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A STATISTICAL THEORY OF STELLAR ENCOUNTERS

S. CHANDRASEKHAR

ABSTRACT

In this paper the principles of a statistical theory of stellar encounters are developed. The fundamental idea of this new method is to describe the fluctuating part of the gravitational field acting on a star in terms of two functions: a function $W(F)$, which gives the probability of occurrence of a field strength F, and a function $T(F)$ which gives the average time during which the field strength F acts. With regard to $W(F)$ it is shown that a probability-distribution function derived by Holtsmark to describe the interionic fields in a discharge tube can be adapted to suit the gravitational case. In a certain approximation this probability of a given field is directly related to the probability of finding the nearest neighbor to a given star at some prescribed distance. In this latter approximation the mean life of the state F can be obtained by using a formula due to Smoluchowski in the theory of Brownian motion. In terms of these functions $W(F)$ and $T(F)$ the probable accelerations which a star will undergo can be determined according to the principles of the theory of random walk.

As an application of the methods of this statistical theory, the problem of the time of relaxation, t_R, of a stellar system has been reconsidered. It is found that

$$t_R = \frac{1}{6}\left(\frac{3}{2\pi}\right)^{3/2} \frac{(\overline{v^2})^{3/2}}{G^2M^2N\left[\log_e\left(\frac{D\overline{v^2}}{3GM}\right) - 0.355\right]},$$

where G is the constant of gravitation, N the number of stars per unit volume, $\overline{v^2}$ their mean square speed, M the mass of a star, and, finally, $D = (4\pi/3N)^{-1/3}$. This formula for the time of relaxation is shown to be in agreement with that derived by the alternative method in which the individual encounters are analyzed in terms of the two-body problem.

1. *Introduction.*—In estimating the influence of a fluctuating stellar distribution on the motions of stars it has invariably been supposed that such effects can be considered as the cumulative result of a large number of separate events, each of which can be idealized as distinct two-body encounters.[1] But a closer examination of the problem along these lines reveals an essential inconsistency in the assumptions made. For, on evaluating any of the desired quantities (e.g., $\Sigma|\Delta E|^2$ or $\Sigma \sin^2 2\Psi$ [cf. I and II]), it appears that the most important of the effects arise from encounters with impact parameters of the same order as the average distance between the stars. In other words, the physical situations most relevant to the problem are precisely those for which the two-body idealization of stellar encounters fails as a satisfactory mode of description. While this results in a divergence of the appropriate integrals as the impact parameter D tends to infinity and has in consequence to be cut off arbitrarily at some value of D, the really serious drawback of the method consists, however, in the essential inadequacy of the model to take account of the inherent *physical* aspects of the problem. A consideration of this and other related difficulties suggests that we abandon the two-body approximation of stellar encounters altogether and devise a more satisfactory *statistical method.* It is the object of this paper to outline the principles of such a statistical theory and to show its practical feasibility by reconsidering the problem of the time of relaxation of a stellar system along these new lines.

[1] For the most recent version of the theory based on these ideas see S. Chandrasekhar, *Ap. J.*, **93**, 285, 1941; R. E. Williamson and S. Chandrasekhar, *Ap. J.*, **93**, 305, 1941; and S. Chandrasekhar, *Ap. J.*, **93**, 323, 1941. These papers will be referred to as I, II, and III, respectively. References to earlier literature will be found in these papers.

2. *The general principles of the statistical method.*—Quite generally the force $\boldsymbol{F}$ acting on a star, per unit mass, is given by

$$\boldsymbol{F} = -G\Sigma \frac{M_n}{|\boldsymbol{r}_n|^3} \boldsymbol{r}_n , \tag{1}$$

where M_n denotes the mass of a typical field star and $\boldsymbol{r}_n$ its position vector relative to the star under consideration. Further, the summation in equation (1) is taken over all the neighboring stars. The actual value of $\boldsymbol{F}$ at any particular instant will depend on the instantaneous positions of all the other stars and is in consequence subject to *fluctuations*. It would therefore be practically impossible to specify the exact dependence of $\boldsymbol{F}$ on the position and/or the time for individual cases. But, on the other hand, we can ask the probability of occurrence of any particular field strength. In evaluating this probability, we can (consistent with the physical situations we have in view) suppose that fluctuations subject only to the restriction of a constant average density occur.

Let

$$W(X, Y, Z)dXdYdZ \tag{2}$$

be the probability that $\boldsymbol{F}$ occurs in the range

$$(X, Y, Z) \leqslant \boldsymbol{F} \leqslant (X + dX, Y + dY, Z + dZ) . \tag{3}$$

From the symmetry of the problem we should expect that

$$W(F) = 4\pi F^2 W(\boldsymbol{F}) , \tag{4}$$

where F stands for $|\boldsymbol{F}|$. The meaning of $W(F)$ is simply that it gives the fraction of a long-enough interval of time during which a force of intensity F acts. A knowledge of $W(F)$ is therefore essential to our problem. It does not, however, provide all the necessary information. For, in order that we may be able to follow the motion of any particular star statistically, we need to know in addition the average length of time during which a given field strength acts *once it has become operative*. In other words we also require a knowledge of the *mean life* of the statistical state defined by F.

