6) contains the variational derivative $\delta u(x)/\delta \epsilon_1(x')$ for $x' \neq x$. Let us proceed with the computation of this derivative.

Using the expression

$$\frac{\partial u(x, \vec{\kappa})}{\partial x} + \frac{i\kappa^2}{2k} u(x, \vec{\kappa}) = \exp\left(-\frac{i\kappa^2 x}{2k}\right) \frac{\partial}{\partial x}\left[u(x, \vec{\kappa}) \exp\left(\frac{i\kappa^2 x}{2k}\right)\right],$$

we transform equation (2) to the form

$$\frac{\partial}{\partial x}\left[u(x, \vec{\kappa}) \exp\left(\frac{i\kappa^2 x}{2k}\right)\right] = \frac{ik}{2} \exp\left(\frac{i\kappa^2 x}{2k}\right) \iint_{-\infty}^{\infty} \epsilon_1(x, \mathrm{p})u(x, \vec{\kappa}-\mathrm{p})d^2p. \quad (7)$$

Integrating this equation over x from x' to x and then multiplying by $\exp\left(-\frac{i\kappa^2 x}{2k}\right)$, we obtain the integral equation

$$u(x, \vec{\kappa}) = u(x', \vec{\kappa}) \exp\left\{-\frac{i\kappa^2(x-x')}{2k}\right\} +$$

$$+ \frac{ik}{2} \int_{x'}^x \exp\left\{-\frac{i\kappa^2(x-\xi)}{2k}\right\} d\xi \iint_{-\infty}^{\infty} \epsilon_1(\xi, \mathrm{p})u(\xi, \vec{\kappa}-\mathrm{p})d^2p. \quad (8)$$

Here $u(x', \vec{\kappa})$ may be regarded as the initial condition. Successive integration of equation (8) yields a series where each term is a linear function of $u(x', \vec{\kappa}')$. The solution of (8) is therefore sought in the form

$$u(x, \vec{\kappa}) = \iint_{-\infty}^{\infty} G(x, \vec{\kappa}; x' \vec{\kappa}')u(x', \vec{\kappa}')d^2\kappa' \quad (9)$$

where G is the Green's function for the problem. Inserting (9) in (8), we obtain for G

$$G(x, \vec{\kappa}; x', \vec{\kappa}') = \delta(\vec{\kappa}-\vec{\kappa}') \exp\left[-\frac{i\kappa^2(x-x')}{2k}\right] +$$

$$+ \frac{ik}{2} \int_{x'}^x d\xi \iint_{-\infty}^{\infty} d^2p \exp\left[-\frac{i\kappa^2(x-\xi)}{2k}\right] \epsilon_1(\xi, \mathrm{p}) G(\xi, \vec{\kappa}-\mathrm{p}; x', \vec{\kappa}') \quad (10)$$

Note that for $x'=x$, equation (10) leads to the expression

$$G(x, \vec{\kappa}; x, \vec{\kappa}') = \delta(\vec{\kappa} - \vec{\kappa}'). \tag{11}$$

Acting on (10) with the operator $\delta/\delta\epsilon_1(x_0, \vec{\kappa}_0)$, where $x'<x_0<x$, we obtain an equation for the variational derivative $\delta G/\delta\epsilon_1$:

$$\frac{\delta G(x, \vec{\kappa}; x', \vec{\kappa}')}{\delta\epsilon_1(x_0, \vec{\kappa}_0)} = \frac{ik}{2} \exp\left[-\frac{i\kappa^2(x-x_0)}{2k}\right] G(x_0, \vec{\kappa} - \vec{\kappa}_0, x', \vec{\kappa}') + $$
$$+ \frac{ik}{2} \int_{x_0}^{x} d\xi \iint_{-\infty}^{\infty} d^2p \exp\left\{-\frac{i\kappa^2(x-\xi)}{2k}\right\} \epsilon_1(\xi, \mathrm{p}) \frac{\delta G(\xi, \vec{\kappa} - \mathrm{p}; x', \vec{\kappa}')}{\delta\epsilon_1(x_0, \vec{\kappa}_0)}. \tag{12}$$

The lower integration limit in (12) has been changed from x' to x_0, since $\delta G(\xi)/\delta\epsilon_1(x_0) = 0$ for $\xi < x_0$. The solution of equation (12) is

$$\frac{\delta G(x, \vec{\kappa}; x', \vec{\kappa}')}{\delta\epsilon_1(x_0, \vec{\kappa}_0)} = \frac{ik}{2} \iint_{-\infty}^{\infty} G(x, \vec{\kappa}; x_0, \vec{\kappa}_0 + \mathrm{q}) G(x_0, \mathrm{q}; x', \vec{\kappa}') d^2q. \tag{13}$$

To show that this is indeed so, we insert (13) in the right-hand side of (12) and apply equation (10):

$$\frac{ik}{2} \exp\left[-\frac{i\kappa^2(x-x_0)}{2k}\right] G(x_0, \vec{\kappa} - \vec{\kappa}_0, x', \vec{\kappa}') + \left(\frac{ik}{2}\right)^2 \int_{x_0}^{x} d\xi \iint_{-\infty}^{\infty} d^2p \iint_{-\infty}^{\infty} d^2q$$
$$\exp\left[-\frac{i\kappa^2(x-\xi)}{2k}\right] \epsilon_1(\xi, \mathrm{p}) G(\xi, \vec{\kappa} - \mathrm{p}; x_0, \vec{\kappa}_0 + \mathrm{q}) G(x_0, \mathrm{q}, x', \vec{\kappa}') =$$
$$= \frac{ik}{2} \iint_{-\infty}^{\infty} d^2q\, G(x_0, \mathrm{q}, x', \vec{\kappa}') \left\{ \exp\left[-\frac{i\kappa^2(x-x_0)}{2k}\right] \delta(\vec{\kappa} - (\vec{\kappa}_0 + \mathrm{q})) + \right.$$
$$\left. + \frac{ik}{2} \int_{x_0}^{x} d\xi \iint_{-\infty}^{\infty} d^2p \exp\left[-\frac{i\kappa^2(x-\xi)}{2k}\right] \epsilon_1(\xi, \mathrm{p}) G(\xi, \vec{\kappa} - \mathrm{p}; x_0, \vec{\kappa}_0 + \mathrm{q}) \right\} =$$
$$= \frac{ik}{2} \iint_{-\infty}^{\infty} d^2q\, G(x_0, \mathrm{q}; x', \vec{\kappa}') G(x, \vec{\kappa}; x_0, \vec{\kappa}_0 + \mathrm{q}).$$

Thus, inserting (13) in (12) yields an identity, and the validity of this expression is thus proved. Incidentally, the Green's function in the coordinate representation (spatial coordinates)

$$G(x, \vec{\rho}; x', \vec{\rho}') = \frac{1}{4\pi^2} \iint_{-\infty}^{\infty} d^2\kappa \iint_{-\infty}^{\infty} d^2\kappa'\, e^{i(\vec{\kappa}\vec{\rho} - \vec{\kappa}'\vec{\rho}')} G(x, \vec{\kappa}; x', \vec{\kappa}')$$

satisfies a relation which is equivalent to (13):*

$$\frac{\delta G(x, \vec{\rho}, x', \vec{\rho}')}{\delta\epsilon_1(x_0, \vec{\rho}_0)} = \frac{ik}{2} G(x, \vec{\rho}; x_0, \vec{\rho}_0) G(x_0, \vec{\rho}_0; x', \vec{\rho}'). \tag{13a}$$

Acting on (9) with the operator $\delta/\delta\epsilon_1(x_0, \vec{\kappa}_0)$ and using (13) and (9), we find

$$\frac{\delta u(x, \vec{\kappa})}{\delta\epsilon_1(x_0, \vec{\kappa}_0)} = \iint_{-\infty}^{\infty} u(x', \vec{\kappa}') d^2x' \frac{ik}{2} \iint_{-\infty}^{\infty} G(x, \vec{\kappa}; x_0, \vec{\kappa}_0 + \mathrm{q}) G(x_0, \mathrm{q}; x', \vec{\kappa}') d^2q =$$
$$= \frac{ik}{2} \iint_{-\infty}^{\infty} G(x, \vec{\kappa}; x_0, \vec{\kappa}_0 + \mathrm{q}) u(x_0, \mathrm{q}) d^2q$$

* An expression for $\delta G/\delta\epsilon_1$ can be found in a straightforward fashion, by solving (10) for G. The solution is conveniently obtained by the diagram method.

i.e., changing the notation and substituting $\vec{\kappa}_0 + q \to q$, we obtain

$$\frac{\delta u(x,\vec{\kappa})}{\delta\epsilon_1(x',\vec{\kappa}')} = \frac{ik}{2}\iint_{-\infty}^{\infty} G(x,\vec{\kappa};x',q)\,u(x',q-\vec{\kappa}')\,d^2q. \tag{14}$$

Note that in the coordinate representation this formula takes the form

$$\frac{\delta u(x,\vec{\rho})}{\delta\epsilon_1(x_0,\vec{\rho}_0)} = \frac{ik}{2} G(x,\vec{\rho};x_0,\vec{\rho}_0)\,u(x_0,\vec{\rho}_0) \tag{14a}$$

and for $x_0 = x$, due to the relation $G(x,\vec{\rho};x,\vec{\rho}_0) = \delta(\vec{\rho}-\vec{\rho}_0)$ (see (11)), it reduces to the previous expression (65.24).

Let us now return to (6). The function $F_\epsilon(x-x',p)$ is a peaked function of the argument $x-x'$, and therefore the main contribution to the integral over $x'$is in the region $x' \sim x$. We will expand the function $\delta u(x,\vec{\kappa})/\delta\epsilon_1(x',\vec{\kappa}')$ in powers of $(x-x')$ to terms of the order $(x-x')^2$. To this end, we need the expansions of the functions $G(x,\vec{\kappa};x',\vec{q})$ and $u(x',\vec{q}-\vec{\kappa}')$ entering the integrand in (14):

$$u(x',q-\vec{\kappa}') = u(x,q-\vec{\kappa}') - (x-x')\frac{\partial u(x,q-\vec{\kappa}')}{\partial x} + \frac{1}{2}(x-x')^2\frac{\partial^2 u(x,q-\vec{\kappa}')}{\partial x^2} + \ldots \tag{15}$$

$$G(x,\vec{\kappa};x',q) = G(x,\vec{\kappa};x,q) - (x-x')\left[\frac{\partial G(x,\vec{\kappa};x',q)}{\partial x'}\bigg|_{x'=x}\right] + \frac{1}{2}(x-x')^2\left[\frac{\partial^2 G(x,\vec{\kappa};x',q)}{\partial x'^2}\bigg|_{x'=x}\right] + \ldots \tag{16}$$

The derivatives entering (15) can be found from equation (2). The derivatives of G can be obtained by differentiating equation (10) (the integral terms are eliminated by setting $x' = x$ after differentiation). Following the substitution of the derivatives, expressions (15) and (16) take on a highly complex form, and they are consequently omitted. A markedly simpler result is obtained when the expressions are inserted in (14) and the integral over $\vec{q}$ is taken. The final expansion for $\delta u/\delta\epsilon_1$ has the form

$$\frac{\delta u(x,\vec{\kappa})}{\delta\epsilon_1(x',\vec{\kappa}')} = \frac{ik}{2}\Big\{u(x,\vec{\kappa}-\vec{\kappa}') + \frac{i(x-x')}{2k}\vec{\kappa}'(\vec{\kappa}'-2\vec{\kappa})u(x,\vec{\kappa}-\vec{\kappa}') - \frac{(x-x')^2}{4}\Big[\frac{(\vec{\kappa}'^2-2(\vec{\kappa}\vec{\kappa}'))^2}{2k^2}u(x,\vec{\kappa}-\vec{\kappa}') + \iint_{-\infty}^{\infty}\vec{\kappa}'q\,\epsilon_1(x,q)\,u(x,\vec{\kappa}-\vec{\kappa}'-q)\,d^2q\Big] + \ldots\Big\} \equiv \frac{ik}{2}R(x,\vec{\kappa};x',\vec{\kappa}'). \tag{17}$$

Inserting expansion (17) in (6) and averaging, we obtain

$$\langle\epsilon_1(x,p)u(x,\vec{\kappa}-p)\rangle = \frac{ik}{2}\int_0^x dx'\,F_\epsilon(x-x',\vec{\kappa})\;\;\bar{u}(x,\vec{\kappa}) + \frac{i(x-x')}{2k}(2\vec{\kappa}p-p^2)\bar{u}(x,\vec{\kappa}) - \frac{(x-x')^2}{4}\Big[\frac{(2\vec{\kappa}p-p^2)^2}{2k^2}\bar{u}(x,\vec{\kappa}) - \iint_{-\infty}^{\infty} p\,q\,\langle\epsilon_1(x,q)u(x,\vec{\kappa}-q)\rangle d^2q\Big] + \ldots \tag{18}$$

We define a new function

$$A_\alpha(x,p) = 2\int_0^x F_\epsilon(x-x',p)\,(x-x')^\alpha\,dx'. \tag{19}$$

Then (18) takes the form

$$\langle \epsilon_1(x, \mathrm{p}) u(x, \vec{\kappa}-\mathrm{p}) \rangle = \frac{ik}{2}\left\{\frac{1}{2} A_0(x, \mathrm{p}) \bar{u}(x, \vec{\kappa}) + \right.$$
$$+ \frac{i}{4k}(2\vec{\kappa}\mathrm{p} - \mathrm{p}^2) A_1(x, \mathrm{p}) \bar{u}(x, \vec{\kappa}) - \frac{(2\vec{\kappa}\mathrm{p} - \mathrm{p}^2)^2}{16k^2} A_2(x, \mathrm{p}) \bar{u}(x, \vec{\kappa}) +$$
$$\left. + \frac{1}{8} A_2(x, \mathrm{p}) \iint_{-\infty}^{\infty} \mathrm{p}\,\mathrm{q} \langle \epsilon_1(x, \mathrm{q}) u(x, \vec{\kappa}-\mathrm{q}) \rangle d^2q + \ldots \right\}. \tag{20}$$

Since the expansion is accurate to terms of the order $(x-x')^2$, the expression described by the first term of (20)

$$\langle \epsilon_1(x, \mathrm{q}) u(x, \vec{\kappa}-\mathrm{q}) \rangle = \frac{ik}{4} A_0(x, \mathrm{q}) \bar{u}(x, \vec{\kappa}) + \ldots$$

may be substituted in the last term in (20). This gives

$$\langle \epsilon_1(x, \mathrm{p}) u(x, \vec{\kappa}-\mathrm{p}) \rangle = \frac{ik}{2}\left\{\frac{1}{2} A_0(x, \mathrm{p}) \bar{u}(x, \vec{\kappa}) + \frac{i(2\vec{\kappa}\mathrm{p} - \mathrm{p}^2)}{4k} A_1(x, \mathrm{p}) \bar{u}(x, \vec{\kappa}) - \right.$$
$$\left. - \frac{(2\vec{\kappa}\mathrm{p} - p^2)^2}{16k^2} A_2(x, \mathrm{p}) \bar{u}(x, \vec{\kappa}) + \frac{ik}{32} A_2(x, \mathrm{p}) \bar{u}(x, \vec{\kappa}) \iint_{-\infty}^{\infty} \mathrm{p}\,\mathrm{q} A_0(x, \mathrm{q}) d^2q + \ldots \right\} \tag{21}$$

We can now insert expansion (21) in equation (3). Integration over $\vec{\rho}$ gives rise to the functions

$$A_{\alpha,\beta}(x) = \iint_{-\infty}^{\infty} A_\alpha(x, \mathrm{p}) |\mathrm{p}|^\beta d^2p. \tag{22}$$

Since $F_\epsilon(x-x', \mathrm{p})$ is an even function of p, $A_\alpha(x, \mathrm{p})$ has the same property. Therefore the following integrals vanish:

$$\iint_{-\infty}^{\infty} A_\alpha(x, \mathrm{p}) p_i d^2p = 0, \quad \iint_{-\infty}^{\infty} A_\alpha(x, \mathrm{p}) \mathrm{p}^2 \mathrm{p}_i d^2p = 0.$$

For the sake of simplicity, let the random field $\epsilon_1(x, \vec{\rho})$ be statistically isotropic in the plane $x =$ const. In this case $F_\epsilon(x-x', \mathrm{p}) = F_\epsilon(x-x', p)$ and $A_\alpha(x, \vec{\mathrm{p}}) = A_\alpha(x, p)$. It is clear from symmetry considerations that for an isotropic field

$$\iint_{-\infty}^{\infty} A_\alpha(x, p) p_i p_j d^2p = \delta_{ij} f(x).$$

Contracting the subscripts i, j, we obtain

$$A_{\alpha,2}(x) = 2f(x)$$

whence

$$\iint_{-\infty}^{\infty} A_\alpha(x, p) p_i p_j d^2p = \frac{1}{2}\delta_{ij} A_{\alpha,2}(x). \tag{23}$$

Integration of (21) over p gives

$$\iint_{-\infty}^{\infty} \langle \epsilon_1(x, \mathrm{p}) u(x, \vec{\kappa}-\mathrm{p}) \rangle d^2p = \frac{ik}{2}\left\{\frac{1}{2} A_{0,0}(x) \bar{u}(x, \vec{\kappa}) - \frac{i}{4k} A_{1,2}(x) \cdot \right.$$
$$\left. \cdot \bar{u}(x, \vec{\kappa}) - \frac{\kappa_i \kappa_j}{4k^2} \cdot \frac{\delta_{ij}}{2} A_{2,2}(x) \bar{u}(x, \vec{\kappa}) - \frac{1}{16k^2} A_{2,4}(x) \bar{u}(x, \vec{\kappa}) + \ldots \right\}. \tag{24}$$

Inserting (24) in (3), we obtain

$$\frac{\partial \bar{u}(x,\vec{\kappa})}{\partial x}+\frac{i\kappa^2}{2k}\bar{u}(x,\vec{\kappa})+\frac{k^2}{8}A_{0,0}(x)\bar{u}(x,\vec{\kappa})=$$
$$=\frac{k^2}{4}\left\{\frac{i}{4k}A_{1,2}(x)\bar{u}(x,\vec{\kappa})+\frac{\kappa^2}{8k^2}A_{2,2}(x)\bar{u}(x,\vec{\kappa})+\frac{A_{2,4}(x)}{16k^2}\bar{u}(x,\vec{\kappa})+\ldots\right\}. \quad (25)$$

In the coordinate representation, this equation takes the form

$$\frac{\partial \bar{u}(x,\vec{\rho})}{\partial x}-\frac{i}{2k}\Delta\bar{u}(x,\vec{\rho})+\frac{k^2}{8}A_{0,0}(x)\bar{u}(x,\vec{\rho})=$$
$$=\frac{k}{16}\left[iA_{1,2}(x)\bar{u}(x,\vec{\rho})-\frac{A_{2,2}(x)}{2k}\Delta\bar{u}(x,\vec{\rho})+\frac{A_{2,4}(x)}{4k}\bar{u}(x,\vec{\rho})\right]. \quad (25a)$$

Let us compare equation (25a) with (65.26)

$$\frac{\partial \bar{u}(x,\vec{\rho})}{\partial x}-\frac{i}{2k}\Delta\bar{u}+\frac{k^2}{8}A(0)\bar{u}(x,\vec{\rho})=0. \quad (26)$$

The $A(0)$ entering (26) is defined by the equality

$$A(0)=\int_{-\infty}^{\infty}B_\epsilon(\xi,0)\,d\xi=2\int_0^{\infty}B_\epsilon(\xi,0)\,d\xi.$$

The $A_{0,0}(x)$ entering (25a) is given by

$$A_{0,0}(x)=\iint_{-\infty}^{\infty}A_0(x,\mathrm{p})\,d^2p=2\int_0^x dx'\iint_{-\infty}^{\infty}F_\epsilon(x-x',\mathrm{p})\,d^2p=2\int_0^x B_\epsilon(\xi,0)\,d\xi.$$

Let L_0 be the characteristic scale of the function $B_\epsilon(\xi,0)$. Then $A_{0,0}(x)\to A(0)$ for $(x/L_0)\to\infty$. Equation (25) thus differs from the previous equation (26) in two respects: first, the number $A(0)$ is replaced with the function $A_{0,0}(x)$, which vanishes for $x=0$ and goes to $A(0)$ for $(x/L_0)\to\infty$, and second, additional terms appear on the right in equation (25).

Equation (25) is easily solved, but there is not much point in solving this equation, since the resulting corrections are valid only if they are small. Therefore the effect of correction terms should be considered only to the first order of the perturbation theory.

The equation

$$\frac{\partial \bar{u}(x,\vec{\kappa})}{\partial x}+\frac{i\kappa^2}{2k}\bar{u}(x,\vec{\kappa})+\frac{k^2}{8}A_{0,0}(x)\bar{u}(x,\vec{\kappa})=0 \quad (27)$$

has the solution

$$\bar{u}(x,\vec{\kappa})=u_0(\vec{\kappa})\exp\left\{-\frac{i\kappa^2x}{2k}-\frac{k^2}{\delta}\int_0^x A_{0,0}(\xi)\,d\xi\right\} \quad (28)$$

For $\xi\ll L_0$, $A_{0,0}(\xi)\approx 2\sigma_\epsilon^2\xi$, so that for $x\approx L_0$, $\int_0^x A_{0,0}(\xi)\,d\xi\approx\sigma_\epsilon^2x^2$, and (28) takes the form

$$\bar{u}(x,\vec{\kappa})\approx u_0(\vec{\kappa})\exp\left\{-\frac{i\kappa^2x}{2k}-\frac{1}{8}k^2\sigma_\epsilon^2x^2\right\}\quad \sigma x\ll L_0.$$

Thus, for $x \ll L_0$, $\bar{u}$ behaves as $\exp(-x^2/2a^2)$, where $a = \lambda/\pi\sigma_\epsilon$. For $\xi \gg L_0$, $A_{0,0}(\xi) \approx A(0) = \sigma_\epsilon^2 L_1$, where L_1 is the one-dimensional integral scale of the correlation function:

$$L_1 = \frac{1}{B_\epsilon(0,0)} \int_{-\infty}^{\infty} B_\epsilon(\xi, 0)\, d\xi.$$

Clearly, if $x \gg L_0$, the region $\xi \approx L_0$ does not make a significant contribution to the integral in (28), and thus

$$\int_0^x A_{0,0}(\xi)\, d\xi \approx \sigma_\epsilon^2 L_1 x, \quad x \gg L_0$$

so that

$$\bar{u}(x, \vec{\kappa}) = u_0(\vec{\kappa}) \exp\left\{-\frac{i\kappa^2 x}{2k}\right\} \cdot \exp\left\{-\frac{1}{8}\sigma_\epsilon^2 k^2 L_1 x\right\}. \tag{29}$$

Expression (29) is a solution of equation (26). The Markov approximation is thus valid only when

$$x \gg L_0.$$

For $x \lesssim L_0$, on the other hand, a "transient" region is observed in which the Markov process "grows" to "steady-state conditions."

Equation (29) may be written in the form

$$\bar{u}(x, \vec{\kappa}) = u^{(0)}(x, \vec{\kappa}) \exp\left(-\frac{1}{2}\alpha x\right) \tag{31}$$

where $u^{(0)}(x, \vec{\kappa}) = u_0(\vec{\kappa}) \exp(-i\kappa^2 x/2k)$ is the solution of the problem in a medium without fluctuations, and

$$\alpha = \frac{1}{4}\sigma_\epsilon^2 k^2 L_1 \tag{32}$$

is the extinction coefficient.

Let us now consider the corrections associated with the appearance of the additional terms on the right side in equation (25a). We will only consider the region $x \gg L_0$ replacing all the functions $A_{\alpha,\beta}(x)$ with their limiting values $A_{\alpha,\beta}(\infty) \equiv A_{\alpha,\beta}$.

To justify the Markov approximation, the terms on the right side in (25a) should be small compared to the corresponding terms on the left in this equation. This condition leads to the inequalities

$$k^2 A(0) \gg A_{2,4}, \frac{1}{k} \gg A_{2,2}. \tag{33}$$

The term $\frac{ik}{16} A_{1,2}\bar{u}$ alters the phase of the mean field and will be considered at a later stage.

The quantities $A_{\alpha,\beta}$ may be expressed in terms of the three-dimensional spectral density of the fluctuations $\phi_\epsilon(\kappa_1, \kappa_2, \kappa_3)$. If $\phi_\epsilon(\kappa_1, \kappa_2, \kappa_3) = \phi_\epsilon(\sqrt{\kappa_1^2 + \kappa_2^2 + \kappa_3^2})$, we have

$$A_{2,2} = 8\pi_2 \int_0^{\infty} \phi_\epsilon(\kappa)\, \kappa\, d\kappa \tag{34}$$

$$A_{2,4} = 16\pi^2 \int_0^\infty \phi_\epsilon(\kappa)\, \kappa^3 d\kappa \tag{35}$$

$$A_{0,0} = A(0) = 4\pi^2 \int_0^\infty \phi_\epsilon(\kappa)\, \kappa d\kappa = \frac{A_{2,2}}{2}. \tag{36}$$

The quantity $A_{2,4}$ is largely determined by small-scale fluctuations, and $A_{2,2}$ by large-scale fluctuations. We will use the following model for the spectrum of the fluctuations of ϵ:

$$\phi_\epsilon(\kappa) = \frac{A C_\epsilon^2}{(\kappa_0^2 + \kappa^2)^{11/6}} e^{-\frac{\kappa^2}{\kappa_m^2}}, \; \kappa_m \gg \kappa_0 \,; \tag{37}$$ 582

here κ_0 and κ_m are the wave numbers characterizing the outer and the internal scale of the inhomogeneities respectively. Inserting (37) in (34), we can ignore the factor $\exp(-\kappa^2/\kappa_m^2)$ in the integration, since the associated corrections are of the order of smallness of $\frac{\kappa_0}{\kappa_m}$. Inserting (37) in (35), we may take to the same order of accuracy

$$\phi_\epsilon(\kappa) = A C_\epsilon^2 \kappa^{-11/3} \exp(-\kappa^2/\kappa_m^2).$$

This gives

$$A_{2,2} = 2A(0) = \frac{24}{5}\pi^2 A C_\epsilon^2 \kappa_0^{-5/3} \tag{38}$$

$$A_{2,4} = 8\pi^2 A \Gamma\left(\frac{5}{6}\right) C_\epsilon^2 \kappa_m^{1/3} \tag{39}$$

where $\sigma_\epsilon^2 = \pi^{3/2} A \Gamma\left(\frac{1}{3}\right)\Gamma^{-1}\left(\frac{11}{6}\right) C_\epsilon^2 \kappa_0^{-2/3}$ (to an accuracy of the order of κ_0/κ_m).

Inserting these expressions in (33) and substituting $\kappa_0 \sim L_0^{-1}$, $\kappa_m \sim l_0^{-1}$, we obtain the conditions

$$k^2 L_0^{5/3} l_0^{1/3} \gg 1, \; C_\epsilon^2 k L_0^{5/3} \ll 1. \tag{40}$$

The first condition implies that the wavelength should be small compared to the scales of the inhomogeneities. Using (32), the second condition may be rewritten in the form

$$\alpha \ll k, \qquad \lambda\alpha \ll 1.$$

This leads to the requirement that the mean field be only slightly attenuated over distances of the order of one wavelength.

Suppose that conditions (40) are satisfied and the right-hand side of equation (25) may be treated as a perturbation. Inserting the solution (31) 583
of the unperturbed equation in the right-hand side of (25), we obtain for $x \gg L_0$, where $A_{\alpha,\beta}(x) = A_{\alpha,\beta} = \text{const}$:

$$\frac{\partial \bar{u}}{\partial x} + \frac{i\kappa^2}{2k}\bar{u} + \frac{\alpha}{2}\bar{u} = f(x, \vec{\kappa})$$

$$f(x, \vec{\kappa}) = \frac{k^2}{4}\left[\frac{iA_{1,2}}{4k} + \frac{\kappa^2 A_{2,2}}{8k^2} + \frac{A_{2,4}}{16k^2}\right] u_0 \exp\left[-\frac{i\kappa^2 x}{2k} - \frac{\alpha x}{2}\right]. \tag{41}$$

The solution of equation (4) with the initial condition $u(0,\vec{\kappa})=u_0(\vec{\kappa})$ has the form

$$\bar{u}(x,\vec{\kappa})=u_0(\vec{\kappa})\exp\left\{-\left(\frac{\alpha}{2}+\frac{i\kappa^2}{2k}\right)x\right\}\cdot\left\{1+\frac{kx}{16}\left(iA_{1,2}+\frac{\kappa^2}{2k}A_{2,2}+\frac{1}{4k}A_{2,4}\right)\right\}. \quad (42)$$

The second term in braces in (42) is associated with the additional terms appearing in the right-hand side of (25). It follows from (42) that the correction increases with increasing x, and in general it may become as large as desired. However, (42) decays exponentially. Therefore it is only natural to require that the correction entering (42) remain small in the region where $\bar{u}$ is still markedly different from zero, i.e., in the region $\alpha x \lesssim 1$ or $x \lesssim \frac{1}{\alpha}$. This leads to the condition

$$\frac{k}{\alpha}\left(iA_{1,2}+\frac{\kappa^2}{2k}A_{2,2}+\frac{1}{4k}A_{2,4}\right)\ll 1. \quad (43)$$

The quantities $A_{2,2}$ and $A_{2,4}$ have been found above. From the definition of $A_{1,2}$ it is clear that $A_{1,2}$ is dimensionless, $A_{1,2}\sim\sigma_\epsilon^2$. Instead of κ^2 in (43) it should be replaced with its largest possible value, which is of the order of l_0^{-2}. This leads to the conditions

$$\frac{k}{\alpha}\sigma_\epsilon^2\ll 1 \quad \text{or} \quad kL_0\gg 1$$

$$\frac{A_{2,2}}{\alpha l_0^2}\ll 1 \quad \text{or} \quad k^2 l_0^2\gg 1$$

$$\frac{A_{2,4}}{\alpha}\ll 1 \quad \text{or} \quad k^2 L_0^{5/3} l_0^{1/3}\gg 1.$$

All these conditions are restrictions on only the wavelength, and they are satisfied if $\lambda\ll l_0$.

In summary, we can say that the Markov approximation is applicable to the computation of the mean field if the following conditions are satisfied:

$$\lambda\ll l_0,\ x\gg L_0,\ \lambda\alpha\ll 1\ (kL_0\sigma_\epsilon^2\ll 1). \quad (44)$$

A similar analysis can be performed for the coherence function $\Gamma_2(x,\vec{\rho}_1,\vec{\rho}_2)$. In this case, we start with the Fourier transform of equation (65.30). The term

$$\iint_{-\infty}^{\infty}\langle\epsilon_1(x,\mathrm{p})\,u(x,\vec{\kappa}_1-\mathrm{p})\,u^*(x,\vec{\kappa}_2)\rangle\, d^2p$$

which appears because of the averaging is expressed, using the Furutsu–Novikov formula, in terms of

$$\left\langle\frac{\delta u(x,\vec{\kappa}_1+\mathrm{q})}{\delta\epsilon_1(x',\mathrm{q})}u^*(x,\vec{\kappa}_2)\right\rangle,\ \left\langle u(x,\vec{\kappa}_1+\mathrm{q})\frac{\delta u^*(x,\vec{\kappa}_2)}{\delta\epsilon_1(x',\mathrm{q})}\right\rangle.$$

The variational derivatives introduced in the process are expanded in powers of $(x-x')$ using (17). It should be remembered that

$$\frac{\delta u^*(x,\vec{\kappa})}{\delta\epsilon_1(x',\vec{\kappa}')}=-\frac{ik}{2}R^*(x,\vec{\kappa};x',-\vec{\kappa}'). \quad (17a)$$

Further computations proceed along the same lines as given above for $\bar{u}$. The equation for $\Gamma_2(x, \vec{\rho}_1, \vec{\rho}_2)$ has the form

$$\frac{\partial \Gamma_2(x, \vec{\rho}_1, \vec{\rho}_2)}{\partial x} + \frac{i}{2k}(\Delta_2 - \Delta_1)\Gamma_2 + \frac{\pi k^2}{4} H(x, \vec{\rho}_1 - \vec{\rho}_2)\Gamma_2 =$$

$$= \frac{\pi k^2}{4}\left\{ \frac{i}{2k}\nabla_1(\Gamma_2 \nabla_1 H_1) - \frac{i}{2k}\nabla_2(\Gamma_2 \nabla_2 H_1) + \frac{\Delta^2 H_2}{8k^2}\Gamma_2 + \frac{1}{4k^2}(\Gamma_2 \nabla_1 \Delta H_2) + \right.$$

$$\left. + \frac{1}{4k^2}\nabla_2(\Gamma_2 \nabla_2 \Delta H_2) + \frac{1}{4k^2}\frac{\partial^2}{\partial \rho_{1i}\partial \rho_{1j}}(\Gamma_2 \frac{\partial^2 H_2}{\partial \rho_{1i}\partial \rho_{1j}}) + \frac{1}{4k^2}\frac{\partial^2}{\partial \rho_{2i}\partial \rho_{2j}}(\Gamma_2 \frac{\partial^2 H_2}{\partial \rho_{2i}\partial \rho_{2j}})\right\}. \quad (45)$$

Here

$$H(x, \vec{\rho}) = \frac{2}{\pi}\int_0^x [B_\epsilon(\xi, 0) - B_\epsilon(\xi, \vec{\rho})]\,d\xi \quad (47)$$

$$H_1 \equiv H_1(x, \vec{\rho}_1 - \vec{\rho}_2) = \frac{2}{\pi}\int_0^x \xi\,[B_\epsilon(\xi, 0) - B_\epsilon(\xi, \vec{\rho}_1 - \vec{\rho}_2)]\,d\xi \quad (48)$$

$$H_2 \equiv H_2(x, \vec{\rho}_1 - \vec{\rho}_2) = \frac{2}{\pi}\int_0^x \xi^2 [B_\epsilon(\xi, 0) - B_\epsilon(\xi, \vec{\rho}_1 - \vec{\rho}_2)]\,d\xi \quad (49)$$

∇_1, ∇_2 are the transverse gradient operators with respect to the variables $\vec{\rho}_1$ and $\vec{\rho}_2$.

