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AN APPLICATION OF THE METHOD OF LEAST SQUARES TO THE
DETERMINATION OF THE PHOTOMETRIC ELEMENTS
OF ECLIPSING BINARIES

PART I. DERIVATION OF FORMULAE

BY

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In their discussion of the darkening at the limb in eclipsing variables, Russell and Shapley¹ showed that it was impossible to determine, from the most accurate observations then available, the coefficient of limb darkening for any eclipsing binary. Their development of a method of solution for the photometric elements on the hypothesis of complete darkening at the limb was "not prompted by the discrepancy between observations and the uniform-disk theory but rather by the reasonableness of the . . . assumption" of limb darkening. Thus it has been customary to assume that the disks are either uniformly illuminated or completely darkened at the limb—at least for the purpose of determining the other elements.

More recently, Pannekoek and Elsa van Dien² investigated the photometric orbit of YZ Cassiopeiae, making use of a light-curve obtained by Huffer³ with the photoelectric cell, in an attempt to determine the darkening coefficient of the primary component. Their results were negative, indicating that even these observations were not accurate enough to yield a reliable value of the darkening constant.

Within the past two years, Kron⁴ has observed the light-curve of the same star, YZ Cassiopeiae, with photoelectric cell and amplifier sensitive enough to increase the previous accuracy four- or five-fold, and his analysis seems to indicate that the observations are of high enough accuracy to permit the reliable determination of

the limb-darkening coefficient for the primary component of the system. It is reasonable to suppose that his investigation can be extended to other favorable cases, especially since he proposes to observe the light-curves of some stars in two different wave-lengths, and thus presumably to make the darkening constants for each wave-length more accurately determinate.

It seems appropriate, therefore, to investigate anew the problem of the determination of the elements of an eclipsing binary, with the limb-darkening coefficient included as one of the unknowns. The method employed by Pannekoek and Elsa van Dien, and by Kron, consisted of arbitrary variations of the constants, (three in number), and the selection of that combination of constants which gave the best representation of the observations. The method is open to certain objections. First, it is very laborious, requiring at least thirty-six combinations of constants, with as many representations of the observations and determinations of residuals. Second, the treatment of primary and secondary minima simultaneously would introduce difficulties; and with the inclusion of another light-curve for a different wave-length, the computational labor would mount greatly, not to mention the theoretical difficulty of combining the results of the different light-curves. Third, the assumption that the depth of minimum (at the middle of the annular eclipse) is known with perfect accuracy limits, to a certain extent, the effects that can be produced by varying the other constants. This fact falsely increases the weights of the unknowns, and leads to smaller mean errors of the constants than are strictly

¹ *Ap. J.* **36**, 239, 1912.

² *B. A. N.* **8**, 141, 1937.

³ *Publ. Washburn Obs.* **15**, 103, 1928.

⁴ *Publ. A. S. P.* **50**, 173, 1938.

correct; the principle is a familiar one in the theory of least squares, and needs no elaboration. This last objection is probably not a vital one, since the calculated mean errors of the elements are probably of the right order of magnitude; but in order to avoid the criticism, it would be necessary to multiply the amount of computation by a factor of at least three. The purpose of the above comments is merely to emphasize the fact that the complete solution of the photometric orbit, including the limb-darkening constants, is rather laborious by *any* method—a fact to be kept in mind throughout the ensuing discussion.

OUTLINE OF THE METHOD

The observed brightness of an eclipsing binary at any time during eclipse can be expressed as a simple function of f , which is defined as that fraction of the total light of the eclipsed star, which is cut off at a given time. (The effects of ellipticity and reflection are neglected for the present.) On the assumption that preliminary photometric elements are already available, it is desired, by comparing the calculated with the observed light-curve, to adjust the elements, including the constants of limb darkening, by the method of least squares; and thus to obtain the most probable values of the elements, and their mean errors. In order to set up the equations of condition, we need the partial derivatives of f with respect to each of the photometric elements. The present investigation is concerned mainly with the question of how these partial derivatives can be obtained for any eclipsing binary. The formulae are developed for Case I₁, namely, for the partial phases of the eclipse of the larger star. The other cases are analogous, and no details are given.

The analytic expression for f is given in equations (3) and (4). The quantity f may be written, $f = g + h$, where g is an integrated term, equal to zero under certain conditions, and where h involves an integral that is not directly integrable. The partial derivatives of g and h with respect to the geometric elements are given in equations (7) and (8) respectively. Equations (8) involve three integrals, which are not directly integrable, but which can be greatly simplified. The two integrals that are finally needed have been evaluated by Dr. G. E. Kron and the writer, using graphical and other methods. The formulae for the equations of condition, and for the calculation of the differential coefficients that are used in them, are given in the synopsis of formulae at the end of the paper.

CIRCULAR ORBIT

In the main, the notation of Russell⁵ and of Russell and Shapley⁶ is here retained. It is necessary in the

⁵ *Ap. J.* **35**, 315; **36**, 54, 1912.

⁶ *Ap. J.* **36**, 239, 385, 1912.

present treatment, however, to employ light units that are different from those of Russell, and therefore the a 's of Russell's method are not used. In addition, the coefficient of limb darkening is here designated by the now conventional " u ," instead of " x ." The following symbols are used repeatedly:

L_1 = light of large star.

L_2 = $1 - L_1$ = light of small star.

r_1 = radius of large star in terms of the distance between centers of the two stars as unit distance.

r_2 = radius of small star (same unit).

k = r_2/r_1 .

δ = apparent distance between centers,
= $\cos^2 i + \sin^2 i \sin^2 \vartheta$,

where i = inclination angle (angle between the line of sight and the normal to the orbit plane),

and ϑ = phase angle = $\frac{2\pi}{P}(t - t_0)$,

where P = period,

t = time of observation,

t_0 = time of mid-eclipse.

u_1 = coefficient of limb darkening of large star in the expression,

$$J = J_1(1 - u_1 + u_1 \cos \varphi).$$

u_2 = coefficient of limb darkening of small star.

$f_1(f_2)$ = fraction of total light of large (small) star that is cut off at any time.

At any time t , during the eclipse of the large star by the small one, the eclipsed fraction of the total light of the large star may be written,

$$f_1 = f_1(u_1, r_1, r_2, i, \vartheta),$$

where it is assumed that ϑ , corresponding to the time of observation, is accurately known, and that u_1 , r_1 , r_2 , and i are the constants of the photometric orbit, for which preliminary values have already been obtained, e.g., by Russell's method. It is desired to set up equations of condition, and to adjust the preliminary constants by the method of least squares.

During the eclipse of the large star, the light of the system, as it is observed at any time, is

$$l = 1 - f_1 L_1.$$

This expression represents the light-curve as computed from the preliminary values of the constants, and it may be compared directly with the observations if the latter are expressed in light units. The equations of condition, for the eclipse under consideration, are of the form,

$$\begin{aligned}\Delta l_{(o-c)} &= -f_1 \Delta L_1 - L_1 \Delta f_1, \\ &= -f_1 \Delta L_1 - L_1 \left(\frac{\partial f_1}{\partial u_1} \Delta u_1 + \frac{\partial f_1}{\partial r_1} \Delta r_1 + \frac{\partial f_1}{\partial r_2} \Delta r_2 + \frac{\partial f_1}{\partial i} \Delta i \right).\end{aligned}$$

A similar set of equations applies to the eclipse of the small star. It is clear that all these equations of condition may be combined into a single set of normal equations, which may be solved for the unknowns, Δu_1 , etc., provided that the differential coefficients, $\frac{\partial f_1}{\partial u_1}$, etc., can be determined.

The present problem evidently consists of defining the differential coefficients, $\frac{\partial f_1}{\partial u_1}$, etc., and of constructing suitable tables to facilitate the determination of the coefficients for use in any particular orbit. In the construction and application of such tables, three separate cases must be considered: Case I₁, the eclipse of the large star by the small one, partial phase; Case I₂, the eclipse of the large star by the small one, annular phase; and Case II, the eclipse of the small star by the large one, partial phase. The three cases will be treated separately in deriving the expressions for the differential coefficients. It is apparent that the tables may be applied as readily to partial eclipses as to total and annular ones, although in the case of partial eclipses it is probably hopeless to try to derive the darkening coefficients, because of the indeterminate nature of the problem. Nevertheless, if it should be desired to adopt any pair of limb-darkening coefficients for an eclipsing binary with partial eclipses, and to adjust the remaining elements by the method of least squares, the proposed tables would be applicable.

