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SPATIALLY HOMOGENEOUS AND EUCLIDEAN COSMOLOGICAL MODELS WITH SHEAR*

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ABSTRACT

Spatially homogeneous universes with the anisotropic, Euclidean metric $ds^2 = dt^2 - A^2(t)dx^2 - B^2(t)dy^2 - W^2(t)dz^2$ are studied in some detail. Matter with the equation of state $p_M = \gamma\rho_M$ and a uniform magnetic field, in various combinations, constitute the stress-energy tensor $T^{\mu\nu}$. The general solution for the case with no magnetic field is derived. This solution is used to construct several semi-realistic cosmological models of our Universe, in which are investigated (*a*) primordial element formation, (*b*) the time when anisotropies become small, and (*c*) possible temperature anisotropy of the observed 3° K cosmic microwave radiation.

I. INTRODUCTION AND SUMMARY

The discovery of the cosmic microwave radiation (Penzias and Wilson 1965; Dicke *et al.* 1965) and the investigation of its isotropy (Partridge and Wilkinson 1967; Conklin and Bracewell 1967; Penzias and Wilson 1967), the problem of the abundance of primordial helium (Wagoner 1967 and references cited therein), and the possibility of large-scale primordial magnetic fields (Thorne 1966*a*) have all stimulated recent investigations of anisotropic cosmological models of our Universe (Doroshkevich 1965, 1966; Hawking and Tayler 1966; Kantowski and Sachs 1966; Thorne 1967; Doroshkevich, Zel'dovich, and Novikov 1967; Misner 1967, 1968; Stewart and Ellis 1967). The work of Doroshkevich (1965) and Thorne (1967) on spatially homogeneous and Euclidean universes with shear but no rotation dealt primarily with the axisymmetric metric

$$ds^2 = dt^2 - A^2(t)(dx^2 + dy^2) - W^2(t)dz^2 . \tag{1}$$

In this paper I extend their work to the most general Bianchi type I metric,

$$ds^2 = dt^2 - A^2(t)dx^2 - B^2(t)dy^2 - W^2(t)dz^2 . \tag{2}$$

In § II the equations governing the time evolution of this metric are derived for a stress-energy tensor $T^{\mu\nu}$ consisting of matter with the equation of state $p_M = \gamma\rho_M$ and a uniform magnetic field. Some general properties of these equations are presented. In § III the general solution when there is no magnetic field present is given and is evaluated explicitly for the "dust universe" ($\gamma = 0$), the "radiation universe" ($\gamma = \frac{1}{3}$), the "hard universes" (some values of γ in the range $\frac{1}{3} < \gamma < 1$), and the "Zel'dovich universe" ($\gamma = 1$). The solution for the "dust-plus-radiation universe" ($\rho_M = \rho_D + \rho_R$, $p_M = \rho_R/3$) is found in § IV by suitably generalizing the equations of § III. § V is devoted to constructing semirealistic cosmological models of our Universe with no magnetic field. In these models I examine (*a*) primordial element formation, (*b*) the time when anisotropies become small, and (*c*) possible temperature anisotropy of the observed 3° K cosmic microwave radiation.

The approach used in this paper was developed independently by Misner (1967,

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1968). The general solution for the axisymmetric case ($A = B$ in eq. [2]) when there is no magnetic field was found independently by Stewart and Ellis (1967). The analytical solution for the "dust universe" ($\gamma = 0$) was found previously by Robinson (1961) and by Heckmann and Schücking (1962). The general solutions for the "radiation universe," the "hard universes," and "Zel'dovich universe," and the "dust-plus-radiation universe" are all new, so far as I know.

II. ANISOTROPIC UNIVERSES

a) The Equations

The context of this paper is the general relativistic, hot big-bang theory of cosmology. With the metric of equation (2), the Einstein field equations with vanishing cosmological constant, $G^{\mu}{}_{\nu} = 8\pi T^{\mu}{}_{\nu}$, give us[1]

$$ab + aw + bw = +8\pi(\rho_M + \rho_B)\,, \tag{3a}$$

$$(b' + w') + (b^2 + w^2) + bw = -8\pi(p_M + \rho_B)\,, \tag{3b}$$

$$(a' + w') + (a^2 + w^2) + aw = -8\pi(p_M + \rho_B)\,, \tag{3c}$$

$$(a' + b') + (a^2 + b^2) + ab = -8\pi(p_M - \rho_B)\,, \tag{3d}$$

where a prime denotes differentiation with respect to proper time t;

$$(a,b,w) = (A'/A,\ B'/B,\ W'/W) \tag{3e}$$

are the Hubble expansion rates along the (x,y,z)-axes; ρ_M and p_M are the total density of mass energy and the pressure of the co-moving matter with the equation of state $p_M = \gamma\rho_M$ $(0 \leq \gamma \leq 1)$; and $\rho_B = B_z^2/8\pi$ is the energy density of the uniform, co-moving magnetic field B_z directed along the z-axis. From the conservation equations, $T^{\mu}{}_{\nu;\mu} = 0$, we find

$$\rho_M/\rho_{M_0} = (A_0B_0W_0/ABW)^{1+\gamma}\,, \tag{4}$$

where the subscript zero denotes the value of a quantity at some fixed proper time. Conservation of magnetic flux in the (x,y)-plane gives us

$$\rho_B/\rho_{B_0} = (A_0B_0/AB)^2\,. \tag{5}$$

To clarify the structure of the system of equations (3)–(5), we make the change of variables

$$[A(t),\ B(t),\ W(t)] = R(t)\exp\,[\alpha(t),\ \beta(t),\ \omega(t)]\,, \tag{6}$$

with

$$\alpha(t) + \beta(t) + \omega(t) = 0\,. \tag{7}$$

We will call (A,B,W) the "expansion functions," R the "mean radius," and (α,β,ω) the "anisotropy functions." Equations (6) and (7) imply

$$ABW = R^3\,. \tag{8}$$

Finally, we define the independent anisotropy functions "perpendicular to" and "in" the (x,y)-plane by

$$(\eta,\sigma) = (\alpha + \beta,\ \alpha - \beta)\,. \tag{9}$$

[1] I use geometrized units, where $c = G = 1$.

Combining equations (3)–(9), we have our final system of field equations:[2]

$$3(R'/R)^2 - (3\eta'^2 + \sigma'^2)/4 = 8\pi[\rho_{M_0}(R/R_0)^{-3(1+\gamma)} + \rho_{B_0}(R/R_0)^{-4}e^{-2(\eta-\eta_0)}]\,, \tag{10a}$$

$$\eta'' + 3(R'/R)\eta' = (32\pi/3)\rho_{B_0}(R/R_0)^{-4}e^{-2(\eta-\eta_0)}\,, \tag{10b}$$

$$\sigma'' + 3(R'/R)\sigma' = 0\,. \tag{10c}$$

b) General Properties of the Equations

Equation (10c), or, equivalently, equations (3b) and (3c), implies

$$(a - b)R^3 = \text{constant}\,. \tag{11}$$

This means that as $R \to \infty$ the expansion rate becomes *isotropic* in the (x,y)-plane (i.e., in the plane perpendicular to the direction of the magnetic field). From equations (3b)–(3d), we find

$$[(a - w)R^3]' = [(b - w)R^3]' = 16\pi\rho_B R^3\,. \tag{12}$$

Since ρ_B varies as R^{-4} as $R \to \infty$, equation (12) implies that we have isotropic expansion in *all* directions as $R \to \infty$. The only exception occurs when all the Hubble rates (a,b,w) are proportional to R^{-3} for all time and the magnetic field vanishes identically for all time. Then we need not, and indeed do not, attain isotropy as $R \to \infty$ (see § III*e* for more details).

When there is no magnetic field, equations (3) give us the simple non-linear differential equation

$$(R''/R) + 2(R'/R)^2 = 4\pi\rho_{M0}(1 - \gamma)(R/R_0)^{-3(1+\gamma)}\,. \tag{13}$$

Solving this equation yields the "mean radius" R and the "mean Hubble expansion rate" (R'/R) as functions of proper time. Equation (13) is very useful in numerical calculations and is easily solved analytically for certain discrete values of γ.

III. SOLUTIONS WITHOUT A MAGNETIC FIELD

a) The General Solution

I have been unable to solve equations (10) analytically with a magnetic field. Consequently, I am at present integrating those equations numerically to see the possible effects of a uniform primordial magnetic field upon the evolution of our Universe (see Thorne 1967).