Like the equation (25) for $\bar{u}$, equation (45) for Γ_2 differs from the corresponding equation (65.31) in that $H(\vec{\rho})$ is replaced with $H(x, \vec{\rho})$ (where $H(\infty, \vec{\rho}) = H(\vec{\rho})$) and additional terms appear on the right.

The first of these two factors is associated with the existence of a transition region, where the Markov regime approaches the "steady-state conditions." However, in contrast to $\bar{u}$, the characteristic scale of the transition region for Γ_2 is determined by the value of ρ, and not L_0. Indeed, (47) can be written in the form

$$H(x, \vec{\rho}_1) = \frac{1}{\pi}\int_0^x [D_\epsilon(\xi, \vec{\rho}) - D_\epsilon(\xi, 0)]\,d\xi \quad (47a)$$

where $D_\epsilon(\xi, \vec{\rho})$ is the structure function of ϵ. If $D_\epsilon(\xi, \vec{\rho}) = G_\epsilon^2(\xi^2 + \rho^2)^{1/3}$, then $H(x; \vec{\rho})$ may be represented in the form

$$H(x\,\vec{\rho}) = \frac{1}{\pi}C_\epsilon^2 \rho^{5/3}\int_0^{\frac{x}{\rho}} [(1 + t^2)^{1/3} - t^{2/3}]\,dt$$

and hence it is clear that H no longer depends on x for $(x/\rho) \gg 1$. Therefore the first condition necessary in order to apply the Markov approximation to Γ_2 has the form

$$x \gg |\vec{\rho}_1 - \vec{\rho}_2|. \quad (50)$$

The effect of the additional terms on the right side in equation (45) will be estimated for the region where the Markov regime is already established. In this region, the functions $H_1(x, \vec{\rho}), H_2(x, \vec{\rho})$ may be replaced with their limiting values

$$H_1(\vec{\rho}) = \frac{2}{\pi} \int_0^\infty \xi [B_\epsilon(\xi, 0) - B_\epsilon(\xi, \vec{\rho})] d\xi \tag{51}$$

$$H_2(\vec{\rho}) = \frac{2}{\pi} \int_0^\infty \xi^2 [B_\epsilon(\xi, 0) - B_\epsilon(\xi, \vec{\rho})] d\xi \,. \tag{52}$$

Although the $H_{1,2}(\vec{\rho})$ are also expressed in terms of $D_\epsilon(\xi, \vec{\rho})$, when we substitute $D_\epsilon(\xi, \vec{\rho}) = C_\epsilon^2(\xi^2 + \rho^2)^{1/3}$, the integrals in (51) and (52) diverge at infinity due to the presence of the additional factors ξ, ξ^2. Therefore, the existence of the outer scale of turbulence also should be taken into consideration. To derive the desired estimates, we use the spectrum (37), and if $\kappa_m \rho \gg 1$, the factor $\exp(-\kappa^2/\kappa_m^2)$ may be neglected. In this case, we obtain

$$H_2(\rho) = -4\pi \int_0^\infty [1 - J_0(\kappa\rho)] \phi_\epsilon'(\kappa) d\kappa =$$

$$= 4\pi A C_\epsilon^2 \kappa_0^{-11/3} [1 - \frac{11(\kappa_0\rho)^{11/6}}{3 \cdot 2^{11/6} \Gamma(17/6)} K_{11/6}(\kappa_0\rho)] \tag{53}$$

where K_ν is the Macdonald function (modified Bessel function). Equation (45) contains the factors $\Delta^2 H$, $\partial\Delta H_2/\partial\rho$, ΔH_2, $\partial H_2/\partial\rho$, in terms of which all the necessary derivatives of H_2 are expressed. We will consider the region $l_0 \ll \rho \ll L_0$, where $\kappa_0\rho \ll 1$. Therefore, expanding (53) in a series in $\kappa_0\rho$, we find the necessary derivatives and retain in each expression the leading term in $\kappa_0\rho$. The results are

$$\frac{\partial H_2}{\partial \rho} = N_1 \kappa_0^{-5/3} C_\epsilon^2 \rho + \ldots \tag{54}$$

$$\frac{\partial^2 H_2}{\partial \rho^2} = N_1 C_\epsilon^2 \kappa_0^{-5/3} + \ldots \tag{55}$$

$$\frac{\partial^2 H_2}{\partial \rho^2} - \frac{1}{\rho}\frac{\partial H_2}{\partial \rho} = N_2 C_\epsilon^2 \rho^{5/3} + \ldots \tag{56}$$

$$\frac{\partial \Delta H_2}{\partial \rho} = -N_3 C_\epsilon^2 \rho^{2/3} + \ldots \tag{57}$$

$$\Delta^2 H_2 = -N_4 C_\epsilon^2 \rho^{-1/3} + \ldots \tag{58}$$

where $N_1, \ldots, N_4$ are positive numerical constants. The first and the second derivative of H_2 depend on κ_0, the third and the fourth derivatives for $\kappa_0\rho \ll 1$ are independent of κ_0.

Let us now estimate the terms on the right in (45). The last term but one on the right in (45) contains an expression of the form

$$\frac{\partial^2 \Gamma_2}{\partial \rho_{1i} \partial \rho_{1j}} \frac{\partial^2 H_2}{\partial \rho_{1i} \partial \rho_{1j}} = \frac{\rho_i \rho_j}{\rho^2} [\frac{\partial^2 H_2}{\partial \rho^2} - \frac{1}{\rho}\frac{\partial H_2}{\partial \rho}] \frac{\partial^2 \Gamma_2}{\partial \rho_{1i} \partial \rho_{1j}} + \frac{1}{\rho}\frac{\partial H_2}{\partial \rho} \Delta_1 \Gamma_2$$

where $\vec{\rho} = \vec{\rho}_1 - \vec{\rho}_2$. This term should be small compared to the term $\frac{1}{k}\Delta_1\Gamma_2$ on the left. Inserting (56) and (54), we obtain the conditions

$$C_\epsilon^2 k \rho^{5/3} \ll 1 \tag{59a}$$

$$C_\epsilon^2 k L_0^{5/3} \ll 1. \tag{59b}$$

Since we are dealing only with the region $\rho \ll L_0$, condition (59b) is the more stringent of the two. It coincides with the previous condition (40), implying small attenuation of the mean value of the field over distances of the order of one wavelength. Comparison of the terms of the form $\Delta^2 H_2 \cdot \Gamma_2$ in the right-hand side of (45) with the $k^2 H\Gamma_2$ on the left leads to the condition

$$k^2\rho^2 \gg 1. \tag{60}$$

Let us now estimate the effect of the right-hand side of (45) by the perturbation technique. We will consider the particular case of a plane wave, when $\Gamma_2(x, \vec{\rho}_1, \vec{\rho}_2) = \Gamma_2(x, \vec{\rho}_1 - \vec{\rho}_2)$. In this case all the functions in (45) depend on $\vec{\rho}_1 - \vec{\rho}_2$, so that we may take $\nabla_1 = -\nabla_2 = \nabla$. Remembering that all the functions are isotropic, we may reduce the equation to the form

$$\frac{\partial\Gamma_2(x,\vec{\rho})}{\partial x} + \frac{\pi k^2}{4} H(\vec{\rho})\Gamma_2(x,\vec{\rho}) = \frac{\pi}{8}\Big[\frac{\Delta^2 H_2(\rho)}{4}\Gamma_2 + \nabla(\Gamma_2\nabla\Delta H_2) + \frac{\partial^2}{\partial\rho_i\partial\rho_j}\Big(\Gamma_2\frac{\partial^2 H_2}{\partial\rho_i\partial\rho_j}\Big)\Big] = \frac{\pi}{8}\Big[\frac{9}{4}\Delta^2 H_2\cdot\Gamma_2 + \Big(\frac{1}{\rho^2}\frac{\partial H_2}{\partial\rho} + 3\frac{\partial\Delta H_2}{\partial\rho}\Big)\frac{\partial\Gamma_2}{\partial\rho} + \frac{\partial^2 H_2}{\partial\rho^2}\frac{\partial^2\Gamma_2}{\partial\rho^2}\Big]. \tag{61}$$

To a first approximation, by equating the right-hand side to zero, we find

$$\frac{\partial\Gamma_2^{(1)}}{\partial x} + \frac{\pi k^2}{4}H(\vec{\rho})\Gamma_2^{(1)} = 0 \qquad \Gamma_2^{(1)}(x,\rho) = |u_0|^2 \exp\left\{-\frac{\pi k^2 H(\rho)}{4}x\right\} \tag{62}$$

where $|u_0|^2$ is the initial value of Γ_2 for $x=0$. Inserting (62) on the right in (61), we obtain the equation for the second order approximation

$$\frac{\partial\Gamma_2^{(2)}}{\partial x} + \frac{\pi k^2}{4}H(\rho)\Gamma_2^{(2)} = \frac{\pi}{8}[f_0(\rho) - f_1(\rho)x + f_2(\rho)x^2]\Gamma_2^{(1)} \tag{63}$$

where

$$f_0(\rho) = \frac{9}{4}\Delta^2 H_2(\rho)$$

$$f_1(\rho) = \frac{\pi k^2}{4}\Big[\frac{\partial^2 H_2}{\partial\rho^2}\frac{\partial^2 H}{\partial\rho^2} + \frac{\partial H}{\partial\rho}\Big(\frac{1}{\rho^2}\frac{\partial H_2}{\partial\rho} + \frac{\partial\Delta H_2}{\partial\rho}\Big)\Big]$$

$$f_2(\rho) = \frac{\pi^2 k^4}{16}\Big(\frac{\partial H}{\partial\rho}\Big)^2\frac{\partial^2 H_2}{\partial\rho^2}.$$

The solution of the inhomogeneous equation (63) has the form

$$\Gamma_2^{(2)}(x,\rho) = \Gamma_2^{(1)}(x,\rho)\left\{1 + \frac{\pi}{8}[f_0(\rho)x - \frac{1}{2}f_1(\rho)x^2 + \frac{1}{3}f_2(\rho)x^3]\right\}. \tag{64}$$

The second term in braces constitutes a correction associated with the additional terms on the right in (45). Like the correction to $\bar{u}$, this term increases with increasing x and may reach values greater than unity.

However, it is only reasonable to require this term to remain small in the region where the main solution $\Gamma_2^{(1)}(x,\rho)$ is markedly different from zero. If we consider the solution $\Gamma_2^{(1)}$ for a fixed value of $\rho \neq 0$, it is seen to vanish exponentially for $x \gg x_0(\rho)$, where $x_0(\rho)$ is determined from the condition $k^2 H(\rho) x_0(\rho) = 1$. Therefore we should impose the condition

$$|f_0(\rho)x_0(\rho) - \frac{1}{2} f_1(\rho) x_0^2(\rho) + \frac{1}{3} f_2(\rho) x_0^3(\rho)| \ll 1$$

or the three conditions

$$|f_0(\rho)x_0(\rho)| \ll 1,\ |f_1(\rho)x_0^2(\rho)| \ll 1,\ |f_2(\rho)x_0^3(\rho)| \ll 1\,. \tag{65}$$

In the region $l_0 \ll \rho \ll L_0$, $H(\rho)$ can be expressed by the previous relation $H(\rho) \sim C_\epsilon^2 \rho^{5/3}$. Using this relation and equations (54)—(58) to compute the functions $f_i(\rho)$ and retaining only the leading terms in $(x_0\rho)$, we obtain the conditions

$$k^2\rho^2 \gg 1 \quad \text{(a)}, \quad k^2 \kappa_0^{5/3} \rho^{11/3} \gg 1 \text{ (b)} \tag{66}$$

(the second and the third inequality in (65) lead to the same condition (66b)). Conditions (66) are a priori satisfied if ρ is replaced with l_0. Then (66a) becomes equivalent to the first condition in (44), and (66b) takes the form

$$\lambda \ll l_0 \left(\frac{l_0}{L_0}\right)^{5/6}. \tag{67}$$

We should consider separately the case $\rho = 0$, when $\Gamma_2^{(1)}(x, 0) = |u_0|^2 = \text{const}$ and does not decrease with increasing x. For $\rho \to 0$, we should remember when computing $f_i(0)$ that in the region $\rho \ll l_0$ the functions $H(\rho)$ and $H_2(\rho)$ are expanded in powers of ρ^2, i.e.,

$$H(\rho) \sim C_\epsilon^2 l_0^{-\frac{1}{3}} \rho^2 + \ldots,\ H_2(\rho) = a_1 C_\epsilon^2 L_0^{5/3} \rho^2 + a_2 C_\epsilon^2 l_0^{-\frac{1}{3}} \rho^4 + \ldots \tag{68}$$

(both expressions are accurate apart from small corrections of the order l_0/L_0). Using (68) to compute $f_i(0)$, we obtain $f_0(0) \sim C_\epsilon^2 l_0^{-\frac{1}{3}}, f_1(0) \sim C_\epsilon^2 k^2 l_0^{-\frac{1}{3}} \cdot C_\epsilon^2 L_0^{5/3}, f_2(0) = 0$. Inserting these results in (64), we obtain the conditions

$$C_\epsilon^2 l_0^{-\frac{1}{3}} x \ll 1 \tag{69}$$

$$C_\epsilon^4 k^2 L_0^{\frac{5}{3}} l_0^{-\frac{1}{3}} x^2 \ll 1. \tag{70}$$

Note that the first of these conditions may be written in the form

$$\sigma_\alpha^2(x) \ll 1 \tag{69a}$$

where $\sigma_\alpha^2(x)$ is the mean square of the fluctuations in the direction of propagation of the wave (see (42.11)). Condition (70) may be written in the form

$$\alpha x \cdot \sigma_\alpha^2(x) \ll 1 \tag{70a}$$

where αx is the attenuation (in nepers) of the mean field over a distance x. This condition may also be interpreted as implying small attenuation of the mean field intensity due to back-scattering (see next section).

§ 68. The limits of application of the parabolic equation in an inhomogeneous random medium

The parabolic equation differs from the complete scalar equation for the function $u(x, \vec{\rho})$

$$2ik\frac{\partial u(x,\vec{\rho})}{\partial x}+\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}+k^2\epsilon_1(x,\vec{\rho})u(x,\vec{\rho})=0$$

in that the term $\partial^2 u/\partial x^2$ is omitted. To establish under what conditions this term may be ignored, we will treat it as a small perturbation. We thus have to consider the equation

$$2ik\frac{\partial u}{\partial x}+\Delta u+k^2\epsilon_1(x,\vec{\rho})u=-\frac{\partial^2 u}{\partial x^2} \tag{1}$$

where $\Delta=\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$, and $\frac{\partial^2 u}{\partial x^2}$ is a small quantity. Let $u(x,\vec{\rho})=u_1(x,\vec{\rho})+$ $+u_2(x,\vec{\rho})+\ldots$ and suppose that $u_2(x,\vec{\rho})$ is of the same order of smallness as $\partial^2 u_1/\partial x^2$. Inserting this expansion in (1) and equating groups of terms of the same order of smallness, we obtain the equations

$$2ik\frac{\partial u_1}{\partial x}+\Delta u_1+k^2\epsilon_1(x,\vec{\rho})u_1=0;\, u_1(0,\vec{\rho})=u_0(\vec{\rho}) \tag{2}$$

$$2ik\frac{\partial u_2}{\partial x}+\Delta u_2+k^2\epsilon_1(x,\vec{\rho})u_2=\frac{\partial^2 u_1}{\partial x^2};\, u_2(0,\vec{\rho})=0\,. \tag{3}$$

We assume, as before, that ϵ_1 is a Gaussian (normal) field which is delta-correlated along the x axis:

$$\langle\epsilon_1(x_1,\vec{\rho}_1)\epsilon_1(x_2,\vec{\rho}_2)\rangle=\delta(x_1-x_2)A(\vec{\rho}_1-\vec{\rho}_2)\,. \tag{4}$$

We will first derive the equations for $\langle u_1\rangle=\bar{u}_1, \langle u_2\rangle=\bar{u}_2$ using the method introduced by Klyatskin /192/. The product $\epsilon_1 u$ is not assumed to be small in this treatment.

Seeing that

$$2ik\frac{\partial f}{\partial x}+k^2\epsilon_1(x,\vec{\rho})f=2ik\; e^{-\frac{ik}{2}\int_0^x\epsilon_1(\xi,\vec{\rho})d\xi}\frac{\partial}{\partial x}[e^{-\frac{ik}{2}\int_0^x\epsilon_1(\xi,\vec{\rho})d\xi}f]$$

we write equations (2) and (3) in the form

$$2ik\frac{\partial}{\partial x}[e^{-\frac{ik}{2}\int_0^x\epsilon_1(\xi,\vec{\rho})d\xi}u_1(x,\vec{\rho})]=-\Delta u_1(x,\vec{\rho})\,e^{-\frac{ik}{2}\int_0^x\epsilon_1(\xi,\vec{\rho})d\xi} \tag{5}$$

$$2ik\frac{\partial}{\partial x}[e^{-\frac{ik}{2}\int_0^x\epsilon_1(\xi,\vec{\rho})d\xi}u_2(x,\vec{\rho})]=-[\Delta u_2(x,\vec{\rho})+\frac{\partial^2 u_1(x,\vec{\rho})}{\partial x^2}]\,e^{-\frac{ik}{2}\int_0^x\epsilon_1(\xi,\vec{\rho})d\xi}\,. \tag{6}$$

We integrate this equation over x from 0 to x and multiply the result by $\exp\,[\frac{ik}{2}\int_0^x \epsilon_1(\xi,\vec{\rho})d\xi]$:

$$2ik\,u_1(x,\vec{\rho}) = 2ik\,u_0(\vec{\rho})\exp\,[\frac{ik}{2}\int_0^x \epsilon_1(\xi,\vec{\rho})d\xi] - \\ - \int_0^x \exp\,[\frac{ik}{2}\int_{x'}^x \epsilon_1(\xi,\vec{\rho})d\xi]\,\Delta u_1(x',\vec{\rho})dx' \tag{7}$$

$$2ik\,u_2(x,\vec{\rho}) = -\int_0^x \exp\,[\frac{ik}{2}\int_{x'}^x \epsilon_1(\xi,\vec{\rho})d\xi]\,[\Delta u_2(x',\vec{\rho}) + \\ + \frac{\partial^2 u_1(x',\vec{\rho})}{\partial x'^2}]dx'. \tag{8}$$

A characteristic feature of equations (7) and (8) is that the function $\epsilon_1(\xi,\vec{\rho})$ on the right is evaluated for $\xi > x'$, whereas the functions u_1, u_2 on the right-hand sides of (7) and (8) are evaluated at the point x'. As we have repeatedly stressed above, u_1(and also u_2) depends only on the preceding values of the function ϵ_1. Since the field ϵ_1 is delta-correlated in x, this indicates that $u(x',\vec{\rho})$ and $\epsilon_1(\xi,\vec{\rho})$ are statistically independent for $\xi > x'$; the same conclusion applies to all functionals of ϵ_1, which contain only the following (relative to x') values of $\epsilon_1(\xi,\vec{\rho})$. Thus,

$$<\exp\,[\frac{ik}{2}\int_{x'}^x \epsilon_1(\xi,\vec{\rho})d\xi]\,\Delta u_1(x',\vec{\rho})> = <\exp\,[\frac{ik}{2}\int_{x'}^x \epsilon_1(\xi,\vec{\rho})d\xi]>\cdot\Delta\bar{u}_1(x',\vec{\rho}).$$

Therefore, averaging (7), we obtain

$$2ik\,\bar{u}_1(x,\vec{\rho}) = 2ik\,u_0(\vec{\rho})<\exp\,[\frac{ik}{2}\int_0^x \epsilon_1(\xi,\vec{\rho})d\xi]> - \\ - \int_0^x <\exp\,[\frac{ik}{2}\int_{x'}^x \epsilon_1(\xi,\vec{\rho})d\xi]>\Delta\bar{u}_1(x',\vec{\rho})dx'. \tag{9}$$

Similarly, averaging (8), we find

$$2ik\,\bar{u}_2(x,\vec{\rho}) = -\int_0^x <\exp\,[\frac{ik}{2}\int_{x'}^x \epsilon_1(\xi,\vec{\rho})d\xi]>\cdot\;[\Delta\bar{u}_2(x',\vec{\rho}) + \frac{\partial^2\bar{u}_1(x',\vec{\rho})}{\partial x'^2}]dx'. \tag{10}$$

Since ϵ_1 is a normal random field, $\int_{x'}^x \epsilon_1(\xi,\vec{\rho})d\xi$ is also a normal field. Therefore, using the equality $<\exp a> = \exp\,(\frac{1}{2}<a^2>)$, which holds true for any normal random variable $a\,(<a>=0)$, we find

$$<\exp\,[\frac{ik}{2}\int_{x'}^x \epsilon_1(\xi,\vec{\rho})d\xi]> = \exp\left\{\frac{1}{2}<[\frac{ik}{2}\int_{x'}^x \epsilon_1(\xi,\vec{\rho})d\xi]^2>\right\} = \\ = \exp\left\{-\frac{k^2}{8}\int_{x'}^x d\xi_1\int_{x'}^x d\xi_2<\epsilon_1(\xi_1,\vec{\rho})\epsilon_1(\xi_2,\vec{\rho})>\right\}.$$

Inserting (4), we find

$$<\exp\,[\frac{ik}{2}\int_{x'}^x \epsilon_1(\xi,\vec{\rho})d\xi]> = \exp\left\{-\frac{k^2 A(0)}{8}(x-x')\right\}. \tag{11}$$

Inserting (11) in (9) and (10) and multiplying both sides of the equality by $\exp\left\{\frac{1}{8}k^2A(0)x\right\}$, we obtain

$$2ik\,\bar{u}_1(x,\vec{\rho})\exp\left[\frac{k^2A(0)x}{8}\right]=2ik\,u_0(\vec{\rho})-\int_0^x e^{\frac{k^2A(0)x'}{8}}\Delta\bar{u}_1(x'\vec{\rho})\,dx' \qquad (12)$$

$$2ik\,\bar{u}_1(x,\vec{\rho})\exp\left[\frac{k^2A(0)x}{8}\right]=-\int_0^x e^{\frac{k^2A(0)x'}{8}}\left[\Delta\bar{u}_2(x',\vec{\rho})+\frac{\partial^2\bar{u}_1(x',\vec{\rho})}{\partial x'^2}\right]dx'. \qquad (13)$$

Differentiating these equations with respect to x and dropping the common factor $\exp\left[\frac{k^2A(0)x}{8}\right]$, we obtain

$$2ik\frac{\partial\bar{u}_1}{\partial x}+\Delta\bar{u}_1+\frac{ik^3A(0)}{4}\bar{u}_1=0 \qquad (14)$$

$$2ik\frac{\partial\bar{u}_2}{\partial x}+\Delta\bar{u}_2+\frac{ik^3A(0)}{4}\bar{u}_2=-\frac{\partial^2\bar{u}_1}{\partial x^2}. \qquad (15)$$

Equation (14) coincides with the previous equation (65.26). Let $u_0(\vec{\rho})=\text{const}=u_0$. Then

$$\bar{u}_1(x)=u_0e^{-\frac{k^2A(0)}{8}x}=u_0e^{-\frac{i}{2}\alpha x}$$

where $\alpha=\frac{1}{4}k^2A(0)$ is the extinction (attenuation) coefficient. Clearly, for $u_0(\vec{\rho})=\text{const}$, $\bar{u}_2=\bar{u}_2(x)$, $\Delta\bar{u}_2=0$. Then (15) takes the form

$$\frac{d\bar{u}_2}{dx}+\frac{k^2A(0)}{8}\bar{u}_2=\frac{i}{2k}\,\frac{k^4A^2(0)}{64}\,u_0e^{-\frac{k^2A(0)}{8}x} \qquad (16)$$

and its solution corresponding to the initial condition $\bar{u}_2(0)=0$ is

$$\bar{u}_2(x)=u_0e^{-\frac{1}{2}\alpha x}\cdot\frac{ik^3A^2(0)}{128}x=\bar{u}_1(x)\cdot\frac{i\alpha^2x}{8k}. \qquad (17)$$

In this case

$$\bar{u}(x)=\bar{u}_1(x)+\bar{u}_2(x)+\ldots=u_0e^{-\frac{1}{2}\alpha x}\left[1+\frac{i\alpha^2x}{8k}+\ldots\right]. \qquad (18)$$

For sufficiently large x, the correction to $\bar{u}_1$, associated with the term $\frac{\partial^2u}{\partial x^2}$, may clearly become very large. However, the field $\bar{u}_1$ should be considered only in the region where it is markedly different from zero, i.e., for $\alpha x\lesssim 1$. Therefore, it is reasonable to require that the correction should be constrained to remain small in this region. We thus arrive at the condition $\frac{\alpha^2}{k}\cdot\frac{1}{\alpha}\ll 1$, i.e.,

$$\alpha\ll k,\quad \lambda\alpha\ll 1. \qquad (19)$$

Condition (19) indicates that the attenuation of the mean field over distances of the order of one wavelength should be small. This condition coincides with one of the conditions for applying the Markov approximation, derived in the previous section.

Let us now consider the corrections to $\Gamma_2(x, \vec{\rho}_1, \vec{\rho}_2)$. If $u = u_1 + u_2 + \dots$, we have

$$\Gamma_2(x, \vec{\rho}_1, \vec{\rho}_2) = \langle u_1(x, \vec{\rho}_1) u_1^*(x, \vec{\rho}_2) \rangle + \langle u_1(x, \vec{\rho}_1) u_2^*(x, \vec{\rho}_2) \rangle + \langle u_2(x, \vec{\rho}_1) u_1^*(x, \vec{\rho}_2) \rangle + \dots$$

Let

$$\Gamma_2^{(1)}(x, \vec{\rho}_1, \vec{\rho}_2) = \langle u_1(x, \vec{\rho}_1) u_1^*(x, \vec{\rho}_2) \rangle$$

$$\Gamma_2^{(2)}(x, \vec{\rho}_1, \vec{\rho}_2) = \langle u_1(x, \vec{\rho}_1) u_2^*(x, \vec{\rho}_2) \rangle .$$

Then

$$\Gamma_2(x, \vec{\rho}_1, \vec{\rho}_2) = \Gamma_2^{(1)}(x, \vec{\rho}_1, \vec{\rho}_2) + \Gamma_2^{(2)}(x, \vec{\rho}_1, \vec{\rho}_2) + \Gamma_2^{*(2)}(x, \vec{\rho}_2, \vec{\rho}_1) + \dots \quad (20)$$

To derive an equation for $\Gamma_2^{(2)}$, we first have to write the equations for $u_1(x, \vec{\rho}_1)$ and $u_2^*(x, \vec{\rho}_2)$:

$$2ik \frac{\partial u_1(x, \vec{\rho}_1)}{\partial x} + \Delta_1 u_1(x, \vec{\rho}_1) + k^2 \epsilon_1(x, \vec{\rho}_1) u_1(x, \vec{\rho}_1) = 0 \quad (21)$$

$$-2ik \frac{\partial u_2^*(x, \vec{\rho}_2)}{\partial x} + \Delta_2 u_2^*(x, \vec{\rho}_2) + k^2 \epsilon_1(x, \vec{\rho}_2) u_2^*(x, \vec{\rho}_2) = -\frac{\partial^2 u_1^*(x, \vec{\rho}_2)}{\partial x^2} \quad (22)$$

Multiplying (21) by $u_2^*(x, \vec{\rho}_2)$ and (22) by $u_1(x, \vec{\rho}_1)$ and subtracting, we obtain

$$2ik \frac{\partial u_1(x, \vec{\rho}_1) u_2^*(x, \vec{\rho}_2)}{\partial x} + (\Delta_1 - \Delta_2) u_1(x, \vec{\rho}_1) u_2^*(x, \vec{\rho}_2) + k^2 [\epsilon_1(x, \vec{\rho}_1) - \epsilon_1(x, \vec{\rho}_2)] u_1(x, \vec{\rho}_1) u_2^*(x, \vec{\rho}_2) = u_1(x, \vec{\rho}_1) \frac{\partial^2 u_1^*(x, \vec{\rho}_2)}{\partial x^2}. \quad (23)$$

Let

$$\epsilon_1(x, \vec{\rho}_1) - \epsilon_1(x, \vec{\rho}_2) = \mu(x, \vec{\rho}_1, \vec{\rho}_2)$$

and

$$\int_0^x \mu(\xi, \vec{\rho}_1, \vec{\rho}_2) d\xi = \nu(x, \vec{\rho}_1, \vec{\rho}_2) .$$

Then equation (23) can be written in the form

$$2ik \frac{\partial u_1(x, \vec{\rho}_1) u_2^*(x, \vec{\rho}_2)}{\partial x} + k^2 \mu(x, \vec{\rho}_1, \vec{\rho}_2) u_1(x, \vec{\rho}_1) u_2^*(x, \vec{\rho}_2) =$$

$$= 2ik \exp\left[\frac{ik}{2} \nu(x, \vec{\rho}_1, \vec{\rho}_2)\right] \frac{\partial}{\partial x} \left\{ \exp\left[-\frac{ik}{2} \nu(x, \vec{\rho}_1, \vec{\rho}_2)\right] u_1(x, \vec{\rho}_1) u_2^* \right\}$$

$$= (\Delta_2 - \Delta_1) u_1(x, \vec{\rho}_1) u_2^*(x, \vec{\rho}_2) + u_1(x, \vec{\rho}_1) \frac{\partial^2 u_1^*(x, \vec{\rho}_2)}{\partial x^2} .$$

or, multiplying by $\exp\,[-\frac{ik}{2}\nu(x,\vec{\rho}_1,\vec{\rho}_2)]$ and integrating,

$$2ik\,\exp\,[-\frac{ik}{2}\nu(x,\vec{\rho}_1,\vec{\rho}_2)]u_1(x,\vec{\rho}_1)u_2^*(x,\vec{\rho}_2)=$$

$$=\int_0^x dx'\exp\,[-\frac{ik}{2}\nu(x',\vec{\rho}_1,\vec{\rho}_2)]\cdot[(\Delta_2-\Delta_1)u_1(x',\vec{\rho}_1)u_2^*(x',\vec{\rho}_2)+u_1(x',\vec{\rho}_1)\frac{\partial^2 u_1^*(x',\vec{\rho}_2)}{\partial x'^2}].$$

