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SYSTEMATIC SUBDIVISION OF TABLES

BY

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SUMMARY

A series of functions may be subdivided systematically to smaller intervals very expeditiously with tabulating machines by a new method, on which a preliminary note was published¹ in 1933. General expressions for the subdivided functions in terms of the interpolating factor n and the original function differences, including the m th order, are set up and differenced. This produces general expressions for all the subdivided differences. We are interested, first, in the "leading" differences, that is, one from each order, up (or down) the diagonal from the first function, and second, in the subdivided differences of the $(m+1)$ th order. It is found that for a given n the $(m+1)$ th subdivided differences opposite an original function are linear functions of the corresponding original $(m+1)$ th difference with the Gauss interpolation formula, or of the corresponding original $(m+1)$ th and $(m+2)$ th differences with the Bessel or Stirling formulas. Exact numerical values of the $(m+1)$ th subdivided differences may be tabulated, or punched on cards, for repeated use with machines. That these exact values require the use of many decimal places finally to be discarded, presents no difficulty since the machines add numbers of any size within their capacity in the same unit of time. The lowest order of original differences that keeps the error of interpolation within the desired limit is chosen as the m th. Then adding the $(m+1)$ th subdivided differences to the leading m th difference in the well known manner produces the column of m th differences; similarly the lower order differences and desired functions in the subdivided table may be obtained. Automatic rounding off is accomplished as usual by initially adding to the leading function 5 units in the first decimal place to be neglected. Since the original

functions are exactly reproduced the method produces by a continuous process the equivalent of proofread printer's copy, at a rate far faster than a copyist can merely copy the functions.

NOTATION

The notation and schedule of differences in the original table to be subdivided are:

Function	ORIGINAL TABLE							
	Differences							
	1st	2nd	3rd	4th	5th	6th	7th	8th
F_{-1}		b_{-1}		d_{-1}		f_{-1}		h_{-1}
F_0	$a_{-\frac{1}{2}}$		$c_{-\frac{1}{2}}$		$e_{-\frac{1}{2}}$		$g_{-\frac{1}{2}}$	
		b_0		d_0		f_0		h_0
F_1	$a_{\frac{1}{2}}$		$c_{\frac{1}{2}}$		$e_{\frac{1}{2}}$		$g_{\frac{1}{2}}$	
		b_1		d_1		f_1		h_1

and, as usual, the mean differences are: $a_0 = \frac{1}{2}(a_{-\frac{1}{2}} + a_{\frac{1}{2}})$, $b_{-\frac{1}{2}} = \frac{1}{2}(b_{-1} + b_0)$, $b_{\frac{1}{2}} = \frac{1}{2}(b_0 + b_1)$, etc.

The notation and schedule of differences in the corresponding subdivided table in the vicinity of F_0 are:

Function	SUBDIVIDED TABLE							
	Differences							
	1st	2nd	3rd	4th	5th	6th	7th	8th
	$(a_0)_{-3\frac{1}{2}}$		$(c_0)_{-3\frac{1}{2}}$		$(e_0)_{-3\frac{1}{2}}$		$(g_0)_{-3\frac{1}{2}}$	
$(F_0)_{-3}$		$(b_0)_{-3}$		$(d_0)_{-3}$		$(f_0)_{-3}$		$(h_0)_{-3}$
	$(a_0)_{-2\frac{1}{2}}$		$(c_0)_{-2\frac{1}{2}}$		$(e_0)_{-2\frac{1}{2}}$		$(g_0)_{-2\frac{1}{2}}$	
$(F_0)_{-2}$		$(b_0)_{-2}$		$(d_0)_{-2}$		$(f_0)_{-2}$		$(h_0)_{-2}$
	$(a_0)_{-1\frac{1}{2}}$		$(c_0)_{-1\frac{1}{2}}$		$(e_0)_{-1\frac{1}{2}}$		$(g_0)_{-1\frac{1}{2}}$	
$(F_0)_{-1}$		$(b_0)_{-1}$		$(d_0)_{-1}$		$(f_0)_{-1}$		$(h_0)_{-1}$
	$(a_0)_{-\frac{1}{2}}$		$(c_0)_{-\frac{1}{2}}$		$(e_0)_{-\frac{1}{2}}$		$(g_0)_{-\frac{1}{2}}$	
F_0		$(b_0)_0$		$(d_0)_0$		$(f_0)_0$		$(h_0)_0$
	$(a_0)_{\frac{1}{2}}$		$(c_0)_{\frac{1}{2}}$		$(e_0)_{\frac{1}{2}}$		$(g_0)_{\frac{1}{2}}$	
$(F_0)_1$		$(b_0)_1$		$(d_0)_1$		$(f_0)_1$		$(h_0)_1$
	$(a_0)_{1\frac{1}{2}}$		$(c_0)_{1\frac{1}{2}}$		$(e_0)_{1\frac{1}{2}}$		$(g_0)_{1\frac{1}{2}}$	
$(F_0)_2$		$(b_0)_2$		$(d_0)_2$		$(f_0)_2$		$(h_0)_2$
	$(a_0)_{2\frac{1}{2}}$		$(c_0)_{2\frac{1}{2}}$		$(e_0)_{2\frac{1}{2}}$		$(g_0)_{2\frac{1}{2}}$	
$(F_0)_3$		$(b_0)_3$		$(d_0)_3$		$(f_0)_3$		$(h_0)_3$
	$(a_0)_{3\frac{1}{2}}$		$(c_0)_{3\frac{1}{2}}$		$(e_0)_{3\frac{1}{2}}$		$(g_0)_{3\frac{1}{2}}$	

¹ *Lick Obs. Bull.* 16, 143, 1933.

DERIVATION OF FORMULAS

The desired formulas are based on the Gauss, Bessel, and Stirling interpolation formulas:

$$\begin{aligned} \text{Gauss: } (F_0)_n &= F_0 + na_1 + \frac{n(n-1)}{2!} b_0 + \frac{n(n^2-1)}{3!} c_1 + \frac{n(n^2-1)(n-2)}{4!} d_0 + \frac{n(n^2-1)(n^2-4)}{5!} e_1 + \frac{n(n^2-1)(n^2-4)(n-3)}{6!} f_0 + \frac{n(n^2-1)(n^2-4)(n^2-9)}{7!} g_1 + \dots \\ \text{Bessel: } (F_0)_n &= F_0 + na_1 + \frac{n(n-1)}{2!} b_{\frac{1}{2}} + \frac{n(n-1)(n-\frac{1}{2})}{3!} c_{\frac{1}{2}} + \frac{n(n^2-1)(n-2)}{4!} d_{\frac{1}{2}} + \frac{n(n^2-1)(n-2)(n-\frac{1}{2})}{5!} e_{\frac{1}{2}} + \frac{n(n^2-1)(n^2-4)(n-3)}{6!} f_{\frac{1}{2}} + \frac{n(n^2-1)(n^2-4)(n-3)(n-\frac{1}{2})}{7!} g_{\frac{1}{2}} + \dots \\ \text{Stirling: } (F_0)_n &= F_0 + na_0 + \frac{n^2}{2!} b_0 + \frac{n(n^2-1)}{3!} c_0 + \frac{n^2(n^2-1)}{4!} d_0 + \frac{n(n^2-1)(n^2-4)}{5!} e_0 + \frac{n^2(n^2-1)(n^2-4)}{6!} f_0 + \frac{n(n^2-1)(n^2-4)(n^2-9)}{7!} g_0 + \dots \end{aligned}$$