In the absence of a magnetic field, equations (10) have the following *general* solution:[3]

$$x_M = \begin{cases} [(1+\gamma)/(1-\gamma)]\int (y^2 - \Omega_0)^{\gamma/(1-\gamma)}dy & (0 \le \gamma < 1) \\ (1+\Omega_0)^{-1/2}(R/R_0)^3 & (\gamma = 1) \end{cases}\,, \tag{14a}$$

$$(\sigma - \sigma_0)/\sigma_0' = (\eta - \eta_0)/\eta_0'$$
$$= \begin{cases} [(1+\gamma)/(1-\gamma)](\tau_M/2\Omega_0^{1/2})\ln|(y - \Omega_0^{1/2})/(y + \Omega_0^{1/2})| & (0 \le \gamma < 1) \\ [3\tau_M(1+\Omega_0)^{-1/2}]\ln(R/R_0) & (\gamma = 1) \end{cases}\,, \tag{14b}$$

[2] This approach was developed independently by Misner (1967, 1968).

[3] The general solution in the axisymmetric case was found independently by Stewart and Ellis (1967).

where

$$y = [(R/R_0)^{3(1-\gamma)} + \Omega_0]^{1/2}, \tag{15a}$$

$$x_M = \text{normalized time} = (t + t_M^*)/\tau_M, \tag{15b}$$

$$\tau_M = \text{time scale} = (1+\gamma)^{-1}(6\pi\rho_{M_0})^{-1/2}, \tag{15c}$$

$$\Omega_0 = \text{anisotropy parameter} = [3^{1/2}(1+\gamma)\tau_M/4]^2(3\eta_0'^2 + \sigma_0'^2). \tag{15d}$$

Note that equation (14a) is the general solution to equation (13). The anisotropy parameter Ω_0 lies in the range $0 \leq \Omega_0 < \infty$. When $\Omega_0 = 0$, we recover the standard (homogeneous, isotropic), Euclidean Friedmann universes:

$$(R/R_0) = x_M^{2/3(1+\gamma)}. \tag{16}$$

The anisotropic universes of equations (14) may be characterized by *two* independent "anisotropy parameters." Using equation (15d), we define the first by

$$\epsilon_M = -2\Omega_0^{1/2} \quad (0 \leq |\epsilon_M| < \infty). \tag{17a}$$

The second anisotropy parameter $\psi (0 \leq \psi < 2\pi/3)$ appears when we satisfy equations (7), (15d), and (17a) by

$$\tau_M(\alpha_0', \beta_0', \omega_0') = \mp[2/3(1+\gamma)]|\epsilon_M|\mathrm{X}, \tag{17b}$$

where

$$\mathrm{X} \equiv \sin(\psi, \psi + 2\pi/3, \psi + 4\pi/3). \tag{17c}$$

Finally, let us express the general solution (14) in terms of our original variables (see eqs. [6]–[9]). Equations (14a) and (15a) give the time dependence of $R(t)$ implicitly. The "expansion functions" (A,B,W) of equation (2) are recovered using equations (6), (14b), and (17):

$$(A/A_0, B/B_0, W/W_0) = (R/R_0)\left\{\frac{[4(R/R_0)^{3(1-\gamma)} + \epsilon_M^2]^{1/2} + |\epsilon_M|}{[4(R/R_0)^{3(1-\gamma)} + \epsilon_M^2]^{1/2} - |\epsilon_M|}\right\}^{\pm[2/3(1-\gamma)]\mathrm{X}} \quad (0 \leq \gamma < 1), \tag{18}$$

$$\log(A/A_0, B/B_0, W/W_0) = \{1 \mp [2|\epsilon_M|/(4+\epsilon_M^2)^{1/2}]\mathrm{X}\}\log(R/R_0) \quad (\gamma = 1).$$

We obtain the Hubble expansion rates (a,b,w) from equations (14), (15), and (18):

$$(a,b,w) = [3(1+\gamma)\tau_M(R/R_0)^3]^{-1}\{[4(R/R_0)^{3(1-\gamma)} + \epsilon_M^2]^{1/2} \mp 2|\epsilon_M|\mathrm{X}\} \quad (0 \leq \gamma \leq 1). \tag{19}$$

In equations (17b)–(19), the upper/lower sign is for $\epsilon_M \gtrless 0$; and $(\psi, \psi + 2\pi/3, \psi + 4\pi/3)$ correspond to (A,B,W). From equation (4) the total density of mass energy ρ_M is given by

$$\rho_M/\rho_{M_0} = (R/R_0)^{-3(1+\gamma)}. \tag{20}$$

In the following four subsections we examine in detail four classes of universes for which the integration of equation (14a) can be performed explicitly (i.e., for which the time dependence of the "mean radius" $R(t)$ may be expressed analytically).

b) The Dust Universe

The "dust universe" is defined by $\gamma = 0$ (i.e., $p_D = 0$); the subscript D, which replaces the subscript M of previous sections, denotes "dust." The explicit time dependence of $R(t)$ may be written as

$$R/R_0 = [x_D(x_D + |\epsilon_D|)]^{1/3} , \tag{21}$$

where

$$x_D = \text{normalized time} = (t + t_D^*)/\tau_D , \tag{22a}$$

$$\tau_D = \text{time scale} = (6\pi\rho_{D_0})^{-1/2} . \tag{22b}$$

The independent anisotropy functions of equation (14b) become

$$\sigma/\sigma_0' = \eta/\eta_0' = (\tau_D/|\epsilon_D|) \ln |x_D/(x_D + |\epsilon_D|)| . \tag{23}$$

The anisotropy parameters of equations (17) satisfy

$$|\epsilon_D| = 2\Omega_0^{1/2} = [3\tau_D^2(3\eta_0'^2 + \sigma_0'^2)/4]^{1/2} \quad (0 \leq |\epsilon_D| < \infty) , \tag{24a}$$

$$\tau_D(\alpha_0', \beta_0', \omega_0') = \mp(2|\epsilon_D|/3)\mathrm{X} . \tag{24b}$$

The expansion functions (A,B,W), the Hubble expansion rates (a,b,w), and the total mass density ρ_D follow from equations (18)–(20) as

$$(A,B,W) = [x_D(x_D + |\epsilon_D|)]^{1/3}[(x_D + |\epsilon_D|)/x_D]^{\pm 2\mathrm{X}/3} , \tag{25a}$$

$$(a,b,w) = [3\tau_D x_D(x_D + |\epsilon_D|)]^{-1}[2x_D + |\epsilon_D|(1 \mp 2\mathrm{X})] , \tag{25b}$$

$$\rho_D/\rho_{D_0} = [x_D(x_D + |\epsilon_D|)]^{-1} . \tag{25c}$$

In equations (24b)–(25b) the upper/lower sign is for $\epsilon_D \gtrless 0$. This solution for the "dust universe" was found previously by Robinson (1961) and by Heckmann and Schücking (1962).

The "dust universe" emerges from a physical singularity (big-bang creation) at $t = -t_D^*$. In the early stages ($x_D \lesssim |\epsilon_D|$) its expansion rate is highly anisotropic, but as $x_D \to \infty$ it becomes isotropic with $(A,B,W) \approx R \approx x_D^{2/3}$. If $\epsilon_D > 0$ and $\psi \neq \pi/6$, or $\epsilon_D < 0$ and $\psi \neq \pi/2$, the initial singularity is of the CIGAR type (e.g., $A \to \infty$, B and $W \to 0$). If $\epsilon_D > 0$ and $\psi = \pi/6$, or $\epsilon_D < 0$ and $\psi = \pi/2$, the singularity is of the PANCAKE type (e.g., $A \to 0$, B and $W \to$ constants). This universe is axisymmetric ($A = B$ at all times; cases studied by Doroshkevich 1965, Thorne 1967, and Stewart and Ellis 1967) if $\psi = \pi/6$ or $\pi/2$ (all ϵ_D); and it is completely isotropic ($A = B = W$ at all times; standard, Euclidean Friedmann case) for $\epsilon_D = 0$ (all ψ).

c) The Radiation Universe

The "radiation universe" is characterized by $\gamma = \frac{1}{3}$ (i.e., $p_R = \rho_R/3$), where the subscript R denotes "radiation." This universe is filled with either massless particles (photons, neutrinos, or gravitons) with velocities distributed isotropically, or an extreme-relativistic gas of massive particles with isotropic velocities. The analytical solution for this universe may be written in two different, but equally useful and equivalent, forms.