Multiplying by $\exp\,[\frac{ik}{2}\nu(x,\vec{\rho}_1,\vec{\rho}_2)]$, we obtain

$$2ik\;u_1(x,\vec{\rho}_1)u_2^*(x,\vec{\rho}_2)=$$

$$\int_0^x dx'\exp\left\{\frac{ik}{2}\int_{x'}^x[\epsilon_1(\xi,\vec{\rho}_1)-\epsilon_1(\xi,\vec{\rho}_2)]d\xi\right\}\cdot$$

$$\cdot[(\Delta_2-\Delta_1)u_1(x',\vec{\rho}_1)u_2^*(x',\vec{\rho}_2)+u_1(x',\vec{\rho}_1)\frac{\partial^2 u_1^*(x',\vec{\rho}_2)}{\partial x'^2}]. \qquad (24)$$

The functions $\epsilon_1(\xi,\vec{\rho})$ entering the argument of the exponential are evaluated for $\xi>x'$, i.e., they are statistically independent from $u_{1,2}(x',\vec{\rho})$. Therefore, we may average the factors containing ϵ_1 and $u_{1,2}$ in (24) independently of one another. Then

$$2ik\,\Gamma_2^{(2)}(x,\vec{\rho}_1,\vec{\rho}_2)=\int_0^x dx'\;\langle\exp\left\{\frac{ik}{2}\int_{x'}^x[\epsilon_1(\xi,\vec{\rho}_1)-\epsilon_1(\xi,\vec{\rho}_2)]\,d\xi\right\}\rangle\cdot$$

$$\cdot[(\Delta_2-\Delta_1)\Gamma_2^{(2)}(x',\vec{\rho}_1,\vec{\rho}_2)+\langle u_1(x',\vec{\rho}_1)\frac{\partial^2 u_1^*(x',\vec{\rho}_2)}{\partial x'^2}\rangle]. \qquad (25)$$

Averaging the exponential, we find

$$\langle\exp\left\{\frac{ik}{2}\int_{x'}^x[\epsilon_1(\xi,\vec{\rho}_1)-\epsilon_1(\xi,\vec{\rho}_2)]d\xi\right\}\rangle=$$

$$=\exp\left\{-\frac{k^2}{8}\int_{x'}^x d\xi_1\int_{x'}^x d\xi_2\,\langle[\epsilon_1(\xi_1,\vec{\rho}_1)-\epsilon_1(\xi_1,\vec{\rho}_2)][\epsilon_1(\xi_2,\vec{\rho}_1)-\epsilon_1(\xi_2,\vec{\rho}_2)]\rangle\right\}$$

$$=\exp\left\{-\frac{k^2}{4}\int_{x'}^x d\xi_1\int_{x'}^x d\xi_2\;\;\delta(\xi_1-\xi_2)\,[A(0)-A(\vec{\rho}_1-\vec{\rho}_2)]\right\}=$$

$$=\exp\left\{-\frac{\pi k^2}{4}H(\vec{\rho}_1-\vec{\rho}_2)\,(x-x')\right\} \qquad (26)$$

where

$$H(\vec{\rho})=\frac{1}{\pi}[A(0)-A(\vec{\rho})]=2\iint_{-\infty}^{\infty}\phi_\epsilon(\vec{\kappa})\,[1-\cos\vec{\kappa}\,\vec{\rho}]d^2\kappa.$$

Inserting (26) in (25) and multiplying both sides of the resulting equality by $\exp\,[\frac{\pi k^2}{4}H(\vec{\rho}_1-\vec{\rho}_2)\,x]$, we obtain

$$2ik\,\Gamma_2^{(2)}(x,\vec{\rho}_1,\vec{\rho}_2)\exp\,[\frac{\pi k^2}{4}H(\vec{\rho}_1-\vec{\rho}_2)\,x]=$$

$$=\int_0^x dx'\exp\,[\frac{\pi k^2}{4}H(\vec{\rho}_1-\vec{\rho}_2)\,x']\cdot[\Delta_2-\Delta_1)\,\Gamma_2^{(2)}(x',\vec{\rho}_1,\vec{\rho}_2)+\langle u_1(x',\vec{\rho}_1)\frac{\partial^2 u_1^*(x',\vec{\rho}_2)}{\partial x^2}\rangle].$$

Differentiating this equation with respect to x and dropping the common exponential factor, we end up with the equation

$$2ik\frac{\partial\Gamma_2^{(2)}(x,\vec{\rho}_1,\vec{\rho}_2)}{\partial x}+(\Delta_1-\Delta_2)\Gamma_2^{(2)}(x,\vec{\rho}_1,\vec{\rho}_2)+$$

$$+\frac{i\pi k^3}{2}H(\vec{\rho}_1-\vec{\rho}_2)\Gamma_2^{(2)}(x,\vec{\rho}_1,\vec{\rho}_2)=\langle u_1(x,\vec{\rho}_1)\frac{\partial^2 u_1^*(x,\vec{\rho}_2)}{\partial x^2}\rangle. \qquad (27)$$

Note that if we were to derive in the same way an equation for $\Gamma_2^{(1)}$, it would not contain the term $\langle u_1\frac{\partial^2 u_1^*}{\partial x^2}\rangle$ on the right (unlike (27)) and would thus coincide with the previously obtained equation (65.31) for Γ_2.

Let us now evaluate the expression entering the right-hand side of (27). Following Klyatskin /195/, we start with the equation

$$\frac{\partial u_1^*(x_2,\vec{\rho}_2)}{\partial x_2}=-\frac{i}{2k}\Delta_2 u_1^*(x_2,\vec{\rho}_2)-\frac{ik}{2}\epsilon_1(x_2,\vec{\rho}_2)u_1^*(x_2,\vec{\rho}_2)$$

multiply it by $u_1(x_1,\vec{\rho}_1)$, where $x_1<x_2$, and average. Setting $\gamma_2(x_1,\vec{\rho}_1;x_2,\vec{\rho}_2)=$ $=\langle u_1(x_1,\vec{\rho}_1)u_1^*(x_2,\vec{\rho}_2)\rangle$, we obtain

$$\frac{\partial\gamma_2(x_1,\vec{\rho}_1;x_2,\vec{\rho}_2)}{\partial x_2}=-\frac{i}{2k}\Delta_2\gamma_2(x_1,\vec{\rho}_1;x_2,\vec{\rho}_2)-$$

$$-\frac{ik}{2}\langle\epsilon_1(x_2,\vec{\rho}_2)u_1(x_1,\vec{\rho}_1)u_1^*(x_2,\vec{\rho}_2)\rangle. \qquad (28)$$

The last term on the right in (28) is found using (65.5):

$$\langle\epsilon_1(x_2,\vec{\rho}_2)u_1(x_1,\vec{\rho}_1)u_1^*(x_2,\vec{\rho}_2)\rangle=$$

$$=\int_0^{x_2}dx'\iint_{-\infty}^{\infty}d^2\rho'\langle\epsilon_1(x_2,\vec{\rho}_2)\epsilon_1(x',\vec{\rho}')\rangle\langle\frac{\delta u_1(x_1,\vec{\rho}_1)u_1^*(x_2,\vec{\rho}_2)}{\delta\epsilon_1(x',\vec{\rho}')}\rangle=$$

$$=\frac{1}{2}\iint_{-\infty}^{\infty}d^2\rho' A(\vec{\rho}_2-\vec{\rho}')\langle\frac{\delta u_1(x_1,\vec{\rho}_1)u_1^*(x_2,\vec{\rho}_2)}{\delta\epsilon_1(x_2,\vec{\rho}')}\rangle.$$

Since $x_2>x_1$, we have $\delta u_1(x_1,\vec{\rho}_1)/\delta\epsilon_1(x_2,\vec{\rho}')=0$. Using (65.24b),

$$\frac{\delta u_1^*(x_2,\vec{\rho}_2)}{\delta\epsilon_1(x_2,\vec{\rho}')}=-\frac{ik}{2}\delta(\vec{\rho}_2-\vec{\rho}')u_1^*(x_2,\vec{\rho}_2)$$

we obtain

$$\langle\epsilon_1(x_2,\vec{\rho}_2)u_1(x_1,\vec{\rho}_1)u_1^*(x_2,\vec{\rho}_2)\rangle=-\frac{ik}{4}A(0)\gamma_2(x_1,\vec{\rho}_1;x_2,\vec{\rho}_2).$$

Inserting this expression in (28), we obtain an equation for γ_2:

$$\frac{\partial\gamma_2(x_1,\vec{\rho}_1;x_2,\vec{\rho}_2)}{\partial x_2}=-\frac{i}{2k}\Delta_2\gamma_2(x_1,\vec{\rho}_1;x_2,\vec{\rho}_2)-\frac{k^2A(0)}{8}\gamma_2(x_1,\vec{\rho}_1;x_2,\vec{\rho}_2). \qquad (29)$$

The initial condition corresponding to equation (29) is

$$\gamma_2(x_1,\vec{\rho}_1;x_1,\vec{\rho}_2)=\Gamma_2^{(1)}(x_1,\vec{\rho}_1,\vec{\rho}_2). \qquad (30)$$

Differentiating (29) with respect to x_2, we express the derivatives on the right using $\partial\gamma_2/\partial x_2$ the same equation (29). The result is

$$\frac{\partial^2\gamma_2(x_1,\vec{\rho}_1;x_2,\vec{\rho}_2)}{\partial x_2^2}=\langle u_1(x_1,\vec{\rho}_1)\frac{\partial^2 u_1^*(x_2,\vec{\rho}_2)}{\partial x_2^2}\rangle=$$

$$=-\frac{1}{4k^2}\Delta_2^2\gamma_2(x_1,\vec{\rho}_1;x_2,\vec{\rho}_2)+\frac{ikA(0)}{8}\Delta_2\gamma_2(x_1,\vec{\rho}_1;x_2,\vec{\rho}_2)+$$

$$+\frac{k^4A^2(0)}{64}\gamma_2(x_1,\vec{\rho}_1;x_2,\vec{\rho}_2).$$

Let $x_2=x_1=x$. Using the initial condition (30), we find

$$\langle u_1(x,\vec{\rho}_1)\frac{\partial^2 u_1^*(x,\vec{\rho}_2)}{\partial x^2}\rangle=\left\{-\frac{1}{4k^2}\Delta_2^2+\frac{ikA(0)}{8}\Delta_2+\frac{k^4A^2(0)}{64}\right\}\Gamma_2^{(1)}(x,\vec{\rho}_1,\vec{\rho}_2).$$

Inserting this expression in (27), we obtain an equation for the function $\Gamma_2^{(2)}$:

$$2ik\frac{\partial\Gamma_2^{(2)}(x,\vec{\rho}_1,\vec{\rho}_2)}{\partial x}+(\Delta_1-\Delta_2)\Gamma_2^{(2)}(x,\vec{\rho}_1,\vec{\rho}_2)+\frac{i\pi k^3}{2}H(\vec{\rho}_1-\vec{\rho}_2)\Gamma_2^{(2)}(x,\vec{\rho}_1,\vec{\rho}_2)=$$

$$=-\frac{1}{4k^2}\Delta_2^2\Gamma_2^{(1)}(x,\vec{\rho}_1,\vec{\rho}_2)+\frac{ikA(0)}{8}\Delta_2\Gamma_2^{(1)}(x,\vec{\rho}_1,\vec{\rho}_2)+\frac{k^4A^2(0)}{64}\Gamma_2^{(1)}(x,\vec{\rho}_1,\vec{\rho}_2). \quad (31)$$

Expression (20) contains the function $\Gamma_2^{(2)}(x,\vec{\rho}_1,\vec{\rho}_2)+\Gamma_2^{(2)*}(x,\vec{\rho}_2,\vec{\rho}_1)=\widetilde{\Gamma}_2(x,\vec{\rho}_1,\vec{\rho}_2)$. Taking the complex conjugate of (31) we interchange the points 1 and 2 and add the result to (31). Seeing that the function $H(\vec{\rho})$ is even and using the relation $\Gamma_2^{(1)*}(x,\vec{\rho}_2,\vec{\rho}_1)=\Gamma_2^{(1)}(x,\vec{\rho}_1,\vec{\rho}_2)$, we obtain the following equation for $\widetilde{\Gamma}_2$:

$$2ik\frac{\partial\widetilde{\Gamma}_2(x,\vec{\rho}_1,\vec{\rho}_2)}{\partial x}+(\Delta_1-\Delta_2)\widetilde{\Gamma}_2(x,\vec{\rho}_1,\vec{\rho}_2)+\frac{i\pi k^3}{2}H(\vec{\rho}_1-\vec{\rho}_2)\widetilde{\Gamma}_2(x,\vec{\rho}_1,\vec{\rho}_2)=$$

$$=\frac{1}{4k^2}(\Delta_1^2-\Delta_2^2)\Gamma_2^{(1)}(x,\vec{\rho}_1,\vec{\rho}_2)+\frac{ikA(0)}{8}(\Delta_1+\Delta_2)\Gamma_2^{(1)}(x,\vec{\rho}_1,\vec{\rho}_2). \quad (32)$$

Let us now estimate the correction $\widetilde{\Gamma}_2$. We will again consider the case of a plane incident wave, when

$$\Gamma_2^{(1)}(x,\vec{\rho}_1,\vec{\rho}_2)=|u_0^2|\exp\left\{-\frac{\pi k^2}{4}H(\vec{\rho}_1-\vec{\rho}_2)x\right\}. \quad (33)$$

In this case, $\widetilde{\Gamma}_2(x,\vec{\rho}_1,\vec{\rho}_2)=\widetilde{\Gamma}_2(x,\vec{\rho})$, where $\vec{\rho}=\vec{\rho}_1-\vec{\rho}_2$. Moreover, $(\Delta_1-\Delta_2)\widetilde{\Gamma}_2=0$, $(\Delta_1^2-\Delta_2^2)\Gamma_2^{(1)}=0$, $(\Delta_1+\Delta_2)\Gamma_2^{(1)}=2\Delta\Gamma_2^{(1)}$. Equation (32) thus takes the form

$$2ik\frac{\partial\widetilde{\Gamma}_2(x,\vec{\rho})}{\partial x}+\frac{i\pi k^3}{2}H(\vec{\rho})\widetilde{\Gamma}_2(x,\vec{\rho})=\frac{ikA(0)}{4}\Delta\Gamma_2^{(1)}(x,\vec{\rho}). \quad (34)$$

Evaluating $\Delta\Gamma_2^{(1)}(x,\vec{\rho})$ and then solving equation (34) with the initial condition $\widetilde{\Gamma}_2(0,\vec{\rho})=0$, we find

$$\widetilde{\Gamma}_2(x,\vec{\rho})=\Gamma_2^{(1)}(x,\vec{\rho})\cdot\frac{\pi k^2A(0)}{64}x^2\left[\frac{\pi k^2x}{3}(H'(\rho))^2-\Delta H(\rho)\right]. \quad (35)$$

Here $H'(\rho)$ denotes the derivative of $H(\rho)$ with respect to the scalar argument. Inserting (35) in (20), we find

$$\Gamma_2(x,\vec{\rho})=\Gamma_2^{(1)}(x,\rho)\cdot\left\{1+\frac{\pi k^2A(0)}{64}x^2\left[\frac{\pi k^2x}{3}H'^2-\Delta H\right]+\ldots\right\}. \qquad (36)$$

As in the case of corrections to the mean field, the correction to $\Gamma_2^{(1)}$ increases with the increasing x. However, for every $\rho\neq 0$, the function $\Gamma_2^{(1)}(x,\rho)$ itself decreases exponentially with increasing x. Therefore, we should naturally constrain the corrections to $\Gamma_2^{(1)}$ to remain small only in the region $x\lesssim x_0(\rho)$, where $x_0(\rho)$ is determined from the equality $k^2H(\rho)x_0(\rho)=$ $=1$. This leads to the condition

$$\left|\frac{A(0)}{k^2H(\rho)}\left[\frac{\pi}{3}\left(\frac{H'}{H}\right)^2-\frac{\Delta H}{H}\right]\right|\ll 1. \qquad (37)$$

Let us again consider a model of the fluctuations ϵ with a three-dimensional spectral density of the form

$$\phi_\epsilon(\kappa)=\frac{AC_\epsilon^2}{(\kappa_0^2+\kappa^2)^{11/6}}\exp(-\kappa^2/\kappa_m^2),\ \kappa_m\gg\kappa_0. \qquad (38)$$

For this model, $A(0)\sim C_\epsilon^2L_0^{5/3}$ (apart from terms of the order of l_0/L_0, see (67.38)). The function $H(\rho)$ has the form $H(\rho)\sim C_\epsilon^2\rho^{5/3}$ for $l_0\ll\rho\ll L_0$ and $H(\rho)\sim C_\epsilon^2 l_0^{-\frac{1}{3}}\rho^2$ for $\rho\ll l_0$.

Consider the region $\rho\gg l_0$. Inserting $A(0)$ and $H(\rho)$ in (37), we obtain the condition

$$\frac{L_0^{5/3}}{k^2\rho^{11/3}}\ll 1. \qquad (39)$$

The smallest value of ρ for which this condition is observed is of the order of magnitude of l_0.. If ρ is replaced with l_0 in (39), we obtain the condition

$$\lambda^2\ll l_0^2\left(\frac{l_0}{L_0}\right)^{5/3} \qquad (40)$$

and (39) is satisfied if (40) is satisfied.

Let us now consider separately the case $\rho=0$, when $\Gamma_2^{(1)}(x,0)=|u_0^2|=\text{const}$. In this case, (36) leads to a certain restriction on the distance x. Since $H'(0)=0$, $\Delta H(0)\sim C_\epsilon^2l_0^{-1/3}$, the requirement of a small correction to the mean intensity $\Gamma_2(x,0)$ leads to the condition

$$k^2C_\epsilon^4L_0^{5/3}l_0^{-1/3}x^2\ll 1. \qquad (41)$$

If we introduce the characteristic length

$$x_m=\frac{\lambda}{C_\epsilon^2L_0^{2/3}}\left(\frac{l_0}{L_0}\right)^{1/6}\sim\frac{\lambda}{\sigma_\epsilon^2}\left(\frac{l_0}{L_0}\right)^{1/6} \qquad (42)$$

condition (41) may be written in the form

$$x^2 \ll x_m^2. \tag{43}$$

Here x_m characterizes the distance over which the mean intensity of the field markedly changes because of back-scattering.

Conditions (40) and (41) coincide with the previously derived conditions necessary for applying the Markov approximation within the framework of the parabolic equation. This also applies to the conditions derived from an analysis ofthe mean field.

Thus, if the conditions $\lambda\alpha \ll 1$, $kl_0 \gg 1$ and the inequalities (40), (41) are violated, both the Markov approximation and the parabolic equation break down simultaneously. Therefore, these two approximations should naturally be applied together.

The conditions necessary to apply the parabolic equation were also considered in /193/, where the solutions of the complete scalar equation $\psi(x, \vec{\rho})$ and of the parabolic equation $u(x, \vec{\rho})$ were represented in the form of continuous integrals. If the refractive index field is a Gaussian random variable, the averaging in these integral representations can be carried out in explicit form. Comparison of the expressions for $\bar{u}\exp(ikx)$ and $\langle\psi\rangle$ makes it possible to find the conditions under which these integrals agree. The results derived for $\bar{u}$ in /193/ lead to the same conditions for the applicability of the parabolic equation as those obtained by the perturbation method above.

§ 69. The Markov approximation for dielectric constant fluctuations with a non-Gaussian probability distribution

In the present section, we will derive equations for the moments of the field $u(x, \vec{\rho})$ assuming the fluctuations of the refractive index are not Gaussian. The method of derivation of these equations was proposed by Klyatskin /192/. This is essentially the method applied in the preceding section to the particular case of Gaussian dielectric constant fluctuations. We also take one further step toward a more generalized application of the method: we consider the case of statistically inhomogeneous fluctuations.

If the field $\epsilon_1(x, \vec{\rho})$ is non-Gaussian, moments of higher order should be used together with the correlation function for its description. We assume, as before, that $\epsilon_1(x, \vec{\rho})$ is a delta-correlated field, and this assumption is now extended to all the moments of the field $\epsilon_1(x, \vec{\rho})$.

Consider any fixed value of the coordinate $x = x_1$ and the points $\xi_1 < x_1, \xi_2 < x_1, \ldots, \xi_n < x_1; \xi_1' > x_1, \ldots, \xi_m' > x_1$. We assume that the following condition is satisfied:

$$\begin{aligned}&\langle \epsilon_1(\xi_1, \vec{\rho}_1) \cdots \epsilon_1(\xi_n, \vec{\rho}_n)\, \epsilon_1(\xi_1', \vec{\rho}_1{}') \cdots \epsilon_1(\xi_m', \vec{\rho}_m{}') \rangle = \\ &= \langle \epsilon_1(\xi_1, \vec{\rho}_1) \cdots \epsilon_1(\xi_n, \vec{\rho}_n) \rangle \cdot \langle \epsilon_1(\xi_1', \vec{\rho}_1{}') \cdots \epsilon_1(\xi_m', \vec{\rho}_m{}') \rangle.\end{aligned} \tag{1}$$

This condition indicates that if all the points $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_n$ lie to the left of the plane $x = x_1$ and all the points $\vec{r}_1{}', \ldots, \vec{r}_m{}'$ lie to the right of this plane, so that there is a "gap" of finite width between the two groups of points, then $\epsilon_1(\vec{r}_1),...,\epsilon_1(\vec{r}_n)$ and $\epsilon_1(\vec{r}'_1),...,\epsilon_1(\vec{r}'_m)$ are statistically independent. This relationship constitutes a generalization of our previous hypothesis regarding the delta-correlation of fluctuations to the case of the higher moments of the field $\epsilon_1(x, \vec{\rho})$.

The parabolic equation is used as the dynamic equation of the problem:

$$2ik\frac{\partial u(x,\vec{\rho})}{\partial x} + \Delta u(x,\vec{\rho}) + k^2\epsilon_1(x,\vec{\rho})u(x,\vec{\rho}) = 0 \qquad u(0,\vec{\rho}) = u_0(\vec{\rho}). \quad (2)$$

We have seen before (see (65.16)) that the solution of equation (2) has the form

$$u(x,\vec{\rho}) = u_0(\vec{\rho}) + \int_0^x dx_1 \iint_{-\infty}^{\infty} d^2\rho_1 G_0(x,\vec{\rho};x_1,\vec{\rho}_1)\,\epsilon_1(x_1,\vec{\rho}_1)\,u_0(\vec{\rho}_1) +$$
$$+ \int_0^x dx_1 \iint_{-\infty}^{\infty} d^2\rho_1 G_0(x,\vec{\rho};x_1,\vec{\rho}_1)\,\epsilon_1(x_1,\vec{\rho}_1) \int_0^{x_1} dx_2 \iint_{-\infty}^{\infty} d^2\rho_2 G_0(x_1,\vec{\rho}_1;x_2,\vec{\rho}_2)\cdot$$
$$\epsilon_1(x_2,\vec{\rho}_2)\,u_0(\vec{\rho}_2) + \ldots \quad (3)$$

We see from this formula that $u(x,\vec{\rho})$ depends only on the values of $\epsilon_1(\xi,\vec{\rho}')$ for $\xi < x$.

Consider the expression

$$\langle u(x,\vec{\rho}_1)u(x,\vec{\rho}_2)\cdots u(x,\vec{\rho}_n)u^*(x,\vec{\rho}'_1)\cdots u^*(x,\vec{\rho}'_m)\cdot$$
$$\cdot\,\epsilon_1(\xi'_1,\vec{\rho}''_1)\cdots\epsilon_1(\xi'_n,\vec{\rho}''_n)\rangle \quad (4)$$

where $\xi'_k > x$ for all $k = 1,...,N$. Using (3), we may represent the product $u(x,\vec{\rho}_1)\cdots u^*(x,\vec{\rho}'_m)$ in the form

$$u(x,\vec{\rho}_1)\cdots u^*(x,\vec{\rho}'_m) = \sum_{n=0}^{\infty} \int_{-\infty}^{\infty}\!\!\cdots\!\int d^2\rho'''_1\cdots d^2\rho'''_e \int_0^x d\xi_1\cdots\int_0^x d\xi_e$$
$$F_e(x;\xi_1,...,\xi_e;\vec{\rho}'''_1,...,\vec{\rho}'''_e)\,\epsilon_1(\xi_1,\vec{\rho}'''_1)\cdots\epsilon_1(\xi_e,\vec{\rho}'''_e) \quad (5)$$

where the integral over each ξ_j is taken between the limits $(0, x)$ (the function F_e may vanish within some subregion of the multidimensional cube $0 < \xi_j < x$). Inserting (5) in (4) and using (1), we obtain

$$\langle u(x,\vec{\rho}_1)\cdots u^*(x,\vec{\rho}'_m)\,\epsilon_1(\xi'_1,\vec{\rho}''_1)\cdots\epsilon_1(\xi'_n,\vec{\rho}''_n)\rangle =$$
$$= \langle u(x,\vec{\rho}_1)\cdots u^*(x,\vec{\rho}'_m)\rangle\langle\epsilon_1(\xi'_1,\vec{\rho}''_1)\cdots\epsilon_1(\xi'_N,\vec{\rho}''_N)\rangle. \quad (6)$$

Note that (3) and the ensuing relation $\delta u(x,\vec{\rho})/\delta\epsilon_1(x',\vec{\rho}') = 0_{npu}$ for $x' > x$ (a mathematical statement of the "causality principle") are independent of the assumption of the delta-correlation of the field ϵ_1, whereas (6) is a direct consequence of this assumption. Equality (6) implies that the fields in the plane x are statistically independent of the succeeding values of ϵ_1. It is a consequence of the "causality principle" and of the delta-correlation of ϵ.

Let us consider in some detail the derivation of the equations for the moments of the field u. Multiplying (2) by $u^*(x,\vec{\rho}')$, and the equation for

$u^*(x, \vec{\rho}')$ by $u(x, \vec{\rho})$ and subtracting, we obtain an equation for the product $u(x, \vec{\rho}_1)u^*(x, \vec{\rho}_2)$. This equation may be transformed into an integral equation (see the analogous transformation in the derivation of 68.24):

$$2ik\, u(x, \vec{\rho}_1)u^*(x, \vec{\rho}_2) = 2ik\, u_0(\vec{\rho}_1)u_0^*(\vec{\rho}_2) \exp\left\{\frac{ik}{2}\int_0^x [\epsilon_1(\xi, \vec{\rho}_1) - \epsilon_1(\xi, \vec{\rho}_2)]d\xi\right\}$$

$$+ \int_0^x dx' \exp\left\{\frac{ik}{2}\int_{x'}^x [\epsilon_1(\xi, \vec{\rho}_1) - \epsilon_1(\xi, \vec{\rho}_2)]d\xi\right\}[(\Delta_2 - \Delta_1)u(x', \vec{\rho}_1)u^*(x', \vec{\rho}_2)]. \quad (7)$$

Here the exponential only contains the succeeding values of $\epsilon_1(\xi, \vec{\rho})$ in relation to $u(x', \vec{\rho})$.

Expanding in a series the exponential in the second term and using (6) when averaging (7), we represent each term of the series as a product of means. Summing the series, we obtain the equation

$$2ik\, \Gamma_2(x, \vec{\rho}_1, \vec{\rho}_2) = 2ik\, \Gamma_2(0, \vec{\rho}_1, \vec{\rho}_2) < \exp\left\{\frac{ik}{2}\int_0^x [\epsilon_1(\xi, \vec{\rho}_1) - \epsilon_1(\xi, \vec{\rho}_2)]d\xi\right\} > +$$

$$+ \int_0^x dx' < \exp\left\{\frac{ik}{2}\int_{x'}^x [\epsilon_1(\xi, \vec{\rho}_1) - \epsilon_1(\xi, \vec{\rho}_2)]d\xi\right\} > \cdot [(\Delta_2 - \Delta_1)\,\Gamma_2(x', \vec{\rho}_1, \vec{\rho}_2)]. \quad (8)$$

This is a closed integral equation for Γ_2. As we have seen in the previous section, for a Gaussian field ϵ_1 this equation reduces to a differential equation. In the general case, however, it remains an integral equation.

Note that in the derivation of the equation for Γ_2 we made use of the delta-correlation of the difference field $[\epsilon_1(\xi, \vec{\rho}_1) - \epsilon_1(\xi, \vec{\rho}_2)]$, and not of the field ϵ_1 itself. A less rigid restriction was thus in effect imposed.

Another remark concerns the transition to the case of a Gaussian field ϵ_1. It would be sufficient to demand that the function $\Delta S = \frac{k}{2}\int_{x'}^x [\epsilon_1(\xi, \vec{\rho}_1) - \epsilon_1(\xi, \vec{\rho}_2)]d\xi$, which is the difference of the phase increments over the path $(x - x')$ between the points $(x, \vec{\rho}_1)$ and $(x, \vec{\rho}_2)$ computed in the first approximation of geometrical optics, be a Gaussian random variable with variance

$$\langle \Delta S^2 \rangle = \frac{1}{4}k^2(x - x')\int_0^\infty [D_\epsilon(\xi, \vec{\rho}_1 - \vec{\rho}_2) - D_\epsilon(\xi, 0)]d\xi \quad (9)$$

(the last expression corresponds to 40.19a). The normal distribution of the geometrical phase difference, however, may result not only from a normal distribution of ϵ_1, but (approximately) also from the central limit theorem. In this case

$$\langle \exp(i\Delta S) \rangle = \exp\left\{-\frac{1}{2}\langle \Delta S^2 \rangle\right\}$$

and insertion of (9) into (8) followed by differentiation with respect to x recovers the previous equation for Γ_2 which corresponds to the case of Gaussian fluctuations of ϵ_1.

Finally, equation (8) is applicable to a fluctuating field ϵ_1 which is not statistically homogeneous. For the sake of simplicity, let ϵ_1 be a normal (Gaussian) field and let

$$\int_{-\infty}^{\infty} \langle \epsilon_1(x, \vec{\rho})\, \epsilon_1(x', \vec{\rho}') \rangle\, dx' = A(x, \vec{\rho} - \vec{\rho}') \quad (10)$$

so that the approximation to the field ϵ_1 with a delta-correlated field has the form

$$\langle \epsilon_1(x, \vec{\rho})\, \epsilon_1(x', \vec{\rho}') \rangle = \delta(x-x')\, A(x, \vec{\rho}-\vec{\rho}'). \tag{11}$$

Then

$$\begin{aligned} &\left\langle \exp\left\{\frac{ik}{2}\int_{x'}^{x} [\epsilon_1(\xi, \vec{\rho}_1) - \epsilon_1(\xi, \vec{\rho}_2)]\, d\xi\right\}\right\rangle = \\ &= \exp\left\{-\frac{k^2}{4}\int_{x'}^{x} [A(\xi, 0) - A(\xi, \vec{\rho}_1-\vec{\rho}_2)]\, d\xi\right\} = \\ &= \exp\left\{-\frac{k^2}{4}\int_{0}^{x} [A(\xi, 0) - A(\xi, \vec{\rho}_1-\vec{\rho}_2)]\, d\xi\right\} \cdot \\ &\quad \exp\left\{+\frac{k^2}{4}\int_{0}^{x'} [A(\xi, 0) - A(\xi, \vec{\rho}_1-\vec{\rho}_2)]\, d\xi\right\}. \end{aligned} \tag{12}$$

Inserting (12) in (8), multiplying by

$$\exp\left\{\frac{k^2}{4}\int_{0}^{x} [A(\xi, 0) - A(\xi, \vec{\rho}_1-\vec{\rho}_2)]\, d\xi\right\}$$

and differentiating with respect to x, we obtain an equation

$$2ik\frac{\partial \Gamma_2(x, \vec{\rho}_1, \vec{\rho}_2)}{\partial x} + (\Delta_1 - \Delta_2)\,\Gamma_2 + \frac{i\pi k^3}{2} H(x, \vec{\rho}_1-\vec{\rho}_2)\,\Gamma_2 = 0 \tag{13}$$

where

$$H(x, \vec{\rho}) = \frac{1}{\pi}[A(x, 0) - A(x, \vec{\rho})] = 2\iint_{-\infty}^{\infty} \phi_\epsilon(x; 0, \vec{\kappa})\,[1 - \cos \vec{\kappa}\vec{\rho}]\, d^2\kappa \tag{14}$$

and $\phi_\epsilon(x; p, \vec{\kappa})$ is the spectral density of the fluctuations ϵ_1, which is a smooth function of the coordinate x (see § 7).