Now the notions of the mean life and the *spontaneous decay* of a given state of fluctuation has been introduced by Smoluchowski in his general investigations on Brownian motion and fluctuation phenomena.[2] According to these ideas of Smoluchowski, the probability $\phi(t)dt$ that a state represented by a well-defined statistical fluctuation continues to exist for a time t and makes a transition to a state of different fluctuation during t and $t + dt$ is expressible in the form

$$\phi(t)dt = e^{-t/T} \frac{dt}{T} , \tag{5}$$

where T is a constant characteristic of the state. Accordingly, we may define T as the mean life of the state under consideration. In our case we need to specify the mean life

$$T(F) \tag{6}$$

[2] Marian von Smoluchowski, *Wien. Ber.*, **123**, II*a*, 2381, 1915; *ibid.*, **124**, II*a*, 339, 1915; see also his papers in *Phys. Zs.*, **16**, 321, 1915, and **17**, 557, 585, 1916. For a general account of Smoluchowski's ideas see R. Furth, *Schwankungserscheinungen in der Physik* (*Sammlung Vieweg*), Heft 48, Braunschweig, 1920.

of a state in which a force of intensity F acts on a particular star (per unit mass). General considerations would suggest that for the average field strengths $\bar{F}$ we should expect

$$T(\bar{F}) \sim \frac{\bar{D}}{\overline{|v|}}\,, \tag{7}$$

where $\bar{D}$ stands for the average distance between the stars and $\overline{|v|}$ their average speeds.

Now, when a state defined by $\boldsymbol{F}$ becomes realized in consequence of fluctuations, the star will be accelerated at the rate F, and, since the mean life of this state is $T(F)$, the average acceleration to be expected during such a state is

$$\Delta \boldsymbol{v} = T(F)\boldsymbol{F}\,. \tag{8}$$

When this state of fluctuation gives place to another, the star will begin to be accelerated at a different rate and in a direction uncorrelated with that in the earlier state. Hence the net acceleration suffered by the star is formally given by

$$\Sigma\Delta \boldsymbol{v} = \Sigma T(F)\boldsymbol{F}\,, \tag{9}$$

where, as we have already indicated, the frequency of occurrence of the different values of $\boldsymbol{F}$ will be governed by $W(\boldsymbol{F})$.

On the basis of equation (9) we cannot, of course, predict the actual acceleration suffered by a star in any specified length of time. On the other hand, according to the principles of the theory of *random walk*[3] we should be able to predict the probability of a star's having been accelerated by a specified amount in a given length of time. This is the principle of our method.

After this general statement of the fundamental ideas we proceed to a more detailed consideration of the various factors which are involved.

3. *The probability of a given field strength. The Holtsmark distribution.*—According to our remarks in § 2, our first problem is to determine the probability of occurrence of a given field strength at some definite point due to a random distribution of centers of inverse square field of forces. This problem is clearly equivalent to finding the probability of a given *electric* field strength acting at a point in a gas composed of simple ions. This latter problem has been considered by J. Holtsmark;[4] and, re-writing his probability function to be appropriate for the gravitational case, we have

$$W(F) = \frac{2F}{\pi}\int_0^\infty e^{-\frac{8}{15}\sqrt{2\pi}\,\pi(GM)^{3/2}N\rho^{3/2}}\,\rho\,\sin F\rho\, d\rho\,, \tag{10}$$

where N stands for the number of stars per unit volume. We can re-write the foregoing formula for $W(F)$ more conveniently if we introduce a normal field strength, defined by

$$Q_H = \left(\tfrac{8}{15}\sqrt{2}\right)^{2/3}\pi GMN^{2/3} = 2.603\,GMN^{2/3}\,, \tag{11}$$

[3] Lord Rayleigh, *Collected Papers*, **1**, 491, Cambridge, England, 1899; **2**, 370, 1903; M. von Smoluchowski, *Bull. Acad. Cracovie*, p. 203, 1906; J. H. Jeans, *An Introduction to the Kinetic Theory of Gases*, p. 219, Cambridge, England, 1940; E. H. Kennard, *Kinetic Theory of Gases*, chap. vii, New York: McGraw-Hill, 1938.