Substituting $b_{\frac{1}{2}} = b_0 + \frac{1}{2}c_1$ in the Bessel formula reproduces the third and fourth terms in the Gauss formula, substituting $d_{\frac{1}{2}} = d_0 + \frac{1}{2}e_1$ reproduces the fifth and sixth terms, etc. Hence it is evident that the Gauss and Bessel formulas yield identical results when the last difference retained is odd, and that the equivalent Bessel formula when the last difference retained is even may be obtained from the Gauss by substituting for the last even difference, whose subscript is "0," the corresponding mean difference with subscript " $\frac{1}{2}$." Similarly by substituting $a_0 = a_{\frac{1}{2}} - \frac{1}{2}b_0$, etc., it is evident that the Gauss and Stirling formulas produce identical results when the last difference retained is even, and that the equivalent Stirling formula, when the last difference retained is odd, may be obtained from Gauss by substituting for the last odd difference, whose subscript is " $\frac{1}{2}$," the corresponding mean difference with subscript "0".

The interpolating factor n is used from 0.0 to 1.0; that is, throughout the interval F_{-1} to F_0 the same set of original differences is used, which changes to a new set for the interval F_0 to F_1 , etc. Using n from -0.5 to $+0.5$ slightly increases the accuracy, but introduces undesirable dissymmetry. (In addition to its usual meaning, n

also designates the interval of the subdivided with respect to the original table).

The interpolates in the vicinity of F_0 are expressed in terms of n including the m th order of differences, and differenced $m+1$ times, the contribution of each original difference being kept separate, except in the first two terms. This operation gives general expressions for the 1st, 2nd, . . . m th differences, and for the m "central" $(m+1)$ th differences symmetrically distributed opposite F_0 in the subdivided table. The other $(m+1)$ th differences, $n-m$ in number, are zero because they are the differences of power series in n of degree m . It is assumed that m will not be taken greater than n , because the resulting "overlapping" of central differences causes undesirable complexity.

GENERAL FORMULAS FOR INTERPOLATION TO n ths

The formulas for the "leading" differences up the diagonal from F_0 in the subdivided table, derived from the Gauss formula, as far as $m=7$ inclusive, are given below. In evaluating these the original differences higher than those to be retained are, of course, taken as zero.

LEADING DIFFERENCES—GAUSS FORMULA

$$\begin{aligned} (a_0)_{-1} &= na_{-1} + \frac{1}{2}(n-n^2)b_0 - \frac{1}{6}(n-n^3)c_{-1} - \frac{1}{24}(2n-n^2-2n^3+n^4)d_0 + \frac{1}{120}(4n-5n^3+n^5)e_{-1} + \frac{1}{720}(12n-4n^2-15n^3+5n^4+3n^5-n^6)f_0 - \frac{1}{5040}(36n-49n^2+14n^3-n^4)g_{-1} \\ (b_0)_{-1} &= n^2b_0 - n^3c_{-1} - \frac{1}{12}(n^2+6n^3-7n^4)d_0 + \frac{1}{4}(n^3-n^5)e_{-1} + \frac{1}{360}(4n^2+45n^3-35n^4-45n^5+31n^6)f_0 - \frac{1}{120}(7n^3-10n^4+3n^5)g_{-1} \\ (c_0)_{-1} &= n^3c_{-1} + \frac{1}{2}(n^3-3n^4)d_0 - \frac{1}{4}(n^3-5n^4)e_{-1} - \frac{1}{8}(n^3-2n^4-5n^5+6n^6)f_0 + \frac{1}{120}(7n^3-50n^4+43n^5)g_{-1} \\ (d_0)_{-2} &= n^4d_0 - 2n^5e_{-1} - \frac{1}{6}(n^4+6n^5-13n^6)f_0 + \frac{1}{3}(2n^5-5n^6)g_{-1} \\ (e_0)_{-2} &= n^5e_{-1} + \frac{1}{2}(n^5-5n^6)f_0 - \frac{1}{3}(n^5-10n^6)g_{-1} \\ (f_0)_{-3} &= n^6f_0 - 3n^7g_{-1} \\ (g_0)_{-3} &= n^7g_{-1} \end{aligned}$$

As before, the equivalent Bessel formulas when retaining even differences can be obtained by substituting for the last even difference, the corresponding mean difference with subscript " $-\frac{1}{2}$." Similarly, the equivalent Stirling formulas when retaining odd differences, can be

obtained by substituting for the last odd difference, the corresponding mean difference with the subscript "0." The formulas for the m central $(m+1)$ th differences opposite any F_0 in the subdivided table, to be used when retaining original m th differences, are, for $m=2, 3, 4, 5, 6, 7$:

BESEL-STIRLING

$$\begin{aligned}
(c_0)_{-1} &= +\frac{1}{2}(n+n^2)c_{-1} \\
(c_0)_1 &= -\frac{1}{2}(n-n^2)c_{-1} \\
(d_0)_{-1} &= -\frac{1}{6}(n-n^3)d_0 \\
(d_0)_0 &= +\frac{1}{6}(2n+4n^3)d_0 \\
(d_0)_1 &= -\frac{1}{6}(n-n^3)d_0 \\
(e_0)_{-1} &= -\frac{1}{24}(2n+n^3-2n^3-n^4)e_{-1} \\
(e_0)_0 &= +\frac{1}{24}(6n+n^3+6n^3+11n^4)e_{-1} \\
(e_0)_1 &= -\frac{1}{24}(6n-n^3+6n^3-11n^4)e_{-1} \\
(e_0)_2 &= +\frac{1}{24}(2n-n^3-2n^3+n^4)e_{-1} \\
(f_0)_{-2} &= +\frac{1}{120}(4n-5n^3+n^5)f_0 \\
(f_0)_{-1} &= -\frac{1}{120}(16n+10n^3-26n^5)f_0 \\
(f_0)_0 &= +\frac{1}{120}(24n+30n^3+66n^5)f_0 \\
(f_0)_1 &= -\frac{1}{120}(16n+10n^3-26n^5)f_0 \\
(f_0)_2 &= +\frac{1}{120}(4n-5n^3+n^5)f_0 \\
(g_0)_{-3} &= +\frac{1}{720}(12n+4n^3-15n^5-5n^7+3n^9)g_{-1} \\
(g_0)_{-1} &= -\frac{1}{720}(60n+12n^3+15n^5+45n^7-75n^9)g_{-1} \\
(g_0)_1 &= +\frac{1}{720}(120n+8n^3+120n^3+50n^5+120n^5+302n^7)g_{-1} \\
(g_0)_3 &= -\frac{1}{720}(120n-8n^3+120n^3-50n^5+120n^5-302n^7)g_{-1} \\
(g_0)_{11} &= +\frac{1}{720}(60n-12n^3+15n^5-45n^7-75n^9)g_{-1} \\
(g_0)_{21} &= -\frac{1}{720}(12n-4n^3-15n^5+5n^7+3n^9)g_{-1} \\
(h_0)_{-3} &= -\frac{1}{5040}(36n-49n^3+14n^5-n^7)h_0 \\
(h_0)_{-2} &= +\frac{1}{5040}(216n-336n^3+120n^5)h_0 \\
(h_0)_{-1} &= -\frac{1}{5040}(540n+441n^3+210n^5-1191n^7)h_0 \\
(h_0)_0 &= +\frac{1}{5040}(720n+784n^3+1120n^5+2416n^7)h_0 \\
(h_0)_1 &= -\frac{1}{5040}(540n+441n^3+210n^5-1191n^7)h_0 \\
(h_0)_2 &= +\frac{1}{5040}(216n-336n^3+120n^5)h_0 \\
(h_0)_3 &= -\frac{1}{5040}(36n-49n^3+14n^5-n^7)h_0
\end{aligned}$$

