THE DYNAMICS OF COLLIDING GALAXIES

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ABSTRACT

The problem of the dynamics of colliding galaxies is considered. A first approximation for the motions of the centers of mass of colliding galaxies is developed. In this approximation, the galaxies are treated as spherically symmetric distributions of constituent point masses (the stars), and the effects of collisions are tentatively assumed to be negligible insofar as changes in the structures and internal energies of the galaxies are concerned. On the basis of these results, an approximate theory has been developed to provide estimates for the exchange of energy between that corresponding to the center-of-mass motions of colliding galaxies and that corresponding to their internal structures and motions. For given initial conditions, the results concerning the energy exchange provide not only a consistency test for the first approximation for the motions of the centers of mass but also approximate lower-limits for the actual energy exchange—regardless of the validity of this first approximation. From the results obtained, it appears that the effect of collisions on the internal energies of galaxies is considerably more important than may previously have been believed and has probably been important in altering the internal energies of non-negligible fraction of the galaxies in the observable universe.

I. INTRODUCTION

The dynamics of colliding galaxies is of interest in several connections in the study of the evolution of galaxies and systems of galaxies. There are two distinct, but related, aspects to the problem: the effects of collisions, or very close encounters, (i) on the center-of-mass motions of galaxies and (ii) on the internal energy and structure of the individual galaxies.

It is clear that these two aspects cannot be treated independently in any detailed description of the problem—except in the limit in which the relative velocity of the centers of mass of the two galaxies greatly exceeds the internal (stellar) motions within the galaxies. Thus, for example, for slower encounters, the variation of the relative velocity of the centers of mass of the two galaxies during the encounter can significantly influence the predicted changes in the internal energies and structures of the galaxies. For the purposes of making these and other estimates, it is of value to develop a first approximation for the motions of the centers of mass of colliding galaxies which neglects changes in the internal structure of the galaxies resulting from the encounter but which goes beyond the point-mass approximation by taking into account in a certain scheme of approximation the extended nature of galaxies. The motions so derived will be referred to as the center-of-mass motions in the *first approximation.* On the basis of the motions derived in this way and under the simplifying assumption that the internal motions of the stars within the colliding galaxies may be neglected during the collision, that is, in the *impulsive approximation*, it is shown how estimates can then be made for the energy interchange between the energy corresponding to the center-of-mass motions for the galaxies as a whole (the *external energy*, say) and the internal energy of the galaxies due to the stars comprising them. Depending upon whether or not the change so derived for the internal energy of a galaxy is small with respect to its initial value for any given initial conditions, conclusions can then be drawn concerning the extent to which the results predicted for the center-of-mass motions in the first approximation are valid and concerning the order of magnitude of the interchanges of external and internal energy due to the encounters.

A basic function that appears in the theory developed here has been tabulated, and the theory for the center-of-mass motions in the first approximation and for the interchange of external and internal energy has been applied in several illustrative cases.

Certain implications of the results obtained concerning the effects of collisions of galaxies are discussed.

II. BASIC THEORY

a) *Motions of the Centers of Mass of Colliding Galaxies*

i) *Assumptions.*—In the theory to be developed here for the center-of-mass motions in the first approximation, it is assumed that the galaxies are spherically symmetric configurations of mass points whose over-all structures remain unchanged during the course of encounters with one another. Implicit here is the assumption that the individual internal energies and angular momenta of interacting galaxies also remain unchanged throughout the encounters. It is further assumed that the only intergalactic forces are the gravitational ones arising from the mutual gravitational interactions of the mass points (i.e., the stars) making up the different galaxies. Under these idealized conditions, it is convenient to express the distribution of matter within each galaxy as a superposition of polytropic distributions having common radii. In practice, it appears that reasonable representations for the distributions of mass in galaxies having approximately spherical symmetry can be achieved through superpositions of the six polytropic distributions corresponding to the integral polytropic indices: $n = 0, 1, 2, 3, 4$, and 5. The numerical work in the present subsection has been carried out for such situations.

ii) *Equations of motion for the centers of mass.*—Under the assumptions made concerning the fixed forms of the galaxies during encounters, the problem of the motion of two spherically symmetric, gravitationally interacting distributions can be treated as one of two gravitationally interacting mass points whose potential energy of interaction is given by

$$\Omega(r) = -\frac{G\mathfrak{M}_1\mathfrak{M}_2}{r}\Psi(r), \tag{1}$$

where r is the distance between the centers of the two configurations having total masses $\mathfrak{M}_1$ and $\mathfrak{M}_2$, and $\Psi(r)$ is a factor which corrects for the fact that one is actually dealing with extended configurations and not with mass points. Clearly, in the present approximation the correction factor will be unity so long as the configurations do not overlap. In cases in which there is overlapping of the two configurations, $\Psi(r)$ can be derived in a straightforward way in terms of the distributions involved. The theory for the derivation of $\Psi(r)$ and the corresponding numerical results are given in the next subsection for the case in which the mass distributions can be represented as superpositions of polytropic distributions.

It is clear that the common center of mass for two interacting galaxies will move with constant velocity. Consequently, we need consider only the relative motion of one of the galaxies with respect to the other. Further, it is clear that the relative motion will take place in a plane. If r and θ are used to denote the polar coordinates in the orbital plane characterizing the relative motion and μ, l, and E, the reduced mass, the angular momentum and the energy, respectively, for the center-of-mass motion, then the Lagrangian equations for the problem give (cf. Goldstein 1950)

$$dt = \frac{dr}{\{2\mu^{-1}[E-\Omega(r)-l^2/2\mu r^2]\}^{1/2}} \tag{2}$$

and

$$d\theta = \frac{l\,dr}{\mu r^2\{2\mu^{-1}[E-\Omega(r)-l^2/2\mu r^2]\}^{1/2}}. \tag{3}$$

These can be used to determine the orbit and the variation of time along the orbit. The magnitude v of the velocity along the orbit is given by

$$\tfrac{1}{2}\mu v^2 = E - \Omega(r), \tag{4}$$

which is simply an expression of the conservation of energy for this problem.

iii) *Determination of the function* $\Psi(\mathfrak{r})$.—Consider two spherically symmetric distributions in such a position that they partially overlap one another. Let us refer to the two configurations as the *configuration 1* and the *configuration 2*, and let the quantities pertaining to these two configurations be characterized by the subscripts 1 and 2, respectively. The gravitational potential energy of the two configurations due only to their mutual interactions on one another is given by

$$\Omega(r) = \int_{\mathfrak{M}_2} V_1(r_1)\, d\,\mathfrak{M}_2\,, \tag{5}$$

where $V_1(r_1)$ is the potential due to *configuration 1* at a distance r_1 from its center, $d\mathfrak{M}_2$ is an element of mass at this point belonging to *configuration 2*, and r is the distance between the centers of the two configurations. Let r_2 denote the distance of a point from the center of *configuration 2*, let $\rho_2(r_2)$ denote the mass density in *configuration 2*, and let β denote the cosine of the polar angle in a spherical polar coordinate system with origin at the center of *configuration 2* and with polar axis directed toward the center of *configuration 1*. If both configurations are considered to have the same radius R (this introduces no significant loss of generality in the present treatment) and if r_1, r_2, r, and (for the subsequent work) r'' when measured in units of R are denoted by $\mathfrak{r}_1$, $\mathfrak{r}_2$, $\mathfrak{r}$, and $\mathfrak{r}''$, respectively, then equation (5) reduces to the following:

$$\Omega(r) = -2\pi R^3 \int_0^1 \int_{-1}^{+1} V_1(\mathfrak{r}_1)\, \rho_2(\mathfrak{r}_2)\, \mathfrak{r}_2^2 d\,\mathfrak{r}_2 d\beta\,. \tag{6}$$

If it is now assumed that both *configuration 1* and *configuration 2* can be expressed as superpositions of polytropic distributions having unit radii in terms of the radii R of the galaxies, then equation (6) can be written as

$$\begin{aligned}\Omega(r) = -2\pi R^3 \sum_{n_1} \sum_{n_2} \int_0^1 \int_{-1}^1 & V_{n_1}[\,\mathfrak{r}_1 = (\mathfrak{r}_2^2 + \mathfrak{r}^2 - 2\,\mathfrak{r}_2 r \beta)^{1/2}] \\ & \times \rho_{n_2}(\mathfrak{r}_2)\, \mathfrak{r}_2^2 d\,\mathfrak{r}_2 d\beta\end{aligned} \tag{7}$$

where $\rho_{n_i}(\mathfrak{r})_i$ is a polytropic distribution of index n_i that is so normalized that its first zero falls at $\mathfrak{r}_i = 1$. From the theory of polytropes (cf. Limber 1961), this expression for $\Omega(r)$ can be expressed as

$$\Omega(r) = -\frac{G}{r} \sum_{n_1} \sum_{n_2} \mathfrak{M}_{n_1} \mathfrak{M}_{n_2} \Psi(\mathfrak{r}, n_1, n_2)\,, \tag{8}$$

$$\Psi(\mathfrak{r}, n_1, n_2) \equiv \frac{3\,\mathfrak{r}}{2} \left(\frac{\rho_{c,n_2}}{\bar{\rho}_{n_2}}\right) \int_0^1 \Theta_{n_2}^{n_2}(\mathfrak{r}_2)\, \mathfrak{r}_2^2 \left[\int_{-1}^1 \Phi_{n_1}(\mathfrak{r}_1)\, d\beta\right] d\,\mathfrak{r}_2\,, \tag{9}$$

$$\left.\begin{aligned}\Phi_{n_1}(\mathfrak{r}_1) &\equiv 1 + \frac{3\,\rho_{c,n_1}/\bar{\rho}_{n_1}}{\xi_{1,n_1}^2}\, \Theta_{n_1}(\mathfrak{r}_1)\,, && (\mathfrak{r}_1 \le 1) \\ &\equiv \frac{1}{\mathfrak{r}_1}\,, && (r_1 \ge 1)\end{aligned}\right\} \tag{10}$$

$$\Theta_{n_1}(\mathfrak{r}_1) \equiv \theta_{n_1}(\xi = \mathfrak{r}_1 \xi_{1,n_1})\,, \tag{11}$$

where $\mathfrak{M}_{n_i}$, $\rho_{c,\,n_i}$, and $\bar{\rho}_{n_i}$ denote the total mass, central density, and mean density, respectively, for the polytropic distribution ρ_{n_i}, where θ is the usual Lane-Emden function,

ξ is the usual polytropic radial variable, and $\xi_{1,n}$ is the value of ξ at which $\theta_n(\xi)$ has its first zero.