[4] *Ann. d. Phys.*, **58**, 577, 1919. See also the papers by the same author in *Phys. Zs.*, **20**, 162, 1919, and **25**, 73, 1924.

and express F in terms of this unit. According to equations (10) and (11), we have

$$W(F) = \frac{2}{\pi F}\int_0^\infty e^{-(Q_H/F)^{3/2}x^{3/2}} x \sin x dx \,, \tag{12}$$

or, if

$$F = \beta Q_H \,, \tag{13}$$

$$W(\beta)d\beta = \frac{2d\beta}{\pi\beta}\int_0^\infty e^{-(x/\beta)^{3/2}} x \sin x dx \,. \tag{14}$$

The function $W(\beta)$ has been evaluated numerically by Holtsmark and more recently by Verweij.[5]

We may note that, according to equation (12),

$$W(F) \to \frac{3}{2}\frac{Q_H^{3/2}}{F^{5/2}} \,, \qquad F \to \infty \,. \tag{15}$$

This corresponds to a relatively slow decrease of the probability for high field strengths. Indeed, the probability distribution (12) gives an infinite value for the mean square field, $\overline{F^2}$. For certain physical problems this is unsatisfactory, and Gans[6] and Holtsmark[7] have modified the law (12), in the electrical case, to take into account the finite sizes of the ions. For the astronomical applications we have in view, the finite sizes of the stars cannot clearly be of any relevance. However, a modification of a different nature must be introduced before we can use the Holtsmark distribution (12). We shall return to this question in § 5.

4. *The probable field strengths produced by the nearest neighbor.*—In a general way it is clear that the main contribution to the field acting on a star must be due to its nearest neighbor. Indeed, as we shall presently see, the probable field strengths produced by the nearest neighbor provides a sufficiently good first approximation to the probability distribution according to equation (12).

To show this, consider first the probability $w(r)dr$ of finding the nearest neighbor to a given star between r and $r + dr$. It is readily seen that if the distribution of the stars is perfectly random (subject only to the restriction of a constant average density N) then $w(r)$ must satisfy the equation[8]

$$\left[1 - \int_0^r w(r)dr\right] 4\pi r^2 N = w(r) \,. \tag{16}$$

From equation (16) we derive

$$\frac{d}{dr}\left[\frac{w(r)}{4\pi r^2 N}\right] = -4\pi r^2 N \frac{w(r)}{4\pi r^2 N} \,. \tag{17}$$

[5] S. Verweij, *Pub. Ap. Inst. Amsterdam*, No. 5, Table 3, 1936.

[6] *Ann. d. Phys.*, **66**, 396, 1921.

[7] *Phys. Zs.*, **25**, 73, 1924.

[8] P. Hertz, *Math. Ann.*, **67**, 387, 1909; R. Gans, *Phys. Zs.*, **23**, 109, 1922; C. V. Raman, *Phil. Mag.*, **47**, 671, 1924.

Hence

$$w(r) = e^{-4\pi r^3 N/3}\, 4\pi r^2 N\,, \tag{18}$$

since, according to equation (16),

$$w(r) \rightarrow 4\pi r^2 N\,, \qquad r \rightarrow 0\,. \tag{19}$$

If we now suppose that the field acting on a star is entirely due to the nearest neighbor, then

$$F = \frac{GM}{r^2}\,, \tag{20}$$

and the law of distribution of the nearest neighbors (eq. [18]) becomes equivalent to

$$W(F)dF = e^{-4\pi (GM)^{3/2} N/3F^{3/2}} 2\pi N (GM)^{3/2} \frac{dF}{F^{5/2}}\,. \tag{21}$$

If we now introduce the normal field

$$Q = (\tfrac{4}{3}\pi)^{2/3} GMN^{2/3} = 2.599 GMN^{2/3}\,, \tag{22}$$

equation (21) becomes

$$W(F)dF = \tfrac{3}{2} Q^{3/2} e^{-Q^{3/2}/F^{3/2}} \frac{dF}{F^{5/2}}\,. \tag{23}$$

According to equation (23),

$$W(F)dF \rightarrow \tfrac{3}{2} Q^{3/2} \frac{dF}{F^{5/2}}\,, \qquad F \rightarrow \infty\,. \tag{24}$$

Comparing equations (11) and (15) with equations (22) and (24), respectively, we conclude that for all practical purposes we may regard them as identical. Moreover, a more detailed comparison of the distributions (12) and (23) shows that even as regards the general dependence on F the two agree sufficiently well. There is an appreciable disagreement between the two distributions only for very small values of F/Q; but, as we shall see later, the weak fields have no significant consequences for the statistical theory. Finally, we may remark that the agreement in the asymptotic behaviors of the two distributions for large values of F implies that the highest field strengths are produced by the nearest neighbor.