Similarly we can derive an equation for the fourth moment

$$\Gamma_4(x; \vec{\rho}_1, \vec{\rho}_2; \vec{\rho}_1', \vec{\rho}_2') = \langle u(x, \vec{\rho}_1)\, u(x, \vec{\rho}_2)\, u^*(x, \vec{\rho}_1')\, u^*(x, \vec{\rho}_2') \rangle.$$

In the derivation of the equation for Γ_4 it is sufficient to assume that the function

$$V(\xi; \vec{\rho}_1, \vec{\rho}_2; \vec{\rho}_1', \vec{\rho}_2') = \epsilon_1(\xi, \vec{\rho}_1) + \epsilon_1(\xi, \vec{\rho}_2) - \epsilon_1(\xi, \vec{\rho}_1') - \epsilon_1(\xi, \vec{\rho}_2')$$

is delta-correlated and its integral is normally distributed. The requirements of delta-correlation and normal distribution in this case need not be imposed on the field $\epsilon_1(\xi, \vec{\rho})$ itself. If the field ϵ_1 is statistically inhomogeneous, the equation for Γ_4 takes the form

$$\frac{\partial \Gamma_4}{\partial x} = \frac{i}{2k}(\Delta_1 + \Delta_2 - \Delta_1' - \Delta_2')\,\Gamma_4 - \frac{\pi k^2}{4} F(x; \vec{\rho}_1, \vec{\rho}_2; \vec{\rho}_1', \vec{\rho}_2')\,\Gamma_4 \tag{15}$$

where

$$\begin{aligned} F(x; \vec{\rho}_1, \vec{\rho}_2; \vec{\rho}_1', \vec{\rho}_2') &= H(x, \vec{\rho}_1 - \vec{\rho}_1') + H(x, \vec{\rho}_2 - \vec{\rho}_2') + H(x, \vec{\rho}_1 - \vec{\rho}_2') + \\ &+ H(x, \vec{\rho}_2 - \vec{\rho}_1') - H(x, \vec{\rho}_2 - \vec{\rho}_1) - H(x, \vec{\rho}_2' - \vec{\rho}_1'). \end{aligned} \tag{16}$$

Equation (15) constitutes a generalization of equation (65.22) to the case of a statistically inhomogeneous field for fluctuations in ϵ_1.

Also note that

$$F(x, \vec{\rho}_1, \vec{\rho}_2, \vec{\rho}_1', \vec{\rho}_2') = \frac{1}{2\pi}\int_{-\infty}^{\infty} \langle V(x; \vec{\rho}_1, \vec{\rho}_2; \vec{\rho}_1', \vec{\rho}_2')\, V(x'; \vec{\rho}_1, \vec{\rho}_2; \vec{\rho}_1', \vec{\rho}_2')\rangle\, dx'$$

so that $F \geqslant 0$.

§70. The second-order mutual coherence function in a turbulent medium

In the present section we solve equation (69.13) for Γ_2:

$$2ik\frac{\partial \Gamma_2(x, \vec{\rho}_1, \vec{\rho}_2)}{\partial x} + (\Delta_1 - \Delta_2)\Gamma_2 + \frac{i\pi k^3}{2} H(x, \vec{\rho}_1 - \vec{\rho}_2)\,\Gamma_2 = 0 \quad (1)$$

$$\Gamma_2(0, \vec{\rho}_1, \vec{\rho}_2) = \Gamma_2^{(0)}(\vec{\rho}_1, \vec{\rho}_2) = u_0(\vec{\rho}_1)\, u_0^*(\vec{\rho}_2). \quad (2)$$

Introducing new variables

$$\mathbf{R} = \frac{1}{2}(\vec{\rho}_1 + \vec{\rho}_2),\; \vec{\rho} = \vec{\rho}_1 - \vec{\rho}_2$$

then $\Delta_1 - \Delta_2 = 2\nabla_\rho \nabla_R$, and equation (1) takes the form

$$2ik\frac{\partial \Gamma_2(x, \mathbf{R}, \vec{\rho})}{\partial x} + 2\nabla_\rho \nabla_R \Gamma_2(x, \mathbf{R}, \vec{\rho}) + \frac{i\pi k^3}{2} H(x, \vec{\rho})\,\Gamma_2 = 0$$

$$\Gamma_2(0, \mathbf{R}, \vec{\rho}) = u_0(\mathbf{R} + \tfrac{1}{2}\vec{\rho})\, u_0^*(\mathbf{R} - \tfrac{1}{2}\vec{\rho}). \quad (3)$$

Equation (3) can be solved by taking the Fourier transform with respect to the variable $\mathbf{R}$. This transformation is feasible if $\Gamma_2(x, \mathbf{R}, \vec{\rho})$ decreases sufficiently fast for $R \to \infty$, i.e., for bounded beams of radiation. The Fourier transformation is formally inapplicable to plane waves, but in this case $\nabla_R \Gamma_2 = 0$ and (3) has the immediate solution

$$\Gamma_2(x, \vec{\rho}) = |u_0^2| \exp\left\{-\frac{\pi k^2}{4}\int_0^x H(\xi, \vec{\rho})\, d\xi\right\}. \quad (4)$$

For bounded beams, we take

$$\Gamma_2(x, \mathbf{R}, \vec{\rho}) = \iint_{-\infty}^{\infty} \check{\Gamma}_2(x, \mathbf{p}, \vec{\rho}) \exp(i\mathbf{p}\mathbf{R})\, d^2 p. \quad (5)$$

Inserting (5) in (3), we obtain

$$\frac{\partial \check{\Gamma}_2(x, \mathbf{p}, \vec{\rho})}{\partial x} + \frac{\mathbf{p}}{k}\nabla_\rho \check{\Gamma}_2(x, \mathbf{p}, \vec{\rho}) + \frac{\pi k^2}{4} H(x, \vec{\rho})\, \check{\Gamma}_2(x, \mathbf{p}, \vec{\rho}) = 0. \quad (6)$$

Equation (6) is a first-order linear partial differential equation, which can be solved by the standard technique of reduction to an ordinary differential equation along the characteristic. The equation of the characteristic $\vec{\rho}=\vec{\rho}(x')$, where x' is the current coordinate, has the form

$$\frac{d\vec{\rho}(x')}{dx'}=\frac{\mathbf{p}}{k} \tag{7}$$

so that the characteristic for $x'=x$ passing through the observation point $(x, \vec{\rho})$ has the form

$$\vec{\rho}(x')=\vec{\rho}-\frac{\mathbf{p}}{k}(x-x'). \tag{8}$$

Equation (6) along this characteristic takes the form

$$\frac{d\check{\Gamma}_2}{dx'}=-\frac{\pi k^2}{4}H(x', \vec{\rho}(x'))\,\check{\Gamma}_2(x', \mathbf{p}, \vec{\rho}(x')) \tag{9}$$

where expression (8) for $\vec{\rho}(x')$ has been substituted for $\vec{\rho}$ on the right.

Integrating (9) and seeing that $\vec{\rho}(0)=\vec{\rho}-\frac{\mathbf{p}x}{k}$, we obtain

$$\check{\Gamma}_2(x, \mathbf{p}, \vec{\rho})=\check{\Gamma}_2(0, \mathbf{p}, \vec{\rho}-\frac{\mathbf{p}}{k}x)\exp\left\{-\frac{\pi k^2}{4}\int_0^x H(x', \vec{\rho}-\frac{\mathbf{p}}{k}(x-x'))\,dx'\right\}. \tag{10}$$

Let us now return to the coordinate representation. Inserting in (10)

$$\check{\Gamma}_2(0, \mathbf{p}, \vec{\rho})=\frac{1}{4\pi^2}\iint_{-\infty}^{\infty} u_0(\mathbf{R}'+\frac{\vec{\rho}}{2})\,u_0^*(\mathbf{R}'-\frac{\vec{\rho}}{2})\exp(-i\,\mathbf{p}\,\mathbf{R}')\,d^2R'$$

and substituting the result in (5), we finally obtain

$$\Gamma_2(x, \mathbf{R}, \vec{\rho})=\frac{1}{4\pi^2}\iint_{-\infty}^{\infty} d^2R'\iint_{-\infty}^{\infty} d^2p\, u_0(\mathbf{R}'+\frac{1}{2}\vec{\rho}-\frac{\mathbf{p}x}{2k})\cdot$$

$$u_0^*(\mathbf{R}'-\frac{1}{2}\vec{\rho}+\frac{\vec{p}x}{2k})\exp\left\{i\,\mathbf{p}(\mathbf{R}-\mathbf{R}')-\frac{\pi k^2}{4}\int_0^x H(x', \vec{\rho}-\frac{\mathbf{p}}{k}(x-x'))\,dx'\right\}. \tag{11}$$

If $u_0=\text{const}$, the integration over $\mathbf{R}'$ can be performed in (11), the result being a δ-function of $\mathbf{p}$, and we obtain solution (4).

The mean field intensity is

$$\langle I(x, \mathbf{R})\rangle=\bar{I}(x, \mathbf{R})=\langle u(x, \mathbf{R})\,u^*(x, \mathbf{R})\rangle=\Gamma_2(x, \mathbf{R}, 0).$$

Setting $\vec{\rho}=0$ in (11) and remembering that $H(x, \vec{\rho})$ is even in $\vec{\rho}$, we find

$$\bar{I}(x, \mathbf{R})=\frac{1}{4\pi^2}\iint_{-\infty}^{\infty} d^2R'\iint_{-\infty}^{\infty} d^2p\, u_0(\mathbf{R}'-\frac{\mathbf{p}x}{2k})\,u_0^*(\mathbf{R}'+\frac{\mathbf{p}x}{2k})\cdot$$

$$\cdot\exp\left\{i\,\mathbf{p}(\mathbf{R}-\mathbf{R}')-\frac{\pi k^2}{4}\int_0^x H(x', \frac{\mathbf{p}}{k}(x-x'))\,dx'\right\}. \tag{12}$$

As we have noted in § 65, equation (1) for the coherence function Γ_2 is connected to the equation of radiative transfer in the small-angle approximation /186/.

We now introduce the spectral density function of Γ_2 with respect to the difference variable $\vec{\rho}$,

$$\Gamma_2(x, \mathbf{R}, \vec{\rho}) = \iint_{-\infty}^{\infty} J(x, \mathbf{R}, \vec{\kappa}) \exp(i\vec{\kappa}\vec{\rho})\, d^2\kappa \tag{13}$$

and use the spectral representation (69.14) of $H(x, \vec{\rho})$:

$$H(x, \vec{\rho}) = 2 \iint_{-\infty}^{\infty} \phi_\epsilon(x; 0, \vec{\kappa}) \,[1 \cdot \cos \vec{\kappa}\vec{\rho}]\, d^2\kappa. \tag{14}$$

Moreover, if we set

$$f(x, \vec{\kappa}) = \frac{\pi k^2}{2} \phi_\epsilon(x; 0, \vec{\kappa}), \quad \alpha(x) = \iint_{-\infty}^{\infty} f(x, \vec{\kappa})\, d^2\kappa$$

then insertion of (13) and (14) in (3) yields the equation of radiative transfer in the small-angle approximation:

$$\frac{\partial J(x, \mathbf{R}, \vec{\kappa})}{\partial x} + \frac{\vec{\kappa}}{k} \nabla_R J + \alpha(x) J = \iint_{-\infty}^{\infty} f(x, \vec{\kappa} - \vec{\kappa}') J(x, \mathbf{R}, \vec{\kappa}')\, d^2\kappa' \tag{15}$$

$\alpha(x)$ is the extinction coefficient, and $f(x, \vec{\kappa})$ is the angular scattering diagram.

The inverse Fourier transform of (11) gives an exact solution of equation (15). In this solution if we express the initial fields u_0, u_0^* in terms of $J_0(\mathbf{R}, \vec{\kappa}) = J(0, \mathbf{R}, \vec{\kappa})$ and the function $H(x, \vec{\rho})$ in terms of $f(x, \vec{\kappa})$, we obtain

$$J(x, \mathbf{R}, \vec{\kappa}) = \iint_{-\infty}^{\infty} d^2R' \iint_{-\infty}^{\infty} d^2\kappa' \, K(x, \mathbf{R}, \vec{\kappa}; \mathbf{R}', \vec{\kappa}') J_0(\mathbf{R}', \vec{\kappa}') \tag{16}$$

where Green's function K is expressed in the form

$$K(x, \mathbf{R}, \vec{\kappa}; \mathbf{R}', \vec{\kappa}') = (2\pi)^{-4} \iint_{-\infty}^{\infty} d^2\mathrm{p} \iint_{-\infty}^{\infty} d^2\rho \exp \Big\{ i(\vec{\kappa}' - \vec{\kappa})\vec{\rho} + $$

$$+ i\mathrm{p}\left(\mathbf{R} - \mathbf{R}' - \frac{\vec{\kappa}' x}{k}\right) - \int_0^x \alpha(x')\, dx' +$$

$$+ \int_0^x dx' \iint_{-\infty}^{\infty} d^2\kappa_1 \, f(x', \vec{\kappa}_1) \cos\left[\vec{\kappa}_1 \vec{\rho} - \frac{\vec{\kappa}_1 \vec{\rho}(x - x')}{k}\right] \Big\}. \tag{17}$$

Solutions (11) and (17) were derived by Dolin /198/. The mean intensity radiation of the $\bar{I}(x, \mathbf{R})$ is related to J by the following expression, which follows from (13):

$$\bar{I}(x, \mathbf{R}) = \iint_{-\infty}^{\infty} J(x, \mathbf{R}, \vec{\kappa})\, d^2\kappa. \tag{18}$$

Let us consider some consequences of equation (3). If the original beam is limited in space, i.e., $\Gamma_2(0, \mathbf{R}, \vec{\rho})$ decreases with increasing R faster than R^{-2}, it can be shown from (11) that for all x, $\Gamma_2(x, \mathbf{R}, \vec{\rho})$ will decrease with increasing R faster than R^{-2}. Assuming that this condition is satisfied, we integrate (3) with respect to $\mathbf{R}$.

Let

$$\iint_{-\infty}^{\infty} \Gamma^2(x, \mathbf{R}, \vec{\rho})\, d^2R = \gamma_2(x, \vec{\rho}).$$

Using Gauss's theorem, we find

$$\iint_{-\infty}^{\infty} \nabla_\rho \nabla_R \Gamma^2(x, \mathbf{R}, \vec{\rho})\, d^2R = 0.$$

Thus,

$$\frac{\partial \gamma_2}{\partial x} + \frac{\pi k^2}{4} H(x, \vec{\rho})\, \gamma_2(x, \vec{\rho}) = 0\,. \tag{19}$$

The function $\gamma_2(x, \vec{\rho})$ is proportional to the mutual coherence function averaged over the entire beam. From (19) it follows that

$$\gamma_2(x, \vec{\rho}) = \gamma_2(0, \vec{\rho}) \exp\left\{-\frac{\pi k^2}{4} \int_0^x H(x', \vec{\rho})\, dx'\right\}. \tag{20}$$

Setting $\vec{\rho} = 0$ in (20) and using the relation $H(x', 0) = 0$ (it follows from the definition of $H(x, \vec{\rho})$), we obtain the conservation law

$$\gamma_2(x, 0) = \gamma_2(0, 0) = \text{const.}$$

i.e.,

$$\iint_{-\infty}^{\infty} \bar{I}(x, \mathbf{R})\, d^2R = \iint_{-\infty}^{\infty} \bar{I}(0, \mathbf{R})\, d^2R\,. \tag{21}$$

This relation is a consequence of the law of conservation of energy for the function Γ^2. (A similar conservation law will be derived for Γ_4 later on.)

For the function

$$j(x, \vec{\kappa}) = \iint_{-\infty}^{\infty} J(x, \mathbf{R}, \vec{\kappa})\, d^2R \tag{22}$$

which is the mean flux (averaged over the entire cross section of the beam) in a direction making an angle $\vec{\kappa}/k$ to the x axis, we obtain from (20)

$$j(x\, \vec{\kappa}) = \iint_{-\infty}^{\infty} W(x, \vec{\kappa} - \vec{\kappa}')\, j_0(\vec{\kappa}')\, d^2\kappa' \tag{23}$$

where

$$W(x, \vec{\kappa} - \vec{\kappa}') = \frac{1}{4\pi^2} \iint_{-\infty}^{\infty} \exp\left\{-i(\vec{\kappa} - \vec{\kappa}')\,\vec{\rho} - \frac{\pi k^2}{4} \int_0^x H(x', \vec{\rho})\, dx'\right\} d^2\rho. \tag{24}$$

Relation (22) corresponds to the equality

$$\iint_{-\infty}^{\infty} j(x, \vec{\kappa})\, d^2\kappa = \iint_{-\infty}^{\infty} j_0(\vec{\kappa})\, d^2\kappa$$

or

$$\iint_{-\infty}^{\infty} d^2R \iint_{-\infty}^{\infty} d^2\kappa\, J(x, \mathbf{R}, \vec{\kappa}) = \iint_{-\infty}^{\infty} d^2R \iint_{-\infty}^{\infty} d^2\kappa\, J_0(\mathbf{R}, \vec{\kappa}). \tag{25}$$

Let us now return to the coherence function Γ_2. We will consider a particular example, when the initial field $u_0(\vec{\rho})$ has the form

$$u_0(\vec{\rho}) = u_0 \exp\left\{-\frac{\rho^2}{2a^2} + \frac{ik\rho^2}{2F}\right\}. \tag{26}$$

Here a is the effective beam radius, since

$$\iint_{-\infty}^{\infty} |u_0(\vec{\rho})|^2 d^2\rho = \pi a^2 u_0^2,$$

F is the radius of curvature of the wavefront. If $F=\infty$, the emergent beam has a plane phase front (a collimated beam), if $F>0$ the beam diverges (from the point $x=-F$), and if $F<0$ the beam is focused at the point $x=|F|$. Inserting (26) in expression (12) for the mean intensity, we obtain an expression in which the exponential contains linear and quadratic terms in $\mathbf{R}'$. Integration over $\mathbf{R}'$ readily gives

$$\bar{I}(x, \mathbf{R}) = \frac{a^2 u_0^2}{4\pi} \iint_{-\infty}^{\infty} d^2 p \exp\left\{ i\mathbf{p}\mathbf{R} - \mathbf{p}^2 \frac{x^2 g^2(x)}{4k^2 a^2} - \frac{\pi k^2}{4} \int_0^x H(x', \frac{\mathbf{p}}{k}(x-x'))dx' \right\} \quad (27)$$

where

$$g(x) = \sqrt{1 + k^2 a^4 (\frac{1}{x} + \frac{1}{F})^2}.$$

Expression (27) gives the mean intensity in a beam whose initial amplitud distribution is Gaussian across the beam and whose initial phase distribution is quadratic.

Let us now connect the integral over H entering the exponential in (27) with the structure function of the complex phase D_1 for a spherical wave. The function $D_1(x, \vec{\rho})$ for a plane wave was introduced in §46 (see (46.23) and (46.33)):

$$D_1(x, \vec{\rho}) = \frac{\pi k^2 x}{2} \cdot 2 \iint_{-\infty}^{\infty} [1 - \cos \vec{\kappa}\vec{\rho}] \phi_\epsilon(\vec{\kappa}) d^2\kappa = \frac{\pi k^2 x}{2} H(\vec{\rho}).$$

For the function $D_{1,\text{sph}}$ corresponding to a spherical wave, we readily obtain a relation

$$D_{1,\text{sph}}(x, \vec{\rho}) = \frac{1}{\rho} \int_0^\rho D_1(x, \rho') d\rho'$$

which is analogous to relation (41.13) for the phase in the geometrical optics approximation. Using this relation, we find

$$D_{1,\text{sph}}(x, \frac{px}{k}) = \frac{k}{px} \int_0^{\frac{px}{k}} D_1(x, \rho') d\rho' = \frac{1}{x} \int_0^x D_1(x, \frac{px'}{k}) dx' = \frac{\pi k^2}{2} \int_0^x H(\frac{px'}{k}) dx'.$$

For a medium with variable statistical characteristics (see §43), we obtain the relation

$$\frac{\pi k^2}{2} \int_0^x H(x', \frac{\mathbf{p}(x-x')}{k}) dx' = D_{1,\text{sph}}(x, \frac{\vec{p}x}{k}) \quad (28)$$

whose derivation is omitted here. Using (28), we can write (27) in the form

$$\bar{I}(x, \mathbf{R}) = \frac{a^2 u_0^2}{4\pi} \iint_{-\infty}^{\infty} d^2 p \exp\left\{ i\mathbf{p}\mathbf{R} - \frac{p^2 x^2 g^2(x)}{4k^2 a^2} - \frac{1}{2} D_{1,\text{sph}}(x, \frac{px}{k}) \right\}. \quad (27a)$$

Note that in our case relation (28) is simply used as a convenient notation demonstrating the physical meaning of (27). In fact, however, relation (28) is valid only in the region where the first approximation of the method of smooth perturbations is applicable. This is irrelevant for our case, since the region of application of (27), with $D_{1,\,sph}$ regarded as a shorthand notation of the corresponding integral, is independent of the conditions under which the structure function of the complex field from a point source truly coincides with the left-hand side of (28).

Let us now consider a particular example, assuming the spectrum of the fluctuations in ϵ_1 has the form

$$\phi_\epsilon(\vec{\kappa}) = A C_\epsilon^2 \kappa^{-11/3} \exp(-\kappa^2/\kappa_m^2). \qquad (29)$$

The function D_1 in this case is (see (47.5))

$$D_1(x,\rho) = \frac{6}{5}\pi^2 A\,\Gamma\left(\frac{1}{6}\right) C_\epsilon^2 k^2 x \kappa_m^{-5/3}\left[{}_1F_1\left(-\frac{5}{6}, 1, -\frac{\kappa_m^2\rho^2}{4}\right)^{-1}\right]. \qquad (30)$$

For $\kappa_m \rho \ll 1$, this function is quadratic in ρ, and for $\kappa_m\rho \gg 1$ it is proportional to $\rho^{5/3}$:

$$D_1(x,\rho) = \begin{cases} M_1 C_\epsilon^2 k^2 x \kappa_m^{1/3} \rho^2, \ \kappa_m\rho \ll 1, M_1 = \dfrac{\pi^2 A}{4}\Gamma\left(\dfrac{1}{6}\right) \\ \\ M_2 C_\epsilon^2 k^2 x \rho^{5/3}, \ \kappa_m\rho \gg 1, M_2 = \dfrac{6}{5}\pi^2 A\,\Gamma\left(\dfrac{1}{6}\right) / \, 2^{5/3}\Gamma\left(\dfrac{1}{6}\right) \end{cases} \qquad (31)$$

(see (47.20) and (47.26)). Hence,

$$D_{1,\,sph}(x,\rho) = \begin{cases} \dfrac{1}{3} M_1 C_\epsilon^2 k^2 x \kappa_m^{1/3}\rho^2 & \kappa_m\rho \ll 1 \quad (a) \\ \\ \dfrac{3}{8} M_2 C_\epsilon^2 k^2 \rho^{5/3} x & \kappa_m\rho \gg 1 \quad (b). \end{cases} \qquad (32)$$

Let us consider the extreme case of very strong fluctuations in the phase difference, when

$$C_\epsilon^2 k^2 x\, l_0^{5/3} \gg 1. \qquad (33)$$

If this condition is satisfied, $D_{1,\,sph}(x, l_0) \gg 1$. Since the contribution to $\bar{I}$ in (27a) from the integral over the region where $D_{1,\,sph} \gg 1$ is negligible, we conclude that in the relevant region $\frac{px}{k} \ll l_0$ and it is possible to use only the first of the asymptotic formulas in (32). Inserting it in (27a) and integrating, we find

$$\bar{I}(x,R) = \frac{a^2 u_0^2}{\dfrac{x^2 g^2(x)}{k^2 a^2} + \dfrac{2M_1}{3} C_\epsilon^2 x^3 \kappa_m^{1/3}} \exp\left\{-\frac{R^2}{\dfrac{x^2 g^2(x)}{k^2 a^2} + \dfrac{2M_1}{3} C_\epsilon^2 x^3 \kappa_m^{1/3}}\right\}. \qquad (34)$$

Note that condition (33), under which relation (34) is valid, is satisfied only over sufficiently long distances in the real atmosphere. The square of the beam width, according to (34), is a sum of the square of the beam width in an unperturbed medium ($\frac{x^2 g^2(x)}{k^2 a^2}$) and the mean square transverse displacement of the beam. Equation (34) is also valid in the other extreme case of relatively weak fluctuations, when

$$C_\epsilon^2 k^2 x \kappa_m^{1/3} \frac{a^2}{g^2(x)} \ll 1. \tag{35}$$

If this condition is satisfied, the effective integration range in (27a) is limited by the factor exp $(-\frac{p^2 x^2 g^2(x)}{4k^2 a^2})$ and the first asymptotic formula in (32) is applicable to the entire range.

Equation (34) is thus valid both for very strong phase different fluctuations (condition (33)) and for very weak fluctuations (condition (35)).

It now remains to consider the most significant case from the viewpoint of applications, namely the case when the inequality (33) is reversed. The region of values of p satisfying the condition $\frac{px}{k} < l_0$ then occupies an insignificant part of the effective integration region, which is described by the condition

$$p < p_{\max} \sim \min\left(\frac{ka}{xg(x)}, (C_\epsilon^2 k^{1/3} x^{8/3})^{-3/5}\right). \tag{36}$$

We therefore impose the inequality

$$\frac{kl_0}{x} \ll p_{\max}. \tag{37}$$

If this inequality is satisfied, the function $D_{1,\,\mathrm{sph}}$ in (27a) may be replaced with its asymptotic expression for $\kappa_m \rho \gg 1$, i.e., expression (32b). Inserting (36) in (37), we obtain the following inequalities:

$$C_\epsilon^2 k^2 x l_0^{5/3} \ll 1 \quad \text{if} \quad C_\epsilon^2 k^2 x \left(\frac{a}{g(x)}\right)^{5/3} \gg 1 \tag{37a}$$

$$l_0 \ll \frac{a}{g(x)} \quad \text{if} \quad C_\epsilon^2 k^2 x \left(\frac{a}{g(x)}\right)^{5/3} \ll 1. \tag{37b}$$

If conditions (37a) or (37b) are satisfied, we insert (32b) in (27a), integrate over the angular variable φ ($\mathrm{p} = \{p\cos\varphi, p\sin\varphi\}$), and introduce a dimensionless integration variable $t = xg(x)p/2ka$ to obtain

$$\bar{I}(x, R) = \frac{2u_0^2 k^2 a^4}{x^2 g^2(x)} \int_0^\infty J_0\left(\frac{2kaR}{xg(x)} t\right) e^{-t^2 - \frac{3M_2}{16} C_\epsilon^2 k^2 x \left(\frac{2a}{g(x)}\right)^{5/3} t^{5/5}} t\,dt. \tag{38}$$

Again using the notation

$$\frac{3}{8} M_2 C_\epsilon^2 k^2 x \left(\frac{2a}{g(x)}\right)^{5/3} = D_{1,\,\mathrm{sph}}\left(x, \frac{2a}{g(x)}\right)$$

we write (38) in the form

$$\bar{I}(x, R) = \frac{2u_0^2 k^2 a^4}{x^2 g^2(x)} \int_0^\infty J_0\left(\frac{2kaR}{xg(x)} t\right) \exp\left\{-t^2 - \frac{1}{2} D_{1,\,\mathrm{sph}}\left(x, \frac{2a}{g(x)}\right) t^{5/3}\right\} t\,dt. \tag{38a}$$

Note that conditions (37a) and (37b) permit quite arbitrary values of the parameter $D_{1,\mathrm{sph}}(x, 2a/g(x))$.

If in (38), we take $R=0, C_\epsilon^2=0$, we obtain the mean value of the intensity on the beam axis in an unperturbed medium:

$$I_0(x, 0)=u_0^2 k^2 a^4/x^2 g^2(x).$$

Using this relation, we write $\bar{I}(x, 0)$ in the form

$$\bar{I}(x, 0)=I_0(x, 0)\cdot f\left(\frac{1}{2}D_{1,\mathrm{sph}}\left(x, \frac{2a}{g(x)}\right)\right) \qquad (39)$$

where

$$f(u)=\int_0^\infty \exp\left\{-z-uz^{5/6}\right\}dz. \qquad (40)$$

The function $f(u)$ may be represented by a series

$$f(u)=\sum_{n=0}^{\infty}\frac{\Gamma(\frac{5n}{6}+1)}{n!}(-u)^n=1-0.94u+0.75u^2-0.564u^3+\ldots$$

which is convergent for all u, and its asymptotic expansion for $u\to+\infty$ is

$$f(u)\sim\frac{6}{5u^{6/5}}\sum_{n=0}^{\infty}\frac{\Gamma(\frac{6(n+1)}{5})}{n!}(-(u^{-6/5}))^n=\frac{1.10}{u^{6/5}}-\frac{1.49}{u^{12/5}}+\ldots$$

The function $f(u)$ is shown in Figure 96. Figure 97 shows the normalized distribution of the mean value of the beam intensity $\bar{I}(x, R)/\bar{I}(x, 0)$ as a function of the argument $\nu=\frac{2kaR}{xg(x)}$; the parameter in this graph is $\mu= =\frac{1}{2}D_{1,\mathrm{sph}}(x, \frac{2a}{g})$. Using the relation $\bar{I}(x, 0)\cdot S_{\mathrm{eff}}(x)=I_0(x, 0)\cdot S_0(x)$, where S_{eff} is the effective area of the beam and S_0 is the effective area of the beam in a medium without fluctuations, we obtain

$$\frac{S_{\mathrm{eff}}(x)}{S_0}=\frac{I_0(x, 0)}{\bar{I}(x, 0)}=\frac{1}{f}.$$

Note that the effective beam area obtained from this relation is close to that computed between the half-value points in Figure 97.

Equations (34) and (38) cover virtually all the possible values of the parameters which characterize the distribution of the mean intensity in the beam.

Note that expression (38) for the case $g(x)=1$ was derived by Kon /199/ using the first and the second approximations of the method of smooth perturbations and a further assumption regarding the log-normal distribution of the field u.

This method is particularly convenient for deriving the corresponding expression for the case of a plane wave. If

$$u=u_0\exp(\phi),$$

we have

$$\Gamma_2(x, \vec{\rho}_1, \vec{\rho}_2)=u_0^2\langle\exp[\phi(x, \vec{\rho}_1)+\phi^*(x, \vec{\rho}_2)]\rangle.$$

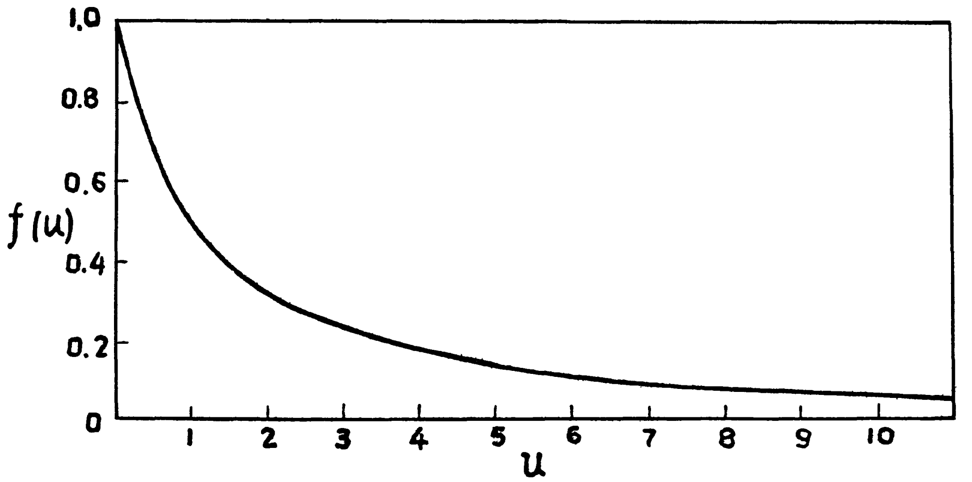


FIGURE 96. The normalized function $f(u)$, related to the mean value of the intensity of a beam as a function of path length.