The value for $\Psi(\mathfrak{r}, n_1 \neq 5, n_2 = 5)$ follows as

$$\Psi(\mathfrak{r}, n_1 \neq 5, n_1 = 5) = \mathfrak{r}\Phi_{n_1}(\mathfrak{r}_1 = \mathfrak{r}) . \tag{12}$$

The tables for $\Theta_n(\mathfrak{r})$ and $\Phi_n(\mathfrak{r})$ given by Limber (1961) have been used to calculate the function $\Psi(\mathfrak{r}, n_1, n_2)$ for the cases of the polytropic distributions having integral indices ranging from 0 through 5. The computational errors appear to be less than 0.5 per cent. The results are given in Table 1.

It will be noted from equation (16) that Ω considered as a function of Ψ and $\mathfrak{r}$ is indeterminate for the case in which $\mathfrak{r} = 0$, since $\Psi(0, n_1, n_2) = 0$. However, in this case, Ω may be conveniently obtained in terms of the dimensionless function $[\Psi(\mathfrak{r}, n_1, n_2)/\mathfrak{r}]_{\mathfrak{r}\to 0}$. This function has been tabulated in Table 2.

b) Exchange of Internal and External Energy

On the basis of the relative motions of the centers of mass of galaxies derived by the procedure discussed above, estimates may be made for the changes in the internal energy of the galaxies resulting from collisions. This change in the internal energies of the galaxies will result in an equal but opposite change in the energy corresponding to the center-of-mass motions of the galaxies and will, therefore, provide a measure for the extent to which the assumption of constant E in the theory for the center-of-mass motions in the first approximation is valid. The desired estimate for the change in the internal energy of the galaxy of mass $\mathfrak{M}_2$ will be obtained from a study of the tidal force experienced by representative stars in it. It is expected that such a treatment should provide at least a rough estimate for the energy exchange.

It is clear that the results will depend, in general, upon the initial conditions assumed for the stars as well as upon the initial conditions assumed for the center-of-mass motions of the two galaxies. The present analysis will be carried through on the assumption that the internal motions of the stars are negligible during the encounter; this is the so-called *impulsive approximation.* This approximation is reasonable and useful for several reasons. First, it appears that the relative speed of colliding galaxies will seldom be smaller than typical internal motions and will very often be considerably larger. In view of this, it appears unlikely that the impulsive approximation will introduce large errors. Second, the calculation for the change in the internal energy according to the impulsive approximation is independent of the velocity distribution of the stars, except insofar as rather general symmetry properties are concerned. Finally, the impulsive approximation greatly simplifies the calculations. For the purpose of the present calculations it will also be assumed that the velocity distribution at any point in the galaxy is an even function of the components of velocity, v_x, v_y, and v_z with respect to the center of mass of the galaxy.

Let us choose a Cartesian coordinate system whose origin is at the center of mass of the galaxy of mass $\mathfrak{M}_2$, whose x-axis is directed toward the center of galaxy of mass $\mathfrak{M}_1$ when the galaxies are at closest approach, and whose x–y plane is the orbital plane. Let the coordinates of the center of the galaxy of mass $\mathfrak{M}_1$ be denoted by x, y, and z (by assumption $z = 0$). Let the coordinates of a representative star in $\mathfrak{M}_2$ be x', y', z', and let its distance from the center of $\mathfrak{M}_2$ be r'. If r'' is used to denote the distance between the galaxy of mass $\mathfrak{M}_1$ and the representative star, then the tidal force, $\boldsymbol{f}$, per unit mass on the star due to $\mathfrak{M}_1$ follows from the relations:

$$f_x = \left(\frac{dV_1}{dr''}\frac{(x - x')}{r''} - \frac{dV_1}{dr}\frac{x}{r}\right), \tag{13}$$

$$f_y = \left(\frac{dV_1}{dr''}\frac{(y - y')}{r''} - \frac{dV_1}{dr}\frac{y}{r}\right), \tag{14}$$

TABLE 1

THE FUNCTION $\Psi(\tau, n_1, n_2)$

τ	$n_2 = 0$					
	$n_1 = 0$	$n_1 = 1$	$n_1 = 2$	$n_1 = 3$	$n_1 = 4$	$n_1 = 5$
0.00	0.000	0.000	0.000	0.000	0.000	0.000
0.10	0.120	0.130	0.138	0.144	0.148	0.150
0.20	0.236	0.257	0.273	0.285	0.293	0.296
0.30	0.348	0.378	0.402	0.419	0.432	0.437
0.40	0.453	0.491	0.522	0.545	0.561	0.568
0.50	0.549	0.593	0.631	0.659	0.680	0.688
0.60	0.636	0.684	0.725	0.759	0.781	0.792
0.70	0.713	0.762	0.804	0.840	0.866	0.879
0.80	0.779	0.827	0.869	0.904	0.929	0.944
0.90	0.836	0.879	0.917	0.948	0.969	0.986
1.00	0.881	0.920	0.950	0.975	0.990	1.000
1.10	0.917	0.949	0.974	0.990	0.997	1.000
1.20	0.947	0.970	0.986	0.995	0.999	1.000
1.30	0.967	0.985	0.995	0.998	1.000	1.000
1.40	0.981	0.991	0.998	0.999	1.000	1.000
1.50	0.990	0.997	0.999	0.999	1.000	1.000
1.60	0.996	0.998	0.999	1.000	1.000	1.000
1.70	0.999	0.999	1.000	1.000	1.000	1.000
1.80	1.000	1.000	1.000	1.000	1.000	1.000
1.90	1.000	1.000	1.000	1.000	1.000	1.000
2.00	1.000	1.000	1.000	1.000	1.000	1.000

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TABLE 1 — Continued

τ	$n_2 = 1$				
	$n_1 = 1$	$n_1 = 2$	$n_1 = 3$	$n_1 = 4$	$n_1 = 5$
0.00	0.000	0.000	0.000	0.000	0.000
0.10	0.149	0.167	0.182	0.193	0.198
0.20	0.293	0.327	0.356	0.378	0.387
0.30	0.429	0.475	0.514	0.543	0.558
0.40	0.551	0.606	0.651	0.687	0.703
0.50	0.659	0.716	0.765	0.802	0.818
0.60	0.749	0.805	0.853	0.885	0.903
0.70	0.822	0.874	0.915	0.943	0.958
0.80	0.880	0.920	0.955	0.975	0.987
0.90	0.924	0.955	0.977	0.990	0.998
1.00	0.952	0.977	0.992	0.997	1.000
1.10	0.972	0.988	0.999	0.999	1.000
1.20	0.989	0.995	1.000	1.000	1.000
1.30	0.994	1.000	1.000	1.000	1.000
1.40	0.997	1.000	1.000	1.000	1.000
1.50	0.999	1.000	1.000	1.000	1.000
1.60	1.000	1.000	1.000	1.000	1.000
1.70	1.000	1.000	1.000	1.000	1.000
1.80	1.000	1.000	1.000	1.000	1.000
1.90	1.000	1.000	1.000	1.000	1.000
2.00	1.000	1.000	1.000	1.000	1.000

TABLE 1 — Continued

τ	$n_2 = 2$			
	$n_1 = 2$	$n_1 = 3$	$n_1 = 4$	$n_1 = 5$
0.00	0.000	0.000	0.000	0.000
0.10	0.198	0.230	0.259	0.275
0.20	0.384	0.442	0.493	0.519
0.30	0.549	0.621	0.681	0.710
0.40	0.686	0.760	0.818	0.844
0.50	0.793	0.859	0.907	0.927
0.60	0.871	0.924	0.957	0.971
0.70	0.924	0.961	0.984	0.991
0.80	0.959	0.982	0.995	0.998
0.90	0.979	0.993	0.999	1.000
1.00	0.990	0.997	1.000	1.000
1.10	0.995	0.999	1.000	1.000
1.20	1.000	1.000	1.000	1.000
1.30	1.000	1.000	1.000	1.000
1.40	1.000	1.000	1.000	1.000
1.50	1.000	1.000	1.000	1.000
1.60	1.000	1.000	1.000	1.000
1.70	1.000	1.000	1.000	1.000
1.80	1.000	1.000	1.000	1.000
1.90	1.000	1.000	1.000	1.000
2.00	1.000	1.000	1.000	1.000

TABLE 1 — Concluded

τ	$n_2 = 3$			$n_2 = 4$		$n_2 = 5$
	$n_1 = 3$	$n_1 = 4$	$n_1 = 5$	$n_1 = 4$	$n_1 = 5$	$n_1 = 5$
0.00	0.000	0.000	0.000	0.000	0.000	0.000
0.10	0.293	0.365	0.416	0.538	0.716	1.000
0.20	0.545	0.651	0.715	0.835	0.935	1.000
0.30	0.734	0.832	0.879	0.949	0.984	1.000
0.40	0.859	0.929	0.955	0.987	0.996	1.000
0.50	0.931	0.972	0.985	0.998	0.999	1.000
0.60	0.969	0.990	0.996	1.000	1.000	1.000
0.70	0.989	0.998	0.999	1.000	1.000	1.000
0.80	0.995	0.999	1.000	1.000	1.000	1.000
0.90	0.999	1.000	1.000	1.000	1.000	1.000
1.00	1.000	1.000	1.000	1.000	1.000	1.000
1.10	1.000	1.000	1.000	1.000	1.000	1.000
1.20	1.000	1.000	1.000	1.000	1.000	1.000
1.30	1.000	1.000	1.000	1.000	1.000	1.000
1.40	1.000	1.000	1.000	1.000	1.000	1.000
1.50	1.000	1.000	1.000	1.000	1.000	1.000
1.60	1.000	1.000	1.000	1.000	1.000	1.000
1.70	1.000	1.000	1.000	1.000	1.000	1.000
1.80	1.000	1.000	1.000	1.000	1.000	1.000
1.90	1.000	1.000	1.000	1.000	1.000	1.000
2.00	1.000	1.000	1.000	1.000	1.000	1.000

and

$$f_z = -\frac{dV_1}{dr''}\frac{z'}{r''}, \tag{15}$$

where V_1 is the potential at the star due to $\mathfrak{M}_1$. It follows from the theory of polytropes used in obtaining equation (8) that

$$\frac{dV_1}{dr} = -\frac{G}{R^2}\sum_{n_1} \mathfrak{M}_{n_1}\frac{d\Phi_{n_1}}{d\mathfrak{r}}. \tag{16}$$

The change in velocity, $\Delta \boldsymbol{v}'$, of the star with respect to the center of $\mathfrak{M}_2$ due to the tidal force as a result of the collision can be obtained from the relation

$$\Delta \boldsymbol{v}' = \int_{-\infty}^{+\infty} \boldsymbol{f} dt. \tag{17}$$

In the most consistent procedure, $\boldsymbol{f}(t)$ would be evaluated on the basis of the center-of-mass motion in the first approximation.