CASE I₁

LARGE STAR ECLIPSED, PARTIAL PHASE

The total light of the large star, uneclipsed, may be expressed as follows. Consider a point P on the surface of a sphere of radius r_1 , such that the radius through P makes an angle φ with the line of sight. Then it is assumed that the surface brightness at the point P is

$$J = J_1 (1 - u_1 + u_1 \cos \varphi),$$

where J_1 is the surface brightness at the center of the disk of the large star, and u_1 is the coefficient of limb darkening. If ρ is the apparent distance from P to the center, then $\rho/r_1 = \sin \varphi$, and $\sqrt{1 - \rho^2/r_1^2} = \cos \varphi$, and the light from an annulus of apparent radius ρ and apparent width $d\rho$, is

$$dI = 2\pi \rho d\rho \cdot J_1 (1 - u_1 + u_1 \sqrt{1 - \rho^2/r_1^2}).$$

Integrating the above expression between the limits, 0 and r_1 , we find

$$I = 2\pi J_1 \int_0^{r_1} \rho (1 - u_1 + u_1 \sqrt{1 - \rho^2/r_1^2}) d\rho,$$

or

$$I = \pi r_1^2 J_1 (1 - u_1/3). \quad (1)$$

This expression will be used presently. The unit of light is, of course, immaterial, since it will later be cancelled out.

We next determine the amount of light, in the same unit as above, that is cut off during partial eclipse by the small star. In the disk of the large star, consider any partially eclipsed annulus of radius ρ and of width $d\rho$, concentric with the large disk. As may be seen in Figure 1, the eclipsed portion of the annulus is measured by the central angle 2θ , where $\cos \theta = \frac{\rho^2 + \delta^2 - r_2^2}{2\delta\rho}$.

Fig. 1. Large star eclipsed, partial phase; $r_2 < \delta$

Now the surface brightness at any point on the annulus is

$$J = J_1 (1 - u_1 + u_1 \sqrt{1 - \rho^2/r_1^2}),$$

and the area of the eclipsed portion of the annulus is

$$2\rho\theta d\rho = 2\rho \cos^{-1} \frac{\rho^2 + \delta^2 - r_2^2}{2\delta\rho} d\rho,$$

wherefore the light of the eclipsed portion of the annulus is

$$dI = 2J_1 (1 - u_1 + u_1 \sqrt{1 - \rho^2/r_1^2}) \rho \cos^{-1} \frac{\rho^2 + \delta^2 - r_2^2}{2\delta\rho} d\rho.$$

Integrating this expression over the eclipsed portion of the larger disk, we have

$$I = 2J_1 \int_{\delta-r_2}^{r_1} \rho (1 - u_1 + u_1 \sqrt{1 - \rho^2/r_1^2}) \cos^{-1} \frac{\rho^2 + \delta^2 - r_2^2}{2\delta\rho} d\rho. \quad (2)$$

Since equations (1) and (2) have the same unit, we divide (2) by (1) to find the eclipsed *fraction* of the light of the large star, namely,

$$f_{11} = \frac{2}{\pi r_1^2 \left(1 - \frac{u_1}{3}\right)} \int_{\delta-r_2}^{r_1} \rho (1 - u_1 + u_1 \sqrt{1 - \rho^2/r_1^2}) \cos^{-1} \frac{\rho^2 + \delta^2 - r_2^2}{2\delta\rho} d\rho. \quad (3)$$

(Subscripts "11" refer to Case I₁, etc.) Equation (3) is fundamental, since it is an analytic expression for f_{11} in terms of the photometric elements of the orbit. We shall return to it shortly.

It is immediately apparent that the above expression for f_{11} is valid only if the limb of the small star does not extend beyond the center of the large one. In case it does, i.e., if $r_2 > \delta$, one of the limits of integration is changed, and an additional term, which is readily integrated, appears in the expression for f_{11} . An appropriate diagram is shown in Figure 2. By a line of reasoning similar to that employed in deriving equation (3), it can be shown that if $r_2 > \delta$, the expression for f_{11} is

$$f_{11} = \frac{(1-u_1) \left(\frac{r_2-\delta}{r_1}\right)^2 + \frac{2u_1}{3} \left\{1 - \left[1 - \left(\frac{r_2-\delta}{r_1}\right)^2\right]^{\frac{3}{2}}\right\}}{1 - \frac{u_1}{3}} + \frac{2}{\pi r_1^2 \left(1 - \frac{u_1}{3}\right)} \int_{r_2-\delta}^{r_1} \rho (1 - u_1 + u_1 \sqrt{1 - \rho^2/r_1^2}) \cos^{-1} \frac{\rho^2 + \delta^2 - r_2^2}{2\delta\rho} d\rho. \quad (4)$$

Before making use of equation (4), we may consider the following expression for f_{11} (see footnote 2):

$$f_{11} = \frac{(1-u_1) f_{11}^U + \frac{2}{3} u_1 f_{11}^D}{1 - \frac{u_1}{3}},$$

where f_{11}^U and f_{11}^D are the values of f_{11} corresponding to Uniform and completely Darkened hypotheses respectively. Differentiating partially with respect to u_1 , we have

$$\frac{\partial f_{11}}{\partial u_1} = \frac{2}{3} \frac{f_{11}^D - f_{11}^U}{\left(1 - \frac{u_1}{3}\right)^2}. \quad (5)$$

Since it is assumed that f_{11}^U and f_{11}^D have already been

$$\frac{\partial f_{11}}{\partial r_1} = \varphi(r_1, u_1) \left[\int_a^b \frac{\partial \psi}{\partial r_1} d\rho + \psi(u_1, r_1, r_2, i, b) \frac{db}{dr_1} \right] + \frac{\partial \varphi}{\partial r_1} \int_a^b \psi(u_1, r_1, r_2, i, \rho) d\rho; \text{ etc.}$$

⁷ Cf. *Advanced Calculus*, by F. S. Woods, pages 141-144.

Fig. 2. Large star eclipsed, partial phase; $r_2 > \delta$

found for the calculation of the preliminary light-curve, the coefficients $\frac{\partial f_{11}}{\partial u_1}$ may be found immediately.

The remaining functions desired are the partial derivatives of f_{11} with respect to the parameters r_1 , r_2 , and i . The first term in (4) is simple enough, and may be neglected for the moment. Considering only the second term, we may write

$$f_{11}(u_1, r_1, r_2, i) = \varphi(r_1, u_1) \int_a^b \psi(u_1, r_1, r_2, i, \rho) d\rho,$$

where the limits a and b are functions of the parameters r_1 , r_2 , and i . It can be shown that $\psi(u_1, r_1, r_2, i, \rho)$ is a continuous function of each of the parameters, r_1 , r_2 , and i , between the limits a and b . It is therefore permissible to differentiate under the integral sign with respect to these parameters, and we have⁷

The partial differential coefficients are thus formally expressed in terms of known quantities. The application of this symbolism to the actual formula for f_{11} is a matter of straightforward algebra and differential calculus. Considering now both terms of f_{11} , we introduce the following auxiliary quantities for Case I₁:

$$\beta_1 = \frac{2}{\pi r_1^2 \left(1 - \frac{u_1}{3}\right)}; \quad \frac{\partial \delta}{\partial i} = -\frac{\sin i \cos i \cos^2 \vartheta}{\delta};$$

$$f_{11} = g_1 + h_{11}; \quad \frac{\partial f_{11}}{\partial r_1} = \frac{\partial g_1}{\partial r_1} + \frac{\partial h_{11}}{\partial r_1}, \quad \text{or} \quad (f_{11}')_{r_1} = (g_1')_{r_1} + (h_{11}')_{r_1}, \text{ etc.};$$

where

$$\left. \begin{aligned} g_1 &= \frac{(1-u_1) \left(\frac{r_2-\delta}{r_1}\right)^2 + \frac{2}{3} u_1 \left\{1 - \left[1 - \left(\frac{r_2-\delta}{r_1}\right)^2\right]^{\frac{3}{2}}\right\}}{1 - \frac{u_1}{3}}, & \text{for } r_2 > \delta; \\ &= 0, & \text{for } r_2 \leq \delta; \\ \text{and} \\ h_{11} &= \beta_1 \int_{r_2-\delta}^{r_1} \rho (1-u_1 + u_1 \sqrt{1-\rho^2/r_1^2}) \cos^{-1} \frac{\rho^2 + \delta^2 - r_2^2}{2\delta\rho} d\rho & \text{for } r_2 > \delta; \\ &= \beta_1 \int_{\delta-r_2}^{r_1} \rho (1-u_1 + u_1 \sqrt{1-\rho^2/r_1^2}) \cos^{-1} \frac{\rho^2 + \delta^2 - r_2^2}{2\delta\rho} d\rho, & \text{for } r_2 \leq \delta. \end{aligned} \right\} (6)$$