$$\begin{aligned}
& +\frac{1}{2}n^2c_{-1}-\frac{1}{4}(n-n^2)d_0 \\
& +\frac{1}{2}n^2c_{-1}+\frac{1}{4}(n+n^2)d_0 \\
& +\frac{1}{12}(n+3n^2+2n^3)d_0-\frac{1}{12}(2n+3n^2+n^3)e_{-1} \\
& -\frac{1}{12}(2n-8n^3)d_0+\frac{1}{12}(4n-4n^3)e_{-1} \\
& +\frac{1}{12}(n-3n^2+2n^3)d_0-\frac{1}{12}(2n-3n^2+n^3)e_{-1} \\
& -\frac{1}{24}(n^2-n^4)e_{-1}+\frac{1}{48}(2n-n^2-2n^3+n^4)f_0 \\
& +\frac{1}{24}(n^2+11n^4)e_{-1}-\frac{1}{48}(6n-n^2+6n^3-11n^4)f_0 \\
& +\frac{1}{24}(n^2+11n^4)e_{-1}+\frac{1}{48}(6n+n^2+6n^3+11n^4)f_0 \\
& -\frac{1}{24}(n^2-n^4)e_{-1}-\frac{1}{48}(2n+n^2-2n^3-n^4)f_0 \\
& -\frac{1}{240}(2n+5n^2-5n^4-2n^6)f_0+\frac{1}{240}(6n+5n^2-5n^4-n^6)g_{-1} \\
& +\frac{1}{240}(8n+10n^2+50n^4+52n^6)f_0-\frac{1}{240}(24n+10n^2+10n^4+50n^4+26n^6)g_{-1} \\
& -\frac{1}{240}(12n-132n^6)f_0+\frac{1}{240}(36n+30n^3-66n^5)g_{-1} \\
& +\frac{1}{240}(8n-10n^2-50n^4+52n^6)f_0-\frac{1}{240}(24n-10n^2+10n^4+50n^4+26n^6)g_{-1} \\
& -\frac{1}{240}(2n-5n^2+5n^4-2n^6)f_0+\frac{1}{240}(6n-5n^2-5n^4-n^6)g_{-1} \\
& +\frac{1}{720}(4n^2-5n^4-n^6)g_{-1}-\frac{1}{1440}(12n-4n^2-15n^3+5n^4+3n^5-n^6)h_0 \\
& -\frac{1}{720}(12n^2+45n^4-57n^6)g_{-1}+\frac{1}{1440}(60n-12n^2+15n^3-45n^4-75n^5+57n^6)h_0 \\
& +\frac{1}{720}(8n^2+50n^4+302n^6)g_{-1}-\frac{1}{1440}(120n-8n^2+120n^3-50n^4+120n^5-302n^6)h_0 \\
& +\frac{1}{720}(8n^2+50n^4+302n^6)g_{-1}+\frac{1}{1440}(120n+8n^2+120n^3+50n^4+120n^5+302n^6)h_0 \\
& -\frac{1}{720}(12n^2+45n^4-57n^6)g_{-1}-\frac{1}{1440}(60n+12n^2+15n^3+45n^4-75n^5-57n^6)h_0 \\
& +\frac{1}{720}(4n^2-5n^4-n^6)g_{-1}+\frac{1}{1440}(12n+4n^2-15n^3-5n^4+3n^5+n^6)h_0 \\
& +\frac{1}{10080}(12n+28n^2-7n^3-35n^5-7n^6+2n^7)h_0-\frac{1}{10080}(48n+28n^2-56n^3-35n^4+7n^5+n^7)h_0 \\
& -\frac{1}{10080}(72n+112n^2+280n^4+168n^6-392n^8-240n^7)h_0+\frac{1}{10080}(288n+112n^2+280n^4-168n^6-392n^8-120n^7)h_0 \\
& +\frac{1}{10080}(180n+140n^2+63n^3+665n^5-105n^6+1715n^8+2382n^7)h_0-\frac{1}{10080}(720n+140n^2+504n^3+665n^4+105n^5+1715n^6+1191n^7)h_0 \\
& -\frac{1}{10080}(240n+112n^3-560n^5-4832n^7)h_0+\frac{1}{10080}(960n-2416n^5-2416n^7)h_0 \\
& +\frac{1}{10080}(180n-140n^2+63n^3-665n^5-105n^6-1715n^8+2382n^7)h_0-\frac{1}{10080}(720n-140n^2+504n^3+665n^4+105n^5-1715n^6+1191n^7)h_0 \\
& -\frac{1}{10080}(72n-112n^2-280n^4+168n^6+392n^8-240n^7)h_0+\frac{1}{10080}(288n-112n^2-280n^4-168n^6-392n^8-120n^7)h_0 \\
& +\frac{1}{10080}(12n-28n^2-7n^3+35n^5-7n^6+2n^7)h_0-\frac{1}{10080}(48n-28n^2-56n^3+35n^4+7n^5-7n^6+n^7)h_0
\end{aligned}$$

The formulae for the leading differences up to diagonal are given, but those for central differences for working down the diagonal. The leading central difference formulae up to diagonal may be found from these down the diagonal, or by changing the signs of all subscripts and all superscripts.

(author's sketch)

Thus any set of $(m+1)$ th central differences depends on the corresponding original $(m+1)$ th difference with the Gauss formula, and on the $(m+1)$ th and $(m+2)$ th original differences with the Bessel-Stirling formulas.

FORMULAS FOR INTERPOLATION TO 4ths, 5ths, 10ths, 20ths, 25ths

The numerical working formulas for the leading and central differences for interpolation to 4ths, 5ths, 10ths, 20ths, 25ths, are given below. The leading difference formulas are based on Gauss; to change to Bessel-Stirling the proper substitution must be made for the last difference retained.

Interpolation to 8ths requires so many decimals (e.g. ten in order to retain 3rd differences) that adapting the work to 5ths or 10ths instead seems more practical at present.

