Triaxiality in elliptical galaxies

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Summary. The existence of a triaxial shape for elliptical galaxies has been considered in recent years to explain the new kinematical and geometrical findings, i.e. (a) the low rotation/velocity dispersion ratio found also in some flat systems, (b) the presence of twisting in the isophotes, (c) the recently found correlation between maximum twisting and maximum flattening, (d) the presence of rotation along the minor axis.

A simple geometrical model of elliptical galaxies having shells with different axial ratios c/a , b/a has been produced to interpret three fundamental key-features of elliptical galaxies: (i) the distribution of the maximum flattening observed; (ii) the percentage of ellipticals showing twisting; and (iii) the correlation between maximum twisting and maximum flattening. The model has been compared with observational data for 348 elliptical systems as given by Strom & Strom. We find that a triaxial ellipsoid with coaxial shells having axial ratios c/a and b/a mutually dependent in a linear way can satisfy the observations.

1 Introduction

Many observations of elliptical galaxies have recently led to the suggestion that their spatial structure cannot be represented by oblate spheroids. This hypothesis has been strongly supported by the need to interpret the following features:

- (a) the low ratio between the rotational velocity and the velocity dispersion found even in flat systems (Bertola & Capaccioli 1975; Illingworth 1977; Peterson 1978; Davies 1979; Schechter & Gunn 1979; Capaccioli 1979);
- (b) the presence of rotation also along the minor axis (Bertola & Capaccioli 1979, private communication; Schechter & Gunn 1979; Davies 1979);
- (c) the presence of an isophotal twisting with the radius (Evans 1951; Liller 1960, 1966; Barbon, Benacchio & Capaccioli 1976; Carter 1978; King 1978; Strom & Strom 1978a, b, c; Williams & Schwarzschild 1978; Bertola & Galletta 1979); and
- (d) the existence of an inverse correlation between the maximum flattening and the maximum twisting observed (Galletta 1980).

The above dynamical properties (a) and (b) have been interpreted as resulting from a prolate or triaxial structure with a complex dynamics for the motion of the stars (Contopoulos 1956; Miller 1978; Aarseth & Binney 1978; Schwarzschild 1979). The photometric properties (c) and (d) also have provided another indirect argument for supposing that a triaxial shape can be present in a number of elliptical galaxies (Binney 1978b; Galletta 1980). A direct argument for the existence of almost prolate structures has been supplied by Bertola & Galletta (1978) with a study of a type of galaxy whose appearance is in most respects that of a normal elliptical galaxy, but in which the presence of a dust lane allows one to define the inclination of the galaxy with respect to the line-of-sight. In these systems the dust lane crosses the galaxy along the minor axis, indicating prolate structure. We then ask ourselves whether the majority of elliptical galaxies, which have unknown dynamics and which do not seem to have appreciable twisting (90 per cent, see Galletta 1980), may also be triaxial structures.

In this paper we discuss the problem from a statistical point of view, analysing in Section 2 the difficulties arising from the observational data in interpreting the apparent frequency of the elongation of the galaxian images. In Section 3 a simple geometrical model is presented. This is then used in Section 4 to compute the predicted frequencies of apparent flattening and the predicted percentage of galaxies showing twists greater than 10° . A comparison with the observations shows that it is possible to explain with a triaxial model not only the presence of twisting, but also the observed flattening distribution and the correlation between the maximum flattening and the maximum twisting observed. In addition, a single intrinsic ellipticity profile can reproduce with reasonable agreement different kinds of isophotal twists observed in elliptical systems.

2 Observational data

From a statistical point of view, considering only the observed frequency of the elongation of the galactic images, the problem of the intrinsic shape of the elliptical galaxies seems to be more complex. A correlation found by Marchant & Olson (1979) between the mean surface brightness and the flattening at a fixed isophote seems to be in agreement with the oblate model, but the authors cannot exclude the presence of a moderate triaxiality. Reproduction of the frequencies of apparent flattening observed assuming a fixed brightness level is in addition possible both with an oblate (Sandage, Freeman & Stokes 1970) and a prolate (Binney 1978a) or triaxial (Binggelli 1980) model.

The above-mentioned statistical works have been always applied to frequencies of apparent flattening measured at a single isophotal level of fixed brightness and for this reason have been confined to constant ellipticity model. This selection involved two important problems: first, the observations show that the ellipticity of the isophotes in elliptical galaxies is changing with the distance from the nucleus with all possible large-scale trends (Bertola & Galletta 1979) and sometime showing strong variations. As a consequence, the flattening observed at a single isophotal level cannot be representative of the whole galaxy. Then, the distributions of the observed flattening is not independent of the selected brightness level.