Then the values of the partial differential coefficients are given by the sums of the appropriate partial derivatives of g_1 and h_{11} , which derivatives are formulated as follows:

$$\left. \begin{aligned} & \begin{array}{cc} r_2 > \delta & r_2 \leq \delta \end{array} \\ (g_1')_{r_2} &= \frac{\partial g_1}{\partial r_2} = \pi \beta_1 (r_2 - \delta) \left[1 - u_1 + u_1 \sqrt{1 - \left(\frac{r_2 - \delta}{r_1}\right)^2}\right]; & 0; \\ (g_1')_{r_1} &= \frac{\partial g_1}{\partial r_1} = -\left(\frac{r_2 - \delta}{r_1}\right) (g_1')_{r_2}; & 0; \\ (g_1')_i &= \frac{\partial g_1}{\partial i} = -\frac{\partial \delta}{\partial i} (g_1')_{r_2}; & 0; \end{array} \right\} (7)$$

$$\left. \begin{aligned} (h_{11}')_{r_1} &= \frac{\partial h_{11}}{\partial r_1} = -\frac{2h_{11}}{r_1} + \beta_1 \left[r_1 (1-u_1) \cos^{-1} \frac{r_1^2 + \delta^2 - r_2^2}{2\delta r_1} + \frac{u_1}{r_1^3} \int_{r_2-\delta}^{r_1} \frac{\rho^3}{\sqrt{1-\rho^2/r_1^2}} \cos^{-1} \frac{\rho^2 + \delta^2 - r_2^2}{2\delta\rho} d\rho \right]; \\ (h_{11}')_{r_2} &= \frac{\partial h_{11}}{\partial r_2} = \frac{\beta_1 r_2}{\delta} \int_{r_2-\delta}^{r_1} \frac{1-u_1 + u_1 \sqrt{1-\rho^2/r_1^2}}{\sqrt{1 - \left(\frac{\rho^2 + \delta^2 - r_2^2}{2\delta\rho}\right)^2}} d\rho - (g_1')_{r_2}; \\ (h_{11}')_i &= \frac{\partial h_{11}}{\partial i} = -\frac{\beta_1}{2\delta^2} \frac{\partial \delta}{\partial i} \int_{r_2-\delta}^{r_1} \frac{(\delta^2 - \rho^2 + r_2^2) (1-u_1 + u_1 \sqrt{1-\rho^2/r_1^2})}{\sqrt{1 - \left(\frac{\rho^2 + \delta^2 - r_2^2}{2\delta\rho}\right)^2}} d\rho - (g_1')_i; \end{aligned} \right\} \begin{array}{l} \text{Change} \\ \text{lower limit} \\ \text{of integra-} \\ \text{tion to} \\ \delta - r_2. \\ (g_1')_{r_2} \text{ and} \\ (g_1')_i \\ \text{vanish} \\ \text{(see} \\ \text{equations} \\ (7) \text{).} \end{array} (8)$$

Similar sets of expressions are found for Case I₂ and Case II, and the differences that appear are discussed later. It is apparent from equations (8), since $f_{11}' = g_1' + h_{11}'$, that

$$\left. \begin{aligned} \frac{\partial f_{11}}{\partial r_2} &= \frac{\beta_1 r_2}{\delta} \int_{|r_2-\delta|}^{r_1} \frac{1-u_1+u_1\sqrt{1-\rho^2/r_1^2}}{\sqrt{1-\left(\frac{\rho^2+\delta^2-r_2^2}{2\delta\rho}\right)^2}} d\rho; \\ \frac{\partial f_{11}}{\partial \delta} &= -\frac{\beta_1}{2\delta^2} \int_{|r_2-\delta|}^{r_1} \frac{(\delta^2-\rho^2+r_2^2)(1-u_1+u_1\sqrt{1-\rho^2/r_1^2})}{\sqrt{1-\left(\frac{\rho^2+\delta^2-r_2^2}{2\delta\rho}\right)^2}} d\rho. \end{aligned} \right\} (8')$$

Likewise the expression for $\frac{\partial f_{11}}{\partial r_1}$ could be written from the second of equations (7) and the first of (8). Professor Henry Norris Russell has called to my attention, however, a device for avoiding the direct evaluation of $\frac{\partial f_{11}}{\partial r_1}$: since f is a function, not of r_1 , r_2 , and δ independently, but only of their ratios, it follows that

$$r_1 \frac{\partial f_{11}}{\partial r_1} + r_2 \frac{\partial f_{11}}{\partial r_2} + \delta \frac{\partial f_{11}}{\partial \delta} = 0. \quad (9)$$

Therefore if the coefficients $\frac{\partial f_{11}}{\partial r_2}$ and $\frac{\partial f_{11}}{\partial \delta}$ can be found, the

remaining one, $\frac{\partial f_{11}}{\partial r_1}$, can be easily obtained from equation (9). (The reason for selecting $\frac{\partial f_{11}}{\partial r_1}$ for calculation by equation (9) is the fact that it is the hardest one of the three to obtain directly.) I am greatly indebted to Professor Russell for suggesting the use of equation (9), as well as for other helpful comments.

The integrals in equations (8') can be reduced by substituting as the unit of length the radius of the small star instead of the distance between centers. If we substitute $\rho' = \rho/r_2$, and then drop the primes for the sake of simplicity, equations (8') become

$$\left. \begin{aligned} (f_{11}')_{r_2} &= \frac{\beta_1 r_2^2}{\delta} \int_{|1-\delta/r_2|}^{1/k} \frac{1-u_1+u_1\sqrt{1-k^2\rho^2}}{\sqrt{1-\left(\frac{\rho^2+(\delta/r_2)^2-1}{2(\delta/r_2)\rho}\right)^2}} d\rho; \\ (f_{11}')_{\delta} &= -\frac{\beta_1 r_2^3}{2\delta^2} \int_{|1-\delta/r_2|}^{1/k} \frac{[(\delta/r_2)^2-\rho^2+1][1-u_1+u_1\sqrt{1-k^2\rho^2}]}{\sqrt{1-\left(\frac{\rho^2+(\delta/r_2)^2-1}{2(\delta/r_2)\rho}\right)^2}} d\rho. \end{aligned} \right\} (10)$$

The integrals in equations (10) may be studied under four type forms, as follows:

$$\int_{|1-a|}^b \frac{dx}{\sqrt{1-\left(\frac{x^2+a^2-1}{2ax}\right)^2}} = \int_{|1-a|}^b \frac{x dx}{\sqrt{1-\left(\frac{x^2+a^2-1}{2a}\right)^2}} = a \int_{-1}^c \frac{dy}{\sqrt{1-y^2}} = a \sin^{-1} y \Big|_{-1}^c, \quad (11)$$

where $c = \frac{b^2-a^2-1}{2a}$.