LEADING AND CENTRAL DIFFERENCES FOR INTERPOLATION TO 4THS

$$\begin{aligned}(a_0)_{-1} &= +.25a_{-1} + .09375b_0 - .0390625c_{-1} - .01708984375d_0 \\ (b_0)_{-1} &= +.0625b_0 - .015625c_{-1} - .0107421875d_0 \\ (c_0)_{-1} &= +.015625c_{-1} + .001953125d_0 \\ (d_0)_{-2} &= +.00390625d_0\end{aligned}$$

GAUSS	BESSEL-STIRLING
$(c_0)_{-1} = +.15625c_{-1}$	$+ .03125c_{-1} - .046875d_0$
$(c_0)_1 = -.09375c_{-1}$	$+ .03125c_{-1} + .078125d_0$
$(d_0)_{-1} = -.0390625d_0$	$+ .0390625d_0 - .05859375e_{-1}$
$(d_0)_0 = +.09375d_0$	$- .03125d_0 + .078125e_{-1}$
$(d_0)_1 = -.0390625d_0$	$+ .0078125d_0 - .02734375e_{-1}$

LEADING AND CENTRAL DIFFERENCES FOR INTERPOLATION TO 10THS

$$\begin{aligned}(a_0)_{-1} &= +.1a_{-1} + .045b_0 - .0165c_{-1} - .0078375d_0 + .00329175e_{-1} + .0015910125f_0 - .00070459125g_{-1} \\ (b_0)_{-1} &= +.01b_0 - .001c_{-1} - .001275d_0 + .0002475e_{-1} + .000225225f_0 - .0000575025g_{-1} \\ (c_0)_{-1} &= +.001c_{-1} + .00035d_0 - .0002375e_{-1} - .0000945f_0 + .0000542025g_{-1} \\ (d_0)_{-2} &= +.0001d_0 - .00002e_{-1} - .0000245f_0 + .0000065g_{-1} \\ (e_0)_{-2} &= +.00001e_{-1} + .0000025f_0 - .00000030g_{-1} \\ (f_0)_{-3} &= +.000001f_0 - .00000003g_{-1} \\ (g_0)_{-3} &= +.00000001g_{-1}\end{aligned}$$

GAUSS	BESSEL-STIRLING
$(c_0)_{-1} = +.055c_{-1}$	$+ .005c_{-1} - .0225d_0$
$(c_0)_1 = -.045c_{-1}$	$+ .005c_{-1} + .0275d_0$
$(d_0)_{-1} = -.0165d_0$	$+ .011d_0 - .01925e_{-1}$
$(d_0)_0 = +.034d_0$	$- .016d_0 + .033e_{-1}$
$(d_0)_1 = -.0165d_0$	$+ .006d_0 - .01425e_{-1}$
$(e_0)_{-1} = -.008625e_{-1}$	$- .0004125e_{-1} + .00391875f_0$
$(e_0)_1 = +.0257125e_{-1}$	$+ .0004625e_{-1} - .01239375f_0$
$(e_0)_2 = -.0247875e_{-1}$	$+ .0004625e_{-1} + .01285625f_0$
$(e_0)_3 = +.0078375e_{-1}$	$- .0004125e_{-1} - .00433125f_0$
$(f_0)_{-2} = +.00329175f_0$	$- .0010395f_0 + .002685375g_{-1}$
$(f_0)_{-1} = -.0134145f_0$	$+ .003773f_0 - .01048025g_{-1}$
$(f_0)_0 = +.0202555f_0$	$- .0049945f_0 + .01512225g_{-1}$
$(f_0)_1 = -.0134145f_0$	$+ .002898f_0 - .00960525g_{-1}$
$(f_0)_2 = +.00329175f_0$	$- .000627f_0 + .002272875g_{-1}$

GAUSS

BESSEL-STIRLING

$$\begin{aligned}(e_0)_{-1} &= -.02197265625e_{-1} - .00244140625e_{-1} + .008544921875f_0 \\ (e_0)_{-1} &= +.07080078125e_{-1} + .00439453125e_{-1} - .031005859375f_0 \\ (e_0)_1 &= -.06201171875e_{-1} + .00439453125e_{-1} + .035400390625f_0 \\ (e_0)_1 &= +.01708984375e_{-1} - .00244140625e_{-1} - .010986328125f_0\end{aligned}$$

LEADING AND CENTRAL DIFFERENCES FOR INTERPOLATION TO 5THS

$$\begin{aligned}(a_0)_{-1} &= +.2a_{-1} + .08b_0 - .032c_{-1} - .0144d_0 + .006336e_{-1} \\ (b_0)_{-1} &= +.04b_0 - .008c_{-1} - .0064d_0 + .00192e_{-1} \\ (c_0)_{-1} &= +.008c_{-1} + .0016d_0 - .00160e_{-1} \\ (d_0)_{-2} &= +.0016d_0 - .00064e_{-1} \\ (e_0)_{-2} &= +.00032e_{-1}\end{aligned}$$

GAUSS

BESSEL-STIRLING

$(c_0)_{-1} = +.12c_{-1}$	$+ .02c_{-1} - .04d_0$
$(c_0)_1 = -.08c_{-1}$	$+ .02c_{-1} + .06d_0$
$(d_0)_{-1} = -.032d_0$	$+ .028d_0 - .044e_{-1}$
$(d_0)_0 = +.072d_0$	$- .028d_0 + .064e_{-1}$
$(d_0)_1 = -.032d_0$	$+ .008d_0 - .024e_{-1}$
$(e_0)_{-1} = -.0176e_{-1}$	$- .0016e_{-1} + .0072f_0$
$(e_0)_{-1} = +.0544e_{-1}$	$+ .0024e_{-1} - .0248f_0$
$(e_0)_1 = -.0496e_{-1}$	$+ .0024e_{-1} + .0272f_0$
$(e_0)_1 = +.0144e_{-1}$	$- .0016e_{-1} - .0088f_0$
$(f_0)_{-2} = +.006336f_0$	$- .002464f_0 + .005632g_{-1}$
$(f_0)_{-1} = -.027264f_0$	$+ .008736f_0 - .022368g_{-1}$
$(f_0)_0 = +.042176f_0$	$- .009824f_0 + .030912g_{-1}$
$(f_0)_1 = -.027264f_0$	$+ .004736f_0 - .018368g_{-1}$
$(f_0)_2 = +.006336f_0$	$- .000864f_0 + .004032g_{-1}$

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$(g_0)_{-2} = +.0017007375g_{-1}$	$+ .0000548625g_{-1} - .00079550625h_0$
$(g_0)_{-1} = -.0085259625g_{-1}$	$- .0001728375g_{-1} + .00409014375h_0$
$(g_0)_1 = +.016953475g_{-1}$	$+ .000118475g_{-1} - .0083582625h_0$
$(g_0)_1 = -.016716525g_{-1}$	$+ .000118475g_{-1} + .0084767375h_0$
$(g_0)_2 = +.0081802875g_{-1}$	$- .0001728375g_{-1} - .00426298125h_0$
$(g_0)_2 = -.0015910125g_{-1}$	$+ .0000548625g_{-1} + .00085036875h_0$
$(h_0)_{-3} = -.00070459125h_0$	$+ .0001457775h_0 - .000498073125i_{-1}$
$(h_0)_{-2} = +.00428505h_0$	$- .00082830h_0 + .002970825i_{-1}$
$(h_0)_{-1} = -.01080217875h_0$	$+ .00193754h_0 - .007338629375i_{-1}$
$(h_0)_0 = +.01444354h_0$	$- .00239146h_0 + .00961323i_{-1}$
$(h_0)_1 = -.01080217875h_0$	$+ .0016462275h_0 - .007047316875i_{-1}$
$(h_0)_2 = +.00428505h_0$	$- .00060060h_0 + .002743125i_{-1}$
$(h_0)_3 = -.00070459125h_0$	$+ .0000909150h_0 - .000443210625i_{-1}$