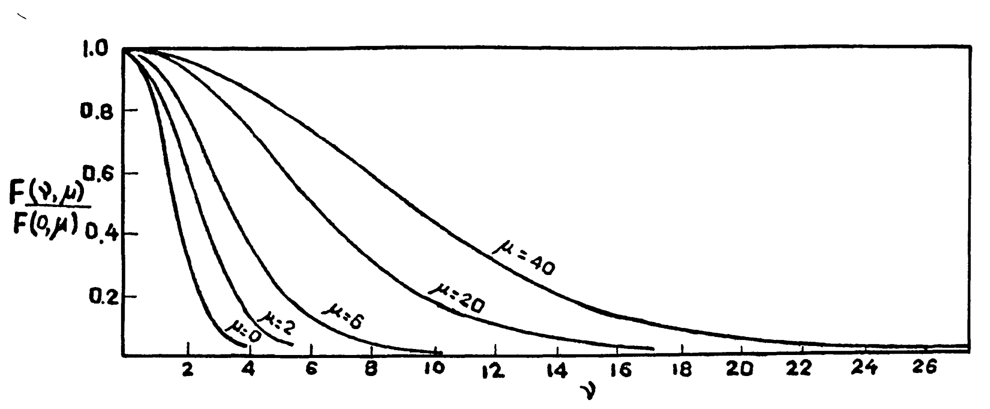


FIGURE 97. The normalized distribution of the mean intensity across the beam.

If it is assumed that ϕ has a normal distribution, we find

$$\Gamma_2 = u_0^2 \exp\left\{ \overline{\phi}(x, \vec{\rho}_1) + \overline{\phi}^*(x, \vec{\rho}_2) + \frac{1}{2}\langle [\phi'(x, \vec{\rho}_1) + \phi'^*(x, \vec{\rho}_2)]^2 \rangle \right\}$$

where $\overline{\phi} = \langle \phi \rangle$, $\phi' = \phi - \overline{\phi}$. For a plane wave

$$\overline{\phi} = \overline{\phi}(x) \text{ and } \overline{\phi}(x, \vec{\rho}_1) + \overline{\phi}^*(x, \vec{\rho}_2) = 2Re\overline{\phi}(x) = 2\overline{\chi}(x).$$

The assumption of a normal distribution for ϕ and the law of energy conservation $k\frac{\partial A^2}{\partial x} + \nabla_\perp(A^2 \nabla_\perp S) = 0$ give for a plane wave $\langle \exp(2\chi) \rangle = 1$, from which $\overline{\chi} = -\sigma_\chi^2$. Using this relation, expressing $\langle [\phi'(x, \vec{\rho}_1) + \phi'^*(x, \vec{\rho}_2)]^2 \rangle$ in terms of D_χ, D_s (see (46.10)—(46.16)), and employing some properties of statistical homogeneity, we find

$$\Gamma_2(x, \vec{\rho}_1, \vec{\rho}_2) = u_0^2 \exp\left\{-\frac{1}{2} D_1(x, \vec{\rho}_1 - \vec{\rho}_2)\right\}$$

which agrees with (4).

It should be emphasized, however, that the above derivation of this formula on the basis of the method of smooth perturbations is essentially a mixture of different "prescriptions": ϕ is assumed to be normally distributed, the resulting relation $\bar{\chi} = \sigma_\chi^2$ is used, and for D_1 the results as calculated from the first approximation are used. Therefore, the derivation of the expression for Γ_2 on the basis of the first Markov approximation is logically more sound. Moreover, it should be noted that the agreement between the method of smooth perturbations and the Markov approximation breaks down for the fourth moments Γ_4. The good fit of the results of the two methods for Γ_2, therefore, does not provide a basis for the application of the method of smooth perturbations to the region of strong intensity fluctuations.

§71. The fourth-order coherence function and intensity fluctuations

The equation for the function

$$\Gamma_4(x; \vec{\rho}_1, \vec{\rho}_2; \vec{\rho}_1', \vec{\rho}_2') = \langle u(x, \vec{\rho}_1) u(x, \vec{\rho}_2) u^*(x, \vec{\rho}_1') u^*(x, \vec{\rho}_2') \rangle$$

was derived in § 66 (see (66.22)) for the case of statistically homogeneous fluctuations, and at a later stage (see (69.15)), it was generalized to the case of a medium with smoothly varying mean characteristics:

$$\frac{\partial \Gamma_4}{\partial x} = \frac{i}{2k}(\Delta_1 + \Delta_2 - \Delta_1' - \Delta_2')\Gamma_4 - \frac{\pi k^2}{4} F(x; \vec{\rho}_1, \vec{\rho}_2, \vec{\rho}_1', \vec{\rho}_2')\Gamma_4 \tag{1}$$

$$\begin{gathered} F(x; \vec{\rho}_1, \vec{\rho}_2; \vec{\rho}_1', \vec{\rho}_2') = H(x, \vec{\rho}_1 - \vec{\rho}_1') + H(x, \vec{\rho}_2 - \vec{\rho}_2') + H(x, \vec{\rho}_1 - \vec{\rho}_2') + \\ + H(x, \vec{\rho}_2 - \vec{\rho}_1') - H(x, \vec{\rho}_2 - \vec{\rho}_1) - H(x, \vec{\rho}_2' - \vec{\rho}_1'). \end{gathered} \tag{2}$$

With equation (1) is associated the initial condition

$$\Gamma_4(0, \vec{\rho}_1, \vec{\rho}_2; \vec{\rho}_1', \vec{\rho}_2') = u_0(\vec{\rho}_1) u_0(\vec{\rho}_2) u_0^*(\vec{\rho}_1') u_0^*(\vec{\rho}_2'). \tag{3}$$

We introduce the new variables

$$\left.\begin{aligned} \mathbf{R} &= \frac{1}{4}(\vec{\rho}_1 + \vec{\rho}_2 + \vec{\rho}_1' + \vec{\rho}_2') \\ \mathbf{r}_1 &= \frac{1}{2}(\vec{\rho}_1 - \vec{\rho}_2 + \vec{\rho}_1' - \vec{\rho}_2') \\ \mathbf{r}_2 &= \frac{1}{2}(\vec{\rho}_1 - \vec{\rho}_2 - \vec{\rho}_1' + \vec{\rho}_2') \\ \vec{\rho} &= \vec{\rho}_1 + \vec{\rho}_2 - \vec{\rho}_1' - \vec{\rho}_2' \end{aligned} \right| \begin{aligned} \vec{\rho}_1 &= \mathbf{R} + \frac{\mathbf{r}_1 + \mathbf{r}_2}{2} + \frac{1}{4}\vec{\rho} \\ \vec{\rho}_2 &= \mathbf{R} - \frac{\mathbf{r}_1 + \mathbf{r}_2}{2} + \frac{1}{4}\vec{\rho} \\ \vec{\rho}_1' &= \mathbf{R} + \frac{\mathbf{r}_1 - \mathbf{r}_2}{2} - \frac{1}{4}\vec{\rho} \\ \vec{\rho}_2' &= \mathbf{R} - \frac{\mathbf{r}_1 - \mathbf{r}_2}{2} - \frac{1}{4}\vec{\rho} \end{aligned} \right\} \tag{4}$$

Then $\Delta_1 + \Delta_2 - \Delta_1' - \Delta_2' = 2(\nabla_R \nabla_\rho + \nabla_{r_1} \nabla_{r_2})$ and equation (1) takes the form

$$\frac{\partial \Gamma_4(x, \mathbf{R}, \mathbf{r}_1, \mathbf{r}_2, \vec{\rho})}{\partial x} = \frac{i}{k}(\nabla_R \nabla_\rho + \nabla_{r_1} \nabla_{r_2})\Gamma_4 - \frac{\pi k^2}{4} F(x, \mathbf{r}_1, \mathbf{r}_2, \vec{\rho})\Gamma_4. \quad (5)$$

The function F expressed in terms of the new variables takes the form (we make use of the evenness of the function $H(x, \vec{\rho})$ in $\vec{\rho}$):

$$F(x; \mathbf{r}_1, \mathbf{r}_2, \vec{\rho}) = F(x, \vec{\rho}_1, \vec{\rho}_2, \vec{\rho}_1', \vec{\rho}_2') = H(x, \mathbf{r}_1 + \frac{1}{2}\vec{\rho}) + H(x, \mathbf{r}_1 - \frac{1}{2}\vec{\rho}) +$$

$$+ H(x, \mathbf{r}_2 + \frac{1}{2}\vec{\rho}) + H(x, \mathbf{r}_2 - \frac{1}{2}\vec{\rho}) - H(x, \mathbf{r}_1 + \mathbf{r}_2) - H(x, \mathbf{r}_1 - \mathbf{r}_2). \quad (6)$$

Inserting the spectral expansion of H, we also find

$$F(x, \mathbf{r}_1, \mathbf{r}_2, \vec{\rho}) = 4 \iint_{-\infty}^{\infty} \phi_\epsilon(x, \vec{\kappa}) [1 + \cos \vec{\kappa}\mathbf{r}_1 \cos \vec{\kappa}\mathbf{r}_2 -$$

$$- \cos \vec{\kappa}\mathbf{r}_1 \cos \frac{\vec{\kappa}\vec{\rho}}{2} - \cos \vec{\kappa}\mathbf{r}_2 \cos \frac{\vec{\kappa}\vec{\rho}}{2}] d^2\kappa. \quad (7)$$

The main advantage of the new variables is that F is independent of $\mathbf{R}$, and the differential operator of the equation takes on a simpler form.

We will consider two cases: a finite beam and an infinite plane wave. For a plane wave, obviously we have $\nabla_R \Gamma_4 = 0$ (statistical homogeneity of the medium and invariance of the solution under translations in the plane $x = \text{const}$). Equation (5) takes the form

$$\frac{\partial \Gamma_4}{\partial x} = \frac{i}{k} \nabla_{r_1} \nabla_{r_2} \Gamma_4 - \frac{\pi k^2}{4} F(x, \mathbf{r}_1, \mathbf{r}_2, \vec{\rho})\Gamma_4 \quad (8)$$

$$\Gamma_4(0, \mathbf{R}, \mathbf{r}_1, \mathbf{r}_2, \vec{\rho}) = |u_0|^4. \quad (9)$$

In (8) the variable $\vec{\rho}$ is introduced as a parameter and it can be set equal to zero. We thus have

$$F(x, \mathbf{r}_1, \mathbf{r}_2, 0) = 2H(x, \mathbf{r}_1) + 2H(x, \mathbf{r}_2) - H(x, \mathbf{r}_1 + \mathbf{r}_2) - H(x, \mathbf{r}_1 - \mathbf{r}_2) =$$

$$= 4 \iint_{-\infty}^{\infty} \phi_\epsilon(x, \vec{\kappa}) [1 - \cos \vec{\kappa}\mathbf{r}_1][1 - \cos \vec{\kappa}\mathbf{r}_2] d^2\kappa = F(x, \mathbf{r}_1, \mathbf{r}_2) \quad (10)$$

$$\frac{\partial \Gamma_4(x, \mathbf{r}_1, \mathbf{r}_2)}{\partial x} = \frac{i}{k} \nabla_{r_1} \nabla_{r_2} \Gamma_4(x, \mathbf{r}_1, \mathbf{r}_2) - \frac{\pi k^2}{4} F(x, \mathbf{r}_1, \mathbf{r}_2)\Gamma_4(x, \mathbf{r}_1, \mathbf{r}_2). \quad (11)$$

The function $\Gamma_4(x, \mathbf{r}_1, \mathbf{r}_2)$ in the new variables is expressed in terms of the fields by the formula

$$\Gamma_4(x, \mathbf{r}_1, \mathbf{r}_2) = \langle u(x, \mathbf{R} + \frac{\mathbf{r}_1 + \mathbf{r}_2}{2}) u(x, \mathbf{R} - \frac{\mathbf{r}_1 + \mathbf{r}_2}{2}) \cdot$$

$$\cdot u^*(x, \mathbf{R} + \frac{\mathbf{r}_1 - \mathbf{r}_2}{2}) u^*(x, \mathbf{R} - \frac{\mathbf{r}_1 - \mathbf{r}_2}{2}) \rangle. \quad (12)$$

The points $\vec{\rho}_1, \vec{\rho}_2, \vec{\rho}_1', \vec{\rho}_2'$ in the case $\vec{\rho} = 0$ lie in the plane x at the vertices of a parallelogram centered at the point $\mathbf{R}$ and with sides $\vec{\rho}_1 - \vec{\rho}_1' = \vec{\rho}_2' - \vec{\rho}_2 = \mathbf{r}_2$, $\vec{\rho}_1 - \vec{\rho}_2' = \vec{\rho}_1' - \vec{\rho}_2 = \mathbf{r}_1$, so that $u(\vec{\rho}_1)$ and $u(\vec{\rho}_2)$ correspond to the two vertices at the opposite ends of a diagonal. Since from (10) it is obvious that

$F(x, \mathbf{r}_1, \mathbf{r}_2) = F(x, \mathbf{r}_2, \mathbf{r}_1)$ and the initial condition for (11) is also symmetrical under the transformation $\mathbf{r}_1 \leftrightarrow \mathbf{r}_2$, we have

$$\Gamma_4(x, \mathbf{r}_1, \mathbf{r}_2) = \Gamma_4(x, \mathbf{r}_2, \mathbf{r}_1). \tag{13}$$

Setting $\mathbf{r}_2 = 0$ in (12), we obtain

$$\Gamma_4(x, \mathbf{r}_1, 0) = \langle |u(x, \mathbf{R} + \frac{1}{2}\mathbf{r}_1)|^2 \cdot |u(x, \mathbf{R} - \frac{1}{2}\mathbf{r}_1)|^2 \rangle$$
$$= \langle I(x, \mathbf{R} + \frac{1}{2}\mathbf{r}_1) \cdot I(x, \mathbf{R} - \frac{1}{2}\mathbf{r}_1) \rangle.$$

Recalling that

$$\Gamma_2(x, \mathbf{R}, \vec{\rho}) = \langle u(x, \mathbf{R} + \frac{1}{2}\vec{\rho}) u^*(x, \mathbf{R} - \frac{1}{2}\vec{\rho}) \rangle, \Gamma_2(x, \mathbf{R}, 0) = \langle I(x, \mathbf{R}) \rangle$$

and taking into account that for a plane wave Γ_2 is independent of $\mathbf{R}$, we may write

$$B_I(x, \mathbf{r}) = \langle I(x, \mathbf{R} + \frac{1}{2}\mathbf{r}) I(x, \mathbf{R} - \frac{1}{2}\mathbf{r}) \rangle - \langle I(x, \mathbf{R} + \frac{\mathbf{r}}{2}) \rangle \cdot$$
$$\cdot \langle I(x, \mathbf{R} - \frac{1}{2}\mathbf{r}) \rangle = \Gamma_4(x, \mathbf{r}, 0) - \Gamma_2^2(x, 0, 0). \tag{14}$$

Thus, $\Gamma_4(x, \mathbf{r}, 0)$ enables us to determine the correlation function $B_I(x, \mathbf{r})$.

Consider the behavior of $\Gamma_4(x, \mathbf{r}_1, \mathbf{r}_2)$ for $|\mathbf{r}_2| \to \infty$. The pairs of points $(\vec{\rho}_1, \vec{\rho}_2')$ and $(\vec{\rho}_1', \vec{\rho}_2)$ move an infinite distance apart. It is clear that the fields corresponding to these pairs of points become statistically independent, and therefore

$$\lim_{|\mathbf{r}_2| \to \infty} \Gamma_4(x, \mathbf{r}_1, \mathbf{r}_2) = \Gamma_2(x, 0, \mathbf{r}_1) \Gamma_2(x, 0, -\mathbf{r}_1) = |\Gamma_2(x, 0, \mathbf{r}_1)|^2 \tag{15}$$

Consider the function $\Gamma_4(x, \mathbf{r}_1, \mathbf{r}_2) - |\Gamma_2^2(x, 0, \mathbf{r}_1)|$. For a plane wave, Γ_2 is expressed by the equation (see (70.3))

$$\frac{\partial \Gamma_2(x, 0, \mathbf{r}_1)}{\partial x} = -\frac{\pi k^2}{4} H(x, \mathbf{r}_1) \Gamma_2(x, 0, \mathbf{r}_1). \tag{15}$$

Owing to the evenness of $H(x, \mathbf{r}_1)$ in $\mathbf{r}_1$, we find

$$\Gamma_2(x, 0, \mathbf{r}_1) = \Gamma_2(x, 0, -\mathbf{r}_1) = \Gamma_2^*(x, 0, \mathbf{r}_1).$$

Multiplying (15) by $2\Gamma_2(x, 0, \mathbf{r}_1)$, we obtain

$$\frac{\partial |\Gamma_2^2(x, 0, \mathbf{r}_1)|}{\partial x} = \frac{\pi k_2}{2} H(x, \mathbf{r}_1) |\Gamma_2^2(x, 0, \mathbf{r}_1)|. \tag{16}$$

Subtracting (16) from (11), we obtain the equation

$$\frac{\partial}{\partial x}[\Gamma_4(x, \mathbf{r}_1, \mathbf{r}_2) - |\Gamma_2^2(x, 0, \mathbf{r}_1)|] = \frac{i}{k} \nabla_{r_1} \nabla_{r_2} [\Gamma_4(x, \mathbf{r}_1, \mathbf{r}_2) - |\Gamma_2^2(x, 0, \mathbf{r}_1)|] -$$

$$-\frac{\pi k^2}{4}F(x, \mathbf{r}_1, \mathbf{r}_2)\,[\Gamma_4(x, \mathbf{r}_1, \mathbf{r}_2) - |\Gamma_2^2(x, 0, \mathbf{r}_1)|] -$$

$$-\frac{\pi k_2}{4}|\Gamma_2^2(x, 0, \mathbf{r}_1)|\cdot[F(x, \mathbf{r}_1, \mathbf{r}_2) - 2H(x, \mathbf{r}_1)]. \tag{17}$$

Let us integrate (17) over $\mathbf{r}_2$ between infinite limits. Applying Gauss's theorem and using (16), we find

$$\iint_{-\infty}^{\infty}\nabla_{r_1}\nabla_{r_2}[\Gamma_4(x, \mathbf{r}_1, \mathbf{r}_2) - |\Gamma_2^2(x, 0, \mathbf{r}_1)|]\,d^2r_2 = 0.$$

Therefore, integration of (17) yields

$$\frac{\partial}{\partial x}\iint_{-\infty}^{\infty}[\Gamma_4(x, \mathbf{r}_1, \mathbf{r}_2) - |\Gamma_2^2(x, 0, \mathbf{r}_1)|]\,d^2r_2 =$$

$$= -\frac{\pi k^2}{4}\iint_{-\infty}^{\infty}F(x, \mathbf{r}_1, \mathbf{r}_2)\,[\Gamma_4(x, \mathbf{r}_1, \mathbf{r}_2) - |\Gamma_2^2(x, 0, \mathbf{r}_1)|]\,d^2r_2 -$$

$$-\frac{\pi k^2}{4}|\Gamma_2^2(x, 0, \mathbf{r}_1)|\cdot\iint_{-\infty}^{\infty}[F(x, \mathbf{r}_1, \mathbf{r}_2) - 2H(x, \mathbf{r}_1)]\,d^2r_2. \tag{18}$$

Condition (15) and expression (10) for F show that all the integrals entering (18) converge at infinity. Let $\mathbf{r}_1 = 0$ in (18). According to the definition of $H(x, \mathbf{r}_1)$ and from (10) we see that $F(x, 0, \mathbf{r}_2) = 0, H(x, 0) = 0$. Therefore,

$$\frac{d}{dx}\iint_{-\infty}^{\infty}[\Gamma_4(x, 0, \mathbf{r}_2) - |\Gamma_2^2(x, 0, 0)|]\,d^2r_2 = 0.$$

Using (13) and (14) we thus find

$$\frac{d}{dx}\iint_{-\infty}^{\infty}B_I(x, \mathbf{r})\,d^2r = 0.$$

For $x = 0$, there are no fluctuations of the field u and $B_I(0, \mathbf{r}) = 0$. Therefore,

$$\iint_{-\infty}^{\infty}B_I(x, \mathbf{r})\,d^2r = 0. \tag{19}$$

Relation (19) is a consequence of the law of energy conservation for the particular case of a plane wave.

Let us now consider the case of a bounded beam, when $u(0, \vec{\rho}) = u_0(\vec{\rho})$ converges sufficiently rapidly to zero for $|\vec{\rho}| \to \infty$.

A solution to equation (5) is sought in the form

$$\Gamma_4(x, \mathbf{R}, \mathbf{r}_1, \mathbf{r}_2, \vec{\rho}) = \iint_{-\infty}^{\infty}\exp(i\mathbf{p}\mathbf{R})\,\check{\Gamma}_4(x; \mathbf{p}; \mathbf{r}_1, \mathbf{r}_2, \vec{\rho})\,d^2p. \tag{20}$$

Inserting (20) in (5), we obtain

$$\frac{\partial\check{\Gamma}_4(x; \mathbf{p}; \mathbf{r}_1, \mathbf{r}_2, \vec{\rho})}{\partial x} + \frac{\mathbf{p}}{k}\Delta_\rho\check{\Gamma}_4 = \frac{i}{k}\nabla_{r_1}\nabla_{r_2}\check{\Gamma}_4 - \frac{\pi k^2}{4}F(x, \mathbf{r}_1, \mathbf{r}_2, \vec{\rho})\,\check{\Gamma}_4. \tag{21}$$

A solution for $\check{\Gamma}_4$ is sought in the form

$$\check{\Gamma}_4(x, \mathbf{p}, \mathbf{r}_1, \mathbf{r}_2, \vec{\rho}) = M(x, \mathbf{p}, \mathbf{r}_1, \mathbf{r}_2, \vec{\rho} - \frac{\mathbf{p}x}{k})$$

$$M(x, \mathbf{p}, \mathbf{r}_1, \mathbf{r}_2, \vec{\rho}) = \check{\Gamma}_4(x; \mathbf{p}, \mathbf{r}_1, \mathbf{r}_2, \vec{\rho} + \frac{\mathbf{p}x}{k}). \tag{22}$$

Inserting (22) in (21), we obtain

$$\frac{\partial M(x;\mathbf{p};\mathbf{r}_1,\mathbf{r}_2,\vec{\rho})}{\partial x}=$$

$$-\frac{i}{k}\nabla_{r_1}\nabla_{r_2}M(x,\mathbf{p},\mathbf{r}_1,\mathbf{r}_2,\vec{\rho})-\frac{\pi k^2}{4}F(x,\mathbf{r}_1,\mathbf{r}_2,\vec{\rho})M(x,\mathbf{p},\mathbf{r}_1,\mathbf{r}_2,\vec{\rho}). \qquad (23)$$

Equation (23) for M coincides with equation (8) for the particular case of a plane wave. From (20), (14), and (22), we conclude that the initial condition for equation (23) is

$$M(0,\mathbf{p},\mathbf{r}_1,\mathbf{r}_2,\vec{\rho})=\frac{1}{4\pi^2}\iint_{-\infty}^{\infty}\exp(-i\mathbf{p}\mathbf{R})u_0(\mathbf{R}+\frac{\mathbf{r}_1+\mathbf{r}_2}{2}+\frac{\vec{\rho}}{4})\cdot$$

$$\cdot u_0(\mathbf{R}-\frac{\mathbf{r}_1+\mathbf{r}_2}{2}+\frac{\vec{\rho}}{4})u_0^*(\mathbf{R}+\frac{\mathbf{r}_1-\mathbf{r}_2}{4}-\frac{\vec{\rho}}{4})u_0^*(\mathbf{R}-\frac{\mathbf{r}_1-\mathbf{r}_2}{2}-\frac{\vec{\rho}}{4})d^2R. \qquad (24)$$

Note that Γ_4 and M are symmetric in $\mathbf{r}_1, \mathbf{r}_2$ as in the case of a plane wave. This follows from the invariance of the equation (23) and the initial condition (24) under the substitution $\mathbf{r}_1 \leftrightarrow \mathbf{r}_2$:

$$\Gamma_4(x,\mathbf{R},\mathbf{r}_1,\mathbf{r}_2,\vec{\rho})=\Gamma_4(x,\mathbf{R},\mathbf{r}_2,\mathbf{r}_1,\vec{\rho}). \qquad (25)$$

Setting $\mathbf{p}=0, \vec{\rho}=0$ in (21), we obtain

$$\frac{\partial\check{\Gamma}_4(x;0;\mathbf{r}_1,\mathbf{r}_2,0)}{\partial x}=\frac{i}{k}\nabla_{r_1}\nabla_{r_2}\check{\Gamma}_4(x;0;\mathbf{r}_1,\mathbf{r}_2,0)-\frac{\pi k^2}{4}F(x,\mathbf{r}_1,\mathbf{r}_2)\check{\Gamma}_4.$$

Let us integrate this equation over $\mathbf{r}_2$ between infinite limits. Since the field outisde the beam falls off sufficiently rapidly, application of Gauss's theoren to the integral over $\nabla_{r_2}\nabla_{r_1}\check{\Gamma}_4$ gives zero. Thus,

$$\frac{\partial}{\partial x}\iint_{-\infty}^{\infty}\check{\Gamma}_4(x;0;\mathbf{r}_1,\mathbf{r}_2,0)d^2r_2=-\frac{\pi k^2}{4}\iint_{-\infty}^{\infty}F(x,\mathbf{r}_1,\mathbf{r}_2)\check{\Gamma}_4(x;0;\mathbf{r}_1,\mathbf{r}_2,0)d^2r_2.$$

Setting $\mathbf{r}_1=0$ and noting that $F(x,0,\mathbf{r}_2)=0$, we find that the integral of equation (5) corresponds to the law of energy conservation:

$$\frac{d}{dx}\iint_{-\infty}^{\infty}\check{\Gamma}_4(x;0;0,\mathbf{r}_2,0)d^2r_2=0. \qquad (26a)$$

Expressing $\check{\Gamma}_4$ in terms of Γ_4, we may write this relation in the form

$$\frac{d}{dx}\iint_{-\infty}^{\infty}d^2R\iint_{-\infty}^{\infty}d^2r_2\Gamma_4(x;\mathbf{R},0,\mathbf{r}_2,0)=0. \qquad (26b)$$

But

$$\Gamma_4(x;\mathbf{R},0,\mathbf{r}_2,0)=\langle I(x,\mathbf{R}+\frac{1}{2}\mathbf{r}_2)I(x,\mathbf{R}-\frac{1}{2}\mathbf{r}_2)\rangle=$$

$$=B_I(x,\mathbf{R},\mathbf{r}_2)+\bar{I}(x,\mathbf{R}+\frac{1}{2}\mathbf{r}_2)\cdot\bar{I}(x,\mathbf{R}-\frac{1}{2}\mathbf{r}_2) \qquad (27)$$

where B_I is the correlation (covariance) function of intensity fluctuations. Inserting (27) in (26b) and using $\mathbf{R}+\frac{1}{2}\mathbf{r}_2$ and $\mathbf{R}-\frac{1}{2}\mathbf{r}_2$ as the new integration variables in the integral containing the product $\bar{I}\cdot\bar{I}$, we find

$$\frac{d}{dx}\iint_{-\infty}^{\infty} d^2R \iint_{-\infty}^{\infty} d^2r_2 B_I(x, \mathbf{R}, \mathbf{r}_2) + \frac{d}{dx}\left[\iint_{-\infty}^{\infty} \bar{I}(x, \vec{\rho})\, d^2\rho\right]^2 = 0.$$

By (10.21), the second term in this equality vanishes, so that

$$\frac{d}{dx}\iint_{-\infty}^{\infty} d^2R \iint_{-\infty}^{\infty} d^2r_2 B_I(x, \mathbf{R}, \mathbf{r}_2) = 0.$$

Since there are no fluctuations in the plane $x=0$ and $B_I(0, \mathbf{R}, \mathbf{r}_2)=0$, we find

$$\iint_{-\infty}^{\infty} d^2R \iint_{-\infty}^{\infty} d^2r_2 B_I(x, \mathbf{R}, \mathbf{r}_2) = 0. \tag{28}$$

The physical meaning of (19) and (28) is that the fluctuations in ϵ only lead to a redistribution of energy in the beam, so that the integral over the fluctuating component of the energy over the entire beam is zero:

$$\iint_{-\infty}^{\infty} I'(x, \mathbf{r})\, d^2r = 0. \tag{29}$$

Here $I' = I - \bar{I}$. The mean square of the last equality is equivalent to (28).

The function Γ_4 is related not only to intensity fluctuations but also to fluctuations in the coordinates of the center of mass of the beam /194/.

The random coordinate of the center of mass of the beam in the plane x is defined by the relation

$$\vec{\xi} = \frac{\iint_{-\infty}^{\infty} \vec{\rho}\, I(x, \vec{\rho})\, d^2\rho}{\iint_{-\infty}^{\infty} I(x, \vec{\rho}\,')\, d^2\rho'}. \tag{30}$$

We will first show that the denominator in (30) is constant. Indeed, multiplying the equation

$$2ik\frac{\partial u(x, \vec{\rho})}{\partial x} + \Delta u(x, \vec{\rho}) + k^2\epsilon_1(x, \vec{\rho})u(x, \vec{\rho}) = 0$$

by $u^*(x, \vec{\rho})$, we obtain

$$2ik\, u^*\frac{\partial u}{\partial x} + u^*\Delta u + k^2\epsilon_1 uu^* = 0.$$

Subtracting the complex conjugate of this equation

$$-2ik\, u\frac{\partial u^*}{\partial x} + u\Delta u^* + k^2\epsilon_1 uu^* = 0$$

we find

$$2ik\frac{\partial I}{\partial x} + (u^*\Delta u - u\Delta u^*) = 0$$

or

$$\frac{\partial I}{\partial x} + \nabla_\perp\left(\frac{u^*\nabla_\perp u - u\nabla_\perp u^*}{2ik}\right) = 0. \tag{31}$$

Integrating (31) over $\vec{\rho}$ between infinite limits and remembering that in a bounded beam u decreases fairly rapidly with distance from the beam axis, we find

$$\frac{d}{dx}\iint_{-\infty}^{\infty} I(x, \vec{\rho})\, d^2\rho = 0$$

or

$$\iint_{-\infty}^{\infty} I(x, \vec{\rho})\, d_2\rho = \text{const} = \iint_{-\infty}^{\infty} I(0, \vec{\rho})\, d^2\rho = P. \qquad (32)$$

Note that (32) is equivalent to (29). Inserting (32) in (30), we find

$$\vec{\xi} = \frac{1}{P}\iint_{-\infty}^{\infty} \vec{\rho}\, I(x, \vec{\rho})\, d^2\rho.$$

The mean square of the coordinate for the center of mass is thus

$$\langle \vec{\xi}^2 \rangle = \frac{1}{P^2}\iint_{-\infty}^{\infty} d^2\rho_1 \iint_{-\infty}^{\infty} d^2\rho_2\, \vec{\rho}_1\vec{\rho}_2 \langle I(x, \vec{\rho}_1) I(x, \vec{\rho}_2)\rangle.$$

Introducing the variables in equation (4), we obtain

$$\langle \vec{\xi}^2 \rangle = \frac{1}{P^2}\iint_{-\infty}^{\infty} d^2R \iint_{-\infty}^{\infty} d^2r_1 \left(R^2 - \frac{r_1^2}{4}\right)\Gamma_4(x; \mathbf{R}, \mathbf{r}_1, 0, 0). \qquad (32)$$

According to the last relation, $\langle \vec{\xi}^2 \rangle$ is expressed in terms of the function Γ_4. Using equation (5), this expression can be represented as a differential equation (see /194/)

$$\frac{d^3 \langle \vec{\xi}^2(x)\rangle}{dx^3} = \frac{\pi}{2P^2}\iint_{-\infty}^{\infty} d^2r_1 \Delta H(x, \mathbf{r}_1) \iint_{-\infty}^{\infty} d^2R\, \Gamma_4(x; \mathbf{R}, \mathbf{r}_1, 0, 0). \qquad (33)$$

If the initial beam is symmetric and collimated, (33) has zero initial conditions.