Second, as the apparent brightness of a shell is a function of its inclination angle with respect to the plane of the sky, if the intrinsic flattening is changing with the radius, the observed flattening at a fixed brightness level is dependent on the intrinsic flattening by means not only of the projection factor, but also of the intrinsic ellipticity profile.

For these reasons we suggest that a useful statistical parameter can be represented by the maximum flattening η_m (i.e. the lower value of the apparent axial ratio b_0/a_0) observed in

the galaxy isophotes. This value represents a lower limit for the intrinsic flattening of the galaxy and is independent of the brightness threshold. For galaxies showing a maximum in the ellipticity trend, adopting η_m we can be sure to consider one of the intrinsically flattest shells. In addition, this value is related to the maximum twists $\Delta\alpha_m$ (Galletta 1980), in the sense that greater twists are observed in less flat galaxies and that there are no galaxies having twists greater than 40° that are flatter than E4. To obtain a complete sample of values, taking into account the existence of a change in the ellipticity and in the orientation of the isophotes, we have selected all the galaxies studied in six clusters by Strom & Strom (1978a, b, c) which are classified ellipticals on the basis of the luminosity profile. We also considered 62 systems for which an intermediate E/S0 luminosity profile is observed, but showing in the η_m – $\Delta\alpha_m$ diagram a statistical distribution similar to that of ellipticals (Galletta 1980). In fact, their inclusion in the sample does not seem to affect the observed flattening and twisting distribution. Then we have a total of 348 systems for which apparent flattening $\eta(a_0)$ and twisting $\Delta\alpha(a_0)$ values are available as a function of the distance a_0 from the centre up to a very faint surface brightness. At the present time, this is the only sample available in the literature containing a large number of elliptical galaxies studied on homogeneous plate material and with position angles and flattening values measured at more than one isophotal level. In addition, the number of systems studied is comparable with the number of elliptical galaxies (338) having measured axial ratio and well determined classification in RC2 (de Vaucouleurs, de Vaucouleurs & Corwin 1976).

Fig. 1(a) shows the representative points of each galaxy in the sample (full circles) plotted in the η_m – $\Delta\alpha_m$ diagram. To avoid confusion, we omit the plot of the representative points for the 319 galaxies with $\Delta\alpha_m \leq 10^\circ$ (dotted strip), which represent 91.7 per cent of the total. For comparison, we plot also the points corresponding to galaxies in other clusters (empty circles), not studied by Strom & Strom (1978a, b, c), and available in the literature (see Galletta 1980).

The distribution of the elongation of the galactic images η_m , given in Table 2, is shown in the histogram of Fig. 2. The values are binned according to the intervals selected by Sandage *et al.* (1970). A comparison with the histogram at $23.1 \text{ mag arcsec}^{-2}$ obtained by Sandage *et al.* (1970) from the *Reference Catalogue of Bright Galaxies* (de Vaucouleurs & de Vaucouleurs 1964) shows two interesting differences: the lack of E galaxies flatter than $\eta_m = 0.36$ (E7) and a smaller percentage (4.9 per cent) of systems rounder than E1. In fact the E0 galaxies represent 23.8 per cent of the sample of Sandage *et al.* (1970), based on the brighter galaxies, and reach an abundance of 35.8 per cent considering all the RC2 galaxies measured at $25 \text{ mag arcsec}^{-2}$. The difference in the abundance of E0 between these and the sample used in this paper partially arises from our selection of the maximum flattening values observed (selection, for instance, of the minimum flattening would have given $N(\text{E0}) = 41.1$ per cent), but this effect persists also if measurements at $25 \text{ mag arcsec}^{-2}$ are considered. Then, as the samples are numerically comparable, we can consider the following hypotheses: (i) the difference originates from some kind of selection effect in the observational data; (ii) the difference is real and due to a greater percentage of flattened systems in the rich clusters considered by Strom & Strom (1978a, b, c). But it is difficult to suppose that, if selection effects are present, they have influenced the observed samples in such a different way. Possible consequences of the hypothesis (ii) on the resulting model shall be considered in Section 5.

5 Geometrical model

We have considered a geometrical model for elliptical galaxies where the surfaces of equal density of light are coaxial ellipsoids with independent values of the principal axes a , b , c .

Figure 1. Comparison between observed (a) and computed (b, c, d, e) distribution of galaxies having a maximum twisting $\Delta\alpha_m$ and a maximum flattening η_m . Most of the galaxies considered (91.7 per cent) fall in the dotted strip ($\Delta\alpha_m \leq 10^\circ$) of (a). Filled circles are from Strom & Strom (1978a, b, c), open circles from other authors (see Galletta 1980). Computed frequencies are shown in (b), (c), (d) and (e) for four models by mean of smoothed isopleths. Percentage of galaxies falling within some particular isopleths are labelled.