TABLE 2

THE FUNCTION $[\Psi(\mathfrak{r}, n_1, n_2)/\mathfrak{r}]_{\mathfrak{r}\to 0}$

n_1	n_2					
	0	1	2	3	4	5
0	1 20	1 30	1 38	1 44	1 48	1 50
1	1 30	1 50	1 68	1 83	1 95	2 00
2	1 38	1 68	2 00	2 34	2.64	2 80
3	1 44	1 83	2 34	3 00	3 81	4 42
4	1 48	1 95	2 64	3 81	6 00	9 33
5	1 50	2 00	2 80	4 42	9 33	

In the computation of the change in its internal energy, the galaxy may be divided into sets of stars, the stars in each set being characterized by their common distance $\boldsymbol{r}'$ from the center. For each such set (consisting of l sample stars, say) the average increase of the kinetic energy per unit mass, which is the same in the impulsive approximation as the average increase of the internal energy of the stars in that set per unit mass, $\langle \Delta u(r') \rangle$ (say), can then be obtained from the relation

$$\langle \Delta u(r') \rangle = \tfrac{1}{2}\langle \Delta v'^2(r') \rangle = \frac{1}{2l}\sum_{i=1}^{l}[(\Delta v_{x\,i}')^2 + (\Delta v_{y\,i}')^2 + (\Delta v_{z\,i}')^2]. \tag{18}$$

It is to be noted that for the assumptions made the change in the internal energy will always be positive; consequently, the change in the external energy will always be negative. The change in the internal energy of the whole galaxy, ΔU (say), is obtained by integrating $\langle \Delta u(r') \rangle$ over the mass of the galaxy. The gravitational potential energy of the galaxy of mass $\mathfrak{M}_2$ follows from equation (8) as

$$\Omega_2 = -\frac{G}{R}\left\{\sum_m \sum_{n>m} \mathfrak{M}_m \mathfrak{M}_n \lim_{\mathfrak{r}\to 0}\left[\frac{\Psi(\mathfrak{r}; m, n)}{\mathfrak{r}}\right] + \sum_m \mathfrak{M}_m^2\left(\frac{3}{5-m}\right)\right\}, \tag{19}$$

where $\mathfrak{M}_m$ is the mass of the polytropic component of index m and where the summations are summations over the different polytropic components. Finally, from the virial theorem, we have the following relation between Ω_2 and the total internal energy, U,

$$U = \tfrac{1}{2}\Omega_2 . \tag{20}$$

The fractional increase of the internal energy, $\Delta U/|U|$ of the galaxy can thus be estimated by means of equations (18)–(20).

It is useful to note that a more crude estimate for the energy interchange can be derived much more easily in practice by assuming uniform rectilinear motion for the relative motion of the galaxies. Suppose that the galaxy $\mathfrak{M}_1$ is moving with a uniform speed V parallel to the y-axis, and its impact parameter, which is the same in this case as the distance of closest approach, is p, and let time, t, be reckoned from the instant of closest approach of the centers of the galaxies. For this case, the expression for the change in the velocity of the star due to the collision simplifies considerably, the resulting integrals being standard integrals whenever the effects of interpenetration can be neglected. In the limit in which the dimensions of the galaxies are negligible with respect to the separation of the galaxies at closest approach, these last mentioned expressions reduce to expressions given by Spitzer (1958) for the change in velocity of a star in a galactic cluster due to the tidal force of a passing interstellar cloud. For such cases, $\Delta U/|U|$ for the galaxy of mass $\mathfrak{M}_2$ is given simply by the relation

$$\frac{\Delta U}{|U|} = \frac{8}{3}\,\frac{G^2\,\mathfrak{M}_1{}^2\,\mathfrak{M}_2\,r_{c,2}{}^2}{p^4 V^2 \Omega_2}, \tag{21}$$

where $r^2{}_{c,2}$ is the mean-square radius of the galaxy of mass $\mathfrak{M}_2$ and potential energy Ω_2.

III. ILLUSTRATIVE EXAMPLES

a) *Mass Distribution within Galaxies*

Although the theory developed in the preceding section in its general form makes use of a superposition of polytropic distributions for the representation of the distribution of mass in galaxies, much preliminary insight into the problem of the motions of colliding galaxies can be gained by representing the mass distributions within galaxies by polytropes of a single index. Limber (1961) has prepared a short table that gives in a convenient form the relative distribution of mass within polytropes corresponding to the indices $n = 0, 1, 2, 3$, and 4. Through a comparison of the results contained in this table with the available observational data on the distribution of mass within galaxies, it is possible to determine that polytrope of integral index that best describes the mass distribution in any particular galaxy. Strictly speaking, since one cannot assign a sharply defined boundary to a galaxy, one can only make several different estimates for its radius and derive on the basis of each of these estimates for R that polytropic distribution of integral index that best represents the corresponding distribution. We can, of course, choose R arbitrarily, for the present purposes, so long as a sphere of radius R contains essentially all of the mass.

De Vaucouleurs (1953) surveys the available evidence concerning the luminosity distribution in elliptical galaxies and gives a model for a globular galaxy which is in reasonable agreement with the luminosity distribution and other known properties of the elliptical nebulae. The mass distribution within the spiral galaxy M31 is available from the work of Brandt (1960). Although spiral galaxies are certainly not spherically symmetric, they can be treated in this way in a first approximation for the present purposes. In Table 3, the fraction of the total mass interior to a sphere of radius r has been tabulated for each of these galaxies for two or three different estimates of their boundaries. The polytropic indices included in this table for each value of r/R are estimates for the indices of polytropic distributions having the same fractional mass interior to r/R as have the observed distributions and have been derived from the table given by Limber which was referred to above. It appears from this comparison that the mass distributions in these astronomical objects can be represented to reasonable accuracy as polytropic distributions of index $n = 4$.

In the remainder of this section the motions of colliding galaxies are determined in the present scheme of approximation for the case in which both galaxies can be represented as polytropic distributions of index $n = 4$ having common radii. The results obtained for this particular form for the mass distributions should be illustrative of the motions predicted in the present approximation for colliding galaxies, and the comparison of these results with those for the case in which the two galaxies are treated simply as mass points should provide useful estimates for the errors that can be expected from the latter procedure.

TABLE 3

DISTRIBUTION OF MASS WITHIN GALAXIES

R	Typical Elliptical Galaxy						Spiral Galaxy M31			
	$R=1$ kpc		$R=1$ 5 kpc		$R=2$ 0 kpc		$R=100$ kpc		$R=125$ kpc	
	$\mathfrak{M}(r)/\mathfrak{M}$	n	$\mathfrak{M}(r)/\mathfrak{M}$	n	$\mathfrak{M}(r)/\mathfrak{M}$	n	$\mathfrak{M}(r)/\mathfrak{M}$	n	$\mathfrak{M}(r)/\mathfrak{M}$	n
0.1	0 30	4 0	0 41	4 2:	0 50	4 3:	0 35	4 1:	0 44	4 2:
.2. .	.50	3 5	64	3 8	0 74	4 0	63	3 8	70	3 9
.4 .	75	3 0	86	3 5	0 93	3 8	84	3 4	87	3 5
.6 .. .	87	2 5	95	2 9	0 99	3 8	92	2 8	93	2 8
0 8. ..	0 95	1.9	0 99	2 7	1 00	4 0	0 97	2 0	0 98	2 3

b) Determination of the Motions

i) *Choice of natural units and parameters.*—For the case in which the two interacting galaxies can be represented as polytropes of index $n = 4$ having common radii, the theory developed in Section II can be readily used to determine the relative motion. We shall determine the motion of the galaxy of total mass $\mathfrak{M}_1$ with respect to that of total mass $\mathfrak{M}_2$. It is convenient to take the conditions at the point of closest approach as the initial conditions, for the purposes of the integration. Let us designate the separation of the centers of the two distributions at their closest approach by r_0 and let us measure the polar angle θ from the line connecting the two galaxies at their closest approach. Finally, let V_0 be the relative speed of the mass $\mathfrak{M}_1$ with respect to the mass $\mathfrak{M}_2$ at the point of closest approach.

In order to reduce the number of independent parameters involved in the integrations, the units of distance, time, and mass may be chosen so that

$$R = 1\,, \qquad V_0 = 1\,, \qquad \mu = 1\,. \tag{22}$$

In the work that follows in the present subsection, it is to be understood that these units are being employed except where an explicit statement to the contrary is made. In the chosen units, the initial conditions are specified once r_0 and a parameter α, where

$$\alpha \equiv G\mathfrak{M}_1\mathfrak{M}_2\,, \tag{23}$$

are specified.