Let us return to equation (23). We seek its solution in the form of a Fourier integral over $\mathbf{r}_2$:

$$M(x; \vec{\rho}; \mathbf{r}_1, \mathbf{r}_2, \vec{\rho}) = \iint_{-\infty}^{\infty} \varphi(x; \vec{\rho}; \mathbf{r}_1, \vec{\kappa}, \vec{\rho})\, e^{i\vec{\kappa}\mathbf{r}_2}\, d^2\kappa. \qquad (34)$$

Inserting (34) and (7) in (23), we obtain after simple manipulations

$$\frac{\partial\varphi(x; \vec{\rho}, \mathbf{r}_1, \vec{\kappa}, \vec{\rho})}{\partial x} + \frac{\vec{\kappa}}{k}\nabla_{r_1}\varphi + \frac{\pi k^2}{4}\left[H\left(x, \mathbf{r}_1 + \frac{\vec{\rho}}{2}\right) + H\left(x, \mathbf{r}_1 - \frac{\vec{\rho}}{2}\right)\right]\varphi =$$

$$= \pi k^2 \iint_{-\infty}^{\infty} \phi_\epsilon(x, \vec{\kappa}')\left[\cos\frac{\vec{\kappa}'\vec{\rho}}{2} - \cos\vec{\kappa}'\vec{r}_1\right]\varphi(x; \vec{\rho}; \vec{r}_1, \vec{\kappa} - \vec{\kappa}', \vec{\rho})\, d^2\kappa'. \qquad (35)$$

From (24) we readily derive the initial condition for this equation:

$$\varphi(0; \vec{\rho}, \mathbf{r}_1, \vec{\kappa}, \vec{\rho}) = (2\pi)^{-4}\iint_{-\infty}^{\infty} d^2R \iint_{-\infty}^{\infty} d^2r_2 \exp(-i\mathbf{p}\mathbf{R} - i\vec{\kappa}\mathbf{r}_2)\cdot$$

$$\cdot u_0\left(\mathbf{R} + \frac{\mathbf{r}_1 + \mathbf{r}_2}{2} + \frac{\vec{\rho}}{4}\right) u_0\left(\mathbf{R} - \frac{\mathbf{r}_1 + \mathbf{r}_2}{2} + \frac{\vec{\rho}}{4}\right) u_0^*\left(\mathbf{R} + \frac{\mathbf{r}_1 - \mathbf{r}_2}{2} - \frac{\vec{\rho}}{4}\right) u_0^*\left(\mathbf{R} - \frac{\mathbf{r}_1 - \mathbf{r}_2}{2} - \frac{\vec{\rho}}{4}\right). \qquad (36)$$

Equation (35) for φ, like the previously considered equation for J, has the form of the equation of radiative transfer in the small-angle approximation. However, unlike this equation for J, the analogy with the equation of radiative transfer is purely formal in this case. Unlike J, φ cannot be identified physically with the energy flux density, and the integral of this quantity over all $\vec{\kappa}$ does not represent the intensity of the radiation. Nevertheless, this formal analogy is quite useful for understanding the structure of the solution of this equation. Also note that, in contrast to the equation for J, the present equation contains "an angular scattering diagram" $\pi k^2 \phi_\epsilon(x, \vec{\kappa}') [\cos \frac{\vec{\kappa}'\vec{\rho}}{2} - \cos \vec{\kappa}' \mathbf{r}_1]$ and "an extinction coefficient" $\frac{1}{4}\pi k^2 [H(x, \mathbf{r}_1 + \frac{1}{2}\vec{\rho}) + H(x, \mathbf{r}_1 - \frac{1}{2}\vec{\rho})]$ which are dependent on the coordinate $\mathbf{r}_1$. This property of equation (35) makes it impossible to solve in analytical form, whereas explicit analytical expressions were obtained for Γ_2 or J.

Let us consider the particular case of a plane incident wave, when $u_0(\vec{\rho}) - u_0 =$ const. Then

$$\varphi(0; \vec{\rho}; \mathbf{r}_1, \vec{\kappa}, \vec{\rho}) = |u_0|^4 \delta(\vec{\kappa}) \delta(\vec{\rho}) \tag{37}$$

and a solution of equation (35) may be sought in the form

$$\varphi(x; \vec{\rho}; \mathbf{r}_1, \vec{\kappa}, \vec{\rho}) = \widetilde{\varphi}(x; \mathbf{r}_1, \vec{\kappa}, \vec{\rho}) \cdot \delta(\vec{\rho}). \tag{38}$$

Inserting (38) in (35), we obtain

$$\frac{\partial \widetilde{\varphi}(x; \mathbf{r}_1, \vec{\kappa}, \vec{\rho})}{\partial x} + \frac{\vec{\kappa}}{k} \nabla_{r_1} \widetilde{\varphi} + \frac{\pi k^2}{4} [H(x, \mathbf{r}_1 + \frac{1}{2}\vec{\rho}) + H(x, \mathbf{r}_1 - \frac{1}{2}\vec{\rho})] \widetilde{\varphi} =$$

$$= \pi k^2 \iint_{-\infty}^{\infty} \phi_\epsilon(x, \vec{\kappa}') [\cos \frac{\vec{\kappa}'\vec{\rho}}{2} - \cos \vec{\kappa}' \mathbf{r}_1] \widetilde{\varphi}(x, \mathbf{r}_1, \vec{\kappa} - \vec{\kappa}', \vec{\rho}) d^2\kappa' \tag{39}$$

$$\widetilde{\varphi}(0, \mathbf{r}_1, \vec{\kappa}, \vec{\rho}) = |u_0|^4 \delta(\vec{\kappa}). \tag{40}$$

Using (38), (34), (22), and (20), we may express $\Gamma_4(x, \mathbf{R}, \mathbf{r}_1, \mathbf{r}_2, \vec{\rho})$ in terms of $\widetilde{\varphi}$:

$$\Gamma_4(x; \mathbf{R}, \mathbf{r}_1, \mathbf{r}_2, \vec{\rho}) = \iint_{-\infty}^{\infty} \widetilde{\varphi}(x, \mathbf{r}_1, \vec{\kappa}, \vec{\rho}) \exp(i\vec{\kappa}\mathbf{r}_2) d^2\kappa. \tag{41}$$

In particular, if we are interested in the value of Γ_4 for $\vec{\rho} = 0$, we may set $\vec{\rho} = 0$ in (39). Writing $\widetilde{\varphi}(x, \mathbf{r}_1, \vec{\kappa}, 0) = \widetilde{\varphi}(x, \mathbf{r}_1, \vec{\kappa})$, we obtain from (39)

$$\frac{\partial \widetilde{\varphi}(x, \mathbf{r}, \vec{\kappa})}{\partial x} + \frac{\vec{\kappa}}{k} \nabla_r \widetilde{\varphi}(x, \mathbf{r}, \vec{\kappa}) + \frac{\pi k^2}{2} H(x, \mathbf{r}) \widetilde{\varphi}(x, \mathbf{r}, \vec{\kappa}) =$$

$$= \pi k^2 \iint_{-\infty}^{\infty} \phi_\epsilon(x, \vec{\kappa}') [1 - \cos \vec{\kappa}' \mathbf{r}] \widetilde{\varphi}(x, \mathbf{r}, \vec{\kappa} - \vec{\kappa}') d^2\kappa' \tag{42}$$

$$\widetilde{\varphi}(0, \mathbf{r}, \vec{\kappa}) = |u_0|^4 \delta(\vec{\kappa}) \tag{43}$$

$$\Gamma_4(x; \mathbf{R}, \mathbf{r}_1, \mathbf{r}_2, 0) = \iint_{-\infty}^{\infty} \widetilde{\varphi}(x, \mathbf{r}_1, \vec{\kappa}) \exp(i\vec{\kappa}\mathbf{r}_2) d^2\kappa. \tag{44}$$

Note that by (42) and (43), the function $\widetilde{\varphi}$ is real: $\widetilde{\varphi}^* = \widetilde{\varphi}$. Equation (42) can be derived directly from (11).

§72. An approximate solution of the equation for the fourth-order coherence function

The equations obtained above for Γ_4 and for φ are applicable to the region of strong intensity fluctuations and the solution of these equations is therefore of considerable interest. Unfortunately, no analytical solutions can be derived, and no numerical solution has been attempted so far. Because of this there is interest in a preliminary qualitative investigation of an approximate solution. We will therefore consider one of the possible approximate solutions of this equation. The approximate solutions will remain of interest only as long as no numerical solution is available.

Our approximate solution is fairly crude. Moreover, the region of its applicability is not established. These points can be settled by a suitable comparison of theoretical and experimental results.

Let us consider the simplest case of a plane wave when the equation for Γ_4 reduces to the equation of radiative transfer of the form (71.42). Consider the equation

$$\frac{\partial \widetilde{\varphi}(x, \mathbf{r}, \vec{\kappa})}{\partial x} + \frac{\vec{\kappa}}{k} \nabla_r \widetilde{\varphi}(x, \mathbf{r}, \vec{\kappa}) + \frac{\pi k^2}{2} H(x, \mathbf{r}) \widetilde{\varphi}(x, \mathbf{r}, \vec{\kappa}) = G(x, \mathbf{r}, \vec{\kappa}). \tag{1}$$

Equation (1) is equivalent to (71.42) if

$$G(x, \mathbf{r}, \vec{\kappa}) = \pi k^2 \iint_{-\infty}^{\infty} \phi_\epsilon(x, \vec{\kappa}') [1 - \cos \vec{\kappa}'\mathbf{r}] \widetilde{\varphi}(x, \mathbf{r}, \vec{\kappa} - \vec{\kappa}') d^2\kappa' \tag{2}$$

and

$$\widetilde{\varphi}(0, \mathbf{r}, \vec{\kappa}) = |u_0|^4 \delta(\vec{\kappa}). \tag{3}$$

We transform this equation to a purely integral form. To do this, we have to solve (1), formally assuming G to be a known function.

Noting that

$$\frac{\partial \widetilde{\varphi}}{\partial x} + \frac{\vec{\kappa}}{k} \nabla_r \widetilde{\varphi} = \exp\left(-\frac{\vec{\kappa} x}{k} \nabla_r\right) \frac{\partial}{\partial x} \left[\exp\left(\frac{\vec{\kappa} x}{k} \nabla_r\right) \cdot \widetilde{\varphi}\right]$$

we write (1) in the form

$$\frac{\partial}{\partial x} \left[\exp\left(\frac{\vec{\kappa} x}{k} \nabla_r\right) \widetilde{\varphi}(x, \mathbf{r}, \vec{\kappa})\right] + \exp\left(\frac{\vec{\kappa} x}{k} \nabla_r\right) \frac{\pi k^2}{2} H(x, \mathbf{r}) \widetilde{\varphi}(x, \mathbf{r}, \vec{\kappa}) =$$
$$= \exp\left(\frac{\vec{\kappa} x}{k} \nabla_r\right) G(x, \mathbf{r}, \vec{\kappa}). \tag{4}$$

As we know, if $\nabla_r \mathbf{a} = 0$,

$$\exp(\mathbf{a} \nabla_r) f(\mathbf{r}) = f(\mathbf{r} + \mathbf{a}) \tag{5}$$

(equation (5) is the Taylor series formula written in operator notation). Therefore, setting

$$\nu(x, \mathbf{r}, \vec{\kappa}) = \widetilde{\varphi}\left(x, \mathbf{r} + \frac{\vec{\kappa} x}{k}, \vec{\kappa}\right), \quad \widetilde{\varphi}(x, \mathbf{r}, \vec{\kappa}) = \nu\left(x, \mathbf{r} - \frac{\vec{\kappa} x}{k}, \vec{\kappa}\right)$$

we obtain from (4) an ordinary differential equation

$$\frac{\partial \nu(x, \mathbf{r}, \vec{\kappa})}{\partial x} + \frac{\pi k^2}{2} H(x, \mathbf{r} + \frac{\vec{\kappa} x}{k}) \nu(x, \mathbf{r}, \vec{\kappa}) = G(x, \mathbf{r} + \frac{\vec{\kappa} x}{k}, \vec{\kappa}). \tag{6}$$

The solution of (6) has the form

$$\nu(x, \mathbf{r}, \vec{\kappa}) = \nu(0, \mathbf{r}, \vec{\kappa}) \exp\left\{-\frac{\pi k^2}{2} \int_0^x H(\xi, \mathbf{r} + \frac{\vec{\kappa}\xi}{k}) d\xi\right\} + \\ + \int_0^x \exp\left\{-\frac{\pi k^2}{2} \int_{x'}^x H(\xi, \mathbf{r} + \frac{\vec{\kappa}\xi}{k}) d\xi\right\} G(x', \mathbf{r} + \frac{\vec{\kappa} x'}{k}, \vec{\kappa}) dx'. \tag{7}$$

Now, $\nu(0, \mathbf{r}, \vec{\kappa}) = \tilde{\varphi}(0, \mathbf{r}, \vec{\kappa})$. Using the relation $\tilde{\varphi}(x, \mathbf{r}, \vec{\kappa}) = \nu(x, \mathbf{r} - \frac{\vec{\kappa} x}{k}, \vec{\kappa})$, we obtain the solution of equation (1):

$$\tilde{\varphi}(x, \mathbf{r}, \vec{\kappa}) = \tilde{\varphi}(0, \mathbf{r} - \frac{\vec{\kappa} x}{k}, \vec{\kappa}) \exp\left\{-\frac{\pi k^2}{2} \int_0^x H(\xi, \mathbf{r} - \frac{\vec{\kappa}(x-\xi)}{k}) d\xi\right\} + \\ + \int_0^x \exp\left\{-\frac{\pi k^2}{2} \int_{x'}^x H(\xi, \mathbf{r} - \frac{\vec{\kappa}(x-\xi)}{k}) d\xi\right\} G(x', \mathbf{r} - \frac{\vec{\kappa}(x-x')}{k}, \vec{\kappa}) dx'. \tag{8}$$

Inserting (2) and (3) on the right in (8), we obtain an integral equation for $\tilde{\varphi}$, which is equivalent to equation (71.42).

Let us consider an approximate solution of the problem which is obtained when the initial value (3) is inserted in expression (2) for G. This is the so-called single-scattering approximation of the theory of radiative transfer. The single-scattering approximation ensures an exact allowance for the extinction of the radiation described by the term $\frac{\pi k^2}{2} H\tilde{\varphi}$.

In the "equation of radiative transfer" (1), the source of scattered waves is the term G on the right, which describes the contribution to the radiation flux propagating in the direction $\vec{\kappa}$ from a radiation flux in the direction $\vec{\kappa} - \vec{\kappa}'$. If the true flux $\tilde{\varphi}(x, \mathbf{r}, \vec{\kappa} - \vec{\kappa}')$ on the right in (1) is replaced with the initial unattenuated flux $\tilde{\varphi}(0, \mathbf{r}, \vec{\kappa} - \vec{\kappa}')$, we overestimate the intensity of the source of the scattered radiation. The solution obtained in the single-scattering approximation therefore gives an exaggerated result for the radiation "intensity" ($\Gamma_4(x, \mathbf{R}, \mathbf{r}_1, 0, 0)$ in our case).

Also note that the term "single scattering" has a slightly different meaning in the theory of radiative transfer compared to its meaning when applied to the wave equation. In the latter case, it corresponds to the first (Born) approximation of the perturbation theory using the parameter ϵ_1, which enters (1) both through the term $\frac{\pi k^2}{2} H\tilde{\varphi}$ and through G. In our case, the dependency of $\frac{\pi k^2}{2} H\tilde{\varphi}$ on ϵ_1 is exact, and the perturbation theory is developed only in relation to G.

We thus insert (3) in (2). The first approximation then takes the form

$$G_1(x, \mathbf{r}, \vec{\kappa}) = \pi k^2 |u_0|^4 \phi_\epsilon(x, \vec{\kappa}) (1 - \cos \vec{\kappa}\mathbf{r}). \tag{9}$$

Inserting (9) and (3) in (8), we find

$$\widetilde{\varphi}_1(x, \mathrm{r}, \vec{\kappa}) = |u_0|^4 \delta(\vec{\kappa}) \exp\left\{-\frac{\pi k^2}{2}\int_0^x H(\xi, \mathrm{r})\, d\xi\right\} + \pi k^2 |u_0|^4 \int_0^x dx'$$

$$\exp\left\{-\frac{\pi k^2}{2}\int_{x'}^x H\left(\xi, \mathrm{r} - \frac{\vec{\kappa}(x-\xi)}{k}\right) d\xi\right\} \phi_\epsilon(x', \vec{\kappa}) \left[1 - \cos\vec{\kappa}\left(\mathrm{r} - \frac{\vec{\kappa}(x-x')}{k}\right)\right]. \quad (10)$$

If, according to (71.44), we take the Fourier transform of (10) with respect to $\vec{\kappa}$, we obtain the first approximation for $\Gamma_4(x, \mathbf{R}, \mathrm{r}_1, \mathrm{r}_2, 0)$. The expression obtained in this way is no longer symmetrical in $\mathrm{r}_1, \mathrm{r}_2$ (i.e., the Fourier transform of the first term in (10) is a function of r_1 only). This is one of the basic shortcomings of our approximation. The correlation function $B_I(x, \mathrm{r})$ is expressed, according to (71.27), either in terms of $\Gamma_4(x, \mathbf{R}, 0, \mathrm{r}, 0)$ or in terms of $\Gamma_4(x, \mathbf{R}, \mathrm{r}, 0, 0)$. The exact choice between the two alternatives is of no consequence for the solution which is symmetrical in $\mathrm{r}_1, \mathrm{r}_2$. In our approximation, however, the symmetry breaks down and the two expressions, in general, may give different results. We will define $B_I(x, \mathrm{r})$ in terms of $\Gamma_4(x, \mathbf{R}, 0, \mathrm{r}, 0)$, basing this only on the considerations that it will ensure a relatively simple final result and that the qualitative conclusions are not in contradiction with experimental findings.

Let $\mathrm{r} = 0$ in (10). From (71.44), $\widetilde{\varphi}_1(x, 0, \vec{\kappa})$ is the spectral density of the function $\Gamma_4(x, \mathbf{R}, 0, \mathrm{r}_2, 0) = B_I(x, \mathrm{r}_2) + (\bar{I})^2$. Seeing that $H(\xi, 0) = 0$ and that $H(\xi, \vec{\rho})$ is even in $\vec{\rho}$, we obtain

$$\widetilde{\varphi}_1(x, 0, \vec{\kappa}) = |u_0|^4 \delta(\vec{\kappa}) + \pi k^2 |u_0|^4 \int_0^x dx'\, \phi_\epsilon(x', \vec{\kappa}) \cdot$$

$$\cdot \left[1 - \cos\frac{\kappa^2(x-x')}{k}\right] \exp\left\{-\frac{\pi k^2}{2}\int_{x'}^x H\left(\xi, \frac{\vec{\kappa}(x-\xi)}{k}\right) d\xi\right\}.$$

The first term in this equality is the spectral density of $(\bar{I})^2$. Therefore the function

$$F_I(x, \vec{\kappa}) = \pi k^2 \bar{I}^2 \int_0^x dx' \exp\left\{-\frac{\pi k^2}{2}\int_{x'}^x H\left(\xi, \frac{\vec{\kappa}(x-\xi)}{k}\right) d\xi\right\} \cdot$$

$$\cdot\, \phi_\epsilon(x', \vec{\kappa}) \left[1 - \cos\frac{\kappa^2(x-x')}{k}\right] \quad (11)$$

in our approximation is the spectral density of the correlation function B_I:

$$B_I(x, \mathrm{r}) = \iint_{-\infty}^{\infty} F_I(x, \vec{\kappa}) \exp(i\vec{\kappa}\mathrm{r})\, d^2\kappa. \quad (12)$$

In particular, the mean square of the relative fluctuations in intensity is given by

$$\beta^2(x) \equiv \frac{\langle (I-\bar{I})^2\rangle}{\bar{I}^2} = \pi k^2 \int_0^x dx' \iint_{-\infty}^{\infty} d^2\kappa\, \phi_\epsilon(x', \vec{\kappa}) \cdot$$

$$\cdot \left[1 - \cos\frac{\kappa^2(x-x')}{k}\right] \exp\left\{-\frac{\pi k^2}{2}\int_{x'}^x H\left(\xi, \frac{\vec{\kappa}(x-\xi)}{k}\right) d\xi\right\}. \quad (13)$$

Equation (13) enables us to compute $\beta^2(x)$ to a first approximation. We will consider two particular examples, when the fluctuations ϵ are produced by turbulence. In the first the quantity C_ϵ^2 is constant, and in the second it vanishes everywhere except along a small section of the path.

If ϕ_ϵ is independent of x, substitution of the variables $x-\xi\to\xi, x-x'\to x'$ reduces (13) to the form

$$\beta^2(x)=\pi k^2\int_0^x dx'\iint_{-\infty}^{\infty} d^2\kappa\,\exp\left\{-\frac{\pi k^2}{2}\int_0^{x'} H\left(\frac{\vec{\kappa}\xi}{k}\right)d\xi\right\}\phi_\epsilon(\vec{\kappa})\left[1-\cos\frac{\kappa^2 x'}{k}\right]. \quad (14)$$

Let us carry out the calculations for the case when

$$\phi_\epsilon(\vec{\kappa})=AC_\epsilon^2|\vec{\kappa}|^{-11/3},\ A=5\sin\frac{\pi}{3}\Gamma(5/3)/12\pi^2\approx 0.033$$

$$H(\vec{\rho})=pC_\epsilon^2\rho^{5/3},\ p=\frac{2M_2}{\pi}=\frac{2^{1/3}\pi^2 A}{\Gamma^2(11/6)}\approx 0.47. \quad (15)$$

Evaluating the integral in the exponential and integrating over the angular variable, we find

$$\beta^2(x)=2\pi^2 AC_\epsilon^2 k^2\int_0^x dx'\int_0^\infty \kappa^{-8/3}\left[1-\cos\frac{\kappa^2 x'}{k}\right]\exp\left\{-\frac{3\pi p}{16}C_\epsilon^2 k^{1/3}\kappa^{5/3}x'^{8/3}\right\}d\kappa. \quad (16)$$

We replace κ with a new integration variable $t=\kappa^2x'/k$ and x' with $u=\frac{x'}{x}$.

We also introduce the mean square of the relative fluctuations in intensity as calculated from the first approximation of the perturbation method (see also (47.31))

$$\beta_0^2=\alpha C_\epsilon^2 k^{7/6}x^{11/6},\ \alpha=\frac{3\pi^3 A}{11\Gamma(11/6)\cos\pi/12}.$$

Equation (16) is then written in the form

$$\beta^2(x)=\frac{\pi^2 A}{\alpha}\beta_0^2\int_0^\infty t^{-11/6}[1-\cos t]\,dt\int_0^1\exp(-g\beta_0^2 t^{5/6}u^{11/6})u^{5/6}du$$

where $g=3\pi p/16\alpha$.

Evaluating the inner integral over u, we obtain

$$\beta^2(x)=f(\beta_0) \quad (17)$$

where

$$f(z)=N\int_0^\infty t^{-8/3}(1-\cos t)\left[1-\exp(-gz^2t^{5/6})\right]dt$$
$$N=\frac{6\pi^2 A}{11\alpha g}=\frac{32\Gamma^2(11/6)}{11\pi\cdot 2^{1/3}}\approx 0.65. \quad (18)$$

By (17), β^2 is a function of β_0, this is a characteristic feature of a spectral density $\phi_\epsilon(\kappa)$ expressed simply by a pure power law. For $z\to\infty$ we obtain

$$\lim_{z\to 0} f(z)=N\int_0^\infty t^{-8/3}(1-\cos t)\,dt=\frac{27}{10}\sin\frac{\pi}{3}\Gamma\left(\frac{4}{3}\right)N\approx 1.36 \quad (19)$$

and, by (17), $\beta^2(x)$ goes to a constant value (19). Note that the limiting value $\beta^{\epsilon}(\infty)$ depends on the form of the function $\phi_{\epsilon}(\kappa)$, i.e., this is not a universal quantity. This conclusion is also a consequence of the adoption of a power law for the spectrum $\phi_{\epsilon}(\kappa)$. Evidently, if the fluctuations ϵ have a finite correlation radius L_0, we see that $\beta^2(x) \to 1$ for $\beta_0^2 \gg 1, \sqrt{\lambda x} \gg L_0$, (as for fluctuations behind a phase screen). Equations (17) and (18) are applicable only when $\sqrt{\lambda x} \gg L_0$, and the limiting region where $\sqrt{\lambda x} \ll L_0$ together with $\beta_0^2 \gg 1$ is therefore not attained. Also note that the amplitude probability distribution in the region $\beta_0^2 \gg 1$ is not universal either, whereas for $\sqrt{\lambda x} \ll L_0$ it is, apparently, described by the Rayleigh function (see footnote on p. 388).

The function $\sqrt{f(z)}$ evaluated by numerical integration is shown in Figure 98. For $z \gg 1$, we have the asymptotic formula

$$f(z) - f(\infty) - \frac{0.907}{z^{4/5}}$$

which ensures an accuracy of no less than 1.5% for $z \geqslant 1.4$.

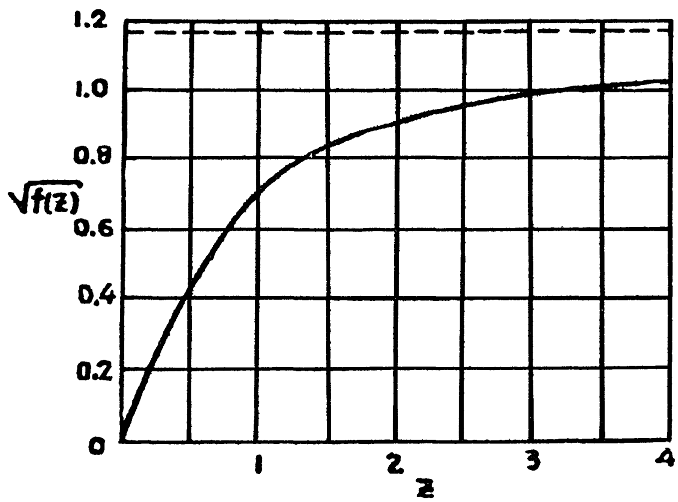


FIGURE 98. Plot of $\sqrt{f(z)}$, relating β_0 to $\beta(x) \equiv \langle I - \bar{I})^2 \rangle / \bar{I}^2$.

The spatial spectrum $F_I(x, \vec{\kappa})$ defined by (11) can be written in the following form for the spectral density given by (15):

$$\frac{\kappa_0^2 F_I(x, \vec{\kappa})}{\bar{I}^2} = \frac{\pi A}{\alpha} \beta_0^2 \left(\frac{\kappa}{\kappa_0}\right)^{-17/3} \int_0^{\left(\frac{\kappa}{\kappa_0}\right)^2} (1 - \cos t) \exp\left(-g\beta_0^2 \left(\frac{\kappa}{\kappa_0}\right)^{-11/3} t^{8/3}\right) dt \tag{20}$$

where $\kappa_0^2 = k/x$.

We see from this relation that the dimensionless spatial spectral density $\kappa_0^2 F_I / \bar{I}^2$ depends on two dimensionless parameters, $\frac{\kappa}{\kappa_0}$ and β_0^2. The function $\kappa_0^2 F_I(x, \vec{\kappa}) / \bar{I}^2 \beta^2(x)$ obtained by numerical integration is shown in Figure 99 (this is the normalized spectral density of intensity fluctuations).

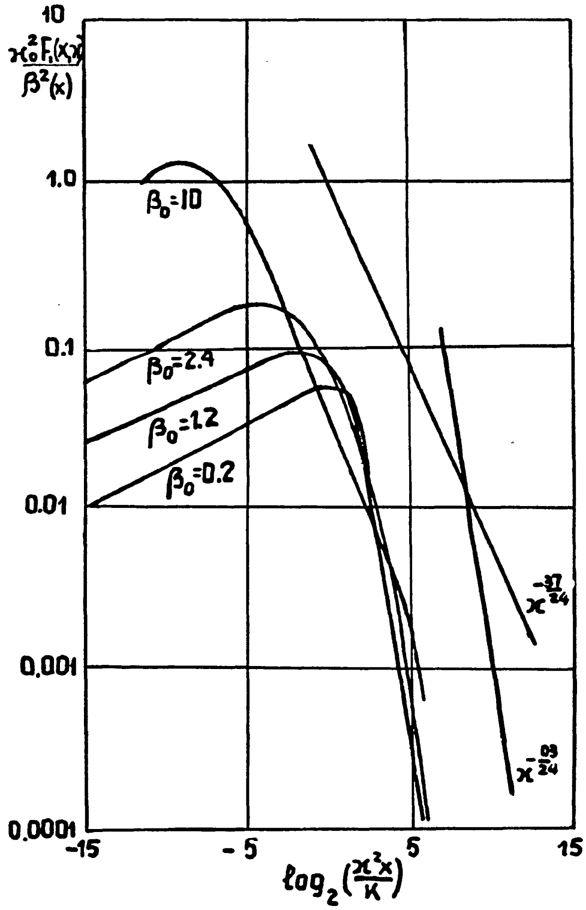


FIGURE 99. The normalized spatial spectrum of the intensity fluctuations.

Equation (20) may be rewritten in an alternative form using a new characteristic wave number
Then the argument of the exponential in (20) may be written in the form

$$\kappa_c = (\alpha g C_\epsilon^2 k^3)^{3/11}. \tag{21}$$

Then the argument of the exponential in (20) may be written in the form

$$g\beta_0^2\left(\frac{\kappa}{\kappa_0}\right)^{-11/3}=\left(\frac{\kappa}{\kappa_c}\right)^{-11/3}$$

and expression (20) itself may be rewritten as

$$\frac{\kappa_c^2 F_I(x,\vec{\kappa})}{\bar{I}^2}=\frac{\pi A}{\alpha g}\left(\frac{\kappa}{\kappa_c}\right)^{-17/3}\int_0^{(\kappa/\kappa_0)^2}(1-\cos t)\exp\left\{-\left(\frac{\kappa}{\kappa_c}\right)^{-11/3}t^{8/3}\right\}dt. \qquad (22)$$

Consider the argument of the exponential at the upper integration limit, i.e., for $t=(\kappa/\kappa_0)^2$. Its value is $\kappa^{5/3}\kappa_c^{11/3}\kappa_0^{-16/3}=\alpha g C_\epsilon^2 k^{1/3}x^{8/3}\kappa^{5/3}$. If this value is large, i.e.,

$$\kappa \gg (\alpha g C_\epsilon^2 k^{1/3}x^{8/3})^{-3/5}=\frac{\kappa_0}{(g\beta_0^2)^{3/5}}\equiv\kappa_*, \qquad (23)$$

the upper integration limit may be changed to infinity, and we get

$$\frac{\kappa_c^2 F_I(x,\vec{\kappa})}{\bar{I}^2}=\frac{\pi A}{\alpha g}\left(\frac{\kappa}{\kappa_c}\right)^{-17/3}\int_0^{\infty}(1-\cos t)\exp\left\{-\left(\frac{\kappa}{\kappa_c}\right)^{-11/3}t^{8/3}\right\}dt,\ (\kappa\gg\kappa_*). \qquad (24)$$

Thus, for wave numbers satisfying condition (23), $F_I(x,\vec{\kappa})$ depends on a single parameter κ_c. For $\beta_0^2\gg 1$, the region where condition (23) is satisfied is very wide, and it is this region that determines the mean square intensity fluctuations. It is readily seen that integration of (24) over all $\vec{\kappa}$ (including the region where condition (23) breaks down) gives the limiting value of β^2 expressed by (19).

Equation (24) thus corresponds to the spatial spectrum of the fluctuations in the limiting case of very large β_0. Note that expression (22) vanishes for $\kappa=0$, which corresponds to the law of conservation of energy (71.28). However, on passing to the limiting spectrum given by (24), this relationship breaks down (since for $\kappa\to 0$ we inevitably reach the region where condition (23) is not satisfied).

The normalized limiting spectral density has the form

$$\tilde{F}_I(\tilde{\kappa})=\frac{\kappa_c^2 F_I(x,\vec{\kappa})}{\bar{I}^2\beta^2(\infty)}=\frac{55}{54\pi\sin\pi/3\,\Gamma(4/3)}\tilde{\kappa}^{-17/3}\int_0^\infty(1-\cos t)\exp\left\{-\frac{t^{8/3}}{\tilde{\kappa}^{11/3}}\right\}dt \qquad (25)$$

where $\tilde{\kappa}=\dfrac{\kappa}{\kappa_c}$ is the dimensionless wave number.