5. *The modification of the distribution function for high field strengths.*—As we have already remarked in § 3, the Holtsmark distribution (12) predicts too high probabilities for high field strengths. The same remark applies also to the distribution (23). In our present case the high probabilities result from the assumption of the randomness of the stellar distribution for all elements of volume. But it is clear that this assumption cannot be valid for the regions in the immediate neighborhoods of the individual stars. For a star with a linear velocity[9] v cannot come closer to another star than a certain critical distance $r(v)$ such that

$$\tfrac{1}{2} M v^2 = \frac{GM^2}{r(v)} \tag{25}$$

[9] At an average distance from the other stars.

or

$$r(v) = \frac{2GM}{v^2} . \tag{26}$$

For otherwise the star should strictly be regarded as the component of a binary system, and this is inconsistent with our original premises. This restriction naturally implies a departure from true randomness for these stars as $r \to r(v)$. However, it appears that under the conditions of our problem these departures become significant only as $r \to 0$. In any case it is apparent that the relatively high probabilities predicted by equation (12) or equation (23) for high field strengths will be reduced if proper account is taken of the increasing lack of randomness in stellar distribution as we approach the centers of attraction. A rigorous treatment of this effect will require a reconsideration of the whole problem in *phase space*[10] and is beyond the scope of the present investigation. However, an elementary treatment of the effect can be given, and this appears to be adequate for our purposes.

We shall first consider the problem along the lines of § 4. If $w(r)$ represents, as before, the probability of finding the nearest neighbor to a given star between r and $r + dr$, then the circumstance that stars with linear velocities v cannot come closer to the center than the limit given by (26) will modify equation (16) to

$$\left[1 - \int_0^r w(r)dr\right] 4\pi r^2 \chi(r) N = w(r) , \tag{27}$$

where the function $\chi(r)$ has been introduced to take account of the lack of randomness at close distances. Quite generally we should expect that

$$\chi(r) \to 0 , \quad r \to 0 ; \qquad \chi(r) \to 1 , \quad r \to \infty . \tag{28}$$

The formal solution of equation (27) can be readily written down. We have

$$w(r) = e^{-4\pi N \int_0^r r^2 \chi(r) dr} 4\pi r^2 \chi(r) N , \tag{29}$$

or, differently, as

$$w(r) = e^{-4\pi N r^3 \bar{\chi}(r)/3} 4\pi r^2 \chi(r) N , \tag{30}$$

where we have written

$$\bar{\chi}(r) = \frac{3}{r^3} \int_0^r r^2 \chi(r) dr . \tag{31}$$

According to equation (28),

$$\bar{\chi}(r) \to 0 , \qquad r \to 0 ; \qquad \bar{\chi}(r) \to 1 , \quad r \to \infty . \tag{32}$$

To make the law of distribution of the nearest neighbors according to equation (30) more definite, we need an explicit expression for $\chi(r)$. As we have already indicated, the exact specification of $\chi(r)$ will require a detailed consideration of the problem in phase

[10] In contrast to Holtsmark's treatment, in which the probability distribution of the centers of attraction in *configuration space* is assumed to be independent of the velocities of the particles.

space. But it appears that in a first approximation we may suppose that *the distribution of stars of any prescribed velocity* v *about a given star is perfectly random for all distances greater than the critical distance* r(v) = 2GM/v². Similarly, we may suppose that *no stars with velocity* v *occur within the sphere of radius* r(v).[11] On these assumptions we can readily write down an explicit expression for $\chi(r)$. We have

$$\chi(r) = \iiint_{|\boldsymbol{v}|=\sqrt{2GM/r}}^{|\boldsymbol{v}|=\infty} f(\boldsymbol{v})dv_x dv_y dv_z \,, \tag{33}$$

where $f(\boldsymbol{v})$ denotes the frequency function of the velocities among the stars. If, for the sake of definiteness, we suppose that $f(\boldsymbol{v})$ is Maxwellian,

$$f(\boldsymbol{v}) = \frac{j^3}{\pi^{3/2}} e^{-j^2|\boldsymbol{v}|^2} \,, \tag{34}$$

then

$$\chi(r) = \frac{4j^3}{\pi^{1/2}} \int_{\sqrt{2GM/r}}^{\infty} e^{-j^2v^2} v^2 dv \,. \tag{35}$$

The foregoing formula for $\chi(r)$ can be expressed more conveniently in the form

$$\chi(r) = \frac{4}{\pi^{1/2}} \int_{a/\sqrt{r}}^{\infty} e^{-y^2} y^2 dy \,, \tag{36}$$

where we have written

$$y = jv \,; \qquad a = \sqrt{2GM}\, j \,. \tag{37}$$

An alternative form for $\chi(r)$ may be noted:

$$\chi(r) = 1 - \frac{2}{\pi^{1/2}} \left[\int_0^{a/\sqrt{r}} e^{-y^2} dy - \frac{a}{\sqrt{r}} e^{-a^2/r} \right] . \tag{38}$$

Again, according to equation (31), we have

$$\bar{\chi}(r) = \frac{12}{\pi^{1/2} r^3} \int_0^r r^2 \left(\int_{a/r^{1/2}}^{\infty} e^{-y^2} y^2 dy \right) dr \tag{39}$$

or, after an integration by parts,

$$\bar{\chi}(r) = \chi(r) - \frac{2a^3}{\pi^{1/2} r^3} \int_0^r r^{1/2} e^{-a^2/r} dr \,. \tag{40}$$

After some further reductions we find that

$$\bar{\chi}(r) = \chi(r) - \frac{4a^6}{3\pi^{1/2} r^3} \left[e^{-a^2/r} \left(\frac{r^{3/2}}{a^3} - 2\frac{r^{1/2}}{a} \right) + 4 \int_{a/r^{1/2}}^{\infty} e^{-y^2} dy \right] . \tag{41}$$

[11] This latter assumption is, however, *necessary* (see the remark immediately following equation [26]).