The relevant range of the parameter r_0 is clearly given by $0 \leq r_0 < 2$, since for

$r_0 \geq 2$ the problem reduces to that of the familiar $1/r^2$ force. As a matter of fact, in the present case where we are dealing with objects having a relatively high degree of central concentration, it is only for values of r_0 significantly smaller than unity that significant deviations from the results predicted by the simple mass-point approximation are obtained. From equations (22) and (23) we see that α can vary between 0 and the value corresponding to circular motion (cf. eq. [31]) depending upon the magnitudes in absolute units of $\mathfrak{M}_1$, $\mathfrak{M}_2$, R, and V_0. For fixed values for the masses and dimensions of the galaxies in absolute units, an increase in α has the effect of decreasing the initial relative speed in absolute units.

ii) *Integration procedure.*—Given values of r_0 and α in the chosen units, the motion is determined by equations (2) and (3) expressed in terms of the chosen units. We note that at the point of closest approach $d\theta/dr$ and dt/dr become infinite, so that the straightforward numerical integration of these is not possible. However, for $r \cong r_0$, $\Psi(r)$ can be expressed to sufficient accuracy in terms of the first two terms in a Taylor series expansion about the point r_0. In terms of this expansion, the equations can be integrated analytically to determine that part of the orbit near the point of closest approach. These series solutions are used until $d\theta/dr$ and dt/dr derived from the exact equations begin to differ appreciably from those derived from the series expansion. Thereafter, the integration is continued by numerical means until $\Psi(r) \cong 1$.

At a distance of about $r = 0.5$, $\Psi(r)$ becomes essentially unity to the accuracy to which we are working. Thus, from this point outward, the motion is that corresponding to the simple two-body problem for the case in which the mass distributions can be treated as mass points. Accordingly, the corresponding motions in this part of the orbit can be obtained analytically. For the present purposes it is convenient to treat mass points as polytropic distributions of index $n = 5$ with finite radii. In this way, the special unit of length being employed introduces no difficulty in the treatment of mass points.

With the above understanding, the basic equations can be integrated analytically for the case $\Psi(r) = 1$, $r \neq 0$, and all such motions can be described in terms of three parameters: the values of r and θ at the points of closest approach, $r_{0,1}$ and $\theta_{0,1}$ (say), and the value for the parameter α, α_1 (say). The relations that give r, θ, and t in this case in terms of $r_{0,1}$, $\theta_{0,1}$ and α_1 are

$$\theta - \theta_{0,1} = \cos^{-1} \pm \left(\frac{r_{0,1}^2 - r\alpha_1}{r r_{0,1} - r\alpha_1} \right), \tag{24}$$

$$t - t_0 = \frac{X^{1/2}}{Y} - \frac{\alpha_1}{Y(-Y)^{1/2}} \cos^{-1} \left(\frac{Yr + \alpha_1}{r_{0,1} - \alpha_1} \right), \qquad (Y < 0, \text{ elliptic orbit})$$

$$t - t_0 = \frac{1}{2\alpha_1^2} \left(\tfrac{1}{3} X^{3/2} + r_{0,1}^2 X^{1/2} \right), \qquad (Y = 0, \text{ parabolic orbit}) \tag{25}$$

$$t - t_0 = \frac{X^{1/2}}{Y} + \frac{\alpha_1}{Y^{3/2}} \ln \left(\frac{r_{0,1} - \alpha_1}{Yr + \alpha_1 + X^{1/2} Y^{1/2}} \right), \qquad (Y > 0, \text{ hyperbolic orbit})$$

where

$$X \equiv r^2 - \frac{2\alpha_1 r^2}{r_{0,1}} + 2\alpha_1 r - r_{0,1}^2, \qquad Y \equiv 1 - \frac{2\alpha_1}{r_{0,1}}, \tag{26}$$

and where t_0 is the arbitrary zero point from which time is being measured at closest approach.

The continuation of the (4-4) orbit (for descriptive simplicity, the case in which both configurations are polytropes of index $n = 4$ will be referred to as the [4-4] case) for those values of r for which $\Psi(r) \cong 1$ is clearly to be found among the orbits included in equation (24). The orbit that does provide the desired continuation can be determined from the requirement that at the fitting point (any point for which $\Psi(r) \cong 1$), the orbital values for $\theta(r)$, $d\theta/dr$, and $d^2\theta/dr^2$ be continuous. It follows from this requirement that

$$\theta_{0,1} = \theta + \cot^{-1}\left[\frac{1}{d\theta/ar}\left(\frac{d^2\theta/dr^2}{d\theta/dr} + \frac{2}{r}\right)\right], \tag{27}$$

$$r_{0,1} = \frac{r^2 \sin(\theta - \theta_{0,1})\, d\theta/dr}{1 - \cos(\theta - \theta_{0,1}) + r \sin(\theta - \theta_{0,1})\, d\theta/dr}, \tag{28}$$

and

$$a_1 = r_{0,1}\left[1 - \frac{1}{1 - \cos(\theta - \theta_{0,1}) + r \sin(\theta - \theta_{0,1})\, d\theta/dr}\right], \tag{29}$$

where the right-hand sides of these last three relations are to be evaluated at the fitting point. The desired continuation of the (4-4) orbit is then obtained from equation (24) for the values for $r_{0,1}$, $\theta_{0,1}$, and a_1 so obtained. The time dependence given by equations (25) can be transformed to the units appropriate to the (4-4) orbit through the requirement that the velocity be continuous across the fitting point.

No numerical integration is required in order to determine the speed along the orbit. In the units adopted, equation (4) reduces to

$$V = \left[1 + 2a\left(\frac{\Psi(r)}{r} - \frac{\Psi(r_0)}{r_0}\right)\right]^{1/2} \tag{30}$$

from which the relative speed can easily be obtained.

iii) *Numerical results.*—The procedure just described has been carried through numerically for eight different pairs of values r_0 and a. The values of r_0 chosen have been: 0.0, 0.1, 0.2, and 0.3. For each of these values of r_0 the calculations have been carried through for two choices for a. One of the values for a was so chosen that it corresponds to the (4-4) escape orbit, i.e., to the (4-4) orbit corresponding to zero total energy. This a has been denoted by a_{esc} (4-4); from the requirement that the total energy E be zero, it can be readily shown that a_{esc} (4-4) has the value $r_0/[2\Psi(r_0; n_1 = 4, n_2 = 4)]$. The second value for a for a given r_0 has been so selected that it corresponds to the escape orbit for the given initial conditions for the corresponding (5-5) case, the case in which the mass distributions are treated as polytropes of index $n = 5$. This latter a has been denoted by a_{esc} (5-5) and has the value $r_0/2$. Finally, it is of some interest to determine for a given value of r_0 the value of a, a_{cir} (say), corresponding to circular motion for the (4-4) orbit. This follows readily by equating the gravitational force to the centripetal force for the case of circular motion. In the units employed here, we obtain in this way

$$a_{cir} = \frac{1}{[\Psi(r_0)/r_0] - (d\Psi/dr)_{r=r_0}}. \tag{31}$$

In Tables 4, 5, 6, and 7 are tabulated the angle θ, time t, and speed V, as functions of the distance r, between the centers of the two configurations for the (4-4) motions for the cases a_{esc} (4-4) and a_{esc} (5-5) for the values: $r_0 = 0.0, 0.1, 0.2, 0.3$. It will be noted that there are two different columns giving the time dependence. They differ only in the choice of the zero point. In the first of these columns the zero point is taken as the point of closest approach, while in the second the zero point is taken as the time when $r = 2$ at first contact.

For each of the eight (4-4) orbits, the corresponding motions for two related (5-5) orbits of particular interest have been derived. For the first of these, the $(5\text{-}5)_I$ case (say), the conditions for the (4-4) and (5-5) motions are taken to be identical at the point of closest approach for the (4-4) orbit. The results for this case follow from equations (24)

TABLE 4

ORBITAL MOTION FOR THE CASE $r_0 = 0.0$

r	$\alpha = \alpha_{esc}(4\text{-}4) = 0.0865$									
	(4-4)				$(5\text{-}5)_I$			$(5\text{-}5)_{II}$		
	θ	t	t'	v	θ	t	v	θ	t	v
0.00	—	0.000	4.61	1.000	—	0.000	1.000	—	4.53	∞
0.01	90°0	0.010	4.60	1.000				90°0	4.53	4.16
0.05	90.0	0.050	4.56	0.991				90.0	4.52	1.86
0.10	90.0	0.101	4.51	0.965				90.0	4.48	1.32
0.20	90.0	0.212	4.40	0.850				90.0	4.39	0.930
0.30	90.0	0.338	4.27	0.740				90.0	4.27	0.759
0.40	90.0	0.482	4.13	0.653				90.0	4.13	0.658
0.50	90.0	0.644	3.97	0.588				90°0	3.97	0.588
0.60	90.0	0.822	3.79	0.537						
0.80	90.0	1.22	3.39	0.465						
1.00	90.0	1.68	2.93	0.416						
1.50	90.0	3.02	1.59	0.340						
2.00	90.0	4.61	0.00	0.294						
3.00	90.0	8.41	-3.80	0.240						
5.00	90.0	18.0	-13.4	0.186						
10.00	90.0	50.8	-46.2	0.132						
50.00	90.0	567	-562	0.059						
∞	90°0	∞	$-\infty$	0.000						

Note: The orbit is physical only at $r = 0$.

TABLE 4 — Concluded

r	$\alpha = \alpha_{esc}(5\text{-}5) = 0.0000$									
	(4-4)				$(5\text{-}5)_I$			$(5\text{-}5)_{II}$		
	θ	t	t'	v	θ	t	v	θ	t	v
0.00	—	0.00	2.00	1.000	—	0.00	1.000	—	2.00	∞
0.01	90°0	0.01	1.99	1.000	180°0	∞	0.000	90°0	1.99	1.000
0.05	90.0	0.05	1.95	1.000	180.0	∞	0.000	90.0	1.95	1.000
0.10	90.0	0.10	1.90	1.000	180.0	∞	0.000	90.0	1.90	1.000
0.20	90.0	0.20	1.80	1.000	180.0	∞	0.000	90.0	1.80	1.000
0.30	90.0	0.30	1.70	1.000	180.0	∞	0.000	90.0	1.70	1.000
0.40	90.0	0.40	1.60	1.000	180.0	∞	0.000	90.0	1.60	1.000
0.50	90.0	0.50	1.50	1.000	180.0	∞	0.000	90°0	1.50	1.000
0.60	90.0	0.60	1.40	1.000	180.0	∞	0.000			
0.80	90.0	0.80	1.20	1.000	180.0	∞	0.000			
1.00	90.0	1.00	1.00	1.000	180.0	∞	0.000			
1.50	90.0	1.50	0.50	1.000	180.0	∞	0.000			
2.00	90.0	2.00	0.00	1.000	180.0	∞	0.000			
3.00	90.0	3.00	-1.00	1.000	180.0	∞	0.000			
5.00	90.0	5.00	-3.00	1.000	180.0	∞	0.000			
10.00	90.0	10.00	-8.00	1.000	180.0	∞	0.000			
50.00	90.0	50.00	-48.0	1.000	180.0	∞	0.000			
∞	90°0	∞	$-\infty$	1.000	180°0	∞	0.000			

Note: θ changes from 90° to 217° on passing through the origin.