The first form follows directly from the results of § III*a*. The analytical form of $R(t)$ is

$$x_R = \tfrac{1}{2}\{(R/R_0)F - (\epsilon_R^2/2) \ln |[2(R/R_0) + F]/|\epsilon_R||\} , \tag{26}$$

where

$$F = [4(R/R_0)^2 + \epsilon_R^2]^{1/2} , \tag{27a}$$

$$x_R = \text{normalized time} = (t + t_R^*)/\tau_R , \tag{27b}$$

$$\tau_R = \text{time scale} = (3/32\pi\rho_R)^{1/2}. \tag{27c}$$

The independent anisotropy functions of equation (14b) are

$$\frac{\sigma - \sigma_0}{\sigma_0'} = \frac{\eta - \eta_0}{\eta_0'} = \frac{2\tau_R}{|\epsilon_R|} \ln \left| \left(\frac{F - |\epsilon_R|}{F + |\epsilon_R|}\right)\left[\frac{(4 + \epsilon_R^2)^{1/2} + |\epsilon_R|}{(4 + \epsilon_R^2)^{1/2} - |\epsilon_R|}\right] \right| . \tag{28}$$

The anisotropy parameters of equations (17) satisfy

$$|\epsilon_R| = 2\Omega_0^{1/2} = [4\tau_R^2(3\eta_0'^2 + \sigma_0'^2)/3]^{1/2} \qquad (0 \leq |\epsilon_R| < \infty) , \tag{29a}$$

$$\tau_R(\alpha_0', \beta_0', \omega_0') = \mp(|\epsilon_R|/2)\mathrm{X} . \tag{29b}$$

From equations (18)–(20), we have

$$(A/A_0, B/B_0, W/W_0) = (R/R_0)\left\{\left(\frac{F + |\epsilon_R|}{F - |\epsilon_R|}\right)\left[\frac{(4 + \epsilon_R^2)^{1/2} - |\epsilon_R|}{(4 + \epsilon_R^2)^{1/2} + |\epsilon_R|}\right]\right\}^{\pm \mathrm{X}} , \tag{30a}$$

$$(a,b,w) = (4\tau_R)^{-1}(R/R_0)^{-3}(F \mp 2|\epsilon_R|\mathrm{X}) , \tag{30b}$$

$$\rho_R/\rho_{R_0} = 3p_R/\rho_{R_0} = (R/R_0)^{-4} . \tag{30c}$$

In equations (29b)–(30b), the upper/lower sign is for $\epsilon_R \gtrless 0$. This solution is new.

The "radiation universe" emerges from a physical singularity at $t = -t_R^*$. In its early stages ($R/R_0 \lesssim |\epsilon_R|$) the expansion rate is highly anisotropic, but as $x_R \to \infty$ it becomes isotropic with $(A/A_0, B/B_0, W/W_0) \approx R/R_0 \approx x_R^{1/2}$. The types of singularities are *exactly the same* as for the "dust universe." The "radiation universe" is axisymmetric (see eqs. [A4] of Thorne 1967) if $\psi = \pi/6$ or $\pi/2$ (all ϵ_R); and it is completely isotropic for $\epsilon_R = 0$ (all ψ).

The second, completely equivalent, form of the solution for the "radiation universe" consists of two parts. The arbitrariness of coordinates in general relativity permits us to set

$$\Omega_0 = +1 , \quad R/R_0 = 2\xi/(\xi^2 - 1) . \tag{31}$$

From equation (31) the range of ξ is $1 \leq \xi < \infty$. This transformation gives us the $\epsilon_R > 0$ part of our previous solution, and equations (26) and (30) take the form

$$x_R = 2\{[\xi(\xi^2 + 1)/(\xi^2 - 1)^2] - \tfrac{1}{2} \ln |(\xi + 1)/(\xi - 1)|\} , \tag{32a}$$

$$(A/A_0, B/B_0, W/W_0) = [\xi/(\xi^2 - 1)]\xi^{+2\mathrm{X}} , \tag{32b}$$

$$(a,b,w) = (16\tau_R)^{-1}[(\xi^2 - 1)/\xi]^3\{[(\xi^2 + 1)/(\xi^2 - 1)] - 2\mathrm{X}\} , \tag{32c}$$

$$\rho_R/\rho_{R_0} = 3p_R/\rho_{R_0} = \tfrac{1}{16}[(\xi^2 - 1)/\xi]^4 . \tag{32d}$$

This solution emerges from a physical singularity at $\xi = \infty$. The expansion rate is highly anisotropic for $\xi \gg 1$, and becomes isotropic as $\xi \to 1$. If $\psi \neq \pi/6$, the initial singularity is of the CIGAR type. If $\psi = \pi/6$, the singularity is PANCAKE. This universe is axisymmetric (see Doroshkevich 1965) if $\psi = \pi/6$ or $\pi/2$. This representation of the solution is peculiar in that it *cannot* describe the limiting case of the standard, isotropic Friedmann solution.

The second part of this representation, in which we reproduce the $\epsilon_R < 0$ part of the "radiation universe" solution, arises when we set

$$\Omega_0 = +1 , \quad R/R_0 = 2\xi/(1 - \xi^2) . \tag{33}$$

The range of ξ is now $0 \leq \xi \leq 1$. Equations (26) and (30) become

$$x_R = 2\{[\xi(\xi^2 + 1)/(1 - \xi^2)^2] - \tfrac{1}{2} \ln |(1 + \xi)/(1 - \xi)|\} , \quad (34a)$$

$$(A/A_0, B/B_0, W/W_0) = [\xi/(1 - \xi^2)]\xi^{\pm 2\mathrm{X}} , \quad (34b)$$

$$(a,b,w) = (16\tau_R)^{-1}[(1 - \xi^2)/\xi]^3\{[(1 + \xi^2)/(1 - \xi^2)] - 2\mathrm{X}\} , \quad (34c)$$

$$\rho_R/\rho_{R_0} = 3p_R/\rho_{R_0} = \tfrac{1}{16}[(1 - \xi^2)/\xi]^4 . \quad (34d)$$

We emerge from a physical singularity at $\xi = 0$. When $\xi \ll 1$, the expansion rate is highly anisotropic, and as $\xi \rightarrow 1$ it becomes isotropic. If $\psi \neq \pi/2$, the singularity is of the CIGAR type. If $\psi = \pi/2$, the singularity is PANCAKE. When $\psi = \pi/6$ or $\pi/2$, this universe is axisymmetric (see Doroshkevich 1965). Again we find that this representation *cannot* describe the limiting isotropic case.

d) The Hard Universes

Equation (14a) could *not* be integrated explicitly for *all* γ in the range $0 \leq \gamma \leq 1$, but two infinite sequences of explicit solutions were found in the range $\frac{1}{3} < \gamma < 1$. I call these the "hard universes." They are characterized by $\rho_H/3 < p_H < \rho_H$, where the subscript H denotes "hard" throughout this subsection.

One infinite sequence of analytical solutions results when we set

$$\gamma/(1 - \gamma) = \text{integer} = n \quad (0 \leq n < \infty) . \quad (35a)$$

Then we have the following sequence of γ's:

$$\gamma = n/(n + 1) = 0, \tfrac{1}{2}, \tfrac{2}{3}, \tfrac{3}{4}, \ldots . \quad (35b)$$

The solution to equations (14) becomes (see Gröbner and Hofreiter 1949, p. 18)

$$x_H = \left(\frac{2n + 1}{4}\right)\left[4\left(\frac{R}{R_0}\right)^{3/(n+1)} + \epsilon_H^2\right]^{1/2} \sum_{\nu=0}^{n} (-1)^\nu \left(\frac{|\epsilon_H|}{2}\right)^{2\nu} \frac{n!}{(n - \nu)!} \times \frac{(n - \nu + \frac{1}{2})!}{(n + \frac{1}{2})!} \left(\frac{R}{R_0}\right)^{3(n-\nu)/(n+1)} , \quad (36a)$$

$$\frac{\sigma}{\sigma_0'} = \frac{\eta}{\eta_0'} = (2n + 1) \frac{\tau_H}{|\epsilon_H|} \ln \left|\frac{[4(R/R_0)^{3/(n+1)} + \epsilon_H^2]^{1/2} - |\epsilon_H|}{[4(R/R_0)^{3/(n+1)} + \epsilon_H^2]^{1/2} + |\epsilon_H|}\right| , \quad (36b)$$

where

$$x_H = \text{normalized time} = (t + t_H^*)/\tau_H , \quad (37a)$$

$$\tau_H = \text{time scale} = [(n + 1)/(2n + 1)](6\pi\rho_{H0})^{-1/2} . \quad (37b)$$

The anisotropy parameters of equations (17) satisfy

$$|\epsilon_H| = 2\Omega_0^{1/2} = [(2n + 1)/(n + 1)][3\tau_H^2(3\eta_0'^2 + \sigma_0'^2)/4]^{1/2} \quad (0 \leq |\epsilon_H| < \infty) , \quad (38a)$$

$$\tau_H(\alpha_0',\beta_0',\omega_0') = \mp[2(n + 1)/3(2n + 1)]|\epsilon_H|\mathrm{X} . \quad (38b)$$

From equations (18)–(20) we find

$$(A/A_0, B/B_0, W/W_0) = (R/R_0)\left\{\frac{[4(R/R_0)^{3/(n+1)} + \epsilon_H^2]^{1/2} + |\epsilon_H|}{[4(R/R_0)^{3/(n+1)} + \epsilon_H^2]^{1/2} - |\epsilon_H|}\right\}^{\pm 2(n+1)\mathrm{X}/3} , \quad (39a)$$

$$(a,b,w) = [(n+1)/3(2n+1)\tau_H](R/R_0)^{-3}\{[4(R/R_0)^{3/(n+1)} + \epsilon_H{}^2]^{1/2} \mp 2|\epsilon_H|\mathrm{X}\}\,, \tag{39b}$$

$$\rho_H/\rho_{H_0} = (n+1)p_H/n\rho_{H_0} = (R/R_0)^{-3(2n+1)/(n+1)}\,. \tag{39c}$$

In equations (38b)–(39b), the upper/lower sign is for $\epsilon_H \gtrless 0$. For $n \geq 1$ this series of solutions is new.