LEADING AND CENTRAL DIFFERENCES FOR INTERPOLATION TO 20THS

$(a_0)_{-1} =$	$+.05a_{-1} + .0237\ 5b_0 - .0083\ 125c_{-1} - .0040\ 5234\ 375d_0 + .0016\ 6146\ 0937\ 5e_{-1} + .0008\ 1688\ 4960\ 9375f_0 - .0003\ 5592\ 8447\ 2656\ 25g_{-1}$
$(b_0)_{-1} =$	$+.0025\ b_0 - .0001\ 25\ c_{-1} - .0002\ 6718\ 75\ d_0 + .0000\ 3117\ 1875\ e_{-1} + .0000\ 4275\ 7421\ 875\ f_0 - .0000\ 0726\ 5644\ 5312\ 5\ g_{-1}$
$(c_0)_{-1} =$	$+.0001\ 25\ c_{-1} + .0000\ 5312\ 5\ d_0 - .0000\ 3085\ 9375\ e_{-1} - .0000\ 1387\ 8906\ 25\ f_0 + .0000\ 0716\ 1738\ 2812\ 5\ g_{-1}$
$(d_0)_{-2} =$	$+.0000\ 0625\ d_0 - .0000\ 0062\ 5\ e_{-1} - .0000\ 0132\ 0312\ 5\ f_0 + .0000\ 0020\ 7031\ 25\ g_{-1}$
$(e_0)_{-2} =$	$+.0000\ 0031\ 25\ e_{-1} + .0000\ 0011\ 7187\ 5\ f_0 - .0000\ 0010\ 1562\ 5\ g_{-1}$
$(f_0)_{-3} =$	$+.0000\ 0001\ 5625\ f_0 - .0000\ 0000\ 2343\ 75\ g_{-1}$
$(g_0)_{-3} =$	$+.0000\ 0000\ 0781\ 25\ g_{-1}$

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$(c_0)_{-1} =$	$+.0262\ 5c_{-1}$	$+.0012\ 5c_{-1} - .0118\ 75d_0$
$(c_0)_1 =$	$-.0237\ 5c_{-1}$	$+.0012\ 5c_{-1} + .0131\ 25d_0$
$(d_0)_{-1} =$	$-.0083\ 125d_0$	$+.0048\ 125d_0 - .0089\ 6875e_{-1}$
$(d_0)_0 =$	$+.0167\ 5\ d_0$	$-.0082\ 5\ d_0 + .0166\ 25\ e_{-1}$
$(d_0)_1 =$	$-.0083\ 125d_0$	$+.0035\ 625d_0 - .0077\ 1875e_{-1}$
$(e_0)_{-1} =$	$-.0042\ 6015\ 625e_{-1}$	$-.0001\ 0390\ 625e_{-1} + .0020\ 2617\ 1875f_0$
$(e_0)_{-1} =$	$+.0126\ 3828\ 125e_{-1}$	$+.0001\ 0703\ 125e_{-1} - .0062\ 1210\ 9375f_0$
$(e_0)_1 =$	$-.0124\ 2421\ 875e_{-1}$	$+.0001\ 0703\ 125e_{-1} + .0063\ 1914\ 0625f_0$
$(e_0)_1 =$	$+.0040\ 5234\ 375e_{-1}$	$-.0001\ 0390\ 625e_{-1} - .0021\ 3007\ 8125f_0$
$(f_0)_{-2} =$	$+.0016\ 6146\ 0937\ 5f_0$	$-.0004\ 6861\ 7187\ 5f_0 + .0012\ 9934\ 7656\ 25g_{-1}$
$(f_0)_{-1} =$	$-.0066\ 7701\ 5625\ f_0$	$+.0017\ 7220\ 3125\ f_0 - .0051\ 1071\ 0937\ 5\ g_{-1}$
$(f_0)_0 =$	$+.0100\ 3142\ 1875\ f_0$	$-.0024\ 9982\ 8125\ f_0 + .0075\ 1553\ 9062\ 5\ g_{-1}$
$(f_0)_1 =$	$-.0066\ 7701\ 5625\ f_0$	$+.0015\ 6126\ 5625\ f_0 - .0048\ 9977\ 3437\ 5\ g_{-1}$
$(f_0)_2 =$	$+.0016\ 6146\ 0937\ 5f_0$	$-.0003\ 6471\ 0937\ 5f_0 + .0011\ 9544\ 1406\ 25g_{-1}$
$(g_0)_{-2} =$	$+.0008\ 4457\ 5976\ 5625g_{-1}$	$+.0000\ 1384\ 5507\ 8125g_{-1} - .0004\ 0844\ 2480\ 4687\ 5h_0$
$(g_0)_{-1} =$	$-.0042\ 1129\ 4335\ 9375g_{-1}$	$-.0000\ 4205\ 6054\ 6875g_{-1} + .0020\ 6359\ 1113\ 2812\ 5h_0$
$(g_0)_{-1} =$	$+.0083\ 8243\ 7109\ 375\ g_{-1}$	$+.0000\ 2821\ 8359\ 375\ g_{-1} - .0041\ 6300\ 0195\ 3125\ h_0$
$(g_0)_1 =$	$-.0083\ 2600\ 0390\ 625\ g_{-1}$	$+.0000\ 2821\ 8359\ 375\ g_{-1} + .0041\ 9121\ 8554\ 6875\ h_0$
$(g_0)_1 =$	$+.0041\ 2718\ 2226\ 5625g_{-1}$	$-.0000\ 4205\ 6054\ 6875g_{-1} - .0021\ 0564\ 7167\ 9687\ 5h_0$
$(g_0)_2 =$	$-.0008\ 1688\ 4960\ 9375g_{-1}$	$+.0000\ 1384\ 5507\ 8125g_{-1} + .0004\ 2228\ 7988\ 2812\ 5h_0$
$(h_0)_{-3} =$	$-.0003\ 5592\ 8447\ 2656\ 25h_0$	$+.0000\ 6635\ 9541\ 0156\ 25h_0 - .0002\ 4432\ 3764\ 6484\ 375i_{-1}$
$(h_0)_{-2} =$	$+.0021\ 4283\ 6328\ 125\ h_0$	$-.0003\ 8509\ 8828\ 125\ h_0 + .0014\ 5651\ 6992\ 1875\ i_{-1}$
$(h_0)_{-1} =$	$-.0053\ 6809\ 3193\ 3593\ 75h_0$	$+.0009\ 2877\ 2529\ 2968\ 75h_0 - .0036\ 1281\ 9125\ 9765\ 625i_{-1}$
$(h_0)_0 =$	$+.0071\ 6237\ 1406\ 25\ h_0$	$-.0011\ 9184\ 7343\ 75\ h_0 + .0047\ 7303\ 3046\ 875\ i_{-1}$
$(h_0)_1 =$	$-.0053\ 6809\ 3193\ 3593\ 75h_0$	$+.0008\ 5849\ 8115\ 2343\ 75h_0 - .0035\ 4254\ 4711\ 9140\ 625i_{-1}$
$(h_0)_2 =$	$+.0021\ 4283\ 6328\ 125\ h_0$	$-.0003\ 2919\ 7265\ 625\ h_0 + .0014\ 0061\ 5429\ 6875\ i_{-1}$
$(h_0)_3 =$	$-.0003\ 5592\ 8447\ 2656\ 25h_0$	$+.0000\ 5251\ 4033\ 2031\ 25h_0 - .0002\ 3047\ 8256\ 8359\ 375i_{-1}$