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10.00	121.0	49.3	-44.7	0.136						
50.00	125.6	547	-543	0.061						

TABLE 5

ORBITAL MOTION FOR THE CASE $r_0 = 0.1$

	$\alpha = \alpha_{esc}(4\text{-}4) = 0.0929$									
	(4-4)				$(5\text{-}5)_I$			$(5\text{-}5)_{II}$		
r	θ	t	t'	v	θ	t	v	θ	t	v
0.05369								-50°.6	4.54	1.86
0.060								-12.8	4.52	1.76
0.080								+19.4	4.50	1.53
0.100	00°.0	0.000	4.57	1.000	00°.0	0.000	1.000	35.1	4.48	1.36
0.105					70.5	0.128	0.955			
0.110	26.4	0.049	4.52	0.988	106.4	0.199	0.912			
0.115					146.0	0.289	0.870			
0.11655					180°.0	0.367	0.858			
0.15	52.2	0.121	4.45	0.940				55.9	4.42	1.11
0.20	65.8	0.192	4.38	0.881				67.0	4.36	0.964
0.30	79.2	0.332	4.24	0.767				79.3	4.23	0.787
0.40	86.4	0.484	4.08	0.677				86.4	4.08	0.682
0.50	91.1	0.650	3.92	0.610				+91°.1	3.92	0.610
0.60	94.6	0.831	3.74	0.557						
0.80	99.3	1.23	3.33	0.482						
1.00	102.6	1.69	2.88	0.431						
1.50	107.6	3.01	1.56	0.352						
2.00	110.5	4.57	0.00	0.305						
3.00	114.0	8.27	-3.70	0.249						
5.00	117.5	17.6	-13.0	0.193						

TABLE 5 — Concluded

r	$\alpha = \alpha_{esc}(5\text{-}5) \approx 0.0500$									
	(4-4)				(5-5)$_{I}$			(5-5)$_{II}$		
	θ	t	t'	v	θ	t	v	θ	t	v
0.07513								-26°1	2.59	1.33
0.08								-00.7	2.56	1.30
0.10	00°0	0.000	2.59	1.000	00°0	0.000	1.000	+26.6	2.52	1.20
0.11	25.5	0.047	2.54	0.994	35.1	0.065	0.953			
0.15	50.2	0.117	2.47	0.968	70.5	0.165	0.817	51.9	2.46	1.06
0.20	62.9	0.183	2.40	0.938	90.0	0.267	0.707	63.3	2.40	0.978
0.30	74.7	0.306	2.28	0.882	109.5	0.471	0.577	74.7	2.28	0.890
0.40	80.6	0.429	2.16	0.842	120.0	0.693	0.500	80.6	2.16	0.843
0.50	84.2	0.555	2.03	0.814	126.9	0.933	0.447	+84°2	2.03	0.814
0.60	86.6	0.683	1.90	0.793	131.8	1.19	0.408			
0.80	89.7	0.944	1.64	0.766	138.6	2.07	0.354			
1.00	91.6	1.21	1.38	0.750	143.1	2.40	0.316			
1.50	94.2	1.89	0.69	0.727	150.1	4.24	0.258			
2.00	95.6	2.59	0.00	0.716	154.2	6.39	0.224			
3.00	96.9	3.99	-1.41	0.704	159.0	11.5	0.183			
5.00	98.0	6.85	-4.27	0.694	163.7	24.3	0.141			
10.00	98.8	14.1	-11.5	0.687	168.5	67.7	0.100			
50.00	99.5	72.4	-69.9	0.681	174.9	748	0.045			
∞	99°7	∞	$-\infty$	0.680	180°0	∞	0.000			

TABLE 6

ORBITAL MOTION FOR THE CASE $r_0 = 0.2$

	$\alpha = \alpha_{esc}(4-4) = 0.1198$									
	(4-4)				$(5-5)_{I}$			$(5-5)_{II}$		
r	θ	t	t'	v	θ	t	v	θ	t	v
0.1665								-24°3	4.30	1.20
0.175								+01.2	4.24	1.17
0.200	00°0	0.000	4.27	1.000	00°0	0.000	1.000	24.0	4.17	1.09
0.21	21.0	0.076	4.19	0.986	28.2	0.102	0.971			
0.25	44.2	0.180	4.09	0.933	59.9	0.245	0.872	46.3	4.07	0.979
0.30	58.8	0.275	3.99	0.870	80.3	0.377	0.775	59.4	3.99	0.894
0.40	75.3	0.448	3.82	0.769	104.3	0.629	0.633	75.4	3.82	0.774
0.50	85.2	0.621	3.65	0.692	119.7	0.900	0.531	+85°2	3.65	0.692
0.60	92.2	0.803	3.47	0.632	131.4	1.21	0.449			
0.80	101.4	1.19	3.07	0.547	150.4	2.02	0.319			
1.00	107.6	1.62	2.65	0.489	173.7	3.73	0.205			
1.0121					180°0	4.28	0.198			
1.50	116.8	2.85	1.42	0.400						
2.00	122.2	4.27	0.00	0.346						
3.00	128.5	7.61	-3.34	0.283						
5.00	134.7	15.9	-11.7	0.219						
10.00	140.9	44.1	-39.8	0.155						
50.00	149.1	484	-480	0.069						
∞	155°7	∞	$-\infty$	0.000						

TABLE 6 — Concluded

	$\alpha = \alpha_{esc}(5\text{-}5) = 0.1000$									
	(4-4)				$(5\text{-}5)_I$			$(5\text{-}5)_{II}$		
r	θ	t	t'	v	θ	t	v	θ	t	v
0.1748								-18°1	3.32	1.14
0.18								+00.3	3.27	1.13
0.20	00°0	0.000	3.29	1.000	00°0	0.000	1.000	21.0	3.21	1.08
0.21	20.4	0.073	3.22	0.989	25.2	0.091	0.976			
0.25	42.7	0.174	3.12	0.944	53.1	0.217	0.894	44.2	3.11	0.982
0.30	56.5	0.264	3.03	0.893	70.5	0.330	0.816	56.9	3.03	0.912
0.40	71.9	0.424	2.87	0.811	90.0	0.533	0.707	71.9	2.87	0.815
0.50	80.8	0.580	2.71	0.752	101.5	0.735	0.632	+80°8	2.71	0.752
0.60	86.9	0.739	2.55	0.706	109.5	0.943	0.577			
0.80	94.7	1.07	2.22	0.644	120.0	1.39	0.500			
1.00	99.7	1.41	1.88	0.604	126.9	1.87	0.447			
1.50	106.6	2.33	0.97	0.546	137.2	3.23	0.365			
2.00	110.3	3.29	0.00	0.515	143.1	4.80	0.316			
3.00	114.2	5.34	-2.04	0.481	150.1	8.48	0.258			
5.00	117.5	9.67	-6.37	0.453	156.9	17.6	0.200			
10.00	120.1	21.1	-17.8	0.430	163.7	48.5	0.141			
50.00	122.3	117	-114	0.411	172.8	530	0.063			
∞	122°8	∞	-∞	0.406	180°0	∞	0.000			

TABLE 7

ORBITAL MOTION FOR THE CASE $r_0 = 0.3$

	$\alpha = \alpha_{esc}(4-4) = 0.1581$									
	(4-4)				$(5-5)_I$			$(5-5)_{II}$		
r	θ	t	t'	v	θ	t	v	θ	t	v
0.2838								-10°.0	3.99	1.06
0.30	00°.0	0.000	3.95	1.000	00°.0	0.000	1.000	+16.9	3.86	1.03
0.31	19.0	0.101	3.85	0.987	21.3	0.114	0.983			
0.35	40.7	0.237	3.72	0.938	45.7	0.266	0.922	41.6	3.71	0.950
0.40	55.0	0.353	3.60	0.883	61.9	0.397	0.858	55.3	3.60	0.889
0.50	72.3	0.553	3.40	0.795	81.1	0.620	0.761	+72°.3	3.40	0.795
0.60	83.1	0.743	3.21	0.726	93.3	0.832	0.688			
0.80	96.9	1.13	2.82	0.629	108.7	1.27	0.584			
1.00	105.7	1.54	2.42	0.562	118.7	1.73	0.512			
1.50	118.5	2.67	1.29	0.459	133.7	3.06	0.396			
2.00	125.8	3.95	0.00	0.398	142.8	4.66	0.323			
3.00	134.2	6.94	-2.98	0.325	154.5	8.82	0.227			
5.00	142.5	14.3	-10.3	0.251	170.7	24.1	0.097			
5.5824					180°.0	39.9	0.054			
10.00	150.6	39.0	-35.1	0.178						
50.00	161.4	423	-419	0.080						
∞	170°.0	∞	-∞	0.000						

TABLE 7 — Concluded

	$\alpha = \alpha_{esc}(5-5) = 0.1500$									
	(4-4)				$(5-5)_{I}$			$(5-5)_{II}$		
r	θ	t	t'	v	θ	t	v	θ	t	v
0.2862								-08°.8	3.68	1.05
0.30	00°.0	0.000	3.65	1.000	00°.0	0.000	1.000	+15.5	3.56	1.03
0.31	18.6	0.100	3.55	0.988	20.7	0.111	0.984			
0.35	39.9	0.232	3.41	0.941	44.4	0.259	0.926	40.6	3.41	0.953
0.40	54.0	0.346	3.30	0.890	60.0	0.385	0.866	±54.1	3.30	0.895
0.50	70.7	0.540	3.11	0.807	78.5	0.599	0.775	+70°.7	3.11	0.807
0.60	81.2	0.723	2.92	0.742	90.0	0.800	0.707			
0.80	94.4	1.09	2.55	0.653	104.5	1.20	0.612			
1.00	102.6	1.47	2.17	0.592	113.6	1.63	0.548			
1.50	114.4	2.51	1.14	0.501	126.9	2.80	0.447			
2.00	120.9	3.65	0.00	0.448	134.4	4.13	0.387			
3.00	128.0	6.17	-2.52	0.389	143.1	7.20	0.316			
5.00	134.5	11.9	-8.27	0.333	151.6	14.8	0.245			
10.00	140.2	28.5	-24.9	0.285	160.2	40.2	0.173			
50.00	145.4	188	-185	0.239	171.1	434	0.077			
∞	146°.9	∞	-∞	0.226	180°.0	∞	0.000			

and (25) for $r_{0,1}$, $\theta_{0,1}$, and α_1 equal to the values of r_0, θ_0, and α for the (4-4) orbit. These results are included in Tables 4-7. The unit of time is that for the corresponding (4-4) motion, and time is measured from the common point, the point of closest approach for the (4-4) orbit.