This first sequence of "hard universes" emerges from an initial physical singularity at $R/R_0 = 0$. In the early stages $(R/R_0 \lesssim |\epsilon_H|^{2(n+1)/3})$, the expansion rate is highly anisotropic, but as $x_H \to \infty$ it becomes isotropic with $(A,B,W) \approx R \approx x_H{}^{2(n+1)/3(2n+1)}$. The types of singularities are *exactly the same* as for the "dust universe" (see § III*b*). These universes are axisymmetric if $\psi = \pi/6$ or $\pi/2$ (all ϵ_H); and they are completely isotropic for all time when $\epsilon_H = 0$ (all ψ).

The second infinite sequence of analytical "hard universes" appears when we set

$$\gamma/(1-\gamma) = \text{integer} + \tfrac{1}{2} = m + \tfrac{1}{2} \quad (0 \leq m < \infty)\,. \tag{40a}$$

This gives the following sequence of γ's:

$$\gamma = (2m+1)/(2m+3) = \tfrac{1}{3}, \tfrac{3}{5}, \tfrac{5}{7}, \ldots\,. \tag{40b}$$

The solution to equations (14) is found to be (see Gröbner and Hofreiter 1949, pp. 36–38)

$$x_H = \frac{2(2m+1)!!}{m!}\Bigg[\!\!\Bigg[\tfrac{1}{4}\Big(\frac{R}{R_0}\Big)^{3/(2m+3)}\Big[4\Big(\frac{R}{R_0}\Big)^{6/(2m+3)} + \epsilon_H{}^2\Big]^{1/2} \times \Big[\sum_{\nu=0}^{m}(-\tfrac{1}{2})^{\nu}\frac{(m-\nu)!}{(2m+1-2\nu)!!}\Big(\frac{|\epsilon_H|}{2}\Big)^{2\nu}\Big(\frac{R}{R_0}\Big)^{6(m-\nu)/(2m+3)}\Big] + (-\tfrac{1}{2})^{m+1}\Big(\frac{|\epsilon_H|}{2}\Big)^{2m+2}\ln\Big\{\tfrac{1}{2}\Big[4\Big(\frac{R}{R_0}\Big)^{6/(2m+3)} + \epsilon_H{}^2\Big]^{1/2} + \Big(\frac{R}{R_0}\Big)^{3/(2m+3)}\Big\}\Bigg]\!\!\Bigg]\,, \tag{41a}$$

$$\frac{\sigma}{\sigma_0{}'} = \frac{\eta}{\eta_0{}'} = 2(m+1)\Big(\frac{\tau_H}{|\epsilon_H|}\Big)\ln\Big|\frac{[4(R/R_0)^{6/(2m+3)} + \epsilon_H{}^2]^{1/2} - |\epsilon_H|}{[4(R/R_0)^{6/(2m+3)} + \epsilon_H{}^2]^{1/2} + |\epsilon_H|}\Big|\,, \tag{41b}$$

where

$$x_H = \text{normalized time} = (t + t_H{}^*)/\tau_H\,, \tag{42a}$$

$$\tau_H = \text{time scale} = [(2m+3)/4(m+1)](6\pi\rho_{H_0})^{-1/2}\,. \tag{42b}$$

The anisotropy parameters of equations (17) satisfy

$$|\epsilon_H| = 2\Omega_0{}^{1/2} = [(m+1)/(2m+3)][12\tau_H{}^2(3\eta_0{}'^2 + \sigma_0{}'^2)]^{1/2} \quad (0 \leq |\epsilon_H| < \infty)\,, \tag{43a}$$

$$\tau_H(\alpha_0{}', \beta_0{}', \omega_0{}') = \mp[(2m+3)/6(m+1)]|\epsilon_H|\mathrm{X}\,. \tag{43b}$$

Equations (18)–(20) for the expansion functions (A,B,W), the Hubble expansion rates (a,b,w), and the total mass density ρ_H are

$$(A/A_0, B/B_0, W/W_0) = (R/R_0)\Big\{\frac{[4(R/R_0)^{6/(2m+3)} + \epsilon_H{}^2]^{1/2} + |\epsilon_H|}{[4(R/R_0)^{6/(2m+3)} + \epsilon_H{}^2]^{1/2} - |\epsilon_H|}\Big\}^{\pm[(2m+3)/3]\mathrm{X}}\,, \tag{44a}$$

$$(a,b,w) = [(2m+3)/12(m+1)\tau_H](R/R_0)^{-3} \times \{[4(R/R_0)^{6/(2m+3)} + \epsilon_H{}^2]^{1/2} \mp 2|\epsilon_H|\mathrm{X}\}\,, \tag{44b}$$

$$\rho_H/\rho_{H_0} = (2m+3)p_H/(2m+1)\rho_{H_0} = (R/R_0)^{-12(m+1)/(2m+3)} . \quad (44c)$$

In equations (43b)–(44b), the upper/lower sign is for $\epsilon_H \gtrless 0$. For all m, this series of analytical solutions is new.

This second sequence of "hard universes" emerges from a physical singularity at $R/R_0 = 0$. The expansion rate is highly anisotropic in the early stages ($R/R_0 \lesssim |\epsilon_H|^{(2m+3)/3}$) but becomes isotropic with $(A,B,W) \approx R \approx x_H^{(2m+3)/6(m+1)}$ as $x_H \to \infty$. We find *exactly the same* behavior as in the first infinite sequence of solutions (see eqs. [35]–[39]) with respect to types of initial singularity, the axisymmetric cases, and the limiting case of isotropy for all time.

e) The Zel'dovich Universe

Zel'dovich (1961; see also Harrison 1965) discussed the possibility of matter with the equation of state $p_Z = \rho_Z$ (i.e., $\gamma = 1$), where the subscript Z denotes "Zel'dovich" throughout this subsection. This is the "hardest" equation of state permitted by causality (Harrison *et al.* 1965). The analytical solution for the "Zel'dovich universe" is greatly simplified if we take as our first anisotropy parameter δ in place of ϵ_Z, where

$$\delta = [\Omega_0/(1+\Omega_0)]^{1/2} = -\epsilon_Z(4+\epsilon_Z^2)^{-1/2} \quad (0 \le |\delta| < 1) . \quad (45)$$

Then the solution to equations (14) is

$$R/R_0 = x_Z^{1/3} , \quad (46a)$$

$$\sigma/\sigma_0' = \eta/\eta_0' = \tau_Z \ln |x_Z| , \quad (46b)$$

where

$$x_Z = \text{normalized time} = (t + t_Z^*)/\tau_Z , \quad (47a)$$

$$\tau_Z = \text{rationalized time scale} = [(1-\delta^2)/24\pi\rho_{Z_0}]^{1/2} . \quad (47b)$$

Our two independent anisotropy parameters now satisfy

$$\delta = \{1 + [4(1-\delta^2)/3\tau_Z^2(3\eta_0'^2 + \sigma_0'^2)]\}^{-1/2} \quad (0 \le |\delta| < 1) , \quad (48a)$$

$$\tau_Z(\alpha_0', \beta_0', \omega_0') = \pm(2|\delta|/3)\mathrm{X} . \quad (48b)$$

From equations (18)–(20), we find

$$(A/A_0,\ B/B_0,\ W/W_0) = x_Z^{1/3} x_Z^{\pm(2|\delta|/3)\mathrm{X}} , \quad (49a)$$

$$(a,b,w) = [3(t+t_Z^*)]^{-1}(1 \pm 2|\delta|\mathrm{X}) , \quad (49b)$$

$$\rho_Z/\rho_{Z_0} = p_Z/\rho_{Z_0} = x_Z^{-2} . \quad (49c)$$

In equations (48b)–(49b), the upper/lower sign is for $\delta \gtrless 0$. This solution for the "Zel'dovich universe" is new.

The "Zel'dovich universe" emerges from an initial singularity (big-bang creation) at $t = -t_Z^*$. The expansion rate is *always* highly anisotropic, even as $t \to \infty$. The expansion rate need not, and indeed does not, become isotropic as $t \to \infty$, because the Hubble rates (a,b,w) of equation (49b) are proportional to $(R/R_0)^{-3}$ for all time (see the discussion of § II*b*). This universe reduces to the standard axisymmetric solution (see Doroshkevich 1965) when $\psi = \pi/6$ or $\pi/2$ (all δ). It is completely isotropic for all time when $\delta = 0$ (all ψ). This universe exhibits initial singularities of the CIGAR, POINT (e.g., A, B, and $W \to 0$), and BARREL (e.g., A = constant, B and $W \to 0$) types, but has *no* PANCAKE singularities. Table 1 displays the possible types of singularities and the ranges of δ and ψ within which each type is found.