Figure 2. Lower panel: distribution of the maximum flattening observed in 348 elliptical galaxies (data from Strom & Strom 1978a, b, c). Dashed line represents the histogram of the corresponding computed distribution. Upper panel: intrinsic abundance of flattening q adopted for the model discussed in this paper. In both panels the extreme of the intervals of flattening labelled are listed in Table 2.

Oblate and prolate spheroids are particular cases of these. Let us assume an intrinsic reference system $OXYZ$ oriented with the X axis along the major axis of the shells, and the Z axis along the smallest. The line-of-sight is represented by the axis X' of a second reference system $OX'Y'Z'$ tilted at angles ϑ and φ with respect to the first, with ϑ representing the zenith angle of X' from Z and φ the azimuthal angle from X along the plane XY . Each ellipsoid is represented in the intrinsic system by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1. \quad (1)$$

The equation of the projected ellipse on the plane $X'Z'$ (plane of the sky) is (see also Stark 1977)

$$jy'^2 + kz'y' + sz'^2 = A \quad (2)$$

where

$$j = \frac{\sin^2 \vartheta}{a^2 b^2} + \frac{\cos^2 \vartheta}{c^2} \left(\frac{\sin^2 \varphi}{a^2} + \frac{\cos^2 \varphi}{b^2} \right), \quad (3a)$$

$$k = \frac{\sin 2\varphi \cos \vartheta}{c^2} \left(\frac{1}{a^2} - \frac{1}{b^2} \right), \quad (3b)$$

$$s = \frac{1}{c^2} \left(\frac{\cos^2 \varphi}{a^2} + \frac{\sin^2 \varphi}{b^2} \right), \quad (3c)$$

$$A = \frac{\cos^2 \varphi}{c^2} + \sin^2 \vartheta c^2 s. \quad (3d)$$

For a real galaxy, if we identify the surfaces of equal density of stars with a series of ellipsoids, the above projection represents an observed isophote of the galaxy. Then, the apparent axial ratio of the ellipsoid and the orientation of its major axis with respect to the Z' axis are

$$\eta = \left(\frac{j + s + \sqrt{(j-s)^2 + k^2}}{j + s - \sqrt{(j-s)^2 + k^2}} \right)^{1/2}, \quad (4)$$

$$\alpha = \arctan \left(\frac{k}{(j-s) - \sqrt{(j-s)^2 + k^2}} \right). \quad (5)$$

In a model having intrinsic axial ratios c/a and b/a changing with distance from the centre, the ellipticity and orientation profiles can be obtained from equations (4) and (5) for a given pair of ϑ and φ values. In this kind of model it is always possible to define a maximum value of the projected flattening η_m and, if a change in the position angles α of each projected isophote is present, a maximum twisting $\Delta\alpha_m$, corresponding to the maximum observed difference in the α values. If the galaxies are randomly oriented in space, all possible combinations of ϑ and φ would be observed and η_m and $\Delta\alpha_m$ are functions also of these angles.

In our statistical approach we are dealing with the converse problem: starting from the observed distribution of η_m and $\Delta\alpha_m$, how is it possible to find the distribution $\Psi(p, q)$ of the intrinsic axial ratios $p = (b/a)_{\min}$, $q = (c/a)_{\min}$ corresponding to the flattest isodensity surfaces?

To reproduce the observations, the function $\Psi(p, q)$ must be weighted for projection factor $P(\vartheta, \varphi) = \sin \vartheta$. But, contrary to the oblate case, for which it is possible to define a direct relation between ϑ and η_m , the angles ϑ and φ are connected in a more complex way to the observed values η_m and $\Delta\alpha_m$ by means of the equations (4) and (5). Then, to solve the problem we obtain the relative frequency of galaxies having flattening and twisting in the intervals η_1, η_2 and $\Delta\alpha_1, \Delta\alpha_2$ respectively by means of the equation:

$$N(\eta_1, \eta_2, \Delta\alpha_1, \Delta\alpha_2) = \int_p \int_q \Psi(p, q) \int_0^{2\pi} \int_0^\pi P(\vartheta, \varphi) \delta d\vartheta d\varphi dp dq. \quad (6)$$

In the above expression $\delta = \delta(\eta_1, \eta_2, \Delta\alpha_1, \Delta\alpha_2, \vartheta, \varphi, p, q)$ is a factor equal to unity when $\eta_1 \leq \eta_m \leq \eta_2$ and $\Delta\alpha_1 \leq \Delta\alpha_m \leq \Delta\alpha_2$, and equal to zero in the other cases. We note that equations (4) and (6) can be reduced to the classical equations (1) and (5) of the Sandage *et al.* (1970) paper setting $b = a$ and $\vartheta = 90^\circ - i$. The factor δ in this case defines the intervals of integration. The goodness of our procedure has been tested by comparing the predicted abundances of flattening for disc galaxies with those obtained by Sandage *et al.* (1970) (Table 1) under the same assumption for $\Psi(p, q)$.