$$\begin{aligned} \int_{|1-a|}^b \frac{x^2 dx}{\sqrt{1-\left(\frac{x^2+a^2-1}{2ax}\right)^2}} &: \text{Integrate by parts, letting } u = x^2, \text{ etc.:} \\ &= a(1+a^2+2ay) \sin^{-1} y \Big|_{-1}^c - 2a^2 \left[y \sin^{-1} y + \sqrt{1-y^2} \right]_{-1}^c. \end{aligned} \quad (12)$$

$$\int_{|1-a|}^b \frac{\sqrt{b^2-x^2}}{\sqrt{1-\frac{(x^2+a^2-1)^2}{2ax}}} dx : \text{Integrate by parts, letting } u=\sqrt{b^2-x^2}, \text{ and making use of (11):}$$

$$= \frac{a\sqrt{2a}}{2} \left[\pi\sqrt{1+c} + \int_{-1}^c \frac{\sin^{-1} y}{\sqrt{c-y}} dy \right]. \quad (13)$$

$$\int_{|1-a|}^b \frac{x^2\sqrt{b^2-x^2}}{\sqrt{1-\frac{(x^2+a^2-1)^2}{2ax}}} dx : \text{Integrate by parts, letting } u=\sqrt{b^2-x^2}, \text{ and making use of (12):}$$

$$= \frac{a\sqrt{2a}}{2} (1+a^2) \left[\pi\sqrt{1+c} + \int_{-1}^c \frac{\sin^{-1} y}{\sqrt{c-y}} dy \right] - a^2\sqrt{2a} \int_{-1}^c \frac{\sqrt{1-y^2}}{\sqrt{c-y}} dy. \quad (14)$$

The substitution of equations (11) to (14) in equations (10) effects a great simplification of the latter. I am obliged to Professor Thomas Buck, of the Berkeley Mathematics Department, for assistance in performing the integrations (11) and (12), and in the simplifications indicated in (13) and (14). If for Case I₁ we adopt the substitutions, $a=\delta/r_2$, $b=1/k$, and define $c=\frac{b^2-a^2-1}{2a}$,

then we may write equations (10) in the following form:

$$(f_{11}')_{r_2} = \beta_1 r_2 \left[(1-u_1) \cos^{-1}(-c) + \frac{u_1\sqrt{2a}}{b} A_{11} \right]; \quad (15)$$

$$(f_{11}')_{\delta} = -\beta_1 r_2 \left[(1-u_1) \sqrt{1-c^2} + \frac{u_1\sqrt{2a}}{b} B_{11} \right]; \quad (16)$$

where

$$A_{11} = \frac{1}{2} \left[\pi\sqrt{1+c} + \int_{-1}^c \frac{\sin^{-1} y}{\sqrt{c-y}} dy \right]; \quad (17)$$

$$\int_{-1}^c \frac{\sin^{-1} y}{\sqrt{c-y}} dy = \int_{-1}^{c-\epsilon} \frac{\sin^{-1} y}{\sqrt{c-y}} dy + \int_{c-\epsilon}^c \frac{\sin^{-1} y}{\sqrt{c-y}} dy$$

$$= \int_{-1}^{c-\epsilon} \frac{\sin^{-1} y}{\sqrt{c-y}} dy + 2\sqrt{\epsilon} \sin^{-1} c - \frac{2}{3(1-c^2)^{3/2}} \epsilon^{3/2} + \frac{c}{5(1-c^2)^{5/2}} \epsilon^{5/2} - \frac{1+2c^2}{21(1-c^2)^{7/2}} \epsilon^{7/2} + \frac{c(3+2c^2)}{36(1-c^2)^{9/2}} \epsilon^{9/2} + \dots \quad (19)$$

where ϵ is small, positive, and finite. Similarly,

$$\int_{-1}^c \frac{\sqrt{1-y^2}}{\sqrt{c-y}} dy = \int_{-1}^{c-\epsilon} \frac{\sqrt{1-y^2}}{\sqrt{c-y}} dy + 2\sqrt{\epsilon} \sqrt{1-c^2} + \frac{2c}{3(1-c^2)^{3/2}} \epsilon^{3/2}$$

$$- \frac{1}{5(1-c^2)^{5/2}} \epsilon^{5/2} + \frac{c}{7(1-c^2)^{7/2}} \epsilon^{7/2} - \frac{1+4c^2}{36(1-c^2)^{9/2}} \epsilon^{9/2} + \dots \quad (20)$$

Both the above series converge rapidly except for c near ± 1 , where it is necessary to use all the terms and to adopt a small value of ϵ . Depending on the value of c , ϵ was taken as 0.1, 0.05, or 0.02. The main parts of the integrals for both A and B , that is, the first terms of (19) and (20), were determined with a planimeter, and the remaining parts were calculated from the series formulae

and

$$B_{11} = \frac{1}{2} \int_{-1}^c \frac{\sqrt{1-y^2}}{\sqrt{c-y}} dy. \quad (18)$$

The quantities A_{11} and B_{11} are functions of c alone, and can therefore be calculated without too much labor by graphical and other methods. They have been evaluated by Dr. Kron and the writer, in the following manner. The integrands involved in both expressions become infinite at the upper limit, and are therefore not manageable in this region by graphical methods. The difficulty is avoided by expanding the numerators of both integrands in the region of $y=c$ by Taylor's series, and integrating term by term:

above. Thus A and B were tabulated as functions of c . For $c=1$, both A and B are directly integrable.

With the tabulation of A and B , and with Russell and Shapley's tables giving f_{11}^U and f_{11}^D , we are now in a position to calculate all four partial differential coefficients for any eclipsing binary by means of algebraic, trigonometric, anti-trigonometric, and other tabular,

functions. The treatment thus far applies only to the partial phase of the eclipse of the large star. The formulae requisite to the calculation of the coefficients are summarized at the end of the paper.

CASE I₂

LARGE STAR ECLIPSED, ANNULAR PHASE

Mathematically, Case I₂ is very similar to Case I₁, the principal difference being that in the expressions for h_{12} and h_{12}' (cf. equations (8)), the upper limit of integration is changed from r_1 to $\delta + r_2$. The partial differential coefficients are derived by methods similar to those of Case I₁. The tables of A and B are a simple extension to the condition that $c > +1$, with the modification that instead of (17) and (18) we have the following:

$$f_2 = \frac{(1-u_2) \left(\frac{r_1-\delta}{r_2} \right)^2 + \frac{2}{3} u_2 \left\{ 1 - \left[1 - \left(\frac{r_1-\delta}{r_2} \right)^2 \right]^{\frac{3}{2}} \right\}}{1 - \frac{u_2}{3}} + \frac{2}{\pi r_2^2 \left(1 - \frac{u_2}{3} \right)} \int_{r_1-\delta}^{r_2} \rho (1-u_2+u_2 \sqrt{1-\rho^2/r_2^2}) \cos^{-1} \frac{\rho^2+\delta^2-r_1^2}{2\delta\rho} d\rho, \text{ if } r_1 > \delta;$$

$$f_2 = \frac{2}{\pi r_2^2 \left(1 - \frac{u_2}{3} \right)} \int_{\delta-r_1}^{r_2} \rho (1-u_2+u_2 \sqrt{1-\rho^2/r_2^2}) \cos^{-1} \frac{\rho^2+\delta^2-r_1^2}{2\delta\rho} d\rho, \text{ if } r_1 \leq \delta.$$

Partial differentiation of f_2 with respect to the geometrical elements gives a set of expressions analogous to equations (7) and (8). The integrals in the h_2' can be simplified, as before, by changing the unit of distance, except that now the unit becomes r_1 instead of r_2 . In fact, if we set $a = \delta/r_1$, $b = k$, and $c = \frac{b^2 - a^2 - 1}{2a}$, we find

that for Case II the expressions required for the differential coefficients are exactly the same in form as those given for Case I₁ in equations (5), (9), (15) and (16), provided that the subscripts "1" or "11" be interchanged with the subscripts "2". It follows that the tables of A and B are applicable for Case II in just the same way as for Case I₁, if the appropriate arguments, which are defined differently for the two cases, are used. The complete expressions are given in the synopsis of formulae.