IV. THE DUST-PLUS-RADIATION UNIVERSE

By suitably generalizing the equations of §§ II and III*a*, we may consider universes with the anisotropic metric of equation (2) which contain non-interacting mixtures of several types of matter with the general equation of state $p_M = \gamma \rho_M$. To illustrate this extension, I have solved the case for universes which contain *both* dust and radiation simultaneously. These universes are anisotropic generalizations of the isotropic model discussed previously by Chernin (1965) and Jacobs (1967) (for similar isotropic models of this type see Alpher and Herman 1949; Alpher, Gamow, and Herman 1967).

TABLE 1

TYPES OF SINGULARITY IN THE ZEL'DOVICH UNIVERSE*

	CIGAR†	POINT‡	BARREL§
$3^{-1/2}<\delta<1$	$\begin{Bmatrix} 0\leq\psi<2\pi/3-\psi_0 \\ \pi/3+\psi_0<\psi<2\pi/3 \end{Bmatrix}$	$2\pi/3-\psi_0<\psi<\pi/3+\psi_0$	$\begin{cases} \psi=2\pi/3-\psi_0 \\ \psi=\pi/3+\psi_0 \end{cases}$
$1/2<\delta\leq 3^{-1/2}$	$\psi_0-\pi/3<\psi<2\pi/3-\psi_0$	$\begin{cases} 0\leq\psi<\psi_0-\pi/3 \\ 2\pi/3-\psi_0<\psi<2\pi/3 \end{cases}$	$\psi=\psi_0-\pi/3$; $\psi=2\pi/3-\psi_0$
$\|\delta\|=1/2$	...	$\begin{cases} \text{All } \psi \text{ except} \\ \psi=\pi/6\ (\delta>0), \\ \psi=\pi/2\ (\delta<0) \end{cases}$	$\psi=\pi/6\ (\delta>0)$; $\psi=\pi/2\ (\delta<0)$
$0\leq\|\delta\|<1/2$	...	All ψ	...
$-3^{-1/2}<\delta<-1/2$	$\psi_0<\psi<\pi-\psi_0$	$\begin{cases} 0\leq\psi<\psi_0 \\ \pi-\psi_0<\psi<2\pi/3 \end{cases}$	$\psi=\psi_0$; $\psi=\pi-\psi_0$
$\delta=-3^{-1/2}$	$\pi/3<\psi<2\pi/3$	$0<\psi<\pi/3$	$\begin{cases} \psi=0 \\ \psi=\pi/3 \end{cases}$
$-1<\delta<-3^{-1/2}$	$\begin{Bmatrix} 0\leq\psi<\pi/3-\psi_0 \\ \psi_0<\psi<2\pi/3 \end{Bmatrix}$	$\pi/3-\psi_0<\psi<\psi_0$	$\begin{cases} \psi=\pi/3-\psi_0 \\ \psi=\psi_0 \end{cases}$

* The effective range of ψ is $0 \leq \psi < 2\pi/3$ We define ψ_0 by $\psi_0 = \arcsin(1/2|\delta|)$, and it has the range $\pi/6 < \psi_0 < \pi/2$

† As we approach the singularity, $A \to \infty$, B and $W \to 0$.

‡ As we approach the singularity, A, B, and W all $\to 0$

§ As we approach the singularity, A = constant, B and $W \to 0$.

In the "dust-plus-radiation universe" the total density of mass energy and the total pressure are given by

$$\rho_M = \rho_D + \rho_R \,, \tag{50a}$$

$$p_M = p_R = \rho_R/3 \,, \tag{50b}$$

where the subscripts M, D, and R denote, respectively, "matter," "dust," and "radiation." Linearity of the conservation equations, $T^{\mu}{}_{\nu;\mu} = 0$, implies

$$\rho_D/\rho_{D_0} = (R/R_0)^{-3} \quad \text{and} \quad \rho_R/\rho_{R_0} = (R/R_0)^{-4} \,. \tag{51}$$

Equations (3), (6)–(9), and (10b)–(12) of § II remain unchanged, and the Einstein field equations for the "dust-plus-radiation universe" can be solved by precisely the same procedures as were used in §§ II and III. In place of equations (14), the general solution becomes (using eqs. [6]–[9], [10b, c], and the generalization of eq. [10a])

$$x = \tfrac{3}{2}\int (R/R_0)^2 \{(R/R_0)^2[(R/R_0) + S_0] + (\epsilon^2/4)\}^{-1/2} d(R/R_0) \,, \tag{52a}$$

$$(\alpha,\beta,\omega) = \mp |\epsilon| X \int (R/R_0)^{-1} \{(R/R_0)^2[(R/R_0) + S_0] + (\epsilon^2/4)\}^{-1/2} d(R/R_0) \,, \tag{52b}$$

where

$$x = \text{normalized time} = (t + t^*)/\tau_D , \tag{53a}$$

$$\tau_D = \text{time scale} = (6\pi\rho_{D_0})^{-1/2} , \tag{53b}$$

$$S_0 = \text{initial-mixture parameter} = \rho_{R_0}/\rho_{D_0} . \tag{53c}$$

In equation (52b), (α,β,ω) are the anisotropy functions defined by equations (6) and (7). The first anisotropy parameter is (see eqs. [15d], [17a], and [24a])

$$|\epsilon| = 2\Omega_0^{1/2} = [3\tau_D^2(3\eta_0'^2 + \sigma_0'^2)/4]^{1/2} \qquad (0 \le |\epsilon| < \infty) . \tag{54}$$

As before, the second anisotropy parameter is ψ $(0 \le \psi < 2\pi/3)$.

The integrals of equations (52) can be evaluated analytically in terms of elliptic integrals (see Abramowitz and Stegun 1965, p. 17; see also Gröbner and Hofreiter 1949, pp. 60–61 and 75 ff.). When this is done, the full analytical solution for the "dust-plus-radiation universe" takes on the complicated form

$$(A/A_0, B/B_0, W/W_0) = (R/R_0) \exp(\alpha,\beta,\omega) , \tag{55a}$$

$$\begin{aligned} x = {} & \{(R/R_0)^2[(R/R_0) + S_0] + (\epsilon^2/4)\}^{1/2} \\ & - S_0\mu[\![(2/\mu^2)\{E(\Phi,k) + [(1+\cos\Phi)/\sin\Phi](1 - k^2\sin^2\Phi)^{1/2}\} \\ & - (r + s\cot\theta)F(\Phi,k)]\!] , \end{aligned} \tag{55b}$$

$$\begin{aligned} (\alpha,\beta,\omega) = {} & \mp|\epsilon|\mathrm{X}\left(\frac{\mu}{s\tan\theta - r}\right)\Big\{F(\Phi,k) \\ & - (1-\zeta)^{-1}\left[\Pi\left(\Phi, \frac{\zeta^2}{1-\zeta^2}, k\right) - \zeta D_4\left(\Phi, \frac{\zeta^2}{1-\zeta^2}, k\right)\right]\Big\} , \end{aligned} \tag{55c}$$

$$\begin{aligned} (a,b,w) = {} & [3\tau_D(R/R_0)^3]^{-1}[\![\{4(R/R_0)^2[(R/R_0) + S_0] + \epsilon^2\}^{1/2} \\ & \mp 2|\epsilon|\mathrm{X}]\!] , \end{aligned} \tag{55d}$$

$$\begin{aligned} \rho_M/\rho_{D_0} &= (R/R_0)^{-4}[(R/R_0) + S_0] \quad \text{and} \\ p_M/\rho_{D_0} &= (S_0/3)(R/R_0)^{-4} , \end{aligned} \tag{55e}$$

where

$$\mu = \left(\frac{\sin 2\theta}{s}\right)^{1/2} ; \quad \Phi = \arccos\left\{\frac{[(R/R_0) - r] - s\cot\theta}{[(R/R_0) - r] + s\tan\theta}\right\} , \quad 0 \le \Phi \le \pi ;$$

$$\theta = \tfrac{1}{2}\arctan\left[\frac{(\Psi_+ - \Psi_-)}{3^{1/2}(\Psi_+ + \Psi_-)}\right] , \quad 0 \le \theta < \frac{\pi}{2} ; \quad r = -\left[\frac{(\Psi_+ + \Psi_-)}{2} + \frac{S_0}{3}\right] ;$$

$$s = \frac{3^{1/2}}{2}(\Psi_+ - \Psi_-) ; \quad k = |\sin\theta| ; \quad \zeta = \left[\frac{s\tan\theta - r}{s\cot\theta + r}\right] ;$$

$$\Psi_\pm = \left\{-\left[\left(\frac{\epsilon^2}{8}\right) + \left(\frac{S_0^3}{27}\right)\right] \pm \left(\frac{|\epsilon|}{2}\right)\left[\left(\frac{\epsilon^2}{16}\right) + \left(\frac{S_0^3}{27}\right)\right]^{1/2}\right\}^{1/3} .$$

In equations (52b) and (55), the upper/lower sign is for $\epsilon \gtrless 0$. This solution for the "dust-plus-radiation universe" is new.