In our model the galaxy is composed of a finite number of shells representing the isodensity surfaces, whose c/a values are spaced by 0.025. Solution of the equation (6) was explored under various assumptions for the values p, q and the function $\Psi(p, q)$.

4 Interpretations

We assume that a model can be considered to be a solution if it is possible to reproduce:

- (1) the observed distribution of flattenings (Table 2);

Table 1. Comparison of predicted abundances of flattening for disc galaxies between Sandage *et al.* (1970) and this paper. (1) Label of the interval in Fig. 2; (2) apparent flattening values of the extremes of the interval; (3) percentage of galaxies computed by Sandage *et al.*; (4) percentage of galaxies computed in this paper.

(1)	(2)	(3)	(4)
i	1.00–0.91	9.7	10.0
ii	0.90–0.83	8.7	8.7
iii	0.82–0.74	9.9	10.1
iv	0.73–0.65	10.0	10.0
v	0.64–0.56	10.3	10.5
vi	0.55–0.47	10.9	10.7
vii	0.46–0.37	13.9	14.2
viii	0.36–0.28	15.0	15.1
ix	0.27–0.19	9.8	9.3
x	0.18–0.10	1.3	1.4
xi	0.09–0.00	0.0	0.0

Table 2. Comparison between observed and computed distribution of the apparent flattening in elliptical galaxies. (1) Label of the interval in Fig. 2; (2) apparent flattening values of the extremes of the interval; (3) observed percentage; (4) computed percentage.

(1)	(2)	(3)	(4)
i	1.00–0.91	4.9	5.9
ii	0.90–0.83	23.3	21.0
iii	0.82–0.74	29.6	32.5
iv	0.73–0.65	21.3	21.9
v	0.64–0.56	12.9	12.0
vi	0.55–0.47	5.7	5.1
vii	0.46–0.37	2.3	1.6
viii	0.36–0.28	0.0	0.2
ix	0.27–0.19	0.0	0.0
x	0.18–0.10	0.0	0.0
xi	0.09–0.00	0.0	0.0

(2) the observed percentage of galaxies showing twisting greater than 10° in respect to the total number (8.3 per cent); and

(3) the $\eta_m - \Delta\alpha_m$ diagram.

For the conditions (1) and (2) it is required also that the differences between the observed values N_0 and the computed values N_c be smaller than $\sqrt{N_0}$.

First of all we remember that rotationally-symmetric oblate or prolate galaxies cannot produce isophote twisting, as we can see from the equations (3) and (5) posing respectively $a = b$ or $b = c$.

In Fig. 1 four typical models are compared with the observations. We note that the shape of the diagram is highly dependent on the kind of model selected (discs, bars, etc.). Their width on the contrary is a function of the particular selection of values for the axial ratios. A model made by a disc ($c/a \sim 0.25$) and a small prolate bar ($c/a \sim 0.3$) similar to the SB0 galaxies produce isophote twisting only in the presence of highly flattened isophotes (Fig. 1c). Two more 'exotic' models with intrinsic distortions of 90° for the principal axes are considered, to explain extreme cases of twisting ($\Delta\alpha_m = 90^\circ$): a 'warped' galaxy (Fig. 1d) whose shells present a rotation of the equatorial plane around to the y axis at angles $\Delta\vartheta = 90^\circ$, and a 'two cigars' model (Fig. 1e) in which the inner and the outer shells present a twisting along the equatorial plane of $\Delta\varphi = 90^\circ$. The diagram $\eta_m - \Delta\alpha_m$ represents in these cases a powerful test which obliges us to exclude the above models as solutions. In fact, in the corresponding diagram 'forbidden regions' are present corresponding to pairs of η_m , $\Delta\alpha_m$ values never reached by the representative points of the model, which on the contrary in the real case are filled by galaxies.

As we can see from Fig. 1(b), only a completely triaxial model can reproduce the triangular shape of the observed diagram, with a greater abundance of galaxies at small twisting.