LEADING AND CENTRAL DIFFERENCES FOR INTERPOLATION TO 25THS

$(a_0)_{-1} =$	$+.04a_{-1} + .0192b_0 - .0066\ 56c_{-1} - .0032\ 6144d_0 + .0013\ 3066\ 752e_{-1} + .0006\ 5646\ 2643\ 2f_0 - .0002\ 8509\ 2347\ 904g_{-1}$
$(b_0)_{-1} =$	$+.0016b_0 - .0000\ 64c_{-1} - .0001\ 6384d_0 + .0000\ 1597\ 44\ e_{-1} + .0000\ 2551\ 6441\ 6f_0 - .0000\ 0372\ 4804\ 096g_{-1}$
$(c_0)_{-1} =$	$+.0000\ 64c_{-1} + .0000\ 2816d_0 - .0000\ 1587\ 20\ e_{-1} - .0000\ 0729\ 9072\ f_0 + .0000\ 0369\ 0725\ 376g_{-1}$
$(d_0)_{-2} =$	$+.0000\ 0256d_0 - .0000\ 0020\ 48\ e_{-1} - .0000\ 0052\ 0192\ f_0 + .0000\ 0006\ 7993\ 60\ g_{-1}$
$(e_0)_{-2} =$	$+.0000\ 0010\ 24\ e_{-1} + .0000\ 0004\ 0960\ f_0 - .0000\ 0003\ 3587\ 20\ g_{-1}$
$(f_0)_{-3} =$	$+.0000\ 0000\ 4096\ f_0 - .0000\ 0000\ 0491\ 52\ g_{-1}$
$(g_0)_{-3} =$	$+.0000\ 0000\ 0163\ 84\ g_{-1}$

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$(c_0)_{-1} =$	$+.0208c_{-1}$	$+.0008c_{-1} - .0096d_0$
$(c_0)_1 =$	$-.0192c_{-1}$	$+.0008c_{-1} + .0104d_0$
$(d_0)_{-1} =$	$-.0066\ 56d_0$	$+.0037\ 44d_0 - .0070\ 72e_{-1}$
$(d_0)_0 =$	$+.0133\ 76d_0$	$-.0066\ 24d_0 + .0133\ 12e_{-1}$
$(d_0)_1 =$	$-.0066\ 56d_0$	$+.0029\ 44d_0 - .0062\ 72e_{-1}$
$(e_0)_{-1} =$	$-.0033\ 9456e_{-1}$	$-.0000\ 6656e_{-1} + .0016\ 3072f_0$
$(e_0)_{-1} =$	$+.0100\ 8384e_{-1}$	$+.0000\ 6784e_{-1} - .0049\ 7408f_0$
$(e_0)_1 =$	$-.0099\ 4816e_{-1}$	$+.0000\ 6784e_{-1} + .0050\ 4192f_0$
$(e_0)_1 =$	$+.0032\ 6144e_{-1}$	$-.0000\ 6656e_{-1} - .0016\ 9728f_0$
$(f_0)_{-2} =$	$+.0013\ 3066\ 752f_0$	$-.0003\ 6661\ 248f_0 + .0010\ 3194\ 624g_{-1}$
$(f_0)_{-1} =$	$-.0053\ 3864\ 448f_0$	$+.0014\ 0055\ 552f_0 - .0040\ 6987\ 776g_{-1}$
$(f_0)_0 =$	$+.0080\ 1605\ 632f_0$	$-.0019\ 9994\ 368f_0 + .0060\ 0797\ 184g_{-1}$
$(f_0)_1 =$	$-.0053\ 3864\ 448f_0$	$+.0012\ 6615\ 552f_0 - .0039\ 3547\ 776g_{-1}$
$(f_0)_2 =$	$+.0013\ 3066\ 752f_0$	$-.0003\ 0005\ 248f_0 + .0009\ 6538\ 624g_{-1}$

LEADING AND CENTRAL DIFFERENCES FOR INTERPOLATION TO 25THS (*Concluded*)

GAUSS

$$\begin{aligned}
(g_0)_{-2\frac{1}{2}} &= +.0006\,7420\,4876\,8g_{-\frac{1}{2}} \\
(g_0)_{-1\frac{1}{2}} &= -.0033\,6148\,2342\,4g_{-\frac{1}{2}} \\
(g_0)_{-\frac{1}{2}} &= +.0066\,9530\,7673\,6g_{-\frac{1}{2}} \\
(g_0)_{\frac{1}{2}} &= -.0066\,5939\,3126\,4g_{-\frac{1}{2}} \\
(g_0)_{1\frac{1}{2}} &= +.0033\,0782\,9657\,6g_{-\frac{1}{2}} \\
(g_0)_{2\frac{1}{2}} &= -.0006\,5646\,2643\,2g_{-\frac{1}{2}} \\
\\
(h_0)_{-3} &= -.0002\,8509\,2347\,904h_0 \\
(h_0)_{-2} &= +.0017\,1427\,8891\,520h_0 \\
(h_0)_{-1} &= -.0042\,9131\,8513\,664h_0 \\
(h_0)_0 &= +.0057\,2426\,4103\,936h_0 \\
(h_0)_1 &= -.0042\,9131\,8513\,664h_0 \\
(h_0)_2 &= +.0017\,1427\,8891\,520h_0 \\
(h_0)_3 &= -.0002\,8509\,2347\,904h_0
\end{aligned}$$

BESSEL-STIRLING

$$\begin{aligned}
&+.0000\,0887\,1116\,8g_{-\frac{1}{2}} - .0003\,2823\,1321\,6h_0 \\
&- .0000\,2682\,6342\,4g_{-\frac{1}{2}} + .0015\,5391\,4828\,8h_0 \\
&+.0000\,1795\,7273\,6g_{-\frac{1}{2}} - .0033\,2969\,6563\,2h_0 \\
&+.0000\,1795\,7273\,6g_{-\frac{1}{2}} + .0033\,4765\,3836\,8h_0 \\
&- .0000\,2682\,6342\,4g_{-\frac{1}{2}} - .0015\,8074\,1171\,2h_0 \\
&+.0000\,0887\,1116\,8g_{-\frac{1}{2}} + .0003\,3710\,2438\,4h_0 \\
\\
&+.0000\,5201\,0090\,496h_0 - .0001\,9455\,6264\,448i_{-\frac{1}{2}} \\
&- .0003\,0356\,4718\,080h_0 + .0011\,6070\,4163\,840i_{-\frac{1}{2}} \\
&+.0007\,3707\,6494\,336h_0 - .0028\,8273\,5751\,168i_{-\frac{1}{2}} \\
&- .0009\,5308\,6296\,064h_0 + .0038\,1521\,8348\,032i_{-\frac{1}{2}} \\
&+.0006\,9229\,2878\,336h_0 - .0028\,3795\,2135\,168i_{-\frac{1}{2}} \\
&- .0002\,6786\,7258\,880h_0 + .0011\,2500\,6704\,640i_{-\frac{1}{2}} \\
&+.0000\,4313\,8973\,696h_0 - .0001\,8568\,5147\,648i_{-\frac{1}{2}}
\end{aligned}$$

FORMATION OF SUBDIVIDED TABLE

For each subdivided table the formulas are rigidly evaluated, for the leading differences but once, and for the m central $(m+1)$ th differences for each F . It is to be noted that the original differences are used directly without modification, except a single mean difference when computing the leading differences with the Bessel-Stirling formula. Since the $(m+1)$ th subdivided differences are linear functions of the corresponding original $(m+1)$ th differences, or of the $(m+1)$ th and $(m+2)$ th, depending on the formulas used, they are readily tabulated for repeated use, for each n and for the practical range in the original differences. Evidently n must have the form 2^i5^j , where i and j are integers, in order that the decimals shall be exact.