ELLIPTICITY AND REFLECTION

From the nature of the problem under discussion, it is assumed that a light-curve of the very highest accuracy is available, and that the star has been adequately observed, not only during, but also between, eclipses. If the observed light-curve shows a small, though perceptible effect of ellipticity or reflection, it would prob-

$$A_{12} = \frac{1}{2} \left[\pi \sqrt{c+1} + \pi \sqrt{c-1} + \int_{-1}^{+1} \frac{\sin^{-1} y}{\sqrt{c-y}} dy \right]; \quad (21)$$

$$B_{12} = \frac{1}{2} \int_{-1}^{+1} \frac{\sqrt{1-y^2}}{\sqrt{c-y}} dy. \quad (22)$$

In the synopsis of formulae at the end of the paper, the two cases so far considered are combined into a single Case I. Equations (28) and (29), which are formally different, are written separately for Case I₁ and Case I₂.

CASE II

SMALL STAR ECLIPSED, PARTIAL PHASE

By the method employed for Case I₁, it can be shown that the eclipsed fraction f_2 of the total light of the small star is as follows:

ably be best to rectify the light-curve by Russell's method before attempting a least squares solution for the remaining elements. While the ellipticity and reflection constants could be included in the general solution, it is desirable to avoid making the work too cumbersome.

It seems best, in general, to restrict the determination of limb-darkening coefficients to stars for which the ellipticity is demonstrably *very small*. Russell's ingenious method of allowing for the effects of ellipticity of figure is based on the assumption that the two components of the binary are similar ellipsoids, and that for darkened disks the contours of equal luminosity are ellipses concentric with the ellipse of figure of the disks. While the assumption has been adequate to permit the determination, in a large number of cases, of the orbital elements corresponding to an *assumed* value of the limb-darkening constant, we have no assurance that the hypothesis is a close enough approximation when the darkening constant is included as an unknown. In other words, the determination of the coefficient of limb darkening makes extremely rigid demands, not only upon the accuracy of the observations, but also upon the exactitude of the method. The satisfactory analysis of the distribution of brightness over the disks of β Lyrae and W Ursae Majoris systems would require elaborate treatment.

ORBITAL ECCENTRICITY

Except for possible variations with time of ellipticity and reflection, which will not be considered, the only dependence of the light-curve on orbital eccentricity results from the change in the distance between centers, both true and apparent. The radii of the two stars, in terms of a variable distance between centers as unit, are functions of the eccentricity and the longitude of periastron; and the apparent distance between centers, δ , is also a function of these quantities since the angular orbital velocity departs from constancy. Therefore it is logically possible to find $\frac{\partial f}{\partial e}$ and $\frac{\partial f}{\partial \omega}$, and thus to include

e and ω as unknowns in the general least squares solution. Whether or not there is any practical method of calculating such differential coefficients has not been investigated. The only case that will be considered here is an approximate method of treatment that is very satisfactory if the eccentricity is small.

The method is equivalent to that employed by Pannekoek and Elsa van Dien,² who considered separately the effects of the radial and the tangential components of the eccentricity. As far as the determination of the elements is concerned, the first-order effect of the tangential component is removed by reflecting one half of the light-curve during eclipse upon the other half. The radial component may be taken into account, to the first order of approximation, by considering that the orbit is the arc of a circle during each eclipse, but that the distance between centers is different for the two eclipses. The difference in the distance between centers manifests itself in two ways: first, in the difference in orbital velocity during the two eclipses, and second, in the difference in the sizes of the two stars in terms of a variable unit.

If e denotes the radial component of the eccentricity, then the distance between centers during, let us say, the eclipse of the large star is $1-e$, and the distance during the other eclipse is $1+e$. Or, for simplicity of computation, we may take the distance during the eclipse of the large star as unity, and that during the other eclipse as $1+2e$. Then during the process of computation, r_1 and r_2 signify the radii of the large and small stars respectively, in terms of the distance between centers during the eclipse of the *large* star as unit distance. During the eclipse of the small star, the radii in terms of the new unit are r_1' and r_2' , where $r_1' \cong r_1(1-2e)$, and $r_2' \cong r_2(1-2e)$. If ϑ be retained to denote the phase angle in terms of the observed period and the distance between centers during the eclipse of the large star, then the corresponding angle during the eclipse of the small star is ϑ' , where

$$\vartheta' = \vartheta(1-4e); \quad \vartheta = \frac{2\pi}{P}(t-t_0).$$

The apparent distance between centers during the eclipse of the small star is given by the expression,

$$\delta'^2 = \cos^2 i + \sin^2 i \sin^2 \vartheta',$$

or

$$\delta'^2 = \cos^2 i + \sin^2 i \sin^2 [\vartheta(1-4e)]. \quad (23)$$

The inclination is, of course, the same for the two eclipses. It follows that

$$\frac{\partial f_1}{\partial e} = 0;$$

$$\frac{\partial f_2}{\partial e} = \frac{\partial f_2}{\partial r_1'} \cdot \frac{\partial r_1'}{\partial e} + \frac{\partial f_2}{\partial r_2'} \cdot \frac{\partial r_2'}{\partial e} + \frac{\partial f_2}{\partial \delta'} \cdot \frac{\partial \delta'}{\partial e}, \quad (24)$$

or

$$\frac{\partial f_2}{\partial e} = -2r_1 \frac{\partial f_2}{\partial r_1'} - 2r_2 \frac{\partial f_2}{\partial r_2'} - \frac{4 \sin^2 i \cdot \vartheta \cdot \sin \vartheta' \cdot \cos \vartheta'}{\delta'} \frac{\partial f_2}{\partial \delta'} \quad (25)$$

If, in the preliminary elements, e has not been assumed equal to zero, it may be necessary to take account of that fact in evaluating the coefficients given by equations (34) to (39). That is to say, the coefficients for the eclipse of the small star should be evaluated for r_1' , r_2' , etc., instead of for the unprimed values. Equation (38) should be replaced by the result of differentiating (23) partially with respect to i , and equation (39) should be replaced by equation (25).

It is much simpler, however, and in most if not all cases perfectly satisfactory (since the present section applies in any event only to cases where the eccentricity is small), to assume in the preliminary elements that $e=0$. In this case, equation (25) reduces to the form

$$\frac{\partial f_2}{\partial e} = -2r_1 \frac{\partial f_2}{\partial r_1} - 2r_2 \frac{\partial f_2}{\partial r_2} - \frac{4 \sin^2 i \cdot \vartheta \cdot \sin \vartheta \cdot \cos \vartheta}{\delta} \frac{\partial f_2}{\partial \delta},$$

and all the equations in the synopsis of formulae are appropriate as they stand.

It is clear that the purpose of selecting the distance units in the above manner is to obviate the necessity of calculating $\frac{\partial f_1}{\partial e}$. In the final tabulation of the elements,

however, it is desirable to adopt as unit distance, the semi-major axis of the orbit. Therefore after the least squares solution has been made, it is necessary to reduce the final values of r_1 and r_2 by the factor $(1-e)$. The other elements are not affected.

If the tangential component of the eccentricity is not zero, its value may be derived from the spacing of the two minima, and may be combined with the radial component to give the true eccentricity and longitude of periastron. In such a case the reduction to the semi-major axis of the orbit can be easily worked out from the geometry of the orbit.

OBSERVATIONS DURING TOTAL PHASE

If the phase of totality during the eclipse of the small star is of fairly long duration, and is well observed, L_2 can be obtained directly from the depth of minimum at total eclipse, with an accuracy that considerably exceeds the accuracy of the other elements. In such an event it is preferable to assume that L_2 is known exactly, that is, that $\Delta L_2 = 0$ in equation (40). This assumption eliminates one unknown from the least squares solution. On the other hand, the total phase may be so short that the observations during totality do not deserve such high weight as the above assumption would give them. In this case, ΔL_2 should be retained as an unknown in equation (40), and equations of condition during totality should be included in the solution. They are of the form, $\Delta l_{(O-C)} = -f_2 \Delta L_2 = -\Delta L_2$.

EFFECT OF A THIRD BODY

It has been assumed thus far that no third body exists, and that $L_1 + L_2 = 1$, where the total light of the system is adopted as unity. From this premise it follows that $\Delta L_1 = -\Delta L_2$: when this substitution is made in the equations of condition, either (32) or (40), one unknown is eliminated. If a third body exists, however, $L_1 + L_2 < 1$, and L_1 and L_2 must be solved separately. It may still be possible to determine L_2 accurately from the depth of one of the minima, as described in the preceding paragraph, so that only ΔL_1 need be included in the least squares solution. For a three-body system with partial eclipses, such as Algol, it is logically necessary to include both ΔL_1 and ΔL_2 as independent unknowns. The degree of determinacy of the problem, where a third body is involved, depends upon the circumstances of the individual case, and can best be judged by future experience.