This solution exhibits qualitatively the same behavior as the solutions of § III: The universe is created at a physical singularity at $R/R_0 = 0$ with a highly anisotropic expansion rate, but as time passes the anisotropy is slowly wiped out. In its late stages ($R/R_0 \gtrsim S_0$ and $x \gtrsim |\epsilon|$), the solution approaches the isotropic, dust-filled Friedmann universe, where $(A,B,W) \approx R \approx x^{2/3}$; and in its very early stages ($R/R_0 \lesssim S_0$), it behaves like the anisotropic "radiation universe" of § III*c*. The types of initial singularities are *exactly the same* as for the "dust universe" of § III*b*. The axisymmetric case occurs when $\psi = \pi/6$ or $\pi/2$ (all ϵ); and we recover Jacobs' (1967) isotropic solution when $\epsilon = 0$ (all ψ).

V. SEMIREALISTIC ANISOTROPIC COSMOLOGICAL MODELS FOR OUR UNIVERSE WITHOUT A MAGNETIC FIELD

Having discussed a number of exact solutions of Einstein's field equations, we now turn our attention to the problem of building semirealistic models for our own Universe out of these solutions.

a) *Constructing the Models*

Our semirealistic cosmological models contain only *dust* with a present mass density $\rho_{D_0} = 2.3 \times 10^{-29}$ g cm^{-3} (the amount necessary to have space be Euclidean rather than open or closed), and the observed 3° K cosmic microwave *radiation* with its present density of mass energy, $\rho_{R_0} = 6.8 \times 10^{-34}$ g cm^{-3}. *I neglect possible contributions due to unobserved neutrinos and gravitons:* Doroshkevich *et al.* (1967) showed that non-interacting neutrinos or gravitons are driven into extremely energetic beams when an anisotropic universe expands out of a CIGAR singularity; but Misner (1967, 1968) subsequently found that the anisotropic heating of the neutrinos is strongly damped by neutrino viscosity in the relativistic electron-positron gas which must exist in the early stages of any hot, big-bang cosmology.

In addition to ignoring the effects of neutrinos and gravitons, our models also ignore the heating of the photon gas when the relativistic electron-positron pairs recombine. Consequently, our models can be valid only in the temperature range $T \lesssim 10^{10}$ ° K; and they might not be valid even there because of our neglect of neutrinos and gravitons. (In the isotropic model of Jacobs 1967, $T \approx 10^{10}$ ° K occurs about *two seconds* after the initial singularity.) We also disregard matter with $\gamma > \frac{1}{3}$, since we expect to encounter it—if ever—only when $\rho_{\text{baryon}} \gtrsim 10^{14}$ g cm^{-3}, and this occurs long before the pairs recombine.

To construct our models we *could* use the dust-plus-radiation solution of equations (55), but this would be extremely cumbersome because the equations are so complicated. Instead, we join smoothly the "dust universe" of § III*b* to the "radiation universe" of § III*c* at the point where $\rho_D = \rho_R$ (i.e., at $R/R_0 = S_0 = \rho_{R_0}/\rho_{D_0} \approx 3 \times 10^{-5}$). The smooth transition is accomplished by making the expansion functions (A,B,W) and the Hubble expansion rates (a,b,w) continuous across the junction. The models resulting from this method differ from the analytical "dust-plus-radiation universe" models only in the immediate vicinity of the transition, and the difference is negligible (cf. Fig. 1 in Jacobs 1967). We obtain the following solution for our model. (i) For $t > t_{\text{transition}}$ the "dust universe" solution of equations (25) applies with the following specifications:

$$0 \leq |\epsilon_D| < \infty \quad \text{and} \quad 0 \leq \psi < 2\pi/3\,, \tag{56a}$$

$$t_D{}^* = \frac{\tau_D}{8}\left[(4S_0{}^3 + \epsilon_D{}^2)^{1/2} - 4|\epsilon_D| + \frac{3\epsilon_D{}^2}{2S_0{}^{3/2}} \ln\left|\frac{2S_0{}^{3/2} + [4S_0{}^3 + \epsilon_D{}^2]^{1/2}}{|\epsilon_D|}\right|\right], \tag{56b}$$

$$\rho_{D_0} = 2.3 \times 10^{-29} \text{ g cm}^{-3} \quad \text{and} \quad \tau_D = (6\pi\rho_{D_0})^{-1/2} = 5.9 \times 10^9 \text{ years}\,. \tag{56c}$$

(ii) For $t < t_{\text{transition}}$ the "radiation universe" solution of equations (26), (27), and (30) holds in the form

$$t/\tau_R = \tfrac{1}{2}[rF - (\epsilon_R{}^2/2) \ln |(2r + F)/|\epsilon_R||] , \tag{57a}$$

$$(A/A_T, B/B_T, W/W_T) = r\left\{\left(\frac{F + |\epsilon_R|}{F - |\epsilon_R|}\right)\left[\frac{(4 + \epsilon_R{}^2)^{1/2} - |\epsilon_R|}{(4 + \epsilon_R{}^2)^{1/2} + |\epsilon_R|}\right]\right\}^{\pm \mathrm{X}} , \tag{57b}$$

$$(a,b,w) = (4\tau_R r^3)^{-1}(F \mp 2|\epsilon_R|\mathrm{X}) , \tag{57c}$$

$$\rho_R/\rho_{RT} = r^{-4} , \tag{57d}$$

where the subscript T denotes "transition" here, and where

$$F = (4r^2 + \epsilon_R{}^2)^{1/2} \quad \text{and} \quad r = (R/R_T) . \tag{57e}$$

The constants which enter into equations (57) are given by

$$\rho_{R_0} = 6.8 \times 10^{-34} \text{ g cm}^{-3} \quad \text{and} \quad S_0 = \rho_{R_0}/\rho_{D_0} = 3.0 \times 10^{-5} , \tag{58a}$$

$$\tau_R = (3S_0{}^{3/2}/4)\tau_D = 730 \text{ years} , \tag{58b}$$

$$(R_T/R_0) = S_0 \quad \text{and} \quad \rho_{RT} = \rho_{D_0}/S_0{}^3 = 8.5 \times 10^{-16} \text{ g cm}^{-3} , \tag{58c}$$

$$|\epsilon_R| = S_0{}^{-3/2}|\epsilon_D| \quad \text{and} \quad \textit{same } \psi \text{ for } \textit{all } t , \tag{58d}$$

$$(A_T,B_T,W_T) = S_0\left[\frac{(\epsilon_D{}^2 + 4S_0{}^3)^{1/2} + |\epsilon_D|}{(\epsilon_D{}^2 + 4S_0{}^3)^{1/2} - |\epsilon_D|}\right]^{\pm 2\mathrm{X}/3} . \tag{58e}$$

Finally, the time of the transition is given by

$$t_{\text{transition}} = \frac{3\tau_D}{8}\left[(4S_0{}^3 + \epsilon_D{}^2)^{1/2} - \left(\frac{\epsilon_D{}^2}{2S_0{}^{3/2}}\right) \ln \left|\frac{2S_0{}^{3/2} + (4S_0{}^3 + \epsilon_D{}^2)^{1/2}}{|\epsilon_D|}\right|\right] . \tag{59}$$

b) Some Representative Semirealistic Models

When we examine the temperature anisotropy of the 3° K cosmic microwave radiation in the following subsection, we will find that, for two opposite extreme assumptions about intergalactic space, the recent observational data limit the range of $|\epsilon_D|$ to

$$0 \le |\epsilon_D| \le \begin{Bmatrix} 10^{-4} & \text{for} & \text{H II} \\ 10^{-7} & \text{for} & \text{H I} \end{Bmatrix} . \tag{60}$$

Here, H II signifies that the "dust" has consisted *almost entirely* of ionized hydrogen since the time $R_0/R \approx 10$ (i.e., a redshift of $z \approx 9$) when galaxies presumably formed, while H I means that the ionized hydrogen recombined when the photon temperature dropped below about 3000° K and the *entire* "dust" content of our Universe has remained neutral hydrogen ever since then. We will also find that observations of the microwave radiation place *no* restriction upon the range of ψ.