Adding the $(m+1)$ th differences to the leading m th difference in the well known manner produces the column of m th differences. Similarly, $(m-1)$ th, $(m-2)$ th, . . . differences, and finally the desired (F) 's are produced. The machines best adapted to this work, however, add the quantities in a different order. The many decimal places finally to be discarded present no difficulty unless the capacity of the machines is exceeded, because the machines add numbers of any size within that capacity in the same unit of time. Rounding off is automatically accomplished by the usual device of initially adding 5 units to the leading F in the first decimal place to be neglected.

The exact reproduction of each original F assures the correctness of the work without further check, and thus produces the equivalent of proofread printer's copy.

RELATIVE ADVANTAGES OF THE TWO FORMULAS

From tables of interpolation coefficients or from their formulas, the maximum errors when retaining a given order of differences are found to be:

Differences retained	Gauss formula Error	at $n =$	Bessel-Stirling formula Error	at $n =$
2nd	.06415c	.58	.00802c	.21, .79
3rd	.02344d	.50	.01042d	.71
4th	.01182e	.54	.00087e	.22, .78
5th	.00488f	.50	.00122f	.68
6th	.00245g	.53	.00013g	.22, .78
7th	.00107h	.50	.00019h	.67
8th	.00054i	.52	.00002i	.22, .78

In comparing the relative advantages of the two formulas which attain a given accuracy in the interpolation, two cases arise, the first less frequently than the second:

Case	Formula	Original differences to be retained	Original differences required	Subdivided differences integrated
1	Gauss	m	$m+1$	$m+1$
	Bessel-Stirling	m	$m+2$	$m+1$
2	Gauss	$m+1$	$m+2$	$m+2$
	Bessel-Stirling	m	$m+2$	$m+1$

In the first case the Gauss formula is far superior because the original $(m+2)$ th differences are not required and the subdivided $(m+1)$ th differences are obtained with a single argument. In the second case, the extra time of finding the Bessel-Stirling subdivided differences with a double argument will generally nearly equal or exceed the time of summing the additional order of differences required by the Gauss formula. Hence, the simpler, uniform procedure with the Gauss formula seems best in general.

EXAMPLES.—INTERPOLATION TO 10THS WITH THE GAUSS FORMULA

The quantities used from the original table, and the leading and last differences derived therefrom for the subdivided table, are in boldface. Until the advent of machines that handle positive and negative numbers as such with equal facility, it is more convenient to consider the sign of the function always positive; other negative numbers will ordinarily be printed as comple-

1934140QB.17...65B

ments. The decimals to be neglected in the interpolated functions need not be recorded, automatic rounding off being accomplished by initially adding 5 units in the first neglected decimal.

A. RETAINING 3RD DIFFERENCES

ORIGINAL TABLE						
GCT 1934	JCT 242	$F=\Delta X_{1934}$	a	b	c	d
Jan -8	7430	-81484		-629		
			- 404		+ 2	
-3	35	-81080		-627		+4
			-1031		+ 6	
2	40	-80049		-621		+7
			-1652		+13	
7	45	-78397		-608		-1

The function ΔX_{1934} , in units of the 7th decimal place, added to the solar coordinate X for the equinox 1934.0 gives X for 1900.0. In practice, however, at an almanac office where previous data are available in a form

for immediate use, much better procedure is available; the solar coordinates for one year at $\frac{1}{2}$ -day intervals (2200 values) may be transferred to 1900.0 and the equivalent of proofread copy printed in 5 hours.

The 4th differences (d) may be taken from a table with argument d :

d	$(d)_{11}$	$(d)_0$
-2	+ .0330	-.0680
-1	+ .0165	-.0340
0	.0000	.0000
+1	-.0165	+.0340
+2	-.0330	+.0680
+3	-.0495	+.1020
+4	-.0660	+.1360
+5	-.0825	+.1700
+6	-.0990	+.2040
+7	-.1155	+.2380
+8	-.1320	+.2720

SUBDIVIDED TABLE

GCT	JCT	$(F)=\Delta X_{1934}$	(a)	(b)	(c)	(d)
					+ .0020	
				-6.2720		-.0660
1934	242		- 68.6480		-.0640	
Jan -3.0	7435.0	81080.5000		6.3360		+ .1360
			74.9840		+ .0720	
2.5	35.5	81005.5160		6.2640		-.0660
			81.2480		+ .0060	
2.0	36.0	80924.2680		6.2580		.0
			87.5060		.0060	
1.5	36.5	80836.7620		6.2520		.0
			93.7580		+ .0060	
1.0	37.0	80743.0040		-6.2460		.0
			-100.0040		.0060	
Jan -0.5	7437.5	80643.0000		6.2400		.0
			106.2440		.0060	
0.0	38.0	80536.7560		6.2340		.0
			112.4780		.0060	
0.5	38.5	80424.2780		6.2280		.0
			118.7060		.0060	
1.0	39.0	80305.5720		6.2220		.0
			124.9280		+ .0060	
1.5	39.5	80180.6440		-6.2160		-.1155
			-131.1440		-.1095	
Jan 2.0	7440.0	80049.5000		6.3255		+ .2380
			137.4695		+ .1285	
2.5	40.5	79912.0305		6.1970		-.1155
			143.6665		+ .0130	
3.0	41.0	79768.3640		6.1840		.0
			149.8505		.0130	
3.5	41.5	79618.5135		6.1710		.0
			156.0215		+ .0130	
4.0	42.0	79462.4920		-6.1580		.0
			-162.1795		.0130	
Jan 4.5	7442.5	79300.3125		6.1450		.0
			168.3245		.0130	
5.0	43.0	79131.9880		6.1320		.0
			174.4565		.0130	
5.5	43.5	78957.5315		6.1190		.0
			180.5755		.0130	
6.0	44.0	78776.9560		6.1060		.0
			186.6815		+ .0130	
6.5	44.5	78590.2745		-6.0930		
			-192.7745			
Jan 7.0	7445.0	78397.5000				

B. RETAINING 7TH DIFFERENCES

ORIGINAL TABLE

x	$F=\sinh x$	a	b	c	d	e	f	g	h
0.00	.000000000		0		0		0		0
		+100166750		+1002503		+10031		+110	
0.10	.100166750		+1002503		+10031		+110		-36
		101169253		1012534		10141		74	
0.20	.201336003		2015037		20172		184		+69
		+103184290		+1032706		+10325		+143	
0.30	.304520293		+3047743		+30497		+327		-78

The 8th differences (h) may be taken from a table with argument h :

h	$(h)_{\pm 3}$	$(h)_{\pm 2}$	$(h)_{\pm 1}$	$(h)_0$
-37	+.02606987625	-.15854685000	+.39968061375	-.53441098000
-36	+.02536528500	-.15426180000	+.38887843500	-.51996744000
-35	+.02466069375	-.14997675000	+.37807625625	-.50552390000
+68	-.04791220500	+.29138340000	-.73454815500	+.98216072000
+69	-.04861679625	+.29566845000	-.74535033375	+.99660426000
+70	-.04932138750	+.29995350000	-.75615251250	+1.01104780000

Interpolation including 6th differences is about the practical working limit and, of course, would have amply sufficed in this case.