CONCLUSION

The method outlined above will serve for the determination, from suitable light-curves of eclipsing binaries, of the coefficients of limb darkening, as well as the other photometric elements. That the light-curve is "suitable" implies that it is of the highest accuracy; that the departures from sphericity of figure and circularity of orbit are small; and (probably) that one of the eclipses is total and the other annular.

The method permits of completely simultaneous treatment of the two minima of the light-curve. It also lends itself readily to the simultaneous analysis of two light-curves representing different effective wave-lengths.

For binary systems that are not suitable for the determination of limb darkening, it may in time become possible to estimate the coefficients of limb darkening from the known properties of the stars, and thus to calculate quite accurately the other photometric elements. The method presented above is adapted to such computations: it is as rigorous for partial eclipses as for total and annular ones. There is little doubt that very accurate observations will reduce the inherent indeterminacy that has hitherto been experienced in the light-curves of partially eclipsed stars; and while it is doubtful whether such stars can yield reliable coefficients of limb darkening, it is not at all improbable that they may provide fairly accurate *differences* in darkening coefficients for two different wave-lengths. In any event, numerous such systems are especially interesting for other reasons, such as accurate mass determinations, and it is believed that much valuable research remains to be done in this field.

The tables that are necessary for the calculation of the differential coefficients are given in the following paper.

MOUNT HAMILTON, CALIFORNIA,
APRIL 11, 1939
ISSUED, SEPTEMBER 1, 1939.

SYNOPSIS OF FORMULAE

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CASE I, LARGE STAR ECLIPSED, PARTIAL₁₁ OR ANNULAR₁₂ PHASE

$$u_1 = \frac{\delta}{r_2}; \quad b_1 = \frac{1}{k} = \frac{r_1}{r_2};$$

$$c_1 = \frac{b_1^2 - a_1^2 - 1}{2a_1}; \quad \left\{ \begin{array}{l} -1 \leq c_1 \leq +1 \text{ for partial phase;} \\ c_1 \geq +1 \text{ for annular phase.} \end{array} \right\}$$

$$\beta_1 = \frac{2}{\pi r_1^2 \left(1 - \frac{u_1}{3}\right)}.$$

$$\frac{\partial f_1}{\partial u_1} = \frac{2}{3} \frac{f_1^D - f_1^U}{\left(1 - \frac{u_1}{3}\right)^2}, \quad \left\{ \begin{array}{l} \text{where } f_1^D \text{ and } f_1^U \text{ have already} \\ \text{been found and used} \\ \text{for the theoretical light-} \\ \text{curve.} \end{array} \right\} \quad (26)$$

$$\frac{\partial f_1}{\partial r_1} = -\frac{r_2}{r_1} \frac{\partial f_1}{\partial r_2} - \frac{\delta}{r_1} \frac{\partial f_1}{\partial \delta}. \quad (27)$$

$$\frac{\partial f_{11}}{\partial r_2} = \beta_1 r_2 (1 - u_1) \cos^{-1}(-c_1) + \frac{\beta_1 r_2 u_1}{b_1} \sqrt{2a_1} A_{11}. \quad (28)_{11}$$

$$\frac{\partial f_{12}}{\partial r_2} = \beta_1 r_2 (1 - u_1) \pi + \frac{\beta_1 r_2 u_1}{b_1} \sqrt{2a_1} A_{12}. \quad (28)_{12}$$

$$\frac{\partial f_{11}}{\partial \delta} = -\beta_1 r_2 (1 - u_1) \sqrt{1 - c_1^2} - \frac{\beta_1 r_2 u_1}{b_1} \sqrt{2a_1} B_{11}. \quad (29)_{11}$$

$$\frac{\partial f_{12}}{\partial \delta} = -\frac{\beta_1 r_2 u_1}{b_1} \sqrt{2a_1} B_{12}. \quad (29)_{12}$$

$$\frac{\partial \delta}{\partial i} = -\frac{\sin i \cos i \cos^2 \vartheta}{\delta}. \quad (30)$$

$$\frac{\partial f_1}{\partial i} = \frac{\partial f_1}{\partial \delta} \cdot \frac{\partial \delta}{\partial i}. \quad (31)$$

CASE II, SMALL STAR ECLIPSED, PARTIAL PHASE

$$a_2 = \frac{\delta}{r_1}; \quad b_2 = k = \frac{r_2}{r_1};$$

$$c_2 = \frac{b_2^2 - a_2^2 - 1}{2a_2}; \quad -1 \leq c_2 \leq +1;$$

$$\beta_2 = \frac{2}{\pi r_2^2 \left(1 - \frac{u_2}{3}\right)}.$$

$$\frac{\partial f_2}{\partial u_2} = \frac{2}{3} \frac{f_2^D - f_2^U}{\left(1 - \frac{u_2}{3}\right)^2}, \quad \left\{ \begin{array}{l} \text{where } f_2^D \text{ and } f_2^U \text{ have already} \\ \text{been found and used for the} \\ \text{theoretical light-curve.} \end{array} \right\} \quad (33)$$

$$\frac{\partial f_2}{\partial r_1} = \beta_2 r_1 (1 - u_2) \cos^{-1}(-c_2) + \frac{\beta_2 r_1 u_2}{b_2} \sqrt{2a_2} A_2. \quad (34)$$

$$\frac{\partial f_2}{\partial r_2} = -\frac{r_1}{r_2} \frac{\partial f_2}{\partial r_1} - \frac{\delta}{r_2} \frac{\partial f_2}{\partial \delta}. \quad (35)$$

$$\frac{\partial f_2}{\partial \delta} = -\beta_2 r_1 (1 - u_2) \sqrt{1 - c_2^2} - \frac{\beta_2 r_1 u_2}{b_2} \sqrt{2a_2} B_2. \quad (36)$$

$$\frac{\partial \delta}{\partial i} = -\frac{\sin i \cos i \cos^2 \vartheta}{\delta}. \quad (37)$$

$$\frac{\partial f_2}{\partial i} = \frac{\partial f_2}{\partial \delta} \cdot \frac{\partial \delta}{\partial i}. \quad (38)$$

$$\frac{\partial f_2}{\partial e} = -2r_1 \frac{\partial f_2}{\partial r_1} - 2r_2 \frac{\partial f_2}{\partial r_2} - \frac{4 \sin^2 i \cdot \vartheta \cdot \sin \vartheta \cdot \cos \vartheta}{\delta} \frac{\partial f_2}{\partial \delta} \quad (39)$$

EQUATIONS OF CONDITION, CASE I

$$\Delta l_{(O-C)} = -f_1 \Delta L_1 - L_1 \frac{\partial f_1}{\partial u_1} \Delta u_1 - L_1 \frac{\partial f_1}{\partial r_1} \Delta r_1 - L_1 \frac{\partial f_1}{\partial r_2} \Delta r_2 - L_1 \frac{\partial f_1}{\partial i} \Delta i. \quad (32)^8$$

EQUATIONS OF CONDITION, CASE II, PARTIAL PHASE

$$\Delta l_{(O-C)} = -f_2 \Delta L_2 - L_2 \frac{\partial f_2}{\partial u_2} \Delta u_2 - L_2 \frac{\partial f_2}{\partial r_1} \Delta r_1 - L_2 \frac{\partial f_2}{\partial r_2} \Delta r_2 - L_2 \frac{\partial f_2}{\partial i} \Delta i - L_2 \frac{\partial f_2}{\partial e} \Delta e. \quad (40)^9$$

EQUATIONS OF CONDITION, CASE II, TOTAL PHASE

(To be used if ΔL_2 is included as an unknown)

$$\Delta l_{(O-C)} = -\Delta L_2. \quad (41)^9$$

⁸ See page 26 regarding the assumption that $\Delta L_1 = 0$.

⁹ See page 26 regarding the assumption that $\Delta L_2 = 0$. If $e \neq 0$ in the preliminary elements, see page 25 regarding the calculation of the coefficients for Case II.