The functions χ and $\bar{\chi}$ are tabulated in Table 1 for different values of the argument $a/r^{1/2}$. An examination of this table shows that to a first approximation we may write

$$\chi(r) = 1 , \quad a \leqslant r^{1/2} ; \qquad \chi(r) = 0 , \quad a > r^{1/2} . \tag{42}$$

In this approximation equation (30) becomes

$$\left.\begin{aligned} w(r) &= e^{-4\pi N(r^3 - r_0^3)/3} 4\pi r^2 N \qquad & (r \geqslant r_0) , \\ &= 0 & (r < r_0) , \end{aligned}\right\} \tag{43}$$

where

$$r_0 = 2GMj^2 . \tag{44}$$

Returning to equation (30), we see that this law of distribution of the nearest neighbors implies a probability of occurrence of a field strength F (assuming that the field arises principally from the first neighbor) given by

$$W(F) = \tfrac{3}{2} Q^{3/2} e^{-Q^{3/2} \bar{\chi}(\sqrt{GM/F})/F^{3/2}} \frac{\chi(\sqrt{GM/F})}{F^{5/2}} , \tag{45}$$

where Q, χ, and $\bar{\chi}$ are defined as in equations (22), (38), and (41). The modification which we have thus effected in the distribution function (23) removes the principal objection to

TABLE 1

$\chi(r)$ AND $\bar{\chi}(r)$

$a/r^{1/2}$	χ	$\bar{\chi}$	$a/r^{1/2}$	χ	$\bar{\chi}$
0	1.000	1.000	1.2	0.410	0.275
0.2	0.994	0.989	1.4	.270	.161
0.4	0.956	0.922	1.6	.163	.086
0.6	0.868	0.787	1.8	.090	.043
0.8	0.734	0.610	2.0	.046	.019
1.0	0.572	0.430	3.0	0.000	0.000

it, namely, the prediction of a nonconvergent value for $\overline{F^2}$, for our present distribution function (45) yields a finite value for the mean square field.

In the approximation (42), equation (45) simplifies to

$$\left.\begin{aligned} W(F) &= \tfrac{3}{2} Q^{3/2} e^{-Q^{3/2}(F^{-3/2} - F_{\max}^{-3/2})} \frac{1}{F^{5/2}} \qquad & (F \leqslant F_{\max}) , \\ &= 0 & (F > F_{\max}) , \end{aligned}\right\} \tag{46}$$

where, according to equation (44),

$$F_{\max} = \frac{1}{4GMj^4} . \tag{47}$$

We shall now consider very briefly how the lack of randomness in the immediate neighborhoods of the centers of attraction can be incorporated into the Holtsmark distribution (12). It appears that it is not an altogether simple matter to modify the Holtsmark distribution rigorously even on the basis of the very simplified assumptions which led to the explicit expressions (38) and (41) for $\chi(r)$ and $\overline{\chi}(r)$. But, remembering that the highest fields are produced by the nearest neighbor and, further, that the lack of randomness becomes significant only as $r \rightarrow 0$, it appears that we may incorporate the main features by considering an approximation corresponding to equation (42), i.e., by supposing that no star has a first neighbor closer than $r_0 = 2GMj^2$ and that the distribution is random but for this restriction. In this last approximation the problem becomes formally the same as when the ions, in the electrical case, have finite dimensions. With suitable changes we can therefore use the results of Gans and Holtsmark,[12] who have modified the distribution function (12), in the electrical case, for the finite sizes of the ions. We have[13]

$$W(F) = \frac{2F}{\pi} e^{4\pi r_0^3 N/3} \int_0^\infty e^{-(Q_H \rho)^{3/2}} K(\rho) \rho \sin F\rho d\rho \, , \tag{48}$$

where $K(\rho)$ is a certain correction factor which is defined in Holtsmark's paper.[14]

6. *The mean life of the state* F.—Our next problem is to determine the mean life of a statistical state defined by F. The totality of statistical complexions which go to make up the state in question are not explicitly defined, and Smoluchowski's ideas cannot be applied without further deep generalizations of them. However, in the approximation in which the fluctuating fields are assumed to arise from the nearest neighbor, the statistical complexion is specified explicitly, and a formula due to Smoluchowski can be directly used.