In Figure 1 we compare our anisotropic model with $\psi = 0$ and $\epsilon_D = +10^{-5}$ to the corresponding isotropic dust-plus-radiation model of Jacobs (1967). In Figure 2 we do the same for the anisotropic model with $\psi = \pi/2$ and $\epsilon_D = -10^{-10}$. These two explicit models demonstrate *all* the essential features of our semirealistic models. Note how (i) the time when anisotropic expansion is important decreases as $|\epsilon_D|$ decreases, (ii) the time when $T \gtrsim 10^{10}\,°$ K decreases rapidly as $|\epsilon_D|$ increases, and (iii) the number density of baryons is much lower at any given time in the anisotropic case than in the isotropic case, during the period of primordial element formation ($T \approx 10^9\,°$ K). Put differently, the average rate of expansion out of the initial singularity is much greater in the anisotropic case. This fact greatly affects primordial element formation.

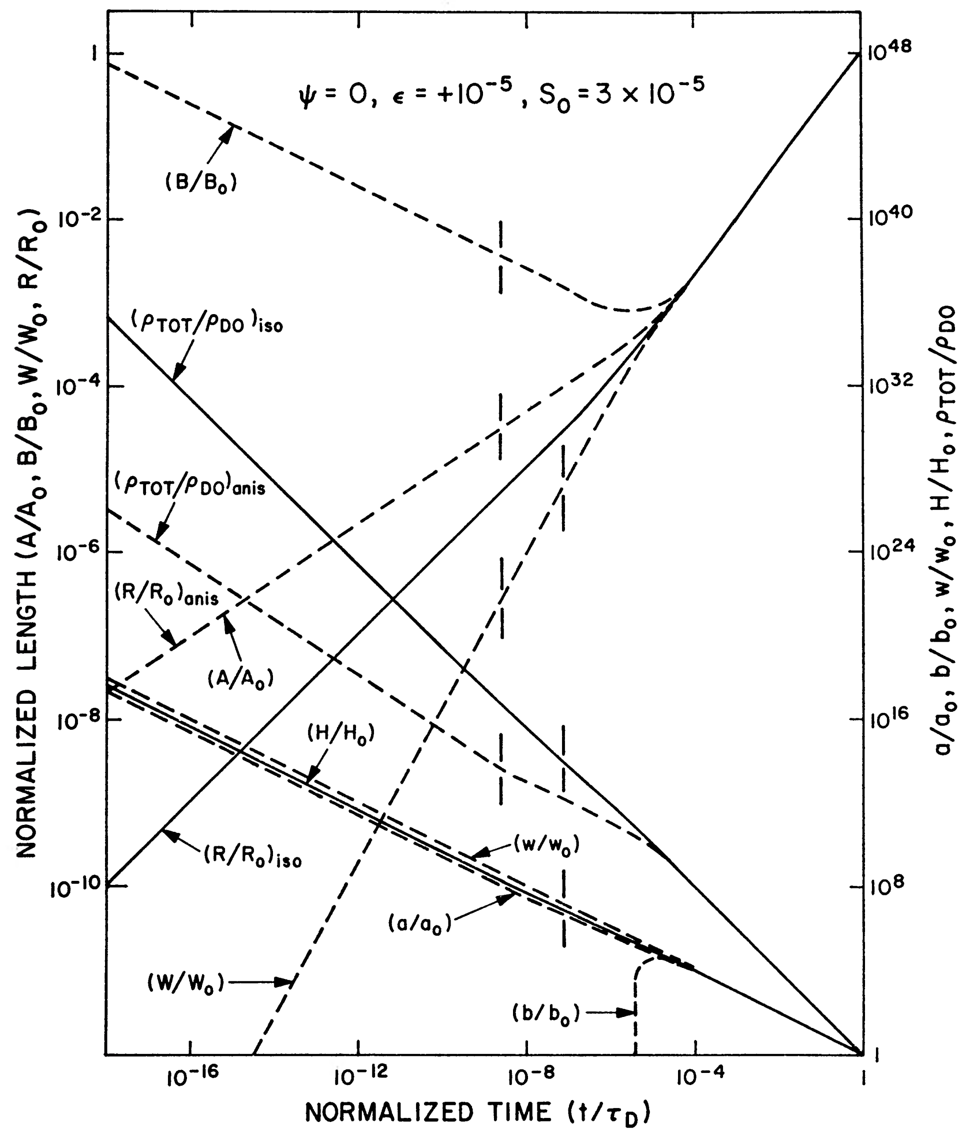

FIG. 1.—Semirealistic anisotropic cosmological model of our Universe with $\psi = 0$ and $\epsilon_D = +10^{-5}$ (*dashed lines*) compared with the isotropic, Euclidean dust-plus-radiation model of Jacobs (1967) (*solid lines*). We show the "expansion functions" (A/A_0, B/B_0, W/W_0), the "mean radius" (R/R_0), the normalized Hubble expansion rates (a/a_0, b/b_0, w/w_0, and H/H_0), and the normalized total density of mass energy ($\rho_{\text{tot}}/\rho_{D_0}$), as functions of normalized time (t/τ_D). The constants which appear are $H_0^{-1} = 8\ 8 \times 10^9$ years, $\rho_{D_0} = 2.3 \times 10^{-29}$ g cm^{-3}, $\tau_D \equiv (6\pi\rho_{D_0})^{-1/2} = 5.9 \times 10^9$ years, and the "initial-mixture parameter," $S_0 \equiv \rho_{R_0}/\rho_{D_0} = 3 \times 10^{-5}$. The relativistic electron-positron pairs recombine at $R/R_0 \approx 10^{-10}$; primordial element formation occurs near $R/R_0 \approx 10^{-9}$–10^{-8}; the anisotropic model enters the "dust universe" phase ($t \gtrsim 20$ years) at the left-hand set of vertical bars, while the isotropic model enters at the right-hand set ($t \approx 730$ years); the anisotropies become small for $t/\tau_D \gtrsim 10^{-5}$.

c) The Temperature Anisotropy of the Cosmic Microwave Radiation

The characteristic temperature of the cosmic microwave radiation depends upon direction in our anisotropic models. Applying Liouville's theorem (see, e.g., Thorne 1966*b*, Appendix B; Thorne 1967) to the propagation of non-interacting photons in our metric (eq. [2]) gives the temperature distribution as a function of the observation direction (in spherical coordinates):

$$T_0(\theta,\phi) = T_S[(A_0/A_S)^2 \sin^2\theta \cos^2\phi + (B_0/B_S)^2 \sin^2\theta \sin^2\phi + (W_0/W_S)^2 \cos^2\theta]^{-1/2}\,. \tag{61}$$