AN APPLICATION OF THE METHOD OF LEAST SQUARES TO THE DETERMINATION OF THE PHOTOMETRIC ELEMENTS OF ECLIPSING BINARIES

PART II. TABLES AND EXAMPLE

BY

A. B. WYSE and G. E. KRON

In the foregoing paper formulae are derived for the solution of the photometric elements of eclipsing binaries, including the coefficients of limb darkening, by the method of least squares. The application to any particular binary system is expedited by the use of tables of the functions A and B (see equations (17), (18), (21), and (22)). The first part of the present paper is concerned with the tabulation of A and B . The second part presents an illustrative example in the form of a portion of the calculation of the elements of the eclipsing binary, YZ Cassiopeiae.

TABLES OF A AND B

For partial phase the quantities tabulated are A and B , which are functions of the parameter c , defined in the synopsis of formulae of the preceding paper. The tables for partial phase, Tables 1a and 2a ($c \leq +1$), are equally applicable to both eclipses, but it must be remembered that c is defined differently in the two cases.

For annular eclipse, the argument c is replaced by $1/c$ for convenience in interpolation, Tables 1b and 2b. When the eclipse is central ($a=0$, $1/c=0$), the quantity A becomes infinite, and therefore a table of A for annular eclipse is not suitable for interpolation. It may be seen, however (equation (28)₁₂), that A is always used with the factor $\sqrt{2a}$. The product $\sqrt{2a} A$ is always finite, and for $1/c=0$, it becomes $\pi\sqrt{b^2-1}$. This fact suggests the tabulation of $\sqrt{2a} A$ as a function of $1/c$ and of b : such a tabulation comprises Table 1b. Unfortunately even Table 1b is not particularly well suited for interpolation, because of the nature of the horizontal differences. Furthermore it is of such a character that if it should be plotted on a single page, the third decimal would be lost. It is suggested that if the third decimal is needed, an effective method for retaining the tabulated accuracy would be to plot on a large scale as much of the table as is required for a given star, and to obtain the functions from the plotted curves. Each curve should consist of the tabulated quantity, $\sqrt{2a} A$, plotted as a function of b , with one curve for each tabulated value of $1/c$. The function B offers no difficulties, and is therefore given directly as for partial phase, Table 2b.

The tables have been checked and smoothed by differencing. It will be noted that the first differences change rather rapidly in certain regions of the tables, so that it may sometimes be necessary to take account of second

differences. The values of A and B are included for $c=+0.95$, and the values of B for $1/c=+0.95$ and $+0.05$. The first differences, when affected by the insertions, are enclosed in parentheses.

EXAMPLE

Inasmuch as the calculation of the differential coefficients is rather complicated, it is considered worth while to publish the coefficients of a concrete case, for the convenience of computers who may wish to have them available for comparison. The method has been applied by Kron to the eclipsing system YZ Cassiopeiae, with results that will soon appear in a *Lick Observatory Bulletin*.

Before attempting a solution by least squares, it is desirable to have a good set of preliminary elements, in order to avoid, if possible, the necessity of repeating the solution. For YZ Cassiopeiae it happened that an unusually good set of preliminary elements was already available,¹ so that the corrections resulting from the solution were very small. While this fortunate circumstance will not ordinarily occur, it should be possible in general to do almost as well. The best procedure would appear to be to find the best "uniform" elements by Russell's method,² and from them to determine the best "darkened" elements from Russell and Shapley's Table C³ and the related equations. Then by assuming a preliminary value for the darkening coefficients, one may find the corresponding preliminary values of the other elements by interpolation, and thus obtain a complete set of elements that may reasonably be expected to require only small corrections in the least squares solution.

The f 's of this and the previous paper are related to Russell's α 's as follows:

LARGE STAR ECLIPSED

$$f_1^U = k^2 a ;$$

$$f_1^D = \frac{1 - (1-k^2)^{\frac{3}{2}}}{1+x} a'' ;$$

SMALL STAR ECLIPSED

$$f_2^U = a ;$$

$$f_2^D = a^1 ;$$

where x is the x in Russell and Shapley's Table 1y, NOT the darkening coefficient. The above expression for f_1^D has been found convenient in the calculation of theo-

¹ *Publ. A. S. P.* **50**, 173, 1938.

² *Ap. J.* **35**, 315; **36**, 54, 1912.

³ *Ap. J.* **36**, 250, 1912.

retical light-curves. The eclipsed fraction f corresponding to any darkening coefficient u is given by the relation

$$f = \frac{(1-u) f^u + \frac{2}{3} u f^D}{1 - \frac{1}{3} u}.$$

The equations of condition, which were used in Kron's paper on YZ Cassiopeiae, mentioned above, and the resulting normal equations, are given below. The differential coefficients were calculated from formulae that were not in the final form listed in the previous paper. In addition, most of them were carried to one decimal place more than is here tabulated. One should therefore not expect to find the published normal equations, which are in exactly the form employed, checking precisely with the published equations of condition.

Each equation of condition was weighted according to the number of observations represented. Since the residuals are expressed in light units, the procedure adopted in weighting is equivalent to the assumption that the probable error of a single observation in light units is the same during eclipse as at maximum light. The assumption has a partial justification in photoelectric observations, in that the errors resulting from instrumental causes are independent of the star's brightness, and are therefore constant over the whole light-curve. Insofar as the errors have their source in bad "seeing" conditions, however, their size is proportional to the star's brightness, and is therefore approximately constant on a *magnitude* scale, just as it is for photographic and visual observations. The particular star in question has such shallow eclipses that the method of weighting makes very little difference. In general, however, it is necessary to investigate the question of weighting the observations: when photographic or visual observations are treated on an intensity scale, observations during eclipse should receive more weight than those at maximum light.⁴

⁴ Compare, for instance, Russell and Shapley, *Ap. J.* **39**, 412, 1914.

TABLE 1a
PARTIAL PHASE

c	A
-1.0	0.000
0.9	+0.112
.8	.225
.7	.340
.6	.456
.5	.575
.4	.695
.3	.817
.2	0.941
-.1	1.067
.0	1.196
+.1	1.328
.2	1.464
.3	1.605
.4	1.750
.5	1.901
.6	2.057
.7	2.220
.8	2.394
.9	2.587
0.95	2.690
+1.0	2.828

TABLE 2a
PARTIAL PHASE

c	B
-1.0	0.000
0.9	+0.110
.8	.214
.7	.313
.6	.408
.5	.499
.4	.586
.3	.667
.2	.742
-.1	.810
.0	.872
+.1	.928
.2	0.978
.3	1.021
.4	1.055
.5	1.078
.6	1.089
.7	1.088
.8	1.072
.9	1.034
0.95	1.002
+1.0	0.943

TABLE 2b
ANNULAR PHASE

$1/c$	B_{12}
+1.0	+0.943
0.95	.873
.9	.836
.8	.755
.7	.691
.6	.629
.5	.568
.4	.505
.3	.433
.2	.353
.1	.251
+.05	+.176
0.0	0.000