SUBDIVIDED TABLE

	(F) = sinh z	(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)
									+ .00001100000
							+ .00007700000		+ .02536528500
						+ .10025500000		+ .02537628500	
						+ .80050000000		+ .02545328500	- .15426180000
						+1003.62705477500		.12570828500	- .12888551500
						+9012.23859697500		.92620828500	- .10343223000
						+10045200.83458808750		1004.55326306000	.02227605500
									+ .25999292000
0.10	.100166750.50000000000	10016.79186003500		.94848434000		+ .15656069000		- .51996744000	
		10055217.62644812250	1005.50174740000	.17883674500		- .25997452000			
.11	.110221968.12644812250	11022.29360743500	1.12732108500	- .10341383000		+ .38887843500			
		10066239.92005555750	1006.62906848500	.07542291500		+ .12890391500			
.12	.120288208.04650368000	12028.92267592000	1.20274400000	+ .02549008500		- .15426180000			
		10078268.84273147750	1007.83181248500	+ .10091300000		- .02535788500			
.13	.130366476.88923515750	13036.75448840500	+1.30365700000	.10104520000		+ .02536528500			
		10091305.59721988250	+1009.13546948500			+ .00000740000			
.14	.140457782.48645504000	+14045.88995789000	1.40470220000	+ .00013960000		.0			
		+10105351.4871777250	1010.54017168500	.10118480000		+ .00000740000			
0.15	.150563133.97363281250	15056.43012957500	1.50588700000	+ .00014700000		.0			
		10120407.91730734750	1012.04605868500	.10133180000		+ .00000740000			
.16	.160683541.89094016000	16068.47618826000	1.60721880000	+ .00015440000		.0			
		10136476.39349560750	1013.65327748500	.10148620000		+ .00000740000			
.17	.170820018.28443576750	17082.12946574500	1.70870500000	+ .00016180000		- .04861679625			
		10153558.52296135250	1015.36198248500	+ .10164800000		- .04860939625			
.18	.180973576.80739712000	18097.49144823000	+1.81035300000	- .04844759625		+ .29566845000			
		10171656.01440958250	+1017.17233548500	.05320040375		+ .24705905375			
.19	.191145232.82180670250	+19114.66378371500	1.86355340375	+ .19861145750		- .74535033375			
		+10190770.67819329750	1019.03588888875	+ .25181186125		- .49829128000			
0.20	.201336003.50000000000	20133.69967260375	2.11536526500	- .29967982250		+ .99660426000			
		10210904.37786590125	1021.15125415375	- .04786796125		+ .49831298000			
.21	.211546907.87786590125	21154.85092675750	2.06749730375	+ .19863315750		- .74535033375			
		10232059.22879265875	1023.21875145750	+ .15076519625		- .24703735375			
.22	.221778967.10665856000	22178.06967821500	2.21826250000	- .04840419625		+ .29566845000			
		10254237.29847087375	1025.43701395750	+ .10236100000		+ .04863109625			
.23	.232033204.40512943375	23203.50669217250	+2.32062350000	+ .00022690000		- .04861679625			
		10277440.80516304625	+1027.75763745750	.10258790000		+ .00001430000			
.24	.242310645.21029248000	+24231.26432963000	2.42321140000	+ .00024120000		.0			
		+10301672.06949267625	1030.18084885750	.10282910000		+ .00001430000			
0.25	.252612317.27978515625	25261.44517848750	2.52604050000	+ .00025550000		.0			
		10326933.51467116375	1032.70688935750	.10308460000		+ .00001430000			
.26	.262939250.79445632000	26294.15206784500	2.62912510000	+ .00026980000		.0			
		10353227.66673900875	1035.33601445750	.10335440000		+ .00001430000			
.27	.273292478.46119532875	27329.48808230250	2.73247950000	+ .00028410000					
		10380557.15482131125	1038.06849395750	+ .10363850000					
.28	.283673035.61601664000	28367.55657626000	+2.83611800000						
		10408924.71139757125	+1040.90461195750						
.29	.294081960.32741421125	+29408.46118821750							
		+10438333.17258578875							
0.30	.304520293.50000000000								

APPLICATION OF TABULATING MACHINES

The processes of differencing the original functions and integrating the subdivided differences to form the desired corresponding functions are readily accomplished with the Ellis² and Hollerith³ tabulating machines. The Ellis tabulators have 4-register 9-figure capacity and 6-register 11 to 17-figure capacity. The Hollerith tabulator 1933 model, 10-register 11-figure capacity, is available only in England. These machines add or subtract the contents of any register, or of the Ellis keyboard or Hollerith card, to or from the contents of any other register or registers, except that the two Ellis tabulators subtract in only 1 or 2 registers respectively.

In differencing with the Ellis 6-register machine, an operator may enter 11-figure functions on the keyboard, depress keys in a routine cycle, and print the functions and all differences to 5th order inclusive at the rate of about 240 functions per hour. With the Hollerith 10-register machine an operator may punch 11-figure functions on cards at the rate of about 1000 per hour, pass them automatically through the machine, and print 6th differences at the rate of 1600 per hour.

In integrating with the Ellis machine an operator may enter 6th differences on the keyboard, depress keys in a routine cycle, and print all differences and the functions at the rate of about 360 functions per hour. With the Hollerith a sufficient stock of cards for repeated use is punched for integrating from m th subdivided differences, and filed in groups of n cards if interpolating to n ths, $n-m$ cards in each group being blank. An operator can assemble groups or sort them after use, at the rate of about 1200 per hour, pass them automatically

² Comrie, *Modern Babbage Machines: Office Machinery Users' Association Bulletin*, London (1932).

³ Comrie, *Hollerith and Powers Tabulating Machines*, London, 1933.

through the machine, and print functions from their 10th differences at the rate of 4000 per hour. When the 11-figure capacity of the registers is exceeded, they can probably be connected in pairs to make five 21-figure registers; then integrating from higher than 5th differences takes two steps, at the rate of 840 functions per hour. Even this much slower speed is faster than a copyist can merely copy the functions.

The earlier Hollerith models³, 3 to 7-register 9 to 10-figure capacity, are not adapted to differencing. For integrating, the registers can be connected into one of 64-figure capacity and a "summary" card punch attached. Then $(m-1)$ th differences are formed by progressively adding the m th differences to the leading $(m-1)$ th difference, and automatically printing and/or punching on cards for use in the next summation. Summation proceeds at the rate of 1.5 per individual difference plus 0.1 for punching each digit, far slower than with the 1933 model. (An obvious improvement is to have the punching of the entire number simultaneous, just as the printing is.) In this method it is more convenient to use the leading differences down the diagonal. The formulas may be found from the corresponding ones above by substituting original differences with subscript " $\frac{1}{2}$ " for those with subscript " $-\frac{1}{2}$," and changing the signs of all even terms.

NOTE.—After the appearance of the preliminary article in *Lick Observatory Bulletin* Number 455, Dr. Comrie of the British Nautical Almanac Office communicated the fact that he was using essentially the same method, but was withholding publication until he had had about a year's experience with it. The writer accordingly has been holding the present article in abeyance. But since the method has probably not been developed in just the same way, and in view of the elapsed time, and of the possibility that others may wish to put it to immediate use or work upon possible improvements, the writer now deems it desirable to publish it.

ERRATUM.—*Lick Obs. Bull.* 16, 143, col. 2, line 5, for +.0100 b₀, read +.0100 b₁.

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