The function $\tilde{F}_I(\tilde{\kappa})$ is normalized by the equality

$$2\pi\int_0^\infty \tilde{F}_I(\tilde{\kappa})\,\tilde{\kappa}\,d\tilde{\kappa}=1.$$

For $\tilde{\kappa}\ll 1$, and expanding the cosine in a series, we obtain

$$\tilde{F}_I(\tilde{\kappa})=a\tilde{\kappa}^{-37/24},\quad a=\frac{55\,\Gamma(9/8)}{288\,\pi\sin\pi/3\,\Gamma(4/3)},\ \tilde{\kappa}\ll 1. \qquad (26)$$

For $\widetilde{\kappa} \gg 1$, on the other hand, the cosine can be dropped in the integral in (25), and we find

$$\widetilde{F}_I(\widetilde{\kappa}) = b\widetilde{\kappa}^{-103/24}, \quad b = \frac{55\,\Gamma(11/8)}{55\,\pi \sin \pi/3\,\Gamma(4/3)}, \quad \widetilde{\kappa} \gg 1. \tag{27}$$

The ratio between the wave numbers κ_c and κ_0 is expressed by the equality

$$\frac{\kappa_c}{\kappa_0} = (\alpha\beta_0^2)^{3/11}.$$

Thus, $\kappa_c \ll \kappa_0$ for $\beta_0 \ll 1$, and $\kappa_c \gg \kappa_0$ for $\beta_0 \gg 1$.

We will now consider qualitatively the form of the spectrum F_I for $\beta_0 \gg 1$. In the region $\kappa \ll \kappa_0(g\beta_0)^{-3/5} = \kappa_*$, we find from (22)

$$\frac{\kappa_c^2 F_I(x, \vec{\kappa})}{\bar{I}^2\beta^2} \sim \frac{\kappa^{1/3}\kappa_c^{17/3}}{\kappa_0^6}.$$

For $\kappa \sim \kappa_*$ (see (23)), this dependence is replaced with (26). As κ is further increased, (26) is replaced by (27) near the point $\kappa \sim \kappa_c$. Thus, for $\beta_0 \gg 1$, we are dealing with three characteristic wave number ranges: $\kappa \ll \kappa_*$, $\kappa_* \ll \kappa \ll \kappa_c$, $\kappa \gg \kappa_c$. The first range does not make a noticeable contribution to the variance of intensity, but the spectral region $\kappa < \kappa_*$ ensures the equality $F_I(x, 0) = 0$, which follows from the law of conservation of energy. Since physically this relation implies that the energy is merely redistributed between different regions in the plane $x = \text{const}$ and that

$$\iint_{-\infty}^{\infty} B_I(x, \vec{\rho})\, d^2\rho = 0$$

the scale κ_*^{-1} is associated with the existence of negative values of $B_I(x, \vec{\rho})$ 4 and, consequently, corresponds to the mean distance between positive and negative intensity peaks. The scale κ_c^{-1} acts in this case as a characteristic size of the inhomogeneities of the intensity field. In the case $\beta_0 \gg 1$, we have $\kappa_* \ll \kappa_c$, i.e., the size of the inhomogeneities of the intensity field is much less than the mean distance between the inhomogeneities. For $\beta_0 \ll 1$, both these distances are on the order of magnitude of κ_0.

Let us now consider the passage of a wave through a layer of finite thickness containing refractive index inhomogeneities.

Let

$$C_\epsilon^2(x') = \begin{cases} C_\epsilon^2 & \text{for} \quad x' < h \\ 0 & \text{for} \quad x' > h. \end{cases} \tag{28}$$

We will compute the value of $\beta^2(x)$ for $x > h$. Inserting (28) in (15) and in (13), we find for $x > h$

$$\beta^\epsilon(x) = \pi k^2 A C_\epsilon^2 \int_0^h dx' \iint_{-\infty}^{\infty} d^2\kappa\, \kappa^{-11/3}\left[1 - \cos\frac{\kappa^2(x-x')}{k}\right] \cdot$$
$$\cdot \exp\left\{-\frac{\pi k^2}{2} p C_\epsilon^2 \frac{\kappa^{5/3}}{k^{5/3}} \int_{x'}^h (x-\xi)^{5/3}\, d\xi\right\}.$$

Integrating over the angular variable and carrying out the integration in the exponential, we find

$$\beta^2(x) = 2\pi^2 k^2 A C_\epsilon^2 \int_0^h dx' \int_0^\infty \kappa^{-8/3}\left[1-\cos\frac{\kappa^2(x-x')}{k}\right]\cdot$$
$$\cdot \exp\left\{-\frac{3\pi p}{16} C_\epsilon^2 k^{1/3}\kappa^{5/3}\left[(x-x')^{8/3}-(x-h)^{8/3}\right]\right\}d\kappa. \qquad (29)$$

Setting $x=h$ in this expression, we return to the previous result. Let us consider the case

$$x \gg h, \qquad (30)$$

i.e., the observation point is far beyond the inhomogeneous layer. In this case

$$(x-x')^{8/3}-(x-h)^{8/3} = x^{8/3}\left[\left(1-\frac{x'}{x}\right)^{8/3}-\left(1-\frac{h}{x}\right)^{8/3}\right] \approx \frac{8}{3}x^{5/3}(h-x').$$

Inserting this expression in (29) and changing integration variables from κ to $t=\frac{\kappa^2(x-x')}{k}$, we obtain

$$\beta^2(x) = \pi^2 A C_\epsilon^2 k^{7/6}\int_0^h (x-x')^{5/6}dx' \int_0^\infty t^{-11/6}(1-\cos t)\cdot$$
$$\exp\left\{-\frac{\pi p}{2}C_\epsilon^2 k^{7/6}x^{5/3}(x-x')^{-5/6}(h-x')\,t^{5/6}\right\}dt. \qquad (31)$$

Using (30), in the last expression we make the reasonable approximation that $(x-x')\approx x$. Then we can integrate over x', which gives

$$\beta^2(x) = \frac{2\pi A}{p}\int_0^\infty t^{-8/3}(1-\cos t)\left[1-\exp\left(-\frac{\pi p}{2}C_\epsilon^2 k^{7/6}x^{5/6}h\,t^{5/6}\right)\right]dt. \qquad (32)$$

The argument in the exponential in (32) may be expressed in terms of the quantity

$$\tilde\beta_0^2(x) = \frac{11}{6}\alpha C_\epsilon^2 k^{7/6}x^{5/6}h \qquad (33)$$

which is equal to β^2 for the problem of an inhomogeneous layer when using the first order perturbation theory for $x \gg h$ ((33) can be derived either from (32), by expanding the exponential function in a series, or from (48.24) using the relation $4\langle\chi^2\rangle$. Expression (32) then takes the form

$$\beta^2(x) = \frac{2\pi A}{p}\int_0^\infty t^{-8/3}(1-\cos t)\left[1-\exp\left(-\frac{16}{11}g\tilde\beta_0^2\,t^{5/6}\right)\right]dt. \qquad (32a)$$

Recalling the definition of $f(z)$ given in (18), we may write (32a) in the form

$$\beta^2(x) = \frac{11}{16}f\left(\sqrt{\frac{16}{11}}\tilde\beta_0\right). \qquad (34)$$

The dependence $\beta^2 = f(\widetilde{\beta}_0)$ for the case of a thin inhomogeneous layer is analogous to the corresponding dependence for a statistically homogeneous medium. For $\widetilde{\beta} \to \infty$, we conclude from (34) and (19) that $\beta^2(x) \to 0.93$.

The region of application of the single-scattering approximation in the problem of a thin inhomogeneous layer is apparently wider than in the problem of fluctuations in a statistically homogeneous medium. Indeed, if the inequality $C_\epsilon^2 k^{7/6} h^{11/6} \ll 1$ is adequately fulfilled, the intensity fluctuations inside the layer are small and there is a greater justification for treating the wave inside the layer as if it were undistorted. The fluctuations become strong only for $x \gg h$, i.e., in the region which is free from inhomogeneities, where the right-hand side of equation (71.42) vanishes. Therefore it is expected that solution (34) holds true when

$$\left(\frac{x}{h}\right)^{5/6} \gg \frac{1}{C_\epsilon^2 k^{7/6} h^{11/6}} \gg 1. \tag{35}$$

The solution of the problem of an inhomogeneous layer may prove useful in estimating the probable error of the previous problem of fluctuations in a statistically homogeneous medium with a constant C_ϵ^2.

Indeed, if the intensity fluctuations at a distance x are large (i.e., $\beta_0^2 \gg 1$, $\beta^2(x) \approx \beta^2(\infty)$) and we artificially remove all the inhomogeneities from the layer (h, x), we see by (34) that the mean square fluctuations diminish by a factor of 11/16. It is clear, from physical considerations that after removing part of the inhomogeneities the fluctuations may only diminish. The solution of the problem of a thin inhomogeneous layer therefore apparently provides an estimate of the lower-bound for the magnitude of fluctuations over a statistically homogeneous path. On the other hand, as we have noted before, the single-scattering approximation yields an exaggerated result for the intensity fluctuations. It therefore seems that for $\beta_0^2 \to \infty$, $\beta^2(x)$ will lie between 0.93 and 1.36.

The latest and the most reliable experimental data about intensity fluctuations /200/ relate to $\sigma_I^2 = \langle \ln I - \langle \ln I \rangle]^2 \rangle$, and not to β^2, and they are difficult to compare with our numerical values. If we assume that for $\beta_0^2 \gg 1$, χ again follows a normal distribution (as is evident from the results of /196/), the conversion of the measured values of σ_I^2 from /200/ into β^2 for the largest values of β_0 yields a result which does not contradict our estimates of $\beta^2(\infty)$.

In conclusion note that the single-scattering approximation can be readily generalized to the case of a bounded beam (see (71.35) and (71.36)), although the calculations are more lengthy.

I. VARIATIONAL DERIVATIVES

This appendix introduces the concept of a variational derivative and describes certain rules of operation. For a more detailed and rigorous treatment, the reader is referred to specialized literature on functional analysis.

A functional $\Phi[f(\xi)]$ is said to be defined if for every function $f(\xi)$ belonging to a certain region there corresponds a number Φ dependent on that function.

For example,

$$\Phi[f(\xi)] = \int_a^b f(\xi)\, a(\xi)\, d\xi, \tag{1}$$

where the function $a(\xi)$ is fixed, is a linear functional defined for such functions $f(\xi)$ for which the integral in (1) converges. An example of a non-linear functional is provided by the bilinear functional

$$\Phi[f(\xi)] = \int_a^b \int_a^b A(\xi, \eta)\, f(\xi)\, f(\eta)\, d\xi\, d\eta, \tag{2}$$

or a functional of the form

$$\Phi[f(\xi)] = \int_a^b F(f(\xi))\, d\xi, \tag{3}$$

where $F(x)$ is a given function and $F(f(\xi))$ is a function of a function.

Let the functional $\Phi[f(\xi)]$ be given. Consider its value for the function $f(\xi) + \delta f(\xi)$, where $\delta f(\xi)$ is zero everywhere except in the neighborhood $\Delta(x)$ of some point x from the interval (a, b). The difference between $\Phi[f(\xi) + \delta f(\xi)]$ and $\Phi[f(\xi)]$ is equal to the functional variation

$$\delta\Phi = \{\Phi[f(\xi) + \delta f(\xi)] - \Phi[f(\xi)]\}, \tag{4}$$

where the braces { } indicate that only the linear (in δf) part of this difference is taken. $\delta\Phi$ vanishes together with $\delta f(\xi)$. Consider the ratio

$$\frac{\delta\Phi}{\int_{\Delta(x)} \delta f(\xi)\, d\xi} \tag{5}$$

and let $\Delta(x)$ approach zero in such a way that the point x always remains inside the diminishing interval. If the corresponding limit of (5) exists, it is called the variational derivative of $\Phi[f(\xi)]$ at the point x and is denoted

$$\frac{\delta\Phi[f(\xi)]}{\delta f(x)}=\lim_{\Delta\to 0}\frac{\{\Phi[f(\xi)+\delta f(\xi)]-\Phi[f(\xi)]\}}{\int\limits_{\Delta(x)}\delta f(\xi)\,d\xi}. \tag{6}$$

As an example, let us find the variational derivative of the functional given in (1)

$$\Phi[f(\xi)+\delta f(\xi)]=\int\limits_a^b[f(\xi)+\delta f(\xi)]\,a(\xi)\,d\xi=\int\limits_a^b f(\xi)\,a(\xi)\,d\xi+\int\limits_a^b\delta f(\xi)\,a(\xi)\,d\xi,$$

$$\Phi[f(\xi)+\delta f(\xi)]-\Phi[f(\xi)]=\int\limits_a^b\delta f(\xi)\,a(\xi)\,d\xi=\int\limits_{\Delta(x)}\delta f(\xi)\,a(\xi)\,d\xi,$$

where in the last equality we made use of the fact that $\delta f(\xi)$ vanishes everywhere except in $\Delta(x)$. In this case $\Phi[f(\xi)+\delta f(\xi)]-\Phi[f(\xi)]$ is linear in δf, so that

$$\frac{\delta\Phi[f(\xi)]}{\delta f(x)}=\lim_{\Delta\to 0}\frac{\int\limits_{\Delta(x)}\delta f(\xi)\,a(\xi)\,d\xi}{\int\limits_{\Delta(x)}\delta f(\xi)\,d\xi}.$$

If $a(\xi)$ is continuous at the point x, we find by applying the theorem of the mean to the integral in the numerator that we have

$$a(x')\int\limits_{\Delta(x)}\delta f(\xi)\,d\xi,$$

where $x'\in\Delta(x)$, so that

$$\frac{\delta\Phi[f(\xi)]}{\delta f(x)}=\lim_{\Delta\to 0}a(x')=a(x),$$

since for $\Delta\to 0$, $x'\to x$. Thus,

$$\frac{\delta}{\delta f(x)}\left[\int\limits_a^b f(\xi)\,a(\xi)\,d\xi\right]=a(x). \tag{7}$$

As another example, let us calculate the derivative of the functional (2):

$$\Phi[f(\xi)+\delta f(\xi)]=\int\limits_a^b\int\limits_a^b A(\xi,\eta)[f(\xi)+\delta f(\xi)][f(\eta)+\delta f(\eta)]\,d\xi\,d\eta=$$

$$=\int\limits_a^b\int\limits_a^b A(\xi,\eta)f(\xi)f(\eta)\,d\xi\,d\eta+\int\limits_a^b\int\limits_a^b A(\xi,\eta)[f(\xi)\,\delta f(\eta)+$$

$$+f(\eta)\,\delta f(\xi)]\,d\xi\,d\eta+\int\limits_a^b\int\limits_a^b A(\xi,\eta)\,\delta f(\xi)\,\delta f(\eta)\,d\xi\,d\eta.$$

The last term in this expression is quadratic in δf, so that it should be omitted from the expression for $\delta\Phi$:

$$\delta\Phi = \int_a^b d\xi \int_{\Delta(x)} d\eta A(\xi, \eta) f(\xi) \delta f(\eta) + \int_a^b d\eta \int_{\Delta(x)} d\xi A(\xi, \eta) f(\eta) \delta f(\xi) =$$

$$= \int_a^b d\xi \int_{\Delta(x)} d\eta [A(\xi, \eta) + A(\eta, \xi)] f(\xi) \delta f(\eta)$$

(in the last equality we used the substitution $\xi \leftrightarrow \eta$). Evaluating the integral over η using the theorem of the mean, we obtain

$$\delta\Phi = \int_a^b f(\xi) [A(\xi, x') + A(x', \xi)] d\xi \int_{\Delta(x)} \delta f(\eta) d\eta,$$

so that

$$\frac{\delta}{\delta f(x)} \int_a^b \int_a^b A(\xi, \eta) f(\xi) f(\eta) d\xi d\eta = \int_a^b [A(\xi, x) + A(x, \xi)] f(\xi) d\xi.$$

Note that $A(\xi, \eta)$ may always be regarded as a symmetric function. Indeed,

$$A(\xi, \eta) = \frac{A(\xi, \eta) + A(\eta, \xi)}{2} + \frac{A(\xi, \eta) - A(\eta, \xi)}{2}.$$

Inserting this expression in (2), we easily find that integration over the second term gives zero:

$$\frac{1}{2} \int_a^b \int_a^b [A(\xi, \eta) - A(\eta, \xi)] f(\xi) f(\eta) d\xi d\eta = 0 \qquad (*)$$

(to prove this, it suffices to make the substitution $\xi \leftrightarrow \eta$ in one of the integrals in (*)).

Therefore in (2) we may take $A(\xi, \eta) = A(\eta, \xi)$. Thus,

$$\frac{\delta}{\delta f(x)} \left[\int_a^b \int_a^b A(\xi, \eta) f(\xi) f(\eta) d\xi d\eta \right] = 2 \int_a^b A(x, \xi) f(\xi) d\xi.$$

Similarly, if $A_n(\xi_1, \xi_2, \ldots, \xi_n)$ is a symmetric function of all its arguments, we have

$$\frac{\delta}{\delta f(x)} \left[\int_a^b \underset{(n)}{\cdots} \int_a^b A_n(\xi_1, \ldots, \xi_n) f(\xi_1) \ldots f(\xi_n) d\xi_1 \ldots d\xi_n \right] =$$

$$= n \int_a^b \underset{(n-1)}{\cdots} \int_a^b A_n(\xi_1, \ldots, \xi_{n-1}, x) f(\xi_1) \ldots f(\xi_{n-1}) d\xi_1 \ldots d\xi_{n-1}. \qquad (8)$$

This expression is an analog of the equality

$$\frac{d}{dx} x^n = nx^{n-1}.$$

An important special case of (7) is the relation

$$\frac{\delta f(x_0)}{\delta f(x)} = \delta(x - x_0). \qquad (9)$$

It can be derived from (7) if we take

$$a(\xi)=\frac{1}{\sqrt{2\pi}\,\sigma}e^{-\frac{\xi^2}{2\sigma^2}},$$

pass to the limit as $\sigma\to 0$ and remember that $\lim\limits_{\sigma\to 0} a(\xi)=\delta(\xi)$. Expression (9) simplifies the differentiation of many functionals.

It is easy to prove the equality

$$\frac{\delta\Phi_1[f(\xi)]\,\Phi_2[f(\xi)]}{\delta f(x)}=\Phi_1[f(\xi)]\frac{\delta\Phi_2[f(\xi)]}{\delta f(x)}+\Phi_2[f(\xi)]\frac{\delta\Phi_1[f(\xi)]}{\delta f(x)}. \qquad (10)$$

As an example, consider the application of (9) and (10) in calculating the derivative of (2):

$$\frac{\delta}{\delta f(x)}\left[\int_a^b\int_a^b A(\xi,\eta)\,f(\xi)\,f(\eta)\,d\xi\,d\eta\right]=\int_a^b\int_a^b A(\xi,\eta)\frac{\delta[f(\xi)\,f(\eta)]}{\delta f(x)}d\xi\,d\eta=$$

$$=\int_a^b\int_a^b A(\xi,\eta)\left[\frac{\delta f(\xi)}{\delta f(x)}f(\eta)+f(\xi)\frac{\delta f(\eta)}{\delta f(x)}\right]d\xi\,d\eta=$$

$$=\int_a^b\int_a^b A(\xi,\eta)\,[\delta(\xi-x)\,f(\eta)+f(\xi)\,\delta(\eta-x)]\,d\xi\,d\eta=$$

$$=\int_a^b A(x,\eta)\,f(\eta)\,d\eta+\int_a^b A(\xi,x)\,f(\xi)\,d\xi=2\int_a^b A(\xi,x)\,f(\xi)\,d\xi.$$

Let us derive an important formula

$$\frac{\delta}{\delta f(x)}\{F(\Phi[f(\xi)])\}=F'(\Phi[f(\xi)])\,\frac{\delta\Phi[f(\xi)]}{\delta f(x)} \qquad (11)$$

for the differentiation of a function of a functional. Let

$$\Psi[f(\xi)]=F(\Phi[f(\xi)]).$$

Then

$$\Psi[f(\xi)+\delta f(\xi)]=F(\Phi[f(\xi)+\delta f(\xi)])=F(\Phi[f(\xi)]+\delta\Phi)=$$
$$=F(\Phi[f(\xi)])+F'(\Phi[f(\xi)])\,\delta\Phi+\ldots,$$
$$\delta\Psi=F'(\Phi[f(\xi)])\,\delta\Phi.$$

Inserting this expression in the definition (6), we obtain (11). For example:

$$\frac{\delta}{\delta f(x)}\left\{e^{\int_a^b f(\xi)a(\xi)d\xi}\right\}=a(x)\,e^{\int_a^b f(\xi)a(\xi)d\xi}$$

We have seen from our examples that $\delta\Phi[f(\xi)]/\delta f(x)$ is again a functional; furthermore it depends on the point x, so that it is also an ordinary function of x.

Second and higher derivatives can be considered:

$$\frac{\delta^2\Phi[f(\xi)]}{\delta f(x_1)\,\delta f(x_2)},\quad \frac{\delta^3\Phi[f(\xi)]}{\delta f(x_1)\,\delta f(x_2)\,\delta f(x_3)}.$$

They are functions of the points x_1, x_2,... For example,

$$\frac{\delta^2}{\delta f(x_1)\,\delta f(x_2)}\left[\int_a^b\int_a^b A(\xi,\eta)\,f(\xi)\,f(\eta)\,d\xi\,d\eta\right]=2A(x_1,x_2).$$

An analog of a Taylor series can be derived for functionals:

$$\Phi[f(\xi)]=\Phi[f_0(\xi)]+\int_a^b\frac{\delta\Phi[f(\xi)]}{\delta f(x)}\bigg|_{f=f_0}[f(x)-f_0(x)]\,dx+$$
$$+\frac{1}{2!}\int_a^b\int_a^b\frac{\delta^2\Phi[f(\xi)]}{\delta f(x_1)\,\delta f(x_2)}\bigg|_{f=f_0}[f(x_1)-f_0(x_1)][f(x_2)-f_0(x_2)]\,dx_1\,dx_2+\ldots \qquad (12)$$

This formula is best obtained by considering the function $\Phi[f(x)]$ for the step-wise function

$$f_n(x)=\begin{cases} f_0 \text{ for } a\leqslant\xi<h=\dfrac{b-a}{n},\\ f_1 \text{ for } h\leqslant\xi<2h,\\ \ldots\ldots\ldots\ldots\ldots\ldots\\ f_{n-1} \text{ for } (n-1)h\leqslant\xi\leqslant nh=b. \end{cases} \qquad (13)$$

In this case

$$\Phi[f_n(x)]=F(f_0,f_1,\ldots,f_{n-1}),$$

i.e., Φ reduces to a function of n variables $f_0,\ldots,f_{n-1}$. We thus have

$$\frac{\delta\Phi[f(\xi)]}{\delta f(x)}=\frac{1}{h}\frac{\partial F(f_0,\ldots,f_{n-1})}{\partial f_k}, \qquad (14)$$

where $f_k=f_n(x_k)$ (see (13)). For $F(f_0,\ldots,f_{n-1})$ we have the Taylor series expansion

$$F(f_0,f_1,\ldots,f_{n-1})=F(f_0^0,f_1^0,\ldots,f_{n-1}^0)+$$
$$+\sum_k\frac{\partial F(f_0^0,f_1^0,\ldots,f_{n-1}^0)}{\partial f_k}(f_k-f_k^0)+$$
$$+\frac{1}{2!}\sum_k\sum_l\frac{\partial^2 F(f_0^0,f_1^0,\ldots,f_{n-1}^0)}{\partial f_k\partial f_l}(f_k-f_k^0)(f_l-f_l^0)+\ldots$$

Inserting (14) and analogous expressions for the higher derivatives, we obtain

$$\Phi[f_n(x)]=\Phi[f_n^0(x)]+\sum_k\frac{\delta\Phi[f_n(\xi)]}{\delta f_n(x_k)}h+\ldots$$

Taking the limit as $n\to\infty$ $(h\to 0)$, we obtain (12).

Let us establish how functional derivatives transform under substitution of variables. Let the function $f(x)$ be replaced by a new function $\varphi(x)$ defined by the equality

$$f(x)=\Psi[\varphi(\xi);x], \qquad (15)$$

where Ψ is some functional of the function $\varphi(\xi)$ which also depends on x. The functional $\Phi[f(\xi)]$ is now a compound functional of φ:

$$\Phi[f(\xi)]=\Phi[\Psi[\varphi(\eta);\ \xi]]\equiv\Phi_1[\varphi(\xi)].$$

Using (12) and (15), we can readily show that

$$\frac{\delta\Phi_1[\varphi(\xi)]}{\delta\varphi(x)}=\int_a^b \frac{\delta\Phi[f(\xi)]}{\delta f(x')}\,\frac{\delta\Psi[\varphi(\eta);x']}{\delta\varphi(x)}\,dx'. \tag{16}$$

As an example consider the change from a functional argument to its Fourier transform. Let

$$f(x)=\int_{-\infty}^{\infty} e^{ikx}\varphi(k)\,dk=\Psi[\varphi(k);x].$$

Then

$$\frac{\delta\Psi[\varphi(k);x]}{\delta\varphi(k')}=e^{ik'x}$$

and by (16)

$$\frac{\delta\Phi[f[\varphi(k);x]]}{\delta\varphi(k')}=\int_{-\infty}^{\infty}\frac{\delta\Phi[f(\xi)]}{\delta f(x')}e^{ik'x'}\,dx'.$$

In conclusion we give (without proof) a formula for deriving the functional $\Phi[f(\xi)]$ from its variational derivative

$$\frac{\delta\Phi[f(\xi)]}{\delta f(x)}=F[f(\xi),x],$$

which is itself a continuous functional /163/:

$$\Phi[f(\xi)]=\int[f(x)-f_0(x)]\,F[\theta(x-\xi)f_0(\xi)+\theta(\xi-x)f(\xi);x]\,dx+\Phi[f_0(\xi)]. \tag{17}$$

Here $\theta(x)=0$ if $x<0$ and $\theta(x)=1$ if $x\geqslant 0$, so that the argument of the functional F in the integrand in (17) is a function equal to $f_0(\xi)$ for $\xi<x$ and to $f(\xi)$ for $\xi>x$. The functional $F[f(\xi);x]$ should satisfy the condition

$$\frac{\delta F[f(\xi);x_1]}{\delta f(x_2)}=\frac{\delta F[f(\xi);x_2]}{\delta f(x_1)}, \tag{18}$$

which follows from the obvious equality

$$\frac{\delta^2\Phi[f(\xi)]}{\delta f(x_1)\,\delta f(x_2)}=\frac{\delta^2\Phi[f(\xi)]}{\delta f(x_2)\,\delta f(x_1)}.$$

Consider one example. Let

$$F[f(\xi);x]=2\int_a^b A(x,\xi)f(\xi)\,d\xi.$$

Condition (18) requires that the following equality is met

$$A(x,\xi)=A(\xi,x),$$

$$F[\theta(x-\xi)f_0(\xi)+\theta(\xi-x)f(\xi);x]=$$

$$=2\int_a^x A(x,\xi)f_0(\xi)\,d\xi+2\int_x^b A(x,\xi)f(\xi)\,d\xi$$

and by (17)

$$\Phi[f(\xi)] = 2\int_a^b [f(x) - f_0(x)]\,dx \left\{\int_a^x A(x,\xi) f_0(\xi)\,d\xi + \right.$$

$$\left. + \int_x^b A(x,\xi) f(\xi)\,d\xi\right\} + \Phi[f_0(\xi)] = 2\int_a^b dx \int_x^b d\xi\, A(x,\xi) f(x) f(\xi) -$$

$$- 2\int_a^b dx \int_a^x d\xi\, A(x,\xi) f_0(x) f_0(\xi) + \Phi[f_0(\xi)] +$$

$$+ 2\int_a^b dx \int_a^x d\xi\, A(x,\xi) f(x) f_0(\xi) - 2\int_a^b dx \int_x^b d\xi\, A(x,\xi) f(\xi) f_0(x).$$

The last two integrals mutually cancel once the order of integration is changed and the substitution $\xi \leftrightarrow x$ is made in one of them. Similarly, it can be readily shown that

$$2\int_a^b dx \int_x^b d\xi A(x,\xi) f(x) f(\xi) = \int_a^b\int_a^b A(x,\xi) f(x) f(\xi)\,dx\,d\xi,$$

so that

$$\Phi[f(\xi)] = \int_a^b\int_a^b A(x,\xi)[f(x) f(\xi) - f_0(x) f_0(\xi)]\,dx\,d\xi + \Phi[f_0(x)].$$

If $f_0(\xi) = 0$ and $\Phi[0] = 0$, we obtain functional (2). Formula (17) can be applied easily in the same way to any n-th degree functional.

In the theory of random functions the main significance is attached to the so-called characteristic functional defined as

$$\Phi[f(\xi)] \equiv \left\langle \exp\left[i\int f(\xi)\,\varepsilon(\xi)\,d\xi\right]\right\rangle, \tag{19}$$

where $\varepsilon(\xi)$ is a random function and $\langle\ \rangle$ signifies averaging over the entire sample space of the function $\varepsilon(\xi)$. The random function $\varepsilon(\xi)$ is completely determined by its characteristic functional. The mean value, the correlation function and higher-order moments can be readily found if the functional (19) is known. For example,

$$\frac{\delta\Phi[f(\xi)]}{\delta f(x)} = \left\langle i\varepsilon(x) \exp\left[i\int f(\xi)\,\varepsilon(\xi)\,d\xi\right]\right\rangle,$$

so that

$$\langle\varepsilon(x)\rangle = \frac{1}{i}\frac{\delta\Phi[f(\xi)]}{\delta f(x)}\bigg|_{f=0}.$$

Similarly

$$\langle\varepsilon(x_1)\,\varepsilon(x_2)\rangle = \left(\frac{1}{i}\right)^2 \frac{\delta^2\Phi[f(\xi)]}{\delta f(x_1)\,\delta f(x_2)}\bigg|_{f=0},$$

etc.

From the characteristic functional one readily obtains characteristic functions of any order n. For example,

$$\Phi[u\delta(\xi - x)] = \langle\exp[iu\varepsilon(x)]\rangle$$

is the characteristic function of the one-dimensional probability distribution. Similarly,

$$\Phi\,[u_1\delta\,(\xi - x_1) + u_2\delta\,(\xi - x_2)] = \langle \exp\{i\,[u_1\,\varepsilon\,(x_1) + u_2\,\varepsilon\,(x_2)]\}\rangle$$

is the characteristic function of the two-dimensional probability distribution, etc.

If $\varepsilon\,(x)$ has a normal probability distribution with $\langle\varepsilon\rangle = 0$, then

$$\Phi\,[f\,(\xi)] = \exp\left\{-\frac{1}{2}\iint B_\varepsilon\,(x_1,\,x_2)\,f\,(x_1)\,f\,(x_2)\,dx_1\,dx_2\right\},$$

where $B_\varepsilon\,(x_1,\,x_2) = \langle\varepsilon\,(x_1)\,\varepsilon\,(x_2)\rangle$.

In conclusion note that the entire treatment of this appendix is easily transferred to the case of functionals of functions of many variables.