Now, according to Smoluchowski, the mean life of a state in which n particles are found in an element of volume σ is given by[15]

$$T = \frac{\sqrt{6\pi}}{\sqrt{\overline{v^2}}\,(n+\nu)} \frac{\sigma}{S_\sigma} \, , \tag{49}$$

where $\overline{v^2}$ denotes the mean square speed of the particles, S_σ the surface area of the element σ, and ν the number of particles which the element σ would contain at the constant average density:

$$\nu = N\sigma \, . \tag{50}$$

For the particular case we have in view

$$\sigma = \tfrac{4}{3}\pi r^3 \; ; \qquad S_\sigma = 4\pi r^2 \; ; \qquad n = 1 \; ; \qquad \nu = \tfrac{4}{3}\pi r^3 N \, , \tag{51}$$

[12] See the references given in nn. 6 and 7.

[13] J. Holtsmark, *Phys. Zs.*, **25**, 73, 1924; see particularly eqs. (104) and (145).

[14] See eq. (124) in the paper referred to in n. 13. Holtsmark has not evaluated this correction factor explicitly for the case of an inverse square field. But an evaluation of this factor along the lines of Holtsmark's analysis for the dipole field is possible.

[15] Smoluchowski, *Phys. Zs.*, **17**, 557, 1916; and see §§ 5, 6, and 7 in this paper and particularly eq. (30). See also Furth, *op. cit.*, pp. 34, 35, and 43.

since we need the mean life of a state in which a particular star continues to exist as the sole occupant of a sphere of radius r. Accordingly,

$$T(r) = \sqrt{\frac{2\pi}{3\overline{v^2}}}\,\frac{r}{\frac{4}{3}\pi r^3 N + 1}\,. \tag{52}$$

In the approximation of § 4

$$r = \sqrt{\frac{GM}{F}}\,, \tag{53}$$

and equation (52) implies for the state F the mean life

$$T(F) = \sqrt{\frac{2\pi GM}{3\overline{v^2}}}\,\frac{F}{Q^{3/2} + F^{3/2}}\,, \tag{54}$$

where Q is defined as in equation (22).

Formula (54) for $T(F)$ is clearly only an approximate one. But since, according to the Holtsmark distribution, the highest fields are produced by the nearest neighbor, the true expression for $T(F)$ must tend to equation (54) for high field strengths. Consequently, we may expect equation (54) to give as good an approximation to the true values of $T(F)$ as the $W(F)$ according to equation (23) or equation (30) provides an approximation to the Holtsmark distribution. This is probably quite sufficient for most purposes.

7. *The acceleration of a star in the fluctuating gravitational field.*—We shall begin our discussion of this problem by considering the following simplified case: Imagine a star's undergoing a series of accelerations during a large number of intervals of constant duration T, in such a way that during each interval it is accelerated at the same rate F but in directions which are uncorrelated from interval to interval. Under these circumstances the star experiences an increase of velocity of amount TF in each of the intervals; but these increments take place along uncorrelated directions in a random manner. We now ask the probability that at the end of s such intervals the star has undergone a net increase of velocity of mFT in some specified direction. According to the principles of the theory of random walk we have[16]

$$P_m = \sqrt{\frac{3}{2\pi s}}\,e^{-3m^2/2s}\,, \tag{55}$$

when s is sufficiently large. Since the net increase in velocity Δv and the time t during which this increase has taken place are related to m and s by

$$\Delta v = mFT\,; \qquad t = sT\,, \tag{56}$$

we have

$$P(\Delta v) = \sqrt{\frac{3T}{2\pi t}}\,e^{-3|\Delta v|^2/(2F^2Tt)}\,. \tag{57}$$

[16] See the references given in n. 3, particularly Kennard, *op. cit.*, pp. 269–72.

Hence, the probability that there occurs an increase in velocity in the range $[\Delta v, \Delta v + d(\Delta v)]$ during a time t in some prescribed direction is given by

$$P(\Delta v)d(\Delta v) = \sqrt{\frac{3}{2\pi F^2 T t}}\, e^{-3|\Delta v|^2/(2F^2Tt)} d(\Delta v) \, . \tag{58}$$

Accordingly,

$$\overline{\Delta v^2} = F^2 T t \, . \tag{59}$$

We shall now generalize the foregoing problem to the case when F does not have a unique value but occurs according to a definite frequency function $W(F)$ and when the average duration of an acceleration at the rate F is given by a function $T(F)$. In view of the addition theorem for the Gaussian error functions, equation (58) becomes modified under these more general circumstances to

$$P(\Delta v) = \sqrt{\frac{3}{2\pi \overline{F^2 T} t}}\, e^{-3|\Delta v|^2/(2\overline{F^2T}t)} \, , \tag{60}$$

where

$$\overline{F^2 T} = \int_0^\infty W(F) F^2 T(F) dF \, . \tag{61}$$

Hence, instead of equation (59) we now have

$$\overline{\Delta v^2} = \overline{F^2 T} t \, . \tag{62}$$

8. *The evaluation of* $\overline{\Delta v^2}$.—According to equations (45), (54), and (61) we have

$$\overline{F^2 T} = \frac{3}{2}\sqrt{\frac{2\pi G M}{3\overline{v^2}}}\, Q^{3/2} \int_0^\infty \frac{F^{1/2}}{Q^{3/2} + F^{3/2}}\, e^{-Q^{3/2}\bar{\chi}(\sqrt{GM/F})/F^{3/2}} \chi(\sqrt{GM/F}) dF \, . \tag{63}$$