The second of the two (5-5) orbits related to a given (4-4) orbit is that which provides the analytic continuation of the (4-4) orbit for $r \geq 2$. The relevant parameters for these (5-5) orbits, the $(5\text{-}5)_{\text{II}}$ orbits (say), were determined in the course of deriving the analytic continuation for the (4-4) orbits. The data describing the $(5\text{-}5)_{\text{II}}$ orbits corresponding to each of the (4-4) orbits are also included in the aforementioned tables. Here, too, the unit of time has been so chosen that for $r \geq 2$, the velocities at corresponding points along the (4-4) and $(5\text{-}5)_{\text{II}}$ orbits are equal. The zero point for time for this case has been taken at the point where $r = 2$ at first contact.

The various orbits referred to above are illustrated in Figures 1 and 2. Since the motions are symmetric about the point of closest approach, the tables contain θ, t, and V as functions of r for only half the orbit, beginning with the point of closest approach.

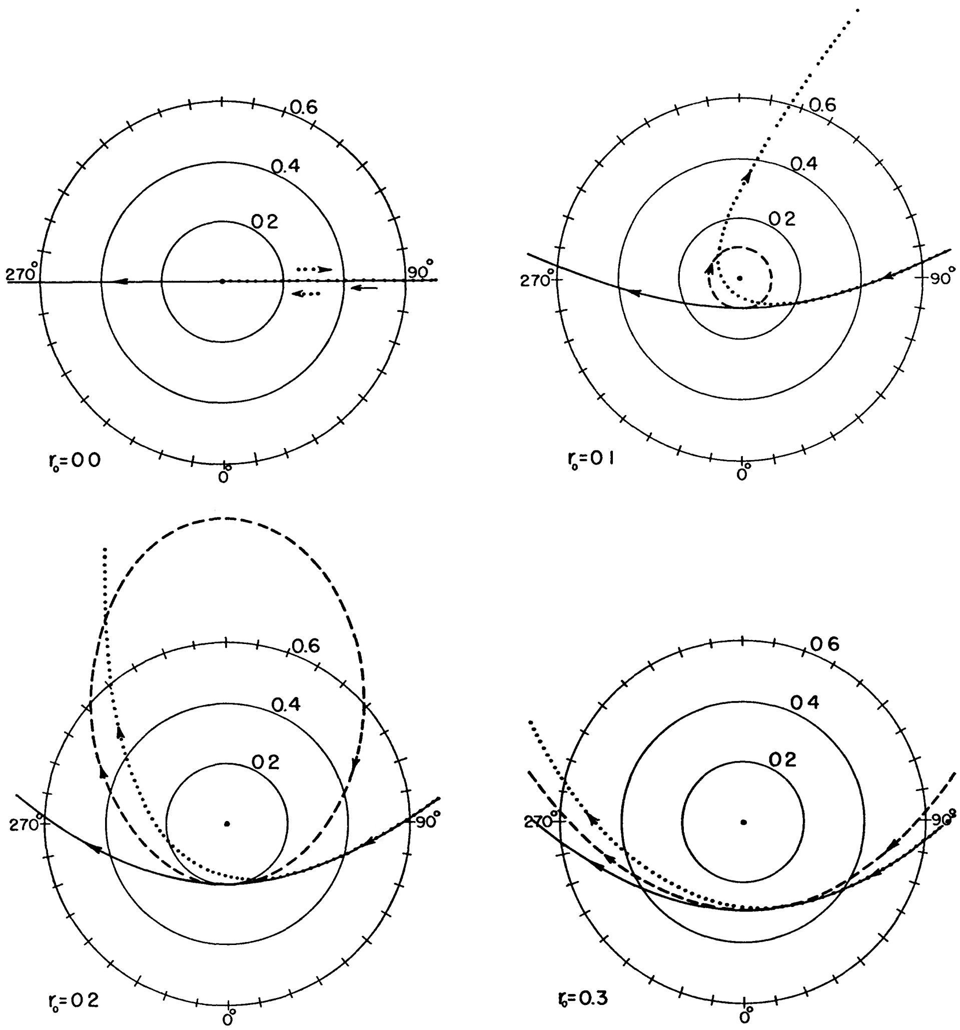

FIG. 1.—Relative orbits of $\mathfrak{M}_1$ about $\mathfrak{M}_2$ for the case $\alpha = \alpha_{\text{esc}}$ (4-4). The (4-4), $(5\text{-}5)_{\text{I}}$, and $(5\text{-}5)_{\text{II}}$ orbits are represented by solid, dashed, and dotted curves, respectively. For the case $r_0 = 0.0$, the $(5\text{-}5)_{\text{I}}$ orbit degenerates to a point at the center.

From the results obtained for each pair of values for r_0 and α, we can derive the corresponding motions in absolute units for a twofold infinity of cases. Let us define the conversion factors for transforming the chosen units to absolute units by means of the following relations;

$$r \text{ (absolute units)} = K_r \times r \text{ (chosen units)} ,$$

and

$$t \text{ (absolute units)} = K_t \times t \text{ (chosen units)} , \tag{32}$$

$$\mathfrak{M} \text{ (absolute units)} = K_{\mathfrak{M}} \times \mathfrak{M} \text{ (chosen units)} .$$

It then follows from the definition of the units chosen that

$$K_r = R(\text{absolute units}), \quad K_t = \frac{R}{V_0}(\text{absolute units}), \quad K_{\mathfrak{M}} = \mu(\text{absolute units}). \tag{33}$$

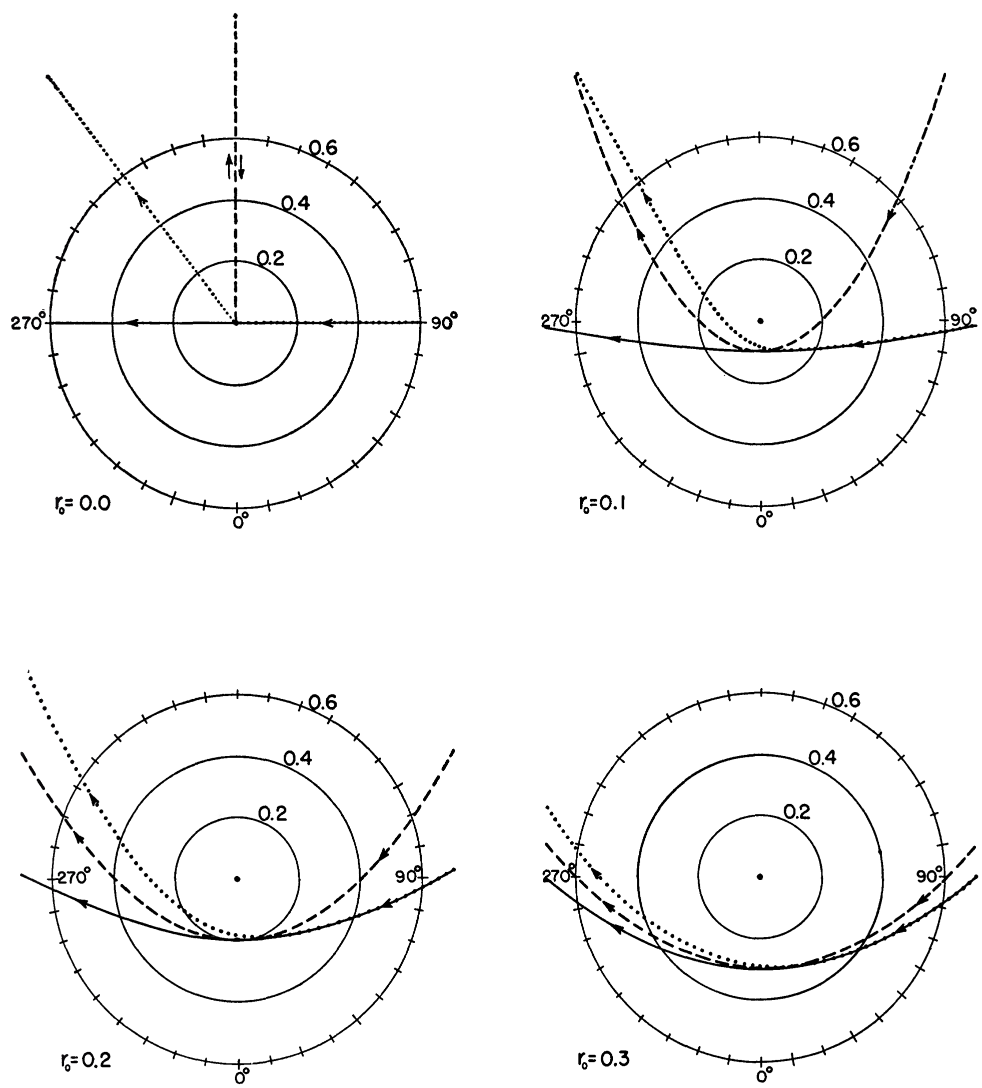

FIG. 2.—Relative orbits of $\mathfrak{M}_1$ about $\mathfrak{M}_2$ for the case $\alpha = \alpha_{esc}$ (5-5). The (4-4), $(5\text{-}5)_{\rm I}$, and $(5\text{-}5)_{\rm II}$ orbits are represented by solid, dashed, and dotted curves, respectively.