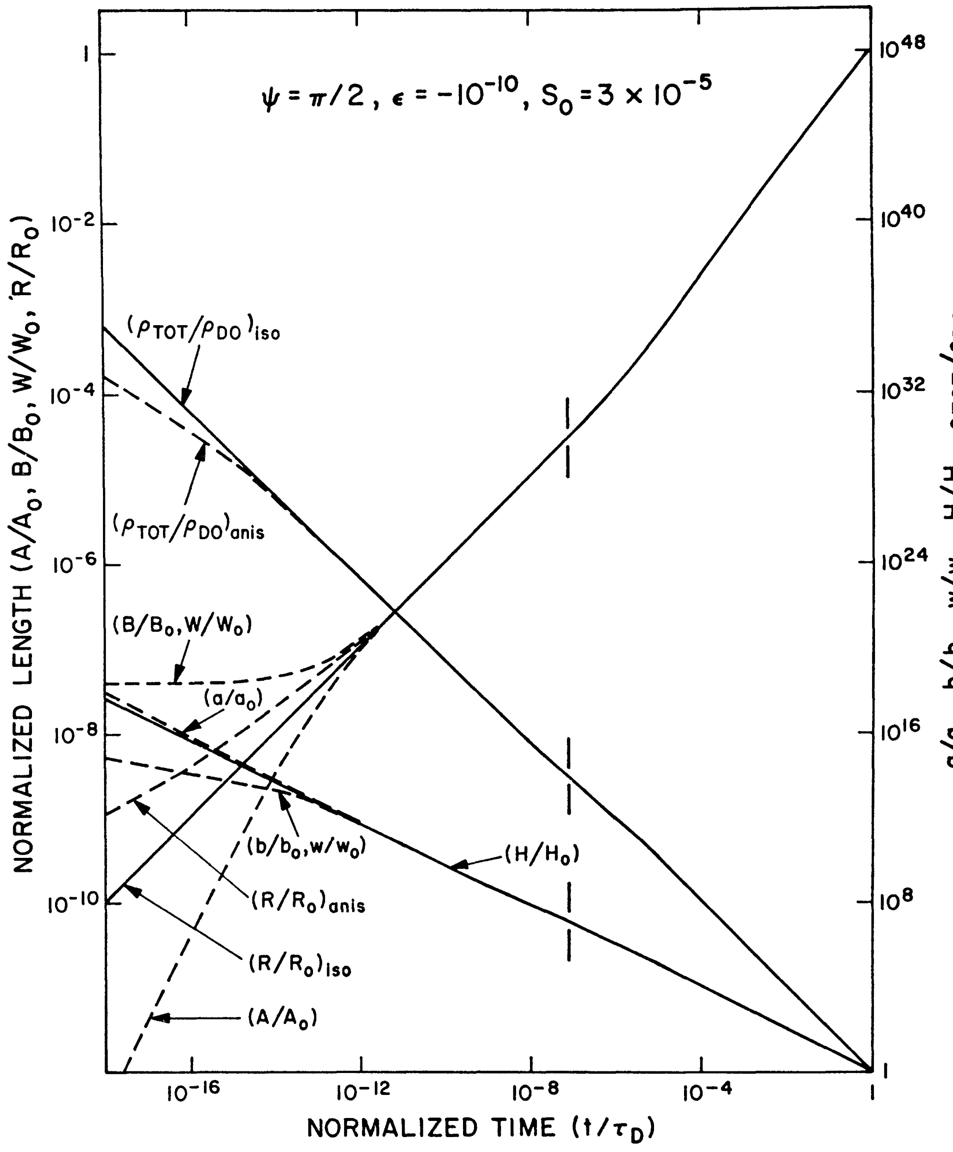

FIG 2.—Comparing the anisotropic model with $\psi = \pi/2$ and $\epsilon_D = -10^{-10}$ (*dashed lines*) to the isotropic model of Jacobs (1967) (*solid lines*). We show the same quantities as in Fig 1, and the normalizing constants are the same as in Fig 1. Anisotropies become small in the "radiation universe" phase (at $t/\tau_D \approx 10^{-14}$) so that the transition to the "dust universe" phase occurs at $t \approx 730$ years for both the isotopic and the anisotropic cases (*vertical bars*). As in Fig. 1, we have recombination of the relativistic electron-positron pairs at $R/R_0 \approx 10^{-10}$ and primordial element formation near $R/R_0 \approx 10^{-9}$–10^{-8}.

Here the subscripts 0 and S denote, respectively, the value of the quantity "today" and "at the time of the last scattering" of the microwave photons by matter, t_S. I define the effective time of the last scattering by

$$\int_{t_S}^{t_0} [\lambda(t)]^{-1} dt = 1 , \qquad (62)$$

where $\lambda(t)$ is the photon mean free path at time t. If our Universe has been filled with ionized hydrogen since galaxy formation at $R_0/R \gtrsim 10$ (case H II), we have Thomson scattering and

$$\lambda(t) = (2.67 \times 10^{-18} \text{ light years})/\rho_D(t)(\text{g cm}^{-3}) . \qquad (63)$$

Since $t_S \gg t_{\text{transition}}$, we use the "dust universe" solution of equations (25) and (56) to obtain, from equations (62) and (63),

$$t_S(\text{H II}) \cong (4.86 \times 10^{-2})\tau_D = 2.86 \times 10^8 \text{ years} , \qquad (64a)$$

$$R_0/R_S(\text{H II}) \cong 7.52 . \qquad (64b)$$

If the ionized hydrogen recombined when the photon temperature dropped below about 3000° K and was never reionized thereafter (case H I), we find

$$t_S(\text{H I}) = (T_0/T_S)^{3/2}\tau_D \cong 1.86 \times 10^5 \text{ years} . \qquad (65)$$

To see what limits the observed anisotropy of the cosmic microwave radiation places upon our anisotropy parameters $|\epsilon_D|$ and ψ, we write

$$T_A \equiv T_0(\theta = \pi/2, \phi = 0) ; \quad T_B \equiv T_0(\theta = \pi/2, \phi = \pi/2) ; \quad T_W \equiv T_0(\theta = 0) . \qquad (66)$$

Then we define the present mean-square temperature anisotropy as

$$(\Delta T/T)_0^2 = 3(T_A + T_B + T_W)^{-2}[(T_A - T_B)^2 + (T_A - T_W)^2 + (T_B - T_W)^2] . \qquad (67)$$

Finally, from equations (25), (56), and (67) we find, *for all* ψ, that

$$|\epsilon_D| \le (\tfrac{3}{2})^{1/2}(t_S/\tau_D)(\Delta T/T)_0 . \qquad (68)$$

Therefore, although the range of $|\epsilon_D|$ is limited by the observed temperature anisotropy and the assumed dust content of our models (cases H I and H II), there are *no restrictions* on ψ. Recently, Partridge and Wilkinson (1967) found that the magnitude of the 12-hour harmonic of the temperature anisotropy around the celestial equator (which corresponds *approximately* to our parameter $(\Delta T/T)_0$) is

$$(\Delta T/T)_0 = (1.6 \pm 0.7) \times 10^{-3} . \qquad (69)$$

It should be noted that equation (67) defines a measure of temperature anisotropy over the *entire* sky, while all observations to date have been performed only over one great circle and small portions of other great circles on the celestial sphere. Also note that our models generate *no* 24-hour harmonics of temperature anisotropy; any observed 24-hour harmonic will probably reflect the Earth's motion relative to the local co-moving frame of the cosmic microwave radiation. Equations (68) and (69) imply that

$$0 \le |\epsilon_D| \lesssim \begin{Bmatrix} (9.6 \pm 4.1) \times 10^{-5} \\ (6.2 \pm 2.7) \times 10^{-8} \end{Bmatrix} \quad \text{for case} \quad \begin{pmatrix} \text{H II} \\ \text{H I} \end{pmatrix} . \qquad (70)$$

Precise observations of the temperature anisotropy have been carried out only on or near the celestial equator (Partridge and Wilkinson 1967) and at declination $\approx 40°$ N. (Conklin and Bracewell 1967). The Partridge-Wilkinson observations lead to the result of equation (70), and the Conklin-Bracewell observations lead to roughly the same limits. Less precise measurements at selected points over the entire sky have been performed by Penzias and Wilson (1967), and they provide the weaker limits *over the entire sky:*

$$0 \leq |\epsilon_D| \lesssim \begin{Bmatrix} 1.8 \times 10^{-3} \\ 1.2 \times 10^{-6} \end{Bmatrix} \quad \text{for case} \quad \begin{pmatrix} \text{H II} \\ \text{H I} \end{pmatrix}. \tag{71}$$

If our Universe is approximately axisymmetric (e.g., $A(t) \cong B(t)$ for all t), there is a 3 per cent probability that the celestial pole is so close to the axis of symmetry that only the weak limits of equation (71) apply to $|\epsilon_D|$. It is easy to see that there is a pressing need for precise experimental investigation of temperature anisotropy along *several* great circles on the celestial sphere.

d) The Time when Anisotropies Ceased To Be Large

From equations (25), (64), and (65) we see that anisotropies are important in our models (e.g., and Hubble expansion rates in different directions (a,b,w) differ by more than a factor $\frac{3}{2}$) for

$$t < |\epsilon_D|\tau_D \tag{72}$$

if anisotropy becomes small during the *dust* phase (i.e., $|\epsilon_D| \gtrsim 6.6 \times 10^{-7}$). Then equation (70) implies that equation (72) applies only to case H II (ionized hydrogen). If anisotropy becomes small during the radiation phase (i.e., $|\epsilon_D| \lesssim 6.6 \times 10^{-7}$), equations (57) and (58) tell us that anisotropies are important for

$$t < (1.5 \times 10^6)\epsilon_D^2\tau_D\,. \tag{73}$$

The criterion of equation (73) applies to *both* cases H I and H II.

e) Primordial Element Formation in Our Models

In a hot, big-bang cosmology, primordial element formation takes place near $T \approx 10^9$ ° K. In our semirealistic anisotropic models, as in the standard isotropic models, this temperature is always encountered in the "radiation universe" phase. To calculate the final relative abundances of the primordial elements formed, we need to know the average expansion rate, R'/R, as a function of the density of total mass energy, ρ_{tot} (cf. Thorne 1967, eq. [B.18] and the associated discussion). Such a relation, together with the equations

$$\rho_D/\rho_{D_0} = (R/R_0)^{-3} \quad \text{and} \quad \rho_{\text{tot}}/\rho_{R_0} = (R/R_0)^{-4}\,, \tag{74}$$

tells us how the number density of baryons varies with time. Solving equation (14) or, equivalently, using equation (57a), and employing equations (25) and (56)–(58), we find

$$(R'/R) = (1.35 \times 10^7)\rho_{\text{tot}}^{3/4}[\epsilon_D^2 + (3.1 \times 10^{-21})\rho_{\text{tot}}^{-1/2}]^{1/2}\ \text{sec}^{-1}\,, \tag{75}$$

where ρ_{tot} is measured in grams per cubic centimeter. But this is *exactly the same* as equation (B.18) of Thorne (1967)! Therefore, all of the results on primordial element formation in axisymmetric universes, as calculated by R. V. Wagoner and reported by Thorne (1967), apply directly to our more general anisotropic models.

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