II. SOME USEFUL FORMULAS

Chapter 1

$$B\,(t_1,\ t_2) = \langle [f\,(t_1) - \langle f\,(t_1)\rangle]\ \ [f^*\,(t_2) - \langle f^*\,(t_2)\rangle]\rangle,$$
$$B^*\,(t_1,\ t_2) = B\,(t_2,\ t_1),\ \ |B\,(t_1,\ t_2)\,|^2 \leqslant B\,(t_1,\ t_1)\cdot B\,(t_2,\ t_2).$$

For stationary processes

$$B\,(t_1,\,t_2) = B\,(t_1 - t_2,\,0) = B\,(\tau)\ \ (\tau = t_1 - t_2),\,B^*\,(\tau) = B\,(-\tau).$$

For stationary and real processes $B\,(-\tau) = B\,(\tau)$,

$$B\,(\tau) = \int_{-\infty}^{\infty} W\,(\omega)\,e^{i\omega\tau}d\omega,\quad W\,(\omega) \geqslant 0.$$

For real stationary processes $W\,(-\omega) = W\,(\omega)$,

$$B\,(\tau) = \int_{-\infty}^{\infty} W\,(\omega)\,e^{i\omega\tau}d\omega = \int_{-\infty}^{\infty} W\,(\omega)\cos\omega\tau\,d\omega = 2\int_{0}^{\infty} W\,(\omega)\cos\omega\tau\,d\omega.$$

For real processes with stationary increments, when $\langle f\,(t)\rangle = \text{const}$,

$$D\,(\tau) = \langle [f\,(t+\tau) - f\,(t)]^2\rangle.$$

If f is a stationary process, we have

$$D\,(\tau) = 2B\,(0) - 2B\,(\tau),$$

and if $B\,(\infty) = 0$, then also

$$B\,(\tau) = \frac{1}{2}\,D\,(\infty) - \frac{1}{2}\,D\,(\tau),$$

$$D\,(\tau) = 2\int_{-\infty}^{\infty} [1 - e^{i\omega\tau}]\,W\,(\omega)\,d\omega = 2\int_{-\infty}^{\infty} [1 - \cos\omega\tau]\,W\,(\omega)\,d\omega.$$

For a structure function of the form

$$D(\tau) = C^2 |\tau|^{\mu} \quad (0 < \mu < 2)$$

the spectral density is

$$W(\omega) = \frac{\Gamma(\mu+1)}{2\pi} \sin\frac{\pi\mu}{2} |\omega|^{-(\mu+1)}.$$

For homogeneous real random fields

$$B(\boldsymbol{\rho}) = \langle [f(\boldsymbol{r}+\boldsymbol{\rho}) - \langle f\rangle][f(\boldsymbol{r}) - \langle f\rangle]\rangle,$$

$$B(\boldsymbol{\rho}) = \iiint\limits_{-\infty}^{\infty} \Phi(\boldsymbol{\varkappa}) e^{i\varkappa\rho} d^3\varkappa = \iiint\limits_{-\infty}^{\infty} \Phi(\boldsymbol{\varkappa}) \cos \boldsymbol{\varkappa\rho}\, d^3\varkappa,$$

$$\Phi(\boldsymbol{\varkappa}) = \frac{1}{8\pi^3} \iiint\limits_{-\infty}^{\infty} e^{-i\varkappa\rho} B(\boldsymbol{\rho})\, d^3\rho = \frac{1}{8\pi^3} \iiint\limits_{-\infty}^{\infty} B(\boldsymbol{\rho}) \cos \boldsymbol{\varkappa\rho}\, d^3\rho.$$

For homogeneous and isotropic fields

$$B(\boldsymbol{\rho}) = B(\rho) = 4\pi \int_0^{\infty} \Phi(\varkappa) \frac{\sin \varkappa\rho}{\varkappa\rho} \varkappa^2\, d\varkappa,$$

$$\Phi(\boldsymbol{\varkappa}) = \Phi(\varkappa) = \frac{1}{2\pi^2} \int_0^{\infty} B(\rho) \frac{\sin \varkappa\rho}{\varkappa\rho} \rho^2\, d\rho.$$

If furthermore

$$B(x) = \int_{-\infty}^{\infty} V(\varkappa) \cos \varkappa x\, d\varkappa,$$

$$V(\varkappa) = \frac{1}{2\pi} \int_{-\infty}^{\infty} B(x) \cos \varkappa x\, dx,$$

the functions $V(\varkappa)$ and $\Phi(\varkappa)$ are related by the equality

$$\Phi(\varkappa) = -\frac{1}{2\pi\varkappa} \frac{dV(\varkappa)}{d\varkappa}.$$

For locally homogeneous real random fields with constant mean

$$D(\boldsymbol{\rho}) = \langle [f(\boldsymbol{r}+\boldsymbol{\rho}) - f(\boldsymbol{r})]^2\rangle,$$

$$D(\boldsymbol{\rho}) = 2 \iiint\limits_{-\infty}^{\infty} [1 - \cos \boldsymbol{\varkappa\rho}]\, \Phi(\boldsymbol{\varkappa})\, d^3\varkappa.$$

For locally isotropic fields

$$D(\boldsymbol{\rho}) = D(\rho), \qquad \Phi(\boldsymbol{\varkappa}) = \Phi(\varkappa),$$

$$D(\rho) = 8\pi \int_0^{\infty} \left(1 - \frac{\sin \varkappa\rho}{\varkappa\rho}\right) \Phi(\varkappa) \varkappa^2\, d\varkappa,$$

$$D(\rho) = 2 \int_{-\infty}^{\infty} [1 - \cos \varkappa\rho]\, V(\varkappa)\, d\varkappa, \quad \Phi(\varkappa) = -\frac{1}{2\pi\varkappa} \frac{dV(\varkappa)}{d\varkappa}.$$

The two-dimensional spectral expansion

$$D(\xi, \eta, \zeta) - D(\xi, 0, 0) = 2 \iint_{-\infty}^{\infty} [1 - \cos(\varkappa_2\eta + \varkappa_3\zeta)] F(\varkappa_2, \varkappa_3, \xi)\, d\varkappa_2\, d\varkappa_3,$$

$$D(0, \eta, \zeta) = 2 \iint_{-\infty}^{\infty} [1 - \cos(\varkappa_2\eta + \varkappa_3\zeta)] F(\varkappa_2, \varkappa_3, 0)\, d\varkappa_2\, d\varkappa_3,$$

$$F(\varkappa_2, \varkappa_3, \xi) = \int_{-\infty}^{\infty} \Phi(\varkappa_1, \varkappa_2, \varkappa_3) \cos \varkappa_1\xi\, d\varkappa_1,$$

$$\Phi(\varkappa_1, \varkappa_2, \varkappa_3) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\varkappa_2, \varkappa_3, \xi) \cos \varkappa_1\xi\, d\xi.$$

For fields which are locally isotropic in the plane $x = \text{const}$

$$D(\xi, \eta, \zeta) = D(\xi, \sqrt{\eta^2 + \zeta^2}) = D(\xi, \rho),$$

$$F(\varkappa_2, \varkappa_3, \xi) = F(\sqrt{\varkappa_2^2 + \varkappa_3^2}, \xi) = F(\varkappa, \xi), \quad \rho^2 = \eta^2 + \zeta^2, \quad \varkappa^2 = \varkappa_2^2 + \varkappa_3^2$$

$$D(\xi, \rho) - D(\xi, 0) = 4\pi \int_0^{\infty} [1 - J_0(\varkappa\rho)] F(\varkappa, \xi)\, \varkappa\, d\varkappa,$$

$$D(0, \rho) = 4\pi \int_0^{\infty} [1 - J_0(\varkappa\rho)] F(\varkappa, 0)\, \varkappa\, d\varkappa.$$

If the random field is statistically homogeneous and isotropic, we have

$$D(\boldsymbol{r}) = 2B(0) - 2B(\boldsymbol{r})$$

and in addition to the previous expansion we have

$$B(r) = 2\pi \int_0^{\infty} J_0(\varkappa\rho) F(\varkappa, \xi)\, \varkappa\, d\varkappa, \qquad r^2 = \xi^2 + \rho^2.$$

For a structure function of the form

$$D(r) = C^2 r^p \qquad (0 < p < 2)$$

the spectral densities are

$$V(\varkappa) = \frac{\Gamma(p+1)}{2\pi} \sin\frac{\pi p}{2} C^2 \varkappa^{-(p-1)},$$

$$\Phi(\varkappa) = \frac{\Gamma(p+2)}{4\pi^2} \sin\frac{\pi p}{2} C^2 \varkappa^{-(p+3)},$$

$$F(\varkappa, x) = \frac{\Gamma\left(\frac{p+2}{2}\right)}{\pi^2} 2^{\frac{p-2}{2}} \sin\frac{\pi p}{2} C^2 \frac{(\varkappa x)^{\frac{p+2}{2}} K_{\frac{p+2}{2}}(\varkappa x)}{\varkappa^{p+2}}.$$

Locally isotropic turbulence

$$D_{ik}(\boldsymbol{r}) = \langle [v_i(\boldsymbol{r} + \boldsymbol{r}') - v_i(\boldsymbol{r}')] [v_k(\boldsymbol{r} + \boldsymbol{r}') - v_k(\boldsymbol{r}')] \rangle,$$

$$D_{ik}(\boldsymbol{r}) = D_{tt}(r)\delta_{ik} + [D_{rr}(r) - D_{tt}(r)] n_i n_k \quad \left(\boldsymbol{n} = \frac{\boldsymbol{r}}{r}\right),$$

$$D_{tt}(r) = \frac{1}{2r} \frac{d}{dr} [r^2 D_{rr}(r)],$$

$$D_{ik}(\boldsymbol{r}) = 2\iiint\limits_{-\infty}^{\infty} [1-\cos \boldsymbol{\varkappa r}]\, \Phi_{ik}(\boldsymbol{\varkappa})\, d^3\varkappa,$$

$$\Phi_{ik}(\boldsymbol{\varkappa}) = \frac{1}{4\pi\varkappa^2}\left(\delta_{ik} - \frac{\varkappa_i\varkappa_k}{\varkappa^2}\right) E(\varkappa),$$

$$D_{rr}(r) = 4\int\limits_0^{\infty}\left[\frac{1}{3} + \frac{\cos\varkappa r}{\varkappa^2 r^2} - \frac{\sin\varkappa r}{\varkappa^3 r^3}\right] E(\varkappa)\, d\varkappa,$$

$$\varepsilon = \frac{\nu}{2}\left\langle\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right)^2\right\rangle = 2\nu\int\limits_0^{\infty} E(\varkappa)\,\varkappa^2\, d\varkappa.$$

$$\left.\begin{aligned} D_{rr}(r) &= \frac{1}{15}\frac{\varepsilon}{\nu} r^2 + \ldots \\ D_{tt}(r) &= \frac{2}{15}\frac{\varepsilon}{\nu} r^2 + \ldots \end{aligned}\right\} \text{ for } r \ll \sqrt[4]{\frac{\nu^3}{\varepsilon}} = l_0,$$

$$\left.\begin{aligned} D_{rr}(r) &= C^2\varepsilon^{2/3} r^{2/3} \\ D_{tt}(r) &= \frac{4}{3} C^2\varepsilon^{2/3} r^{2/3} \end{aligned}\right\} \quad (l_0 \ll r \ll L_0),$$

$$E(\varkappa) = 0.76\, C^2\varepsilon^{2/3}\varkappa^{-5/3} \quad \left(\frac{2\pi}{L_0} \ll \varkappa \ll \frac{2\pi}{l_0}\right).$$

Temperature field

$N = \chi\langle(\nabla T)^2\rangle$ (χ is the molecular thermal conductivity),

$$l_1 = \sqrt[4]{\frac{\chi^3}{\varepsilon}}, \qquad \lambda_0 = (3a^2)^{\frac{3}{4}} l_1,$$

$$D_T(r) = \frac{1}{3}\frac{N}{\chi} r^2 + \ldots = C_T^2 \lambda_0^{2/3}\left(\frac{r}{\lambda_0}\right)^2 + \ldots \quad (r \ll \lambda_0),$$

$$D_T(r) = a^2\frac{N}{\varepsilon^{1/3}} r^{2/3} = C_T^2 r^{2/3}, \text{ where } C_T^2 = \frac{a^2 N}{\varepsilon^{1/3}} \quad (\lambda_0 \ll r \ll L_0),$$

$$\Phi_T(\varkappa) = A C_T^2 \varkappa^{-11/3} \quad \left(\frac{2\pi}{L_0} \ll \varkappa \ll \frac{2\pi}{\lambda_0}\right),$$

$$A = 5\Gamma\left(\frac{5}{3}\right)\sin\frac{\pi}{3} \Big/ 12\pi^2 = 0.033.$$

For all $\varkappa \gg \frac{2\pi}{L_0}$ we use the approximation

$$\Phi_T(\varkappa) = A C_T^2 \varkappa^{-11/3} e^{-\varkappa^2/\varkappa_m^2}, \quad \varkappa_m\lambda_0 = 5.92.$$

Chapter 2

The effective scattering cross section per unit volume is

$$\sigma_0 = \frac{\pi}{2} k^4 \tilde{\Phi}_\varepsilon(\boldsymbol{k}_0 - \boldsymbol{k}_s) \sin^2\chi,$$

where $k = \frac{\omega}{c} = \frac{2\pi}{\lambda}$; $\boldsymbol{k}_0$ and $\boldsymbol{k}_s$ are the wave vectors of the incident and the scattered wave; $\Phi_\varepsilon(\boldsymbol{\varkappa})$ is the three-dimensional spatial spectral density of the dielectric constant fluctuations; χ is the angle between the electric vector of the incident wave and the vector $\boldsymbol{k}_s$; $\tilde{\Phi}_\varepsilon(\boldsymbol{\varkappa})$ is the function $\Phi_\varepsilon(\boldsymbol{\varkappa})$

averaged in the space of the vector $\varkappa$ over a volume $8\pi^3/V_{\text{ef}}$ around the point $\varkappa = k_0 - k_s$,

$$|k_0 - k_s| = 2k \sin\frac{\theta}{2},$$

where θ is the scattering angle ($k_0 k_s = k^2 \cos\theta$), $l(\theta) = \frac{2\pi}{|k_0 - k_s|} = \frac{\lambda}{2\sin\frac{\theta}{2}}$ is the spatial period of the Fourier component of the ε field producing the scattering. In the inertial range of the turbulence spectrum

$$\Phi_\varepsilon(\varkappa) = 0.033 C_\varepsilon^2 \varkappa^{-11/3} \quad \left(\frac{2\pi}{L_0} \ll \varkappa \ll \frac{2\pi}{\lambda_0}\right).$$

The correlation angle of the scattered field is $\Delta\theta \sim \lambda/L$, where L is the dimension of the scattering volume in the plane of the angle $\Delta\theta$.

The spatial correlation radius of the scattered field at two transversally separated points is $\Delta r \sim r\Delta\theta = \lambda r/L$, where r is the distance from the scattering volume to the observation point.

The frequency band width over which fields of different frequencies ($|f_1 - f_2| < \Delta f$) scattered at the same angle θ are correlated is given by

$$\Delta f \sim \frac{c}{2L \sin\left(\frac{\theta}{2}\right)}.$$

Δf also determines the passband of a transmission channel using scattering.

The effective scattering volume for narrow-beam antenna is

$$V_{\text{ef}} \approx \frac{d^3 \gamma_1^2 \gamma_2}{8\theta} \quad (\gamma_1, \gamma_2 \ll \theta \ll 1).$$

Here d is the distance from the transmitter to the receiver, γ_1, γ_2 are the effective beam widths between half-power points in the vertical and the horizontal planes.

For wide-beam antennas the effective scattering volume is $V_{\text{ef}} \sim d^3\theta_0^2$, where θ_0 is the minimum scattering angle corresponding to the lower part of the scattering volume. For narrow- and wide-beam antennas the L in the formulas for the correlation radii of the scattered field should be replaced by the appropriate expressions for V_{ef}.

Chapter 3

Geometrical optics

The conditions of applicability $\lambda \ll \lambda_0$, $\sqrt{\lambda L} \ll \lambda_0$, $\langle\chi^2\rangle \ll 1$, where L is the path length in the inhomogeneous medium, $\chi = \ln\frac{A}{A_0}$ is the log-amplitude.

General relations. Plane wave. The structure function of the eikonal $\theta = \frac{S}{k}$:

$$D_\theta(L, \eta, \xi) = \langle[\theta(L, \eta, \xi) - \theta(L, 0, 0)]^2\rangle =$$

$$= \frac{L}{2}\int_0^\infty [D_\varepsilon(\xi, \eta, \zeta) - D_\varepsilon(\xi, 0, 0)]\, d\xi = \pi L \iint_{-\infty}^{\infty} [1 - e^{i(\varkappa_2\eta + \varkappa_3\xi)}]\, \Phi_\varepsilon(0, \varkappa_2, \varkappa_3)\, d\varkappa_2\, d\varkappa_3.$$

For isotropic dielectric constant fluctuations

$$D_\theta(L, \eta, \zeta) = D_\theta(L, \rho), \quad \rho^2 = \eta^2 + \zeta^2,$$
$$D_\theta(L, \rho) = 2\pi^2 L \int_0^\infty [1 - J_0(\varkappa\rho)]\, \Phi_\varepsilon(\varkappa)\, \varkappa\, d\varkappa.$$

The mean square fluctuation of the direction of propagation in the x,y plane at a distance L from the source is

$$\langle\alpha^2\rangle = \frac{1}{2} \frac{\partial^2 D_\theta(L, \eta, 0)}{\partial\eta^2}\Big|_{\eta=0}.$$

For isotropic fluctuations

$$\langle\alpha^2\rangle = \frac{L}{4} \int_0^\infty \frac{D'_\varepsilon(\xi)}{\xi}\, d\xi.$$

The "longitudinal" correlation function of the fluctuations in the direction of propagation (the angle is reckoned in the x, y plane, the separation between the points is observed in the direction of the y axis)

$$B_\alpha(L, \rho) = \frac{1}{2} \frac{\partial^2 D_\theta(L,\rho)}{\partial\rho^2}.$$

The "transverse" correlation function of the fluctuations in the direction of propagation (the angle is reckoned in the x,y plane, the separation between the points is observed in the direction of the z axis)

$$B_\beta(L, \rho) = \frac{1}{2\rho} \frac{\partial D_\theta(L,\rho)}{\partial\rho}.$$

The cross correlation $B_{\alpha\beta}(\rho) = 0$. The correlation function of log-amplitude fluctuations in the plane $x = L$ is

$$B_\chi(L, \eta, \zeta) = \langle[\chi(L, \eta, \zeta) - \langle\chi\rangle][\chi(L, 0, 0) - \langle\chi\rangle]\rangle =$$
$$= -\frac{L^3}{48} \int_0^\infty \Delta_\perp^2 D_\varepsilon(\xi, \eta, \zeta)\, d\xi = -\frac{L^2}{24} \Delta_\perp^2 D_\theta(L, \eta, \zeta).$$

S p h e r i c a l w a v e. If the two observation points are at the same distance L from the point source and a distance b from each other, and the angle between the two rays is small, we have

$$\langle[\theta_1 - \theta_2]^2\rangle = \frac{1}{b} \int_0^b D_\theta(L, \rho)\, d\rho,$$

where $D_\theta(L,\rho)$ is the eikonal structure function for a plane wave. If the distance between the rays at one end of the path is b_1 and at the other end it is b_2 (the point source is situated outside the random medium), the length of each ray is L and the angle between the rays is small, we have

$$\langle[\theta_1 - \theta_2]^2\rangle = \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} D_\theta(L, |\rho|)\, d\rho$$

(the formula is applicable to intersecting rays, in which case b_1 is negative).

The log-amplitude correlation function of a spherical wave in the case of a small angle between the rays can be expressed in terms of the corresponding plane-wave correlation function:

$$[B_\chi(L, \rho)]_{\text{сф}} = 3\int_0^1 q^2(1-q)^2 B_\chi(L, \rho q)\,dq.$$

Relations for a turbulent medium. Plane wave.

$$D_\theta(L, \rho) = BC_\varepsilon^2 L\varkappa_m^{-5/3}\left[{}_1F_1\left(-\frac{5}{6}, 1, -\frac{\varkappa_m^2\rho^2}{4}\right)-1\right],$$

$$B = -\frac{5}{12}\Gamma\left(\frac{5}{3}\right)\Gamma\left(-\frac{5}{6}\right)\sin\frac{\pi}{3} = 2.2.$$

The asymptotic expression for $\rho \gg \lambda_0$ and $\rho \ll \lambda_0$:

$$D_\theta(L, \rho) \approx 0.73\, C_\varepsilon^2 L\rho^{5/3} \text{ and } D_\theta(L, \rho) \approx 0.82\, C_\varepsilon^2 L\lambda_0^{-1/3}\rho^2.$$

The mean square and the correlation functions of the fluctuations in the direction of propagation:

$$\langle\alpha^2\rangle = 0.82\, C_\varepsilon^2 L\lambda_0^{-1/3},$$

$$b_\beta(\rho) = \frac{B_\beta(L, \rho)}{B_\beta(L, 0)} = {}_1F_1\left(\frac{1}{6}, 2, -\frac{\varkappa_m^2\rho^2}{4}\right),$$

$$b_\alpha(\rho) = \frac{B_\alpha(L, \rho)}{B_\alpha(L, 0)} = {}_1F_1\left(\frac{1}{6}, 2, -\frac{\varkappa_m^2\rho^2}{24}\right) - \frac{\varkappa_m^2\rho^2}{24}\,{}_1F_1\left(\frac{7}{6}, 3, -\frac{\varkappa_m^2\rho^2}{4}\right).$$

The mean square and the correlation function of log-amplitude fluctuations:

$$\langle\chi^2\rangle = 0.80\, C_\varepsilon^2 L^3\lambda_0^{-1/3},$$

$$b_\chi(\rho) = \frac{B_\chi(L, \rho)}{B_\chi(L, 0)} = {}_1F_1\left(\frac{7}{6}, 1, -\frac{\varkappa_m^2\rho^2}{4}\right).$$

The mean square of the transverse displacement of the beam:

$$\langle(\delta x)^2\rangle = \frac{2}{3}L^2\langle\alpha^2\rangle = 0.55\, C_\varepsilon^2 L^3\lambda_0^{-1/3}.$$

Spherical wave.

$$[D_{\theta,\,\text{sph}}(L, \rho)] = \frac{1}{3}D_{\theta,\,\text{pl}}(L, \rho) \quad (\rho \ll \lambda_0),\ \langle\alpha_{\text{sph}}^2\rangle = \frac{1}{3}\langle\alpha_{\text{pl}}^2\rangle,$$

$$D_{\theta,\,\text{sph}}(L, \rho) = \frac{3}{8}D_{\theta,\,\text{pl}}(L, \rho) = 0.27 C_\varepsilon^2 L\rho^{5/3} \quad (\rho \gg \lambda_0),$$

$$\langle\chi_{\text{sph}}^2\rangle = \frac{1}{10}\langle\chi_{\text{pl}}^2\rangle = 0.080 C_\varepsilon^2 L^3\lambda_0^{-7/3}.$$

The limits of applicability of the first approximation: $\lambda^3 L \ll \lambda_0^4, \langle\chi^2\rangle \ll 1$. The relations for a plane wave in a turbulent medium are:

$$D_\chi(L,\rho) = \langle[\chi(L,\eta,\zeta) - \chi(L,0,0)]^2\rangle = \frac{D_1(L,\rho) + \operatorname{Re} D_2(L,\rho)}{2}$$
$$(\rho^2 = \eta^2 + \zeta^2),$$
$$D_S(L,\rho) = \langle[S(L,\eta,\zeta) - S(L,0,0)]^2\rangle = \frac{D_1(L,\rho) - \operatorname{Re} D_2(L,\rho)}{2},$$
$$D_{\chi S}(L,\rho) = \langle[\chi(L,\eta,\zeta) - \chi(L,0,0)]\,[S(L,\eta,\zeta) - S(L,0,0)]\rangle = \frac{1}{2}\operatorname{Im} D_2(L,\rho),$$
$$D_1(L,\rho) = \frac{6}{5}\pi^2 A C_\varepsilon^2 k^2 L \varkappa_m^{-5/3}\left[{}_1F_1\left(-\frac{5}{6}, 1, -g\right) - 1\right],$$
$$D_2(L,\rho) = -\frac{36}{55}\pi^2\Gamma\left(\frac{1}{6}\right) i A C_\varepsilon^2 k^2 L \varkappa_m^{-5/3}\frac{1}{D}\left\{{}_1F_1\left(-\frac{11}{6}, 1, -g\right) - 1 - (1+iD)^{11/6}\left[{}_1F_1\left(-\frac{11}{6}, 1, -\frac{g}{1+iD}\right) - 1\right]\right\},$$

where $g = \frac{\varkappa_m^2\rho^2}{4} = 8.8\frac{\rho^2}{\lambda_0^2}$, $D = \frac{\varkappa_m^2 L}{k} = 5.6\frac{\lambda L}{\lambda_0^2}$, $A = 0.033$.

For $D \ll 1$, we obtain the equations of geometrical optics

For

$$D \gg 1,\ g \ll 1\quad (\rho \ll \lambda_0)$$
$$D_1(L,\rho) = \frac{1}{4} A\pi^2\Gamma\left(\frac{1}{6}\right) C_\varepsilon^2 k^2 L \varkappa_m^{1/3}\rho^2 + \ldots,$$
$$D_2(L,\rho) = \frac{1}{4}\cdot\frac{6}{5} A\pi^2\Gamma\left(\frac{1}{6}\right) e^{11i\pi/12} C_\varepsilon^2 k^{13/6} L^{5/6}\rho^2 + \ldots,$$
$$D_\chi(L,\rho) \approx 0.41 C_\varepsilon^2 k^2 L \lambda_0^{-1/3}\left[1 - 0.87\left(\frac{\lambda_0^2}{\lambda L}\right)^{1/6}\right]\rho^2 + \ldots,$$
$$D_S(L,\rho) \approx 0.41\, C_\varepsilon^2 k^2 L \lambda_0^{-1/3}\left[1 + 0.87\left(\frac{\lambda_0^2}{\lambda L}\right)^{1/6}\right]\rho^2 + \ldots,$$
$$D_{\chi S}(L,\rho) \approx 0.071 C_\varepsilon^2 k^{13/6} L^{5/6}\rho^2 + \ldots$$

For

$$D \gg 1,\ g \gg 1\quad (\rho \gg \lambda_0)\qquad \langle\chi^2\rangle = 0.077 C_\varepsilon^2 k^{7/6} L^{11/6},$$
$$b_\chi(L,\rho) = \operatorname{Re}{}_1F_1\left(-\frac{11}{6}, 1, \frac{ik\rho^2}{4L}\right) - \cot\frac{\pi}{12}\operatorname{Im}{}_1F_1\left(-\frac{11}{6}, 1, \frac{ik\rho^2}{4L}\right) - \frac{1}{\frac{6}{11}\Gamma\left(\frac{11}{6}\right)\sin\frac{\pi}{12}}\left(\frac{k\rho^2}{4L}\right)^{5/6},$$
$$b_\chi(L,\rho) = 1 - 2.36\left(\frac{k\rho^2}{L}\right)^{5/6} + 1.71\frac{k\rho^2}{L} - 0.024\left(\frac{k\rho^2}{L}\right)^2 + \ldots$$
$$(\lambda_0 \ll \rho \ll \sqrt{\lambda L}),$$
$$b_\chi(L,\rho) = -0.122\left(\frac{k\rho^2}{L}\right)^{-7/6}\quad (\rho \gg \sqrt{\lambda L},)$$
$$D_S(L,\rho) = 0.73 C_\varepsilon^2 k^2 L \rho^{5/3} - 0.154 C_\varepsilon^2 k^{7/6} L^{11/6}\,[1 - b_\chi(L,\rho)],$$
$$D_S(L,\rho) = 0.73 C_\varepsilon^2 k^2 L \rho^{5/3}\quad (\rho \gg \sqrt{\lambda L}),$$
$$D_S(L,\rho) = \frac{1}{2}\cdot 0.73 C_\varepsilon^2 k^2 L \rho^{5/3}\quad (\lambda_0 \ll \rho \ll \sqrt{\lambda L}),$$
$$\langle\chi S\rangle = \cot\frac{\pi}{12}\langle\chi^2\rangle = 0.28\, C_\varepsilon^2 k^{7/6} L^{11/6},\quad \langle\alpha^2\rangle = 0.41 C_\varepsilon^2 L \lambda_0^{-1/3}.$$

For variable C_ε^2:

$$1.\ D \ll 1 \qquad (\sqrt{\lambda L} \ll \lambda_0), \quad \langle \chi^2 \rangle = 2.4\,\lambda_0^{-7/3} \int_0^L C_\varepsilon^2(x)\,(L-x)^2\,dx,$$

$$D_S(\rho) = \frac{6}{5}\pi^2 \Gamma\left(\frac{1}{6}\right) A k^2 \varkappa_m^{-5/3} \left[{}_1F_1\left(-\frac{5}{6}, 1, -g\right) - 1 \right] \int_0^L C_\varepsilon^2(x)\,dx,$$

$$2.\ D \gg 1 \qquad (\sqrt{\lambda L} \gg \lambda_0), \langle \chi^2 \rangle = 0.141\, k^{/6} \int_0^L C_\varepsilon^2(x)\,(L-x)^{5/6} dx.$$

$$D_S(\rho) = 0.73\, k^2 \rho^{5/3} \int_0^L C_\varepsilon^2(x)\,dx \quad (\rho \gg \sqrt{\lambda L}),$$

Chapter 4

Frequency spectra. Assuming the approximation of "frozen turbulence," which holds true for $f > \frac{v_\perp}{L_0}$, we get

$$R_\chi(\tau) = B_\chi(v_\perp \tau) = \int_0^\infty \cos(2\pi f \tau)\, W_\chi(f)\, df,$$

$$H_S(\tau) = D_S(v_\perp \tau) = 2 \int_0^\infty [1 - \cos 2\pi f \tau]\, W_S(f)\, df,$$

$$W_{\chi,S}(f) = \frac{8\pi}{v_\perp} \int_0^\infty F_{\chi,S}\left(\sqrt{\varkappa^2 + \frac{4\pi^2 f^2}{v_\perp^2}}, L \right) d\varkappa.$$

For a turbulent medium

$$W_{\chi,S}(f) = \frac{\pi}{2} A C_\varepsilon^2 k^{7/6} L^{11/6} f_0^{-1} \Omega^{-8/3} \left\{ N \mp \operatorname{Im} \frac{\sqrt{\pi} e^{i\Omega^2}}{\Omega^2} \times \right.$$
$$\left. \times \left[\frac{\Gamma\left(\frac{7}{3}\right)}{\Gamma\left(\frac{17}{6}\right)} {}_1F_1\left(\frac{1}{2}, -\frac{4}{3}, -i\Omega^2\right) + (-i\Omega^2)^{/3} \frac{\Gamma\left(-\frac{7}{3}\right)}{\sqrt{\pi}} \times {}_1F_1\left(\frac{17}{6}, \frac{10}{3}, -i\Omega^2\right) \right] \right\},$$

where $\Omega = \frac{f}{f_0}$, $f_0 = \frac{v_\perp}{\sqrt{2\pi\lambda L}}$, $N = \frac{\sqrt{\pi}\,\Gamma\left(\frac{4}{3}\right)}{\Gamma(11/6)} = 1.69$, $A = 0.033$.

The asymptotic expansions for high and low frequencies are

$$W_\chi(f) = 0.15 \langle \chi^2 \rangle \frac{1}{f_0} [1 + 0.48\, \Omega^{4/3} + \ldots] \quad (\Omega \ll 1),$$

$$W_\chi(f) = 1.14 \langle \chi^2 \rangle f_0^{-1} \Omega^{-8/3} \quad (\Omega \gg 1),$$

$$W_S(f) = 8.2 \cdot 10^{-3} C_\varepsilon^2 k^2 L v_\perp^{5/3} f^{-8/3} \quad (\Omega \ll 1),$$

$$W_S(f) = 4.1 \cdot 10^{-3} C_\varepsilon^2 k^2 L v_\perp^{5/3} f^{-8/3} \quad (\Omega \gg 1).$$

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EXPLANATORY LIST OF ABBREVIATED NAMES OF USSR INSTITUTIONS AND PERIODICALS APPEARING IN THIS BOOK

Abbreviation	Full name (transliterated)	Translation
DAN SSSR	Doklady Akademii Nauk SSSR	Reports of the Academy of Sciences of the USSR
GAO AN SSSR	Glavnaya Astronomicheskaya Observatoriya (Akademii Nauk SSSR)	Main Astronomical Observatory (of the Academy of Sciences of the USSR)
GeoFIAN	Geofizicheskii Institut Akademii Nauk SSSR	Geophysical Institute of the Academy of Sciences of the USSR
IFA AN SSSR	Institut Fiziki Atmosfery (Akademii Nauk SSSR)	Institute of Atmospheric Physics (of the Academy of Sciences of the USSR)
Izv.	Izvestiya	Bulletin
PMM	Prikladnaya Matematika i Mekhanika	Applied Mathematics and Mechanics
TsAO	Tsentral'naya Aerologicheskaya Observatoriya	Central Aerological Observatory
UMN	Uspekhi Matematicheskikh Nauk	Advances in Mathematical Sciences
ZhETF	Zhurnal Eksperimental'noi i Teoreticheskoi Fiziki	Journal of Experimental and Theoretical Physics
ZhTF	Zhurnal Tekhnicheskoi Fiziki	Journal of Technical Physics