If α in absolute units is expressed in terms of its value in the chosen units by

$$\alpha\ (\text{absolute units}) = K_\alpha \times \alpha\ (\text{chosen units})\,, \tag{34}$$

then from equations (23) and (33) the conversion factor for α follows as

$$K_\alpha = R\mu V_0^2\ (\text{absolute units})\,. \tag{35}$$

c) Estimate for the Energy Interchange

The most self-consistent procedure for estimating the energy interchange on the basis of the foregoing work is to derive the energy interchange on the basis of the center-of-mass motions in the first approximation. However, for the purpose of the present preliminary survey of the problem of energy interchange, it appears more reasonable and useful to emphasize a somewhat different approach. As was noted earlier, the derivation of estimates for the energy interchange on the basis of uniform rectilinear center-of-mass motions is a much simpler problem in practice than is the corresponding derivation that uses the center-of-mass motions derived according to the first approximation. Further, it is readily seen that results derived for $\Delta U/|U|$ on the basis of uniform rectilinear relative motions for the galaxies for given p/R scale in a very simple way insofar as the masses $\mathfrak{M}_1$ and $\mathfrak{M}_2$, the common radius R, and the initial relative speed V of the two galaxies are concerned; this is not the case for calculations of $\Delta U/|U|$ based upon center-of-mass motions in the first approximation. Thus, for example, if the two colliding galaxies are each treated as polytropes of index 4 having common radii, then insofar as the numerical calculations are concerned $\Delta U/|U|$ needs be calculated only as a function of the one parameter p/R. It follows that if for given initial conditions, the value of $\Delta U/|U|$ that would be derived on the basis of the center-of-mass motions in the first approximation can be approximated by the value of $\Delta U/|U|$ derived on the basis of a uniform rectilinear center-of-mass motion related in some simple way to the motion in the first approximation, then a considerable reduction in computational work will have been achieved. Since the greatest contribution to $\Delta U/U$ is expected to result from the motion near the point of closest approach, it seems reasonable to expect that the values for $\Delta U/U$ as derived on the basis of the first approximation and on the basis of uniform rectilinear center-of-mass motion will be approximately the same so long as the same separation and relative speed at closest approach are used in the two calculations. This prediction will be tested for one particular case of a very close encounter and will be found to be satisfied to a reasonable approximation.

Here, as in the preceding subsection, it will be assumed that the two galaxies can be treated as polytropes of index 4 having common radii. Let us first derive estimates for $\Delta U/|U|$ on the basis of uniform rectilinear motions. Representative stars are chosen in sets of six in such a way that the stars in each set lie on the points of intersection of a sphere of given radius centered at the center of galaxy $\mathfrak{M}_2$ with the axes of the Cartesian coordinate system introduced earlier. The stars are chosen at intervals of 0.1 R, and the results for $\Delta U/|U|$ are given numerically for the case in which the two galaxies have identical masses and radii equal to $10^{11}\,\mathfrak{M}_\odot$ and 10 kpc, respectively, and where the relative velocity of the centers of mass is 1000 km/sec. For each of seven different values of the impact parameter, the root-mean-square values for the change in the velocity of the stars in the different representative sets are given in Table 8; also given in the table are the corresponding values for $\Delta U/|U|$ for the galaxy as a whole. The circular velocities corresponding to the distances from the center of these representative sets are included in the table for purposes of comparison. It is clear from the work in Section II that for a given p/R, the scaling of $\Delta U/|U|$ in $\mathfrak{M}_1$, $\mathfrak{M}_2$, R, and V, goes as

$$\frac{\Delta U}{|U|} \propto \frac{\mathfrak{M}_1^2\,\mathfrak{M}_2}{V^2R^2}\,\frac{1}{\Omega_2} \propto \frac{\mathfrak{M}_1^2}{\mathfrak{M}_2}\,\frac{1}{V^2R}\,, \tag{36}$$

$\Delta U_{p/R}/\Delta U_{\infty}$	1.00	1.00	0.99	0.97	0.93	0.35	0.00	

TABLE 8

$(\Delta v')_{R.M.S.}$ in kms/sec

a/R \ p/R	10	5	2	1	0.6	0.2	0.0	Circular Velocity
0.0	0	0	0	0	0	0	0	0
0.1	0.070	0.28	1.8	7.1	20	100	330	360
0.2	0.14	0.56	3.5	14	41	220	300	400
0.3	0.21	0.84	5.3	22	68	370	220	360
0.4	0.28	1.1	7.7	31	100	360	170	320
0.5	0.35	1.4	9.1	43	120	340	140	290
0.6	0.42	1.7	11	59	88	330	120	270
0.7	0.49	2.0	13	83	240	320	100	250
0.8	0.56	2.3	16	120	220	310	88	230
0.9	0.63	2.6	18	140	190	310	78	220
1.0	0.71	2.9	22	53	170	310	70	210
$\frac{\Delta U}{\lvert U \rvert}$	1.4×10^{-7}	2.2×10^{-6}	8.8×10^{-5}	1.6×10^{-3}	1.3×10^{-2}	3.1×10^{-1}	8.6×10^{-1}	$\lvert U \rvert = 1.29\times10^{59}$
$\frac{\Delta U}{\lvert U \rvert}$ [eqn. (37)]	1.4×10^{-7}	2.2×10^{-6}	8.5×10^{-5}	1.4×10^{-3}	1.1×10^{-2}	8.5×10^{-1}	∞	c.g.s. units
$\Delta U_{p/R}/\Delta U_{\infty}$	1.00	1.00	0.99	0.97	0.93	0.35	0.00	

so that the numerical values given can be readily converted to other choices of $\mathfrak{M}_1$, $\mathfrak{M}_2$, R, and V by means of equation (36).

The dependence of energy exchange on the impact parameter for values of p/R that are not too small can be obtained from equation (21). If the galaxies are represented as polytropes of index 4, then $r_c = 0.188\, R$, and equation (21) reduces to

$$\frac{\Delta U}{|U|} = 0.0314 \frac{G\,\mathfrak{M}_1{}^2 R^3}{\mathfrak{M}_2 p^4 V^2}. \tag{37}$$

For the case in which $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are equal, equation (37) can be rewritten in terms of the maximum circular velocity, v_{cir}', for the galaxies as

$$\frac{\Delta U}{|U|} \propto \frac{v_{\text{cir}}'^2 R^4}{V^2 p^4}. \tag{38}$$

In Table 8 the dependence upon p of the values of $\Delta U/|U|$ are compared with those predicted by equation (37). It is seen that the dependence of $\Delta U/|U|$ on p is described with reasonable accuracy by equation (37) or (38) for $p/R \gtrsim 0.5$ and is in error by less than a factor of 3 for $p/R \gtrsim 0.2$.

From the results derived here on the basis of uniform rectilinear relative motion, it can be readily shown that it is the very close encounters, in spite of their low frequency of occurrence, that contribute most to $\Delta U/|U|$ in the life of a typical galaxy. Thus, the contribution to ΔU by encounters corresponding to values of the impact parameter less than or equal to a specified value for p/R in units of the change in U due to encounters for all value of p/R is given by

$$\frac{\Delta U_{p/R}}{\Delta U_\infty} = \frac{\int_0^{p/R} (p'/R)\Delta U(p'/R)\, d(p'/R)}{\int_0^\infty (p'/R)\Delta U(p'/R)\, d(p'/R)}. \tag{39}$$

Table 8 also gives $\Delta U_{p/R}/\Delta U_\infty$ for several different values for p/R for the case of uniform rectilinear center-of-mass motion. The importance of very close encounters for the interchange of energy is readily apparent from the results given in this table.

The foregoing calculations have been carried through on the assumption that the relative motion of the centers of mass of the two galaxies is uniform rectilinear motion. These calculations, therefore, do not attempt to include the effects of the change in the relative velocity during an encounter. To examine the importance of such effects, it seems reasonable to carry through calculations for $\Delta U/|U|$ on the basis of the center-of-mass motions according to the first approximation. This has been done for the case $r_0 = 0.2$, $a = 0.1$ of the preceding subsection. This case corresponds to a value of the impact parameter of about 0.5 R. If the masses of the two galaxies are taken to be 10^{11} solar masses and their common radii to be 10 kpc, it follows that the relative speed at closest approach and at infinity are about 940 km/sec and 380 km/sec, respectively. Representative stars have been chosen here just as for the calculations of $\Delta U/|U|$ on the basis of uniform rectilinear motion, and for the representative stars at 0.1 R and 0.3 R, the Cartesian components for the change in velocity, $\Delta \boldsymbol{v}'$ have been derived. It has not seemed necessary to carry out the calculations for sets of representative stars beyond 0.3 R for the present purposes since the contribution by such stars to $\Delta U/|U|$ is small, more than 90 per cent of the total mass lying interior to 0.3 R. The results so obtained are given in Table 9. For the purposes of comparison, the corresponding values for Cartesian components of $\Delta \boldsymbol{v}'$ for the case of that uniform rectilinear relative motion corresponding to the same separation and relative velocity at closest approach (i.e., p/R = 0.2, V = 940 km/sec) have also been included in the table. It is seen that both for the

TABLE 9

$\Delta v'$ in kms/sec for the case $r_0=0.2$, $\alpha=0.1$

Star Number	Position of Star	Uniform Rectilinear Motion			Computed Orbits		
		v'_x	v'_y	v'_z	v'_x	v'_y	v'_z
			a = 0.1 R				
1	+ x axis	+49	0	0	- 5.3	0	0
2	- x axis	-94	0	0	-96	0	0
3	+ y axis	0	0	0	+11	+64	0
4	- y axis	0	0	0	+11	-64	0
5	+ z axis	-62	0	-160	-69	0	-210
6	- z axis	-62	0	+160	-69	0	+210
$\langle \Delta u(a) \rangle$		5.95x10^3			9.05x10^3		
			a = 0.3 R				
7	+ x axis	-820	0	0	-900	0	0
8	- x axis	-200	0	0	-150	0	0
9	+ y axis	0	0	0	+ 67	+210	0
10	- y axis	0	0	0	+ 67	-210	0
11	+ z axis	-250	0	-210	-260	0	-270
12	- z axis	-250	0	+210	-260	0	+270
$\langle \Delta u(a) \rangle$		7.65x10^4			1.02x10^5		

representative stars at 0.1 R and 0.3 R, the contribution to $\Delta U/|U|$ for the two cases differ by a factor of about 1.5 or less. This result supports the prediction made above concerning the values for $\Delta U/|U|$ derived according to the two different center-of-mass motions.