TABLE 1b
 $\sqrt{2a_1} A_{12}$, ANNULAR PHASE

	b	1.0	1.005	1.01	1.02	1.04	1.06	1.08	1.10	1.2	1.4	1.6	1.8	2.0	2.2	2.4	2.6	2.8	3.0
1/c																			
+1.0	0.0	+.283	.400	.566	.800	0.980	1.131	1.265	1.789	2.530	3.098	3.578	4.000	4.382	4.733	5.060	5.367	+5.657	
0.9	0.0	.293	.414	.586	.830	1.019	1.177	1.317	1.870	2.660	3.273	3.794	4.254	4.671	5.056	5.414	5.751	6.070	
.8	0.0	.299	.422	.599	.848	1.040	1.203	1.347	1.918	2.746	3.395	3.950	4.444	4.894	5.309	5.697	6.063	6.408	
.7	0.0	.303	.428	.608	.861	1.056	1.222	1.370	1.958	2.820	3.504	4.094	4.624	5.107	5.556	5.976	6.373	6.749	
.6	0.0	.307	.433	.615	.871	1.069	1.239	1.390	1.993	2.887	3.606	4.233	4.799	5.320	5.806	6.263	6.695	7.106	
.5	0.0	.310	.437	.621	.879	1.080	1.253	1.406	2.022	2.944	3.697	4.360	4.965	5.526	6.052	6.550	7.024	7.476	
.4	0.0	.312	.439	.625	.885	1.089	1.263	1.418	2.044	2.991	3.774	4.472	5.115	5.718	6.288	6.831	7.350	7.849	
.3	0.0	.313	.441	.627	.890	1.096	1.270	1.427	2.063	3.028	3.837	4.566	5.246	5.890	6.504	7.096	7.666	8.217	
.2	0.0	.314	.443	.629	.893	1.100	1.275	1.433	2.074	3.056	3.884	4.639	5.350	6.030	6.687	7.324	7.945	8.551	
+.1	0.0	.315	.444	.630	.895	1.102	1.279	1.437	2.080	3.072	3.914	4.686	5.418	6.124	6.810	7.483	8.143	8.795	
0.0	0.0	+.315	.445	.631	.897	1.104	1.282	1.440	2.084	3.078	3.924	4.702	5.441	6.156	6.854	7.540	8.216	+8.886	

EQUATIONS OF CONDITION

		PRIMARY MINIMUM											
Number	θ (Radians)	Weight											
1	0.0058	1	-0.0805	$\Delta u_1 + 4.01$	$\Delta r_1 - 7.54$	$\Delta r_2 - 0.327$	$\Delta i + 0.0000$	$\Delta e + 0.0000$	$\Delta u_2 - 0.0016$			=0	
2	.0167	1	.0772	3.98	7.51	.332			-	23		=0	
3	.0267	1	.0714	4.08	7.44	.341			-	18		=0	
4	.0374	1	.0629	3.79	7.31	.357			-	9		=0	
5	.0461	1	.0531	3.64	7.19	.382			-	17		=0	
6	.0570	1	.0407	2.83	6.38	.728			-	4		=0	
7	.0657	1	.0297	2.10	5.65	.934			-	11		=0	
8	.0736	1	.0215	1.58	5.20	.971			+	4		=0	
9	.0783	1	.0177	1.33	4.98	.966			-	35		=0	
10	.0872	1	.0107	0.92	4.61	.931			+	5		=0	
11	.0937	1	.0060	.67	4.37	.892			-	26		=0	
12	.1004	1	-0.0018	.42	4.11	.845			-	45		=0	
13	.1090	1	+0.0035	+0.12	3.85	.784			-	30		=0	
14	.1155	1	.0067	-0.08	3.66	.738			-	24		=0	
15	.1231	1	.0103	.29	3.43	.683			-	30		=0	
16	.1300	$\frac{3}{4}$	.0114	.40	2.80	.635			-	33		=0	
17	.1402	1	.0155	.67	2.94	.562			-	23		=0	
18	.1473	1	.0169	.79	2.74	.517			-	30		=0	
19	.1540	1	.0172	.89	2.55	.473			-	25		=0	
20	.1622	1	.0181	.99	2.32	.421			-	17		=0	
21	.1674	1	.0166	1.00	2.17	.388			-	9		=0	
22	.1768	1	.0153	1.04	1.89	.331			-	2		=0	
23	.1861	1	.0132	1.04	1.61	.276			-	21		=0	
24	.1938	1	.0110	0.98	1.36	.229			+	1		=0	
25	.2013	1	.0078	0.88	1.11	.183			+	1		=0	
26	.2080	1	.0041	0.74	0.85	.138			-	3		=0	
27	0.2177	1	+0.0004	$\Delta u_1 - 0.36$	$\Delta r_1 - 0.37$	$\Delta r_2 - 0.062$	$\Delta i + 0.0000$	$\Delta e + 0.0000$	$\Delta u_2 + 0.0009$			=0	

SECONDARY MINIMUM

1	0.0556	2	+0.0000	$\Delta u_1 - 0.123$	$\Delta r_1 + 0.119$	$\Delta r_2 - 0.068$	$\Delta i - 0.0061$	$\Delta e - 0.00006$	$\Delta u_2 - 0.0007$			=0
2	.0649	2		.405	.312	.196	.0246	.00177	+	9		=0
3	.0725	2		.513	.418	.225	.0466	.00315	+	17		=0
4	.0796	2		.602	.461	.242	.0661	.00365		25		=0
5	.0866	2		.668	.458	.249	.0824	.00409		28		=0
6	.0958	4		1.035	.605	.351	.1481	.00614		19		=0
7	.1098	4		1.113	.467	.333	.1911	.00540		34		=0
8	.1243	4		1.134	.303	.302	.2312	.00348		8		=0
9	.1382	4		1.107	+0.112	.268	.2577	-0.00084		15		=0
10	.1552	4		1.018	-0.114	.221	.2730	+0.00156		32		=0
11	.1721	4		0.877	.292	.174	.2669	.00324		38		=0
12	.1843	2		.527	.268	.098	.1741	.00246		32		=0
13	.1943	2		.432	.285	.077	.1526	.00203		21		=0
14	.2042	2		.322	.255	.055	.1213	.00161		20		=0
15	0.2121	2	+0.0000	$\Delta u_1 - 0.213$	$\Delta r_1 - 0.190$	$\Delta r_2 - 0.035$	$\Delta i - 0.0853$	$\Delta e + 0.00076$	$\Delta u_2 + 0.0024$			=0

NORMAL EQUATIONS

+0.030121	$\Delta u_1 - 1.73469$	$\Delta r_1 + 2.87381$	$\Delta r_2 + 0.15739$	$\Delta i + 0.00000$	$\Delta e + 0.00000$	$\Delta u_2 + 0.0004024$	=0
-1.73469	+111.507	-180.202	-9.50659	+1.76983	+0.017229	-0.053486	=0
+2.87381	-180.202	+547.684	+60.9806	-0.12751	-0.015729	+0.18629	=0
+0.15739	-9.50659	+60.9806	+10.3344	+0.45232	+0.006772	+0.020211	=0
+0.00000	+1.76983	-0.12751	+0.45232	+0.41364	+0.0014424	-0.0052083	=0
+0.00000	$\Delta u_1 + 0.017229$	$\Delta r_1 - 0.015729$	$\Delta r_2 + 0.006772$	$\Delta i + 0.0014424$	$\Delta e + 0.00014905$	$\Delta u_2 - 0.00002708$	=0

Experience has shown that it would have been advisable to reduce the number of equations of condition by combining the normal places in pairs for the straight portion of the light-curve during partial eclipse at principal minimum.

After the calculation of the differential coefficients, numerical checks are of course desirable. We have found it useful to plot the coefficients as functions of ϑ or of $\sin \vartheta$, in order to disclose any errors that may apply to an individual equation of condition. This procedure does not, however, reveal systematic errors of computation that may affect the whole set of coefficients. Such errors

are not likely to occur in the quantities $\frac{\partial f}{\partial u}$ or $\frac{\partial \delta}{\partial i}$, because

the formulae are so simple. Partial checks should be made on some of the other quantities. The computations for the theoretical light-curve from the preliminary elements provide a rough check on $\frac{\partial f}{\partial \delta}$, because they con-

tain a column of f and a column of δ . Either or both of the coefficients $\frac{\partial f}{\partial r_1}$ and $\frac{\partial f}{\partial r_2}$ should be checked by calcu-

lating f for a few normal places with an arbitrary, slightly different, value of r_1 or r_2 (varying only one element at a time, of course), and by comparing the resulting values of f with the values found for the theoretical light-curve.

Since the checks are at best only approximate, it is just as well in checking $\frac{\partial f}{\partial r_1}$ or $\frac{\partial f}{\partial r_2}$ to compare the f'' 's only,

and thereby to save the unnecessary work of computing additional f'' 's.

In conclusion it may be stated that the amount of labor required for the determination of the photometric elements in the above manner is very moderate; and that especially in view of certain simplifications that were found after the computation of the orbit of YZ Cassiopeiae, the problem has turned out to be much simpler than had been expected.

MOUNT HAMILTON, CALIFORNIA

APRIL 14, 1939

ISSUED, SEPTEMBER 1, 1939.