When we introduce a new variable x defined by

$$\frac{Q^{3/2}}{F^{3/2}} = x \, , \tag{64}$$

equation (63) becomes

$$\overline{F^2 T} = 2\left(\frac{2\pi}{3}\right)^{3/2} \frac{G^2 M^2 N}{\sqrt{\overline{v^2}}} \int_0^\infty e^{-x\bar{\chi}(\sqrt{GM/Q}\, x^{1/3})} \left(\frac{1}{x} - \frac{1}{x+1}\right) \chi(\sqrt{GM/Q}\, x^{1/3}) dx \, . \tag{65}$$

Substituting for Q from equation (22) in the argument for the functions χ and $\bar{\chi}$ in the foregoing equation, we obtain

$$\sqrt{\frac{GM}{Q}}\, x^{1/3} = \left(\frac{x}{\frac{4}{3}\pi N}\right)^{1/3} = D x^{1/3} \, , \tag{66}$$

where we have written

$$D = \frac{1}{(\frac{4}{3}\pi N)^{1/3}} . \tag{67}$$

Now, according to equations (38) and (41), the functions χ and $\bar{\chi}$ depend on r only through the combination $a/r^{1/2}$. From equations (37) and (66) we now find that

$$\frac{a}{r^{1/2}} = \sqrt{\frac{2GMj^2}{D}}\, x^{-1/6} . \tag{68}$$

Since

$$\frac{D}{2GM} = 2.33 \times 10^4 \frac{(D/\text{parsec})}{(M/\odot)(10 \text{ km/sec})^2} , \tag{69}$$

it follows that under most stellar conditions $2\,GMj^2/D \sim 10^{-4}$. Hence, only for values of $x < 10^{-11}$ do the functions χ and $\bar{\chi}$ deviate appreciably from unity (see Table 1). We can therefore replace χ by unity whenever it does not occur multiplied by a factor which diverges at $x = 0$. Similarly, we can replace $\bar{\chi}$ also by unity; but this we can always do since $\bar{\chi}$ occurs in the exponent multiplied with x. Thus, to a high degree of accuracy, the integral on the right-hand side of equation (63) is the same as

$$\int_0^\infty \frac{e^{-x}}{x} \chi(Dx^{1/3})dx - \int_0^\infty \frac{e^{-x}}{x+1} dx = J \quad \text{(say)} . \tag{70}$$

Substituting for χ according to (36) in the foregoing equation we obtain

$$J = \frac{4}{\pi^{1/2}} \int_0^\infty \frac{e^{-x}}{x} \left[\int_{\sqrt{2GMj^2/D}\, x^{-1/6}}^\infty e^{-y^2} y^2 dy \right] dx - \int_1^\infty \frac{e^{-(x-1)}}{x} dx . \tag{71}$$

Writing

$$- E(-x) = \int_x^\infty \frac{e^{-x}}{x} dx , \tag{72}$$

we have

$$J = \frac{4}{\pi^{1/2}} \int_0^\infty \frac{d}{dx} (E(-x)) \int_{\sqrt{2GMj^2/D}\, x^{-1/6}}^\infty e^{-y^2} y^2 dy dx + eE(-1) . \tag{73}$$

Integrating by parts the integral on the right-hand side of equation (73), we find

$$J = -\frac{4}{\pi^{1/2}} \int_0^\infty E\left(-\left[\frac{2GMj^2}{D} \right]^3 z^{-6} \right) e^{-z^2} z^2 dz - 0.5963 . \tag{74}$$

The argument for the exponential integral occurring under the integral sign in equation (74) is seen to be extremely small for the values of x which are at all relevant to the value of the integral. Hence we can use the asymptotic expansion for $E(-x)$ valid for $x \to 0$. We have

$$E(-x) = \log x + 0.5772 + O(x) , \tag{75}$$

where the constant on the right-hand side is the Euler-Mascheroni constant. Using the foregoing expansion for $E(-x)$ in equation (74), we readily find that

$$J = 3 \log\left(\frac{D}{2GMj^2}\right) - 0.5772 + \frac{6}{\pi^{1/2}} \int_0^\infty e^{-x} x^{1/2} \log x dx - 0.5963 . \qquad (76)$$

On the other hand, we have

$$\left.\begin{aligned} \int_0^\infty e^{-x} x^{1/2} \log x dx &= \Gamma(\tfrac{3}{2}) \left[\frac{d \log \Gamma(x)}{dx}\right]_{x=3/2} , \\ &= \frac{\pi^{1/2}}{2} \times 0.03649 . \end{aligned}\right\} \qquad (77)$$