We have found that a solution of the problem accounting for all of the observational constraints exists, and it is obtained with a triaxial model having shells whose axial ratios are mutually dependent in a linear way. The best agreement with the observational data is obtained with the following assumptions: the intrinsic abundance function $\Psi(p, q)$ is a Gaussian with $\bar{q} = 0.62$, $\sigma_q = 0.13$ and $\bar{p} = 0.81$, $\sigma_p = 0.04$; the relation between p and q

Figure 3. (a) A picture for the proposed galaxy model. Although our conclusions are independent of the intrinsic ellipticity profile, in this figure we assumed an almost spherical core with a progressively flatter halos shells and a rounder outer corona. (b) Representation, in the q, p plane of the relation found between the axial ratios of the model Oblate and prolate sequences also are shown, together with the intrinsic flattening found by Sandage *et al.* (1970) for spiral and S0 galaxies (the asterisk represents the peak of the distribution, at $p = 1, q = 0.25$).

is $p = 0.31q + 0.62$ with $0.25 \leq q \leq 0.9$. In Fig. 2 the Gaussian intrinsic abundance is shown in the upper panel, considering for simplicity only the dependence on q . Fig. 3(b) illustrates in the p, q plane the mutual dependence of the axial ratios.

For this model, whose $\eta_m - \Delta\alpha_m$ diagram is shown in Fig. 1(b), the expected number of galaxies showing different flattening are listed in Table 2, together with the observed abundances. The corresponding histograms are shown in Fig. 2, lower panel. As we might expect, this result is completely independent of the internal ellipticity trend. A picture of this model is shown in Fig. 3(a). The rounder shells (assumed for simplicity to be the central ones in the figure) have axial ratios c/a in the range 0.8–0.9. The percentage of galaxies showing twisting of the isophotes greater than 10° is in this case 11.2 per cent, in agreement with the observations. This value is strongly dependent on the mean axial ratios of the rounder shells.

The projected ellipticity and orientation profiles of the model are shown in Fig. 4 at different orientations of the line-of-sight. The single profiles can be compared with those observed in four galaxies (Bertola & Galletta 1979; Strom & Strom 1978a, c) and shown in Fig. 5. It is possible to see that a monotonic change of flattening in triaxial, coaxial shells can also produce as a projection effect peculiar twisting or ellipticity trends, like that observed in the galaxy CR32 in the Perseus Cluster.

5 Conclusions

The proposed model can explain in a simple way not only the observed abundances of flattening, assuming the $\sqrt{N_0}$ criterion, but also the observed percentage of galaxies showing twisting greater than 10° and the correlation between twisting and flattening. We remember that the above conclusions are referred to a sample of elliptical galaxies belonging to rich clusters. As previously discussed, some differences in the relative abundance of almost round

Figure 5. A sample of observed profiles of ellipticity and orientation for NGC 205 and CR 32 (Bertola & Galletta 1979), NGC 4816 (Strom & Strom 1978a) and A2199-45 (Strom & Strom 1978c). A comparison with the predicted profiles of Fig. 4 shows that these different features can be explained by a single model.

Figure 4. Ellipticity (solid line) and orientation (dashed line) profiles for the galaxy model of Fig. 3 at various ϑ and φ angles. The intrinsic c/a profiles are shown in the fourth and sixth sections of the figure ($\vartheta = 90^\circ$, $\varphi = 90^\circ$, XZ plane) respectively for the model without and with the outer rounder corona. Note that regular intrinsic profiles can produce peculiar projected ellipticity and twisting trends.

systems seems to be present in comparison with the RC2 sample of bright galaxies. Unfortunately, as the flattening in the RC2 is referred only to a fixed isophote, we cannot apply our analysis to this sample. But, if these differences are real, only the intrinsic abundance function $\Psi(p, q)$ will be affected from these. The conclusion concerning the $\eta_m - \Delta\alpha_m$ relation (Fig. 1) are still valid. Since the maximum flattening found in the model ($b/a = 0.7$, $c/a = 0.25$) reflects a slight oblate shape, we can expect that it could generate a slight correlation between apparent flattening and mean surface brightness, but not so strong as found by Marchant & Olson (1979). Of course, we cannot expect that all the properties of elliptical galaxies can be explained by our simple geometrical model; nevertheless, the fact that a solution is found with triaxial model is another step which supports the hypothesis of a triaxial structure in elliptical galaxies.

A study of the dynamical stability for this model would be very interesting, but out of the goals of this paper. For models having more enhanced deviation from the oblate shape than in our case, the problem of dynamical stability has been investigated by Aarseth & Binney (1978), while dynamical equilibrium has been studied by Schwarzschild (1979).

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