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THE BINARY ORBIT WHOSE PLANE IS PERPENDICULAR TO THE
TANGENTIAL PLANE OF PROJECTION

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This is the special case of which 42 *Comae Berenices* is an example. This type of orbit has been solved very neatly by Laves.¹ He makes use of a simple geometric relation to derive the equations for a and ω . The object of this paper is to derive these elements by another equally simple geometric relation and to put the final equations into a somewhat simpler form. This method is much shorter than that of Henroteau.² For the sake of completeness this new work will be combined here with that of Aitken,³ so as to have the whole theory in one place.

The revolution period can then be read directly from the curve. In general, the elements e , the eccentricity, and T , the time of periastron passage, are easily determined.

In the figure let the ellipse represent the true orbit. Let the primary be at the focus S , and let C be the center of the ellipse. Let the Earth be in the direction indicated, and $T' C' T_1$ be the projected orbit. The point C' is known for $T' C'$ must equal half the amplitude of the curve of distances. The point S' is the projected position of the primary and hence is known.

Figure 1. Apparent and True Orbits and Interpolating Curve of Observed Distances.

The element $i=90^\circ$. The element Ω is known for this is the mean of the observed position angles. The remaining elements must be determined from the curve representing the observed distances. This is obtained by plotting the times as abscissae and the corresponding observed distances as ordinates and drawing the most probable smooth curve to represent the plotted points.

¹ *A. J.* **37**, 97, 1926.

² *Handbuch der Astrophysik* **6**, 338, 1928.

³ Aitken: *The Binary Stars*, pp. 95-96.

The points on the curve of distances which correspond to the points P and P_1 of the true orbit must be separated by exactly half of the revolution period and their distances from the line $C'CE$ must be equal in length and of opposite sign. The point corresponding to periastron, P , must lie on the same side of this line as S' , and on the steeper branch of the curve as the rate of change of distance is greater near periastron than near apastron. In practice these two points are readily found by cutting a rectangular slip of paper to a width

equal to half that of the period on the adopted scale and sliding it along the curve until the edges, kept perpendicular to the line C'CE, cut equal ordinates (with respect to C'CE) on the curve. Laves cleverly suggests "draw the central axis in the graph and make a tracing of the time graph of distances, turn it through 180° about the central time-axis and then advance the tracing by half the period of the orbit. Of the four points of intersection of graph and tracing it is easy to find that pair which is separated by half the period." In either of these ways the positions of the points P' and P₁ are obtained.

Since ratios are not altered in the projection we have

$$e = \frac{C'S'}{C'P'} \quad (1)$$

The remaining elements a and ω are simply derived. Laves does this by making use of a theorem relating to perpendiculars dropped from the foci of an ellipse on the tangent to the ellipse. Here a relation is used involving the perpendicular from the center of the ellipse on a tangent to the ellipse. In the figure let CH be such a perpendicular and call its length p . Let ω (which is here our element ω) be the angle the tangent line makes with the minor axis. Then, if a and b are the major and minor semi-axes respectively, from the properties of the conic it can be shown that

$$p^2 = a^2 \cos^2 \omega + b^2 \sin^2 \omega \quad (2)$$

which can be written

$$p^2 = a^2 - (a^2 - b^2) \sin^2 \omega = a^2 - a^2 e^2 \sin^2 \omega \quad (3)$$

But from the figure

$$CK \equiv m = \overline{CS} \cos \omega = ae \cos \omega$$

$$\text{Hence } \cos \omega = \frac{m}{ae} \quad (4)$$

and

$$\sin^2 \omega = 1 - \frac{m^2}{a^2 e^2} \quad (5)$$

Substituting (5) into (3) and solving for a we have

$$a = \sqrt{\frac{p^2 - m^2}{1 - e^2}} \quad (6)$$

p is C'T', and m is C'S' both of which are known.

The order of solution is, then, equations (1), (6), and (4), which give e , a , and ω respectively.

By a simple transformation Laves' (2) may be put into the same form as (6), which is slightly simpler. Equation (4) is decidedly simpler than his (3) from which ω is determined.

In the *special case* $\omega = 90^\circ$, the points C', P', and S' coincide (this means that the major axis is pointed toward the Earth) and (1) becomes indeterminate. Further, in this case $p = b$ and $m = 0$, Hence from (6)

$$a = \sqrt{\frac{b^2}{1 - e^2}} \quad (7)$$

To determine e , let τ be the ratio of the intervals of time from elongation to elongation such that $\tau < 1$. Then from the law of areas we have τ equal to the ratio of the area described by the radius vector in the true orbit in the shorter interval to the area described by the radius vector in the longer interval, or

$$\tau = \frac{\frac{1}{2} \pi a b - a b e}{\frac{1}{2} \pi a b + a b e} = \frac{\frac{1}{2} \pi - e}{\frac{1}{2} \pi + e}$$

$$\text{whence } e = \frac{\pi}{2} \cdot \frac{1 - \tau}{1 + \tau} \quad (8)$$

If the intervals are equal then $\tau = 1$ and from (8) $e = 0$ and the orbit is a circle.

The periastron passage time, T, (in the case of the ellipse) is the middle instant of the shorter interval between elongation times.

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