Hence,

$$J = 3 \log\left(\frac{D}{2GMj^2}\right) - 1.0640 . \qquad (78)$$

Finally, substituting for $\overline{F^2T}$ according to equations (65) and (78) in equation (62) we obtain

$$\overline{\Delta v^2} = 6\left(\frac{2\pi}{3}\right)^{3/2} \frac{GM^2N}{\sqrt{\overline{v^2}}} \left[\log\left(\frac{D\overline{v^2}}{3GM}\right) - 0.355\right] t . \qquad (79)$$

We may note that if we had used the approximation (46) for $W(F)$ (instead of the more accurate formula [45]) we should have obtained

$$\overline{F^2T} = 2\left(\frac{2\pi}{3}\right)^{3/2} \frac{G^2M^2N}{\sqrt{\overline{v^2}}} \int_{(2GMj^2/D)^3}^\infty \frac{e^{-x}}{x(x+1)} dx \qquad (80)$$

instead of equation (65). On evaluating the integral on the right-hand side of (80), we find

$$\int_{(2GMj^2/D)^3}^\infty \frac{e^{-x}}{x(x+1)} dx = 3 \log\left(\frac{D}{2GMj^2}\right) - 1.1735 , \qquad (81)$$

which should be compared with equation (78). We thus see that approximations based on the assumption (43) (or their equivalents) are likely to provide sufficient accuracy for most purposes. In particular, the modification of the Holtsmark distribution suggested on page 519 to take account of the lack of randomness in stellar distribution in the immediate neighborhoods of stars can be justified on these grounds.

9. *The time of relaxation of a stellar system.*—An immediate application of the fundamental formula (79) is to the problem of the *time of relaxation* of a stellar system. According to the general ideas outlined in I, § 1, we can definite this as the time required for $\overline{\Delta v^2}$ to become of the same order as $\overline{v^2}$. Thus, if t_R denotes this time, we have

$$t_R = \frac{1}{6}\left(\frac{3}{2\pi}\right)^{3/2} \frac{(\overline{v^2})^{3/2}}{G^2M^2N\left[\log\left(\frac{D\overline{v^2}}{3GM}\right) - 0.355\right]} . \qquad (82)$$

We can now compare this formula with that obtained on the basis of the two-body idealization of stellar encounters. We have[17]

$$t_E = \frac{1}{16}\left(\frac{3}{\pi}\right)^{1/2}\frac{(\overline{v^2})^{3/2}}{G^2M^2N\log\left(\frac{\overline{D}\overline{v^2}}{3GM}\right)}, \tag{83}$$

where $\overline{D}$ is the average distance between the stars.[18] We notice that the two equations (82) and (83) are of identical forms; further it is found that the numerical factors in the two formulae differ only by a factor 1.11. This agreement, while confirming the general correctness of our statistical method, exhibits also its immense superiority over the earlier treatments of the same problem both in the appropriateness of the physical ideas and in the simplicity of the mathematical treatment.

10. *Concluding remarks.*—The perfectly natural way in which the solution to the problem of the time of relaxation appears on the present theory suggests the extension of these methods to solve other problems of stellar dynamics. Thus the evolution of wide binaries in a fluctuating gravitational field is a problem to which the principles of the statistical theory are particularly well adapted. For, while on the classical methods the treatment of this problem would require the analysis of individual encounters considered strictly as three-body problems, on the statistical theory all such detailed considerations would be eliminated. Again, the application of the fundamental theorem of statistical dynamics due to Planck and Fokker[19] to problems of stellar dynamics is another field to which the method of the present paper can be used. We shall consider these problems in later papers.

In conclusion I wish to record my indebtedness to Messrs. G. Randers and R. E. Williamson for valuable discussions.

YERKES OBSERVATORY
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NOTE ADDED IN PROOF.—Since the foregoing paper was written it has been found possible to solve rigorously the question of the half-life treated approximately in section 6. While this exact treatment leads to substantially the same results, it enables a more complete visualization of the phenomenon in question. It is hoped to publish these newer results in the near future.

[17] See. S. Chandrasekhar, *The Principles of Stellar Dynamics*, chap. ii, University of Chicago Press. (In Press.)

[18] According to eq. (18), the average distance $\overline{D}$ between the stars is given by

$$\overline{D} = \int_0^\infty e^{-4\pi r^3 N/3} 4\pi r^3 N dr \tag{84}$$

or, after some elementary reductions,

$$\overline{D} = \frac{1}{(\frac{4}{3}\pi N)^{1/3}}\int_0^\infty e^{-x} x^{1/3} dx\ . \tag{85}$$

Hence, comparing eqs. (67) and (83), we have

$$\overline{D} = \Gamma(\tfrac{4}{3})D = 0.8930D\ , \tag{86}$$

a result due to Hertz (see the reference in n. 8).

[19] M. Planck, *Sitzungsber. der preuss. Akad.*, p. 324, Berlin, 1917; A. Fokker, *Ann. d. Phys.*, **45**, 812, 1914.