We are now in a position to make a few further comments concerning the usefulness and accuracy of the first approximation for the center-of-mass motions. First, although in the discussion just presented, it has been shown that the motion in the first approximation need not be used directly in the calculation of the energy exchange, a knowledge of the motion in the first approximation has been necessary in order to determine that uniform rectilinear motion corresponds to the same separation and relative speed at closest approach as does the motion in the first approximation. Now, although the center-of-mass motion in the first approximation is not the true motion, it appears to provide a better approximation to the true motion than does the motion derived on the basis of point masses. In any case, the difference between the motions predicted by the first approximation and by the treatment according to point masses should indicate for the first time the order of the uncertainties involved in the latter approach. With the computed estimates for the energy exchange, as well, we should be able to go well beyond this in drawing conclusions concerning the true motion.

IV. IMPLICATIONS OF THE RESULTS AND DISCUSSION

a) Effects of Collisions on the Structure of Galaxies

It will be seen from the results obtained for the energy exchange in the previous section that for close encounters the increase in internal energy of a galaxy due to an encounter with a second galaxy can easily be of the same order as its initial internal energy. Whenever this is the case, the theory for the center-of-mass motions in the first approximation will not provide an accurate description of the center-of-mass motion since the assumption that the energy corresponding to the center-of-mass motion of the galaxies is constant would not then be even approximately correct, in general. The above theory, however, does provide something of a lower limit for the energy exchange in the sense that a more accurate treatment, that takes into account the decrease in the external energy due to this energy transfer, will lead to still greater values for the energy exchange. The above results, therefore, show that the effects of increases in the internal energy of galaxies as a result of close encounters can markedly influence the structure of galaxies. They also support Zwicky's (1957) contention that on close encounters galaxies can disrupt each other and populate the intergalactic space not only with gas and dust but also with stars. When it is realized that double and multiple systems of galaxies are quite common, it appears likely that major changes in the internal structure of galaxies due to collisions are important for a non-negligible number of the galaxies in the observable part of the universe.

Previous rough estimates for the energy interchange have been carried out by Spitzer and Baade (1951). It is of considerable interest to compare their results with those obtained here. Making a very rough estimate for the fact that the galaxies are not mass points, Spitzer and Baade concluded that in a hundred collisions the internal energy of an elliptical galaxy would increase by about 2 per cent, so that the effects of one hundred collisions on the size and shape of a galaxy would be negligible. In their calculations, Spitzer and Baade assumed that the velocities of the stars in the colliding galaxies range up to 200 km/sec and that the relative velocity of the galaxies was 2400 km/sec (as in the Coma cluster). Since $\Delta U/|U|$ varies as $v_{\text{cir}}'^2/V^2$, it follows that the results given in Table 8 should be reduced by a factor of about 20 in order that a comparison may be made with Spitzer and Baade's estimate. When this is done, the present treatment predicts for the above velocities an increase in internal energy of about 3 per cent in a hundred collisions, even if a collision be defined in the most liberal way so as to include the situations of negligible interpenetration. Now, a collision in Spitzer and Baade's sense is an encounter in which there is an appreciable overlapping of the distributions of the interstellar matter within the two galaxies and hence, presumably, of the stars. Since for a polytrope of index $n = 4$, about 90 per cent of the mass is inclosed within a radius equal to 0.3 R, it is perhaps more appropriate for the purposes of the present comparison to define a collision as an encounter in which the spheres containing this fraction of the mass interpenetrate. If a collision be defined in this way, it follows that an increase of $\Delta U/|U|$ by about 30 per cent is to be expected in one hundred collisions. Thus, insofar as a comparison of Spitzer and Baade's very rough model with the presently suggested (and still highly idealized) model is meaningful, it appears that the changes in the internal energy that result from collisions of galaxies may be considerably larger than might have been inferred from a superficial examination of the results given by Spitzer and Baade.

b) Dynamics of Double Galaxies

It is known from classical dynamics that an encounter between two mass points subject only to their mutual gravitational interaction cannot lead to the formation of a closed binary system without the aid of a third body. It will be seen from the foregoing discussion, however, that two colliding galaxies can form a binary system by virtue of the decrease in their center-of-mass energy during an encounter. For the case in which the collision is one involving two galaxies with common mass $10^{11}\,\mathfrak{M}_\odot$ and common radius 10 kpc and for the initial conditions such that the relative speed at infinite separation is 380 km/sec and the impact parameter is 0.5 R, the energy, E, of the center-of-mass motion is only about half the computed increase in the internal energies of the two galaxies. Thus, although the two galaxies will clearly be moving on hyperbolic orbits with respect to one another after the encounter if it is assumed that energy of center-of-mass motion is conserved, a more accurate treatment is likely to show that the two galaxies with the given initial conditions will form a double system as a result of the encounter. It also follows that double galaxies, however formed, will revolve around each other with mean separations that will continually decrease in time. Given long enough, the components of a double galaxy will disrupt one another and give rise to a single loose system. Natural extensions of the present work should provide estimates for the rate at which this evolution will proceed.

c) Dynamics of Clusters of Galaxies

One application of the present theory involves the study of the escape of galaxies from clusters as a result of close encounters with other galaxies within the cluster. In order to decide whether or not a galaxy can escape from a cluster by virtue of the change in velocity it has undergone as a result of an encounter, one has simply to compare the velocity of escape from the cluster at that point with the velocity of the galaxy after the collision. Since the treatment of galaxies as mass points leads to predictions of greater changes in the relative velocities as a result of encounters than does the more accurate first approximation developed here, predictions based upon the former treatment may require significant modifications. The differences may well be important in the study of escape of galaxies from at least certain clusters. Here, however, it must be emphasized that the first approximation for the center-of-mass motions will, itself, be a reasonably accurate approximation only if the value for $\Delta U/|U|$ calculated for this case is of a smaller order than unity. From the above discussion it is clear that if $\Delta U/|U|$ is of order unity, then the increase in the internal energy will take an appreciable amount of energy out of the center-of-mass motion and will act further to reduce the probability that a galaxy will escape from a cluster as a result of a close encounter.

It may also be of interest to note that, if one is dealing with a cluster in which significant fractions of the galaxies are interpenetrating one another at any typical instant, then the virial theorem for a cluster in quasi-equilibrium,

$$\langle T\rangle = \tfrac{1}{2}\left\langle\frac{\partial\Omega}{\partial r}\, r\right\rangle \tag{40}$$

(cf. Goldstein 1950, eqs. [3–28]), reduces to

$$\sum_i \mathfrak{M}_i V_i^2 = \sum_{\text{pairs}} \frac{G\,\mathfrak{M}_i\,\mathfrak{M}_j}{r_{ij}}\left[\Psi(\mathfrak{r}_{ij}) - \mathfrak{r}_{ij}\,\frac{d\Psi(\mathfrak{r}_{ij})}{d\,\mathfrak{r}_{ij}}\right], \tag{41}$$

where the sums in the first and second terms are to be taken over all the galaxies and over all pairs of galaxies, respectively, and where r_{ij} denotes the distance between the centers of the ith and jth galaxies, where $\mathfrak{M}_i$ is the mass of the ith galaxy, and V_i is the magnitude of the velocity of the ith galaxy with respect to the center of mass of the cluster. For a pair of interpenetrating galaxies, $\Psi(\mathfrak{r}_{ij})$ lies between 0 and 1, and $d\Psi(\mathfrak{r})_{ij}/d\mathfrak{r}_{ij}$ is a positive quantity. Hence it follows from equation (41) that in equilibrium the members of the cluster will have smaller velocities than those predicted in the approximation that treats the galaxies as mass points. As a rule, the number of galaxies in a cluster that are undergoing interpenetrating collisions at any instant appears to be very small; hence the modified form of the virial theorem given by equation (41) will probably hardly differ from the familiar one in most cases.

It appears that the equations for the relaxation time of a cluster derived by Chandrasekhar (1942) will give fair estimates when applied to clusters of galaxies, at least insofar as the order of magnitude is concerned. Nevertheless, the derivation of the expression for the relaxation times is significantly less good when applied to galaxies than when applied to stars for three different kinds of reasons. First, of course, the theory for the relaxation time has been developed for encounters involving the gravitational interaction of mass points. Since the mean separation of neighboring galaxies in clusters is not more than about an order of magnitude greater than the dimensions of the galaxies, encounters involving the overlapping of galaxies and hence deviations from inverse square forces will be important in clusters of galaxies. Second, the conversion of center-of-mass energy into internal energy may be an important additional factor in the case of a cluster of galaxies. Finally, because of the differences between the masses, mean separations, and mean relative velocities for the case of clusters of galaxies and for the case of clusters of stars, the neglect of "non-dominant" terms at three different points in Chandrasekhar's derivation of the relaxation time, T_E, is not justified insofar as applications to clusters of galaxies are concerned. For values for the relevant parameters appropriate to clusters of galaxies, the neglected terms are of the same order as the "dominant" term, the only term retained in Chandrasekhar's analysis. For these reasons, the usual expression for the time of relaxation must be used with some caution in considering clusters of galaxies. It seems unlikely, however, that the effects described above will change the relaxation time by more than an order of magnitude.

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REFERENCES

Brandt, J. D. 1960, *Ap. J.*, **131**, 293.
Chandrasekhar, S. 1942, *Principles of Stellar Dynamics* (Chicago: University of Chicago Press).
Goldstein, H. 1950, *Classical Mechanics* (Reading, Mass.: Addison-Wesley Publishing Co.).
Limber, D. N. 1961, *Ap. J.*, **134**, 537.
Spitzer, L. 1958, *Ap. J.*, **127**, 17.
Spitzer, L., and Baade, W. 1951, *Ap. J.*, **113**, 413.
Vaucouleurs, G. de. 1953, *M.N.*, **113**, 134.
Zwicky, F. 1957, *Morphological Astronomy* (Berlin: Springer